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FROM AMPLITUDES TO FORM FACTORS IN THE $\mathcal{N}$=4 SYM THEORY

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We discuss the latest progress in understanding the infrared behavior of inclusive cross sections and the form factors for the so-called half-BPS operators in the $\mathcal{N}$=4 SYM theory. In both cases, we observe the exponentiation of the infrared divergences while the finite parts do not have such a simple behavior. We show how the infrared divergences cancel in inclusive cross sections when the emission of soft and collinear quanta is taken into account. We calculate the finite parts of the form factors in the two-loop approximation. They are rather complicated and are expressed in terms of generalized polylogarithms of several variables. But the principle of maximal transcendentality is nevertheless still satisfied.

Keywords: $\mathcal{N}$=4 maximally supersymmetric Yang–Mills theory, form factor, $\mathcal{N}$=1 superspace, infrared-finite observable, maximally helicity-violating amplitude

1. Introduction

In the past few years, much attention has been given to studying the planar limit for the scattering amplitudes in the $\mathcal{N}$=4 supersymmetric Yang–Mills (SYM) theory. It is believed that the hidden symmetries responsible for integrability properties of the $\mathcal{N}$=4 SYM theory completely fix the structure of the amplitudes (the S-matrix of the theory) [1]. In particular, this is manifested in the fact that the answers for the amplitudes are expressed in terms of pseudoconformal integrals in the momentum space [2]. Moreover, the so-called dual conformal symmetry in the weak-coupling regime can be extended to the $\mathcal{N}$=4 supersymmetric version and together with the usual $\mathcal{N}$=4 superconformal symmetry forms the so-called Yangian symmetry, which is governed by the Yangian infinite-dimensional algebra. Another remarkable property of the considered amplitudes is the amplitude/Wilson loop duality [3], [4].

In the strong-coupling regime, the amplitudes in the $\mathcal{N}$=4 SYM theory can be obtained using the AdS/CFT correspondence by computing the open-string scattering amplitudes in the AdS_5 space with strings ending on the D3 brane positioned at some fixed value $z \neq 0$ of the radial AdS_5 coordinate in the semiclassical regime [3]. This problem can be reformulated as the problem of finding the minimum surface in the AdS_5 space with a special boundary condition and was recently reduced to solving a set of functional equations for conformally invariant cross ratios as functions of the spectral parameters, the so-called Y-systems [5]. The Y-systems usually appear in integrable systems, and this is another hint that the $\mathcal{N}$=4 amplitudes contain an integrable structure.

The hidden symmetries of the amplitudes of the theory in the planar limit[1] were first established for

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[1]The planar limit corresponds to the case where the gauge coupling constant $g \to 0$ and the rank of the gauge group $N_c \to \infty$ such that the t'Hooft coupling constant $\lambda = g^2 N_c/16\pi^2$ remains fixed.

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the so-called maximally helicity-violating (MHV) amplitudes,[2] which are color-ordered and defined through the group structure decomposition

$$\mathcal{A}_n^{1-\text{loop}} = g^{n-2}\lambda^l \sum_{\text{perm}} \text{Tr}\big(T^{a_{\rho(1)}}\cdots T^{a_{\rho(n)}}\big) A_n^{(l)}\big(p_{\rho(1)},\ldots,p_{\rho(n)}\big),$$

where $\mathcal{A}_n$ are the physical n-point amplitudes, A_n are the color-ordered amplitudes, $T^{a(i)}$ are the generators of the gauge group $SU(N_\text{c})$, $a_{\rho(i)}$ is the color index of the $\rho(i)$th external particle, $p_{\rho(i)}$ is its momentum, and $\lambda = g^2 N_\text{c}/16\pi^2$. These amplitudes are ultraviolet (UV) finite but have infrared (IR) divergences. Bern, Dixon, and Smirnov (BDS) recently formulated an ansatz [6] for the all-loop n-point MHV amplitudes, which was confirmed at the three-loop level for the four-point amplitude and in the framework of dimensional regularization has the form

$$\begin{aligned}\mathcal{M}_n \equiv \frac{A_n}{A_n^{\text{tree}}} &= 1 + \sum_{L=1}^{\infty} \lambda^L M_n^{(L)}(\epsilon) = \\ &= \exp\bigg[-\frac{1}{8}\sum_{l=1}^{\infty}\lambda^l\bigg(\frac{\gamma_{\text{cusp}}^{(l)}}{(l\epsilon)^2} + \frac{2G_0^{(l)}}{l\epsilon}\bigg)\sum_{i=1}^{n}\bigg(\frac{\mu^2}{-s_{i,i+1}}\bigg)^{l\epsilon} + \frac{1}{4}\sum_{l=1}^{\infty}\lambda^l\gamma_{\text{cusp}}^{(l)}F_n^{(1)}(0) + C(g)\bigg],\end{aligned}$$

where $\gamma_{\text{cusp}}(g) = \sum_l \lambda^l \gamma_{\text{cusp}}^{(l)}$ is the so-called cusp anomalous dimension [7] and $G_0(g) = \sum_l \lambda^l G_0^{(l)}$ is the coupling constant function depending on the IR regularization. These functions completely define the IR behavior of the amplitude.

It is unsurprising that all IR divergences of the amplitudes are factorable and exponentiate [8]. It is much less obvious that this is also true for the finite part. According to the BDS ansatz, the finite part of the amplitude is defined by the cusp anomalous dimension and a function of kinematic parameters specified at the one-loop level. For a four-gluon amplitude, we have

$$F_4^{(1)}(0) = \frac{1}{2}\log^2\bigg(\frac{-t}{s}\bigg) + 4\zeta_2. \tag{1}$$

For $n = 4, 5$, the BDS ansatz passed several nontrivial tests: the amplitudes were calculated up to four loops for four gluons and up to three loops for five gluons. But it fails starting from $n = 6$, although the abovementioned duality with the Wilson loop still holds. The finite part $\text{Fin}[\log\mathcal{M}_n]$ for the four- and five-point amplitudes is totally fixed by the dual conformal symmetry; it is a solution of the anomalous Ward identities for this symmetry [9],

$$\sum_{i=1}^{n}(2x_i^\nu x_i\partial_i - x_i^2\partial_i^\nu)\text{Fin}[\log\mathcal{M}_n] = \frac{1}{2}\gamma_{\text{cusp}}\sum_{i=1}^{n}\log\frac{x_{i,i+2}^2}{x_{i-1,i+1}^2}x_{i,i+1}^\nu,$$

where $x_{i,i+1}^\mu = x_i^\mu - x_{i+1}^\mu = p_i^\mu$. For a number of legs greater than five, the solution of the anomalous Ward identities in addition to the BDS terms contains a function of cross ratios of kinematic variables. The exact form of this function is still unknown.

While all the UV divergences in the $\mathcal{N}$=4 SYM theory are absent from the scattering amplitudes, the IR divergences remain but presumably cancel in properly defined physical observables. Regularized expressions thus serve as a kind of scaffolding that should be absent from the final answer. Just the

[2] The MHV amplitudes are the amplitudes where all particles are outgoing and the total helicity is equal to $n-4$, where n is the number of particles. For gluon amplitudes, MHV amplitudes are defined as the amplitudes in which all but two gluons have positive helicities.

physical observables are the goal of our calculation. Although the Kinoshita–Lee–Nauenberg theorem [10] in principle tells how to construct such observables, this procedure is not simple to realize explicitly, and various possibilities can be considered. In particular, we can consider the so-called energy flow functions defined in terms of the energy–momentum tensor correlators considered in the weak-coupling regime in [11] and in the strong-coupling regime in [12]. We concentrated on the inclusive cross sections [13], [14], hoping that they retain the factorization properties present in the amplitudes. Similar questions were discussed in [15], where IR-safe observables of the type of the inclusive cross section in the $\mathcal{N}$=4 SYM theory were constructed. Here, we consider the IR structure of the MHV gluon amplitudes in the planar limit for the $\mathcal{N}$=4 SYM theory in the next-to-leading-order approximation. We show the explicit cancellation of the IR divergence in properly defined inclusive cross sections in the $\mathcal{N}$=4 SYM theory theory and calculate the finite parts of the amplitudes analytically. Unfortunately, in contrast to the virtual corrections, the finite parts do not reveal any simple structure and are not obviously factorable.

In the strong-coupling regime, the natural generalization of the Y-system for the amplitudes is the Y-system for the form factors [16], i.e., the matrix elements of the form

$$\langle 0|\mathcal{O}|p_1^{\lambda_1}, \dots, p_n^{\lambda_n}\rangle, \tag{2}$$

where $\mathcal{O}$ is a gauge-invariant operator acting on the vacuum and producing some state $|p_1^{\lambda_1}, \dots, p_n^{\lambda_n}\rangle$ with the momenta $p_1, \dots, p_n$ and helicities $\lambda_1, \dots, \lambda_n$. In the dual string theory, this matrix element can be described using the amplitudes of strings with ends on the $D3$ brane positioned at some fixed value of the radial AdS$_5$ coordinate z in the presence of closed strings [16].

It is interesting to understand whether these objects in the weak-coupling region have the same features as the amplitudes or, in other words, whether form factors are governed by the Yangian symmetry (or its analogue) and whether they are determined by it. In [17], we established that up to the second order of the perturbation theory, all the answers are expressed in terms of the pseudoconformal integrals, and this hints at a hidden integrability.

Inspired by the two-loop calculation of the form factor associated with the operator $\mathcal{V}_X$ from the energy–momentum tensor superconformal multiplet of the $\mathcal{N}$=4 SYM theory performed long ago by van Neerven [15], we systematically study the simplest types of form factors in the planar $\mathcal{N}$=4 SYM theory in the weak-coupling regime for the half-BPS operators $\mathcal{O}_I^{(n)}$ and the Konishi operator $\mathcal{K}$. Our results resemble the answers obtained for the amplitudes. In particular, the form factor for the operator with three legs is similar to the six-point amplitude, and so on. Similar problems were discussed in [18] using the unitary-based technique and extending the amplitude/Wilson loop duality to the case with form factors.

2. IR-safe observables in the $\mathcal{N}$=4 SYM theory

Our aim is to evaluate the next-to-leading-order correction to the inclusive differential cross section of polarized scattering in the weak-coupling regime in the planar limit of the $\mathcal{N}$=4 SYM theory in an analytic form and to obtain the cancellation of the IR divergences.

We start with the 2→2 MHV scattering amplitude with two incoming positively polarized gluons and two outgoing positively polarized gluons and consider the differential cross section $d\sigma_{2\to2}(g^+g^+ \to g^+g^+)/d\Omega$ as a function of the scattering solid angle. The total cross section is divergent at the angle zero. Treating all the particles as outgoing in this amplitude, we let $(--++)$ denote the MHV amplitude. At the tree level, the cross section is given by

$$\frac{d\sigma_{2\to2}}{d\Omega_{13}} = \frac{1}{J}\int d\phi_2\, |\mathcal{M}_4^{(\mathrm{tree})}|^2 \mathcal{S}_2, \tag{3}$$

where J is the flux factor ($J = s$ in our case), s is the standard Mandelstam variable, and the phase volume of a two-particle process (we the use dimensional regularization $D = 4 - 2\epsilon$) is

$$d\phi_2 = \frac{d^D p_3\, \delta^+(p_3^2)}{(2\pi)^{D-1}} \frac{d^D p_4\, \delta^+(p_4^2)}{(2\pi)^{D-1}} (2\pi)^D \delta^D(p_1 + p_2 - p_3 - p_4).$$

In accordance with (3), $\mathcal{S}_n$ ($n = 2$ in this case) is the so-called measurement function and specifies the detected state. In this particular case,

$$\mathcal{S}_2 = \delta_{+,h_3} \delta^{D-2}(\Omega_{\text{Det}} - \Omega_{13}),$$

where $\delta^{D-2}(\Omega_{\text{Det}} - \Omega_{13})$ means that our observable is the differential cross section $d\sigma_{2\to2}/d\Omega_{13}$, $d\Omega_{13} = d\phi_{13}\, d\cos\theta_{13}$, θ_{13} is the scattering angle of the particles with the momenta $\mathbf{p}_3$ with respect to $\mathbf{p}_1$ in the center-of-mass frame, and δ_{+,h_3} means that we detect a particle with positive helicity.[3] The matrix element is obtained from the color-ordered amplitudes by summation

$$|\mathcal{M}_4^{(\text{tree})}|^2 = g^4 \sum_{\text{colors}} |\mathcal{A}_4^{(\text{tree})}|^2 = g^4 N_{\text{c}}^2 (N_{\text{c}}^2 - 1) \sum_{\sigma \in P_3} \left| A_4^{(\text{tree})}\big(p_1, p_{\sigma(1)}, \ldots, p_{\sigma(3)}\big) \right|^2,$$

where P_n is the set of all permutations of n objects ($n = 3$ in this case), and hence

$$|\mathcal{M}_4^{(\text{tree})(--++)}|^2 = g^4 N_{\text{c}}^2 (N_{\text{c}}^2 - 1) \sum_{\sigma \in P_3} \frac{s_{12}^4}{s_{1\sigma(1)} s_{\sigma(1)\sigma(2)} s_{\sigma(2)\sigma(3)} s_{\sigma(3)1}}, \tag{4}$$

where we take $s_{ij} = (p_i + p_j)^2$ in all expressions.

In dimensional regularization (reduction), the cross section is

$$\left(\frac{d\sigma_{2\to2}}{d\Omega_{13}}\right)_0^{(--++)} = \frac{\alpha^2 N_{\text{c}}^2}{2E^2} \left(\frac{s^4}{t^2 u^2} + \frac{s^2}{t^2} + \frac{s^2}{u^2}\right) \left(\frac{\mu^2}{s}\right)^{\epsilon} = \frac{\alpha^2 N_{\text{c}}^2}{E^2} \left(\frac{\mu^2}{s}\right)^{\epsilon} \frac{4(3 + c^2)}{(1 - c^2)^2},$$

where s, t, and u are the Mandelstam variables, E is the total energy in the center-of-mass frame, $\alpha = g^2 N_{\text{c}}/4\pi$, $c = \cos\theta_{13}$, and μ and ϵ are the parameters of the dimensional regularization (reduction). In the center-of-mass frame, the Mandelstam variables satisfy the usual relations $s = E^2$, $t = -(E^2/2)(1 - c)$, and $u = -(E^2/2)(1 + c)$.

The next step is to calculate the next-to-leading-order corrections, which includes calculating the virtual and real parts together with the splitting counterterms, which appear because of the indistinguishability of the collinear particles in the initial and final states.

Virtual part. We start with the virtual contribution. To obtain the one-loop contribution to the differential cross section, we use the already known expression for the one-loop color-ordered amplitude

$$M_4^{1-\text{loop}}(\epsilon) = \frac{A_4^{1-\text{loop}}}{A_4^{(\text{tree})}} = -\frac{1}{2} st\, I_4^{1-\text{loop}}(s, t),$$

where $I_4^{(1)}(s, t)$ is the scalar "box" diagram

$$I_4^{1-\text{loop}}(s, t) = -\frac{2}{st} \frac{\Gamma(1 + \epsilon)\Gamma(1 - \epsilon)^2}{\Gamma(1 - 2\epsilon)} \left[\frac{1}{\epsilon^2} \left(\left(\frac{\mu^2}{s}\right)^{\epsilon} + \left(\frac{\mu^2}{-t}\right)^{\epsilon}\right) + \frac{1}{2} \log^2\left(\frac{s}{-t}\right) + \frac{\pi^2}{2}\right] + O(\epsilon).$$

[3] More accurately, in dimensional regularization (reduction), we have $d\Omega_{13}^{\epsilon} = d\phi_{13} \sin\phi_{13}^{-2\epsilon}\, d\cos\theta_{13}\, \sin\theta_{13}^{-2\epsilon}$.

The square of the matrix element ($|\mathcal{M}_4^{1-\text{loop}}|^2 = \sum_{\text{colors}}(\mathcal{A}_4^{(\text{tree})}\mathcal{A}_4^{1-\text{loop}*} + \text{c.c.})$) is

$$|\mathcal{M}_4^{1-\text{loop}(--++)}|^2 = -g^4 N_c^2(N_c^2-1)\left(\frac{g^2 N_c}{16\pi^2}\right)\times$$
$$\times\left[\frac{s^4}{s^2t^2}stI_4^{1-\text{loop}}(s,t) + \frac{s^4}{s^2u^2}suI_4^{1-\text{loop}}(s,u) - \frac{s^4}{t^2u^2}tuI_4^{1-\text{loop}}(-t,u)\right]. \qquad (5)$$

Real emission. The next step is to calculate the amplitude with three outgoing particles. Here, we must define which is the process of interest. There are several possibilities for three particles in the final state:

1. Three gluons with positive helicities, $g^+g^+ \to g^+g^+g^+$: this is the MHV amplitude.
2. Two gluons with positive helicities and the third gluon with negative helicity, $g^+g^+ \to g^+g^+g^-$: this is the anti-MHV amplitude.[4]
3. One of three final particles is the gluon with positive helicity and the other two are the quark-antiquark pair, $g^+g^+ \to g^+q^-\bar{q}^+$ or $g^+g^+ \to g^+q^+\bar{q}^-$: this is an anti-MHV amplitude.[5]
4. One of three final particles is the gluon with positive helicity and the other two are scalars, $g^+g^+ \to g^+\Lambda\Lambda$: this is an anti-MHV amplitude.

If we fix one gluon with positive helicity scattered at the angle θ and sum over all the other particles, then all the abovementioned processes contribute. In the case where we fix two gluons with positive helicity and seek the rest, only the first two processes are allowed.

The cross section of these processes can be written as

$$\frac{d\sigma_{2\to3}}{d\Omega_{13}} = \frac{1}{J}\int d\phi_3\, |\mathcal{M}_5^{(\text{tree})}|^2 \mathcal{S}_3, \qquad (6)$$

where $d\phi_3$ is the three-particle phase volume. It can be written in a form more convenient for our calculations,

$$d\phi_3 = \frac{d^D p_3\, \delta^+(p_3^2)}{(2\pi)^{D-1}}\frac{d^D p_4\, \delta^+\big((p_4-k)^2\big)}{(2\pi)^{D-1}}\frac{d^D k\, \delta^+(k^2)}{(2\pi)^{D-1}}(2\pi)^D\delta^D(p_1+p_2-p_3-p_4),$$

and $\mathcal{S}_3$ is the measurement function that constrains the phase space and defines a particular observable. To simplify the integration in what follows, we choose the universal measurement function

$$\mathcal{S}_3(p_3,p_4,p_5) = \Theta\left(p_3^0 - \frac{1-\delta}{2}E\right)\delta^{D-2}(\Omega_{\text{Det}}-\Omega_3), \qquad (7)$$

where we take $\delta = 1/3$ in the case of identical particles and $\delta = 1$ in the other cases. Therefore, the registration of one fastest gluon corresponds to $\delta = 1/3$ for the MHV and anti-MHV amplitudes and $\delta = 1$ for the matter–antimatter amplitude, while the registration of two fastest gluons corresponds to $\delta = 1/3$ for the MHV amplitude and $\delta = 1$ for the anti-MHV amplitude.[6] We verified that the IR and collinear divergences cancel in observables for any value of δ.

[4]There is also a $g^+g^+ \to g^+g^-g^+$ helicity configuration, but the amplitude in this case is the same. We let $(--++-)$ denote both configurations.

[5]The $\mathcal{N}$=4 supermultiplet consists of a gluon g, four fermions (quarks) q^A, and six real scalars Λ^{AB}, where A and B are $SU(4)_R$ indices and Λ is an antisymmetric tensor. It is implied that all squared amplitudes with quarks and scalars are summed over these indices.

[6]These are not precisely the needed requirements but are pretty close to them. Satisfying the exact requirements for the fastest particles is technically more involved but does not change the general picture.

Splitting. Taking the emission of additional soft quanta into account allows canceling the IR divergences (double poles in ϵ), but the single poles originating from collinear divergences remain. Indeed, in the case of massless particles, the asymptotic states (both the initial and final states) are ill defined because massless quanta can split into two parallel quanta indistinguishable from the original one. To take this into account, we introduce the notion of distribution of the initial particle (gluon) with respect to the fraction of the carried momentum z: $g(z)$. Then the initial distribution corresponds to $g(z) = \delta(1-z)$, and the emission of a gluon leads to a splitting: the gluon carries the fraction of momentum equal to z, while the collinear gluon carries $1-z$. The probability of this event is given by the so-called splitting functions $P_{gg}(z)$ [19]. For a final-state gluon, this corresponds to the fragmentation of the gluon into a pair of gluons or quarks or scalars.

Additional contributions from collinear particles in the initial and final states to inclusive cross sections have the respective forms

$$d\sigma_{2\to 2}^{\text{In Split}} = \frac{\alpha}{2\pi}\frac{1}{\epsilon}\left(\frac{\mu^2}{Q_{\text{f}}^2}\right)^{\epsilon}\int_0^1 dz\, P_{gg}(z) \sum_{\substack{i,j=1,2,\\ i\neq j}} d\sigma_{2\to 2}(zp_i, p_j, p_3, p_4)\, \mathcal{S}_2^{\text{In Split}}(z) \tag{8}$$

and

$$d\sigma_{2\to 2}^{\text{Fn Split}} = \frac{\alpha}{2\pi}\frac{1}{\epsilon}\left(\frac{\mu^2}{Q_{\text{f}}^2}\right)^{\epsilon} d\sigma_{2\to 2}(p_1, p_2, p_3, p_4)\int_0^1 dz \sum_{l=g,q,\Lambda} P_{gl}(z)\mathcal{S}_2^{\text{Fn Split}}(z), \tag{9}$$

where the scale Q_{f}^2, sometimes called the factorization scale, belongs to the definition of the coherent asymptotic state and restricts the value of transverse momenta. The dependence of the parton distribution on Q_{f}^2 is governed by the DGLAP equation [19], [20]. The splitting function P_{ij} for each helicity configuration can be obtained as a collinear limit of the corresponding partial amplitude.

Final result. In the next-to-leading-order approximation, there are two sets of amplitudes (MHV and anti-MHV amplitudes) that contribute to the observables. The leading-order four-gluon amplitude is both MHV and anti-MHV, and we split it into two parts. We can then construct three types of IR-safe quantities in the next-to-leading order:

1. the purely gluonic MHV amplitude

$$A^{\text{MHV}} = \frac{1}{2}\left(\frac{d\sigma_{2\to 2}}{d\Omega_{13}}\right)^{(--++)}_{\text{Virt}} + \left(\frac{d\sigma_{2\to 3}}{d\Omega_{13}}\right)^{(--+++)}_{\text{Real}} + \left(\frac{d\sigma_{2\to 3}}{d\Omega_{13}}\right)^{(--+++)}_{\text{In Split}} + \left(\frac{d\sigma_{2\to 3}}{d\Omega_{13}}\right)^{(--+++)}_{\text{Fn Split}},$$

2. the purely gluonic anti-MHV amplitude

$$B^{\text{antiMHV}} = \frac{1}{2}\left(\frac{d\sigma_{2\to 2}}{d\Omega_{13}}\right)^{(--++)}_{\text{Virt}} + \left(\frac{d\sigma_{2\to 3}}{d\Omega_{13}}\right)^{(--++-)}_{\text{Real}} + \left(\frac{d\sigma_{2\to 3}}{d\Omega_{13}}\right)^{(--++-)}_{\text{In Split}} + \left(\frac{d\sigma_{2\to 3}}{d\Omega_{13}}\right)^{(--++-)}_{\text{Fn Split}},$$

3. the anti-MHV amplitude with fermions or scalars, which together with gluons form the full $\mathcal{N}=4$ supermultiplet,

$$C^{\text{Matter}} = \left(\frac{d\sigma_{2\to 3}}{d\Omega_{13}}\right)^{(ggg, q\bar{q}+\Lambda\Lambda)}_{\text{Real}} + \left(\frac{d\sigma_{2\to 3}}{d\Omega_{13}}\right)^{(ggg, q\bar{q}+\Lambda\Lambda)}_{\text{In Split}}.$$

We again stress that in each expression, *all IR divergences cancel* for arbitrary δ and only the finite part remains.

Defining the physical condition for the observation, we now obtain several IR-safe inclusive cross sections:

1. registration of *two fastest* gluons of positive helicity

$$A^{\mathrm{MHV}}\big|_{\delta=1/3} + B^{\mathrm{antiMHV}}\big|_{\delta=1}, \tag{10}$$

2. registration of *one fastest* gluon of positive helicity

$$A^{\mathrm{MHV}}\big|_{\delta=1/3} + B^{\mathrm{antiMHV}}\big|_{\delta=1/3} + C^{\mathrm{Matter}}\big|_{\delta=1}, \tag{11}$$

3. the anti-MHV cross section

$$B^{\mathrm{antiMHV}}\big|_{\delta=1} + C^{\mathrm{Matter}}\big|_{\delta=1}. \tag{12}$$

The relative simplicity of virtual contribution (1), which contains logarithms and no other special functions, suggests a similar structure of the real part. But this is not the case. While the singular terms are sufficiently simple and cancel completely, the finite parts are usually cumbersome and contain polylogarithms. The only expression where they cancel corresponds to the δ=1 case, which is possible only for the last set of observables, namely, for anti-MHV cross section (12). Choosing $Q_{\mathrm{f}} = E$ as the factorization scale, we obtain

$$\begin{aligned}
\left(\frac{d\sigma}{d\Omega_{13}}\right)_{\mathrm{antiMHV}} = \frac{4\alpha^2 N_{\mathrm{c}}^2}{E^2}\Bigg\{ & \frac{3+c^2}{(1-c^2)^2} - \\
& - \frac{\alpha}{4\pi}\Bigg[2\frac{(c^4+2c^3+4c^2+6c+19)\log^2\big((1-c)/2\big)}{(1-c)^2(1+c)^4} + \\
& + 2\frac{(c^4-2c^3+4c^2-6c+19)\log^2\big((1+c)/2\big)}{(1-c)^4(1+c)^2} - \\
& - 8\frac{(c^2+1)\log\big((1+c)/2\big)\log\big((1-c)/2\big)}{(1-c^2)^2} + \frac{6\pi^2(3c^2+13)-5(61c^2+99)}{9(1-c^2)^2} - \\
& - 2\frac{(11c^3-31c^2-47c-133)\log\big((1-c)/2\big)}{3(1+c)^3(1-c)^2} + \\
& + 2\frac{(11c^3+31c^2-47c+133)\log\big((1+c)/2\big)}{3(1-c)^3(1+c)^2}\Bigg]\Bigg\}.
\end{aligned}$$

It can be seen that even this expression does not repeat the form of the Born amplitude and lacks a simple structure. The reason for this might be that in constructing the IR-finite observable, we mix the MHV and anti-MHV amplitudes and thus lose the fine properties of the former. Another reason might be that the MHV amplitudes themselves for a number of legs exceeding five lack the exponentiation property for the finite parts.

3. Form factors

We now consider the Lagrangian $\mathcal{L}_{\mathcal{N}=4}(\mathcal{W})$ for the $\mathcal{N}$=4 SYM theory theory coupled to an external classical current J by a gauge-invariant local operator $\mathcal{O}[\mathcal{W}]$,

$$\mathcal{L}_{\mathcal{N}=4}(\mathcal{W}) \to \mathcal{L}_{\mathcal{N}=4}(\mathcal{W}) + \mathcal{O}[\mathcal{W}]J,$$

where $\mathcal{W}$ denotes the whole $\mathcal{N}$=4 on-shell multiplet. We tacitly assume the planar limit in the further calculations.

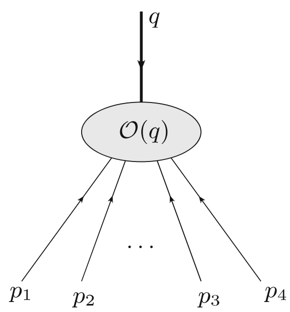

Fig. 1. The Feynman diagram for the form factor with the operator $\mathcal{O}$.

We consider process (2), where $\mathcal{O}$ acts on the vacuum and produces some state $|p_1^{\lambda_1}, \ldots, p_n^{\lambda_n}\rangle$ with the momenta $p_1, \ldots, p_n$ and helicities $\lambda_1, \ldots, \lambda_n$. It is shown schematically in Fig. 1. In the language of the dual string theory, this process can be described as an insertion of some closed-string state (corresponding to the local operator $\mathcal{O}$) on the worldsheet in addition to n open-string states (corresponding to the state $|p_1^{\lambda_1}, \ldots, p_n^{\lambda_n}\rangle$ in the dual theory).

For further calculations, we consider the set of gauge-invariant operators $\mathcal{C}_{IJ} = \mathrm{Tr}(\phi_I \phi_J)$ and $\mathcal{V}_I^J = \mathrm{Tr}(\bar{\phi}^J \phi_I)$ with the naive mass dimension $\Delta_0 = 2$, which coincides with the conformal dimension because of the absence of quantum corrections. These operators can be viewed as the lowest members of the energy–momentum tensor multiplet

$$T^{AB} = \mathrm{Tr}\left(W^A W^B - \frac{1}{6}\delta^{AB} W^C W_C\right),$$

where $A, B, C = 1, \ldots, 6$ are the $SO(6)_R \simeq SU(4)_R$ indices, $I, J = 1, 2, 3$ are the indices of the $SU(3)$ subgroup of $SU(4)_R$, and W^A is some constrained chiral superfield in the $\mathcal{N}$=4 superspace containing all physical fields of the $\mathcal{N}$=4 supermultiplet.

The other set of operators that we consider are the half-BPS operators $\mathcal{O}_I^{(n)} = \mathrm{Tr}(\phi_I^n)$, whose naive mass dimension coincides with the conformal dimension $\Delta_0 = n$, being protected from the quantum corrections, and the lowest component of the Konishi supermultiplet

$$\mathcal{K} = \sum_{I=1}^{3} \mathrm{Tr}(\bar{\phi}^I \phi_I)$$

with the naive mass dimension $\Delta_0 = 2$ and a nonzero anomalous dimension because of the presence of UV divergences. The mass dimension of the Konishi operator has radiation corrections, and the corresponding form factors consequently contain UV divergences. This means that we must consider the renormalized form factor $\langle 0|\mathcal{K}_\mathrm{R}|p_1^{\lambda_1}, \ldots, p_n^{\lambda_n}\rangle$, where $\mathcal{K}_\mathrm{R} = Z_K^{-1}\mathcal{K}_B$. Here, Z_K is the renormalization constant, which appears because of the UV divergences and should be calculated to the same order of the perturbation theory as the form factors. After such a UV renormalization, only the IR divergences remain. All the statements concerning the Konishi operator hold for the renormalized operator. The same is true for other operators that acquire UV divergences.

For simplicity, we choose only scalars as components of the state $|p_1^{\lambda_1}, \ldots, p_n^{\lambda_n}\rangle$ produced by the operator $\mathcal{O}$ and can then omit the helicity indices from $|p_1, \ldots, p_n\rangle$. We also consider the states with the number of particles equal to the naive mass dimension of the operator $\mathcal{O}$, i.e., we consider the states consisting of Δ_0 scalars.

Calculation strategy. For our calculations, it is convenient to use the $\mathcal{N}$=1 formulation of the $\mathcal{N}$=4 SYM theory and compute explicitly in terms of the $\mathcal{N}$=1 superfields in the momentum space. The operators $\mathcal{O} = \{\mathcal{C}_{IJ}, \mathcal{V}_I^J, \mathcal{O}_I^{(n)}, \mathcal{K}\}$ can be identified with the lowest components of the $\mathcal{N}$=1 local operators

$$\begin{aligned} &\mathcal{C}_{IJ} = \mathrm{Tr}(\Phi_I \Phi_J), \qquad \mathcal{V}_I^J = \mathrm{Tr}(e^{-gV}\bar{\Phi}^J e^{gV}\Phi_I), \quad I \neq J, \\ &\mathcal{O}_I^{(n)} = \mathrm{Tr}(\Phi_I^n), \qquad \mathcal{K} = \sum_I \mathrm{Tr}(e^{-gV}\Phi^I e^{gV}\Phi_I), \end{aligned} \tag{13}$$

where Φ_I are chiral $\mathcal{N}$=1 superfields and V is a real vector $\mathcal{N}$=1 superfield. The operators $\mathcal{C}_{IJ}$ and $\mathcal{O}_I^{(n)}$ are chiral, and $\mathcal{V}_I^J$ and $\mathcal{K}$ are nonchiral from the $\mathcal{N}$=1 superfield standpoint.

We let $\mathcal{F}(p_1, \ldots, p_n) = \langle p_1, \ldots, p_n|\mathcal{O}(q)|0\rangle$ denote the form factor of the corresponding operator and expect that $\mathcal{F}$ has the property

$$\mathcal{F}(p_1, \ldots, p_n) = \mathcal{F}_{\mathrm{tree}}(p_1, \ldots, p_n)(1 + \text{“loops”}),$$

where $\mathcal{F}_{\mathrm{tree}}(p_1, \ldots, p_n)$ denotes the tree-level contribution and “loops” schematically denotes the contributions of the next orders of the perturbation theory. It is convenient to define the ratio

$$\mathcal{M}(p_1, \ldots, p_n) = \frac{\mathcal{F}(p_1, \ldots, p_n)}{\mathcal{F}_{\mathrm{tree}}(p_1, \ldots, p_n)} = (1 + \text{“loops”}) = \sum_{l=0} \lambda^l \mathcal{M}^{(l)}.$$

We first consider the chiral case. To calculate the form factors, we use the generating functional for the strong-coupling Green's functions $\Gamma[\Phi^{\mathrm{cl}}, J]$ in the $\mathcal{N}$=1 superspace. It can be obtained from the generating functional

$$Z[j, J] = \int \mathcal{D}(\Phi_I, V, \ldots) \exp\Big\{S^{\mathcal{N}=4} + \int d^6z\, J(z)\mathcal{O}(z) + \int d^6z\, \mathrm{Tr}\big(j(z)\Phi(z)\big)\Big\}$$

using the Legendre transformation with respect to chiral sources j (we note that the source J is unaffected by the Legendre transformation). After performing the D-algebra, each supergraph gives a contribution local in the θ, and $\Gamma[\Phi^{\mathrm{cl}}, J]$ can be written as (we assume the mass-shell condition $p_i^2 = 0$ when performing the D-algebra)

$$\begin{aligned} \Gamma[\Phi^{\mathrm{cl}}, J] = \sum_{l=0} \lambda^l \Gamma^{(l)}[\Phi^{\mathrm{cl}}, J] = \sum_{l=0} \lambda^l \int d^4p_1 \cdots d^4p_n\, d^6z\, J(-q, \theta) \times \\ \times \mathrm{Tr}\big(\Phi^{\mathrm{cl}}(-p_1, \theta) \cdots \Phi^{\mathrm{cl}}(-p_n, \theta)\big)\mathcal{M}^{(l)}(p_1, \ldots, p_n) + O(J^2), \end{aligned}$$

where $d^6z = d^4q\, d^2\theta$ and $\mathcal{M}^{(l)}(p_1, \ldots, p_n)$ is given by the sum of scalar integrals over “bosonic” variables. Then

$$\mathcal{M}^{(l)}(p_1, \ldots, p_n) = \frac{\delta^{n+1}\Gamma^{(l)}}{\delta\Phi^{\mathrm{cl}} \cdots \delta\Phi^{\mathrm{cl}}\, \delta J}\bigg|_{\substack{p_i^2=0,\, \theta=0,\\ \Phi^{\mathrm{cl}}=0,\, J=0}}.$$

We again stress that the on-shell condition $p_i^2 = 0$ and momentum conservation $q + p_1 + \cdots + p_n = 0$ should be taken into account when obtaining this expression.

The situation is a bit more complicated in the nonchiral case. All the integrals in $\Gamma[\Phi^{\mathrm{cl}}, \bar{\Phi}^{\mathrm{cl}}, \mathcal{J}]$ ($\mathcal{J}$ is a nonchiral source) are now integrals over the full $\mathcal{N}$=1 superspace $\int d^8z\, (\ldots)$, where $d^8z = d^4q\, d^4\theta$ (and not only over the chiral subspace $\int d^6z\,(\dots)$ as in the chiral case), and the expansion for $\Gamma[\Phi^{\rm cl},\bar\Phi^{\rm cl},\mathcal{J}]$ contains extra terms,

$$\begin{aligned}\Gamma[\Phi^{\rm cl},\overline{\Phi}^{\rm cl},\mathcal{J}] =& \sum_{l=0}\lambda^l\Gamma^{(l)}[\Phi^{\rm cl},\overline{\Phi}^{\rm cl},\mathcal{J}] = \\ =& \sum_{l=0}\lambda^l\int d^4p_1\cdots d^4p_n\,d^8z\,\mathcal{J}(-q,\theta,\bar\theta)\times \\ &\times\big[\mathrm{Tr}\big(\overline{\Phi}^{\rm cl}(-p_1,\bar\theta)\cdots\Phi^{\rm cl}(-p_n,\theta)\big)\mathcal{M}^{(l)}(p_1,\dots,p_n)+ \\ &+\mathrm{Tr}\big(\overline{D}^{\dot\beta}\overline{\Phi}^{\rm cl}(-p_1,\bar\theta)\cdots D^\alpha\Phi^{\rm cl}(-p_n,\theta)\big)\mathcal{M}^{(l)}_{\dot\beta\alpha}(p_1,\dots p_n)+ \\ &+\mathrm{Tr}\big(\overline{D}^2\overline{\Phi}^{\rm cl}(-p_1,\bar\theta)\cdots D^2\Phi^{\rm cl}(-p_n,\theta)\big)\mathcal{M}^{(l)}_2(p_1,\dots,p_n)\big]+O(\mathcal{J}^2).\end{aligned}\tag{14}$$

From the $\mathcal{N}$=1 superspace standpoint, the additional terms correspond to the operators of other mass dimensions, and we have a situation with mixed operators. But from the "component" standpoint, we can always choose a concrete projection on a particular component of a superfield. We consider only scalar components. Correspondingly, the last terms in (14) do not contribute to our consideration and can be dropped.

We calculate in the $\mathcal{N}$=1 superspace formalism and take the projection on the scalar component with $\theta=\bar\theta=0$ at the end of the calculations. There are advantages and disadvantages to this approach. A major advantage is the drastic reduction in the number of diagrams compared with the component case together with the simplified form of the scalar integrals. The disadvantage of the method is the lack of an explicit $\mathcal{N}$=4 covariance for the answer. It would be desirable to use modern $\mathcal{N}$=4 covariant methods "on the mass shell," which are used to calculate amplitudes, but they require some modifications for application to form factors.

Form factors with $\Delta_0 = 2$. For the operators $\mathcal{C}_{IJ}$ and $\mathcal{V}_I^J$ and with certain distinctions for the operator $\mathcal{K}$, the form factors have the same form, which corresponds to the general structure of the form factor $\mathcal{M}$ with two external legs in the gauge theory with a zero beta function[7]

$$\log\mathcal{M} = \frac{1}{2}\sum_{i=1}^{2}\widehat{M}\left(\frac{s_{i,i+1}}{\mu^2}\right)+O(\epsilon),$$

where we introduce

$$\widehat{M}\left(\frac{s_{i,i+1}}{\mu^2}\right) = -\frac{1}{2}\sum_l\left(\frac{\lambda}{16\pi^2}\right)^l\left(\frac{\gamma^{(l)}_{\rm cusp}}{(l\epsilon)^2}+\frac{G^{(l)}}{l\epsilon}+C^{(l)}\right)\left(\frac{s_{i,i+1}}{\mu^2}\right)^{l\epsilon},\tag{15}$$

$\gamma^{(l)}_{\rm cusp}$ are the coefficients of the perturbative expansion $\gamma_{\rm cusp}(\lambda)=\sum_l\gamma^{(l)}_{\rm cusp}\lambda^l$ of the cusp anomalous dimension, which we encountered previously when considering the scattering amplitudes, $G^{(l)}$ are the coefficients of the perturbative expansion $G(\lambda)=\sum_l G^{(l)}\lambda^l$ of the collinear anomalous dimension, and $C^{(l)}$ are some constants. The quantities $G^{(l)}$ and $C^{(l)}$ are regularization and scheme dependent in contrast to the cusp anomalous dimension. For the considered form factors, we obtain

$$\log\mathcal{M} = a\left(\frac{s_{12}}{\mu^2}\right)^{-\epsilon}\left(\frac{-2}{\epsilon^2}+\zeta_2\right)+a^2\left(\frac{s_{12}}{\mu^2}\right)^{-2\epsilon}\left(\frac{\zeta_2}{\epsilon^2}+\frac{\zeta_3}{\epsilon}\right)+O(a^3),$$

[7] A very similar structure of the form factors in QCD was first established in [8].

where ζ_n are the corresponding values of the Riemann zeta function and $a = \lambda e^{-\epsilon\gamma_E}$. Comparing the obtained answer with (15), we can find the values of the first two coefficients of the expansion for the anomalous dimensions $\gamma^{(l)}_{\text{cusp}}$ and $G^{(l)}$:

$$\begin{aligned} &\gamma^{(1)}_{\text{cusp}} = 4, \qquad \gamma^{(2)}_{\text{cusp}} = -8\zeta_2, \\ &G^{(1)} = 0, \qquad G^{(2)} = -\zeta_3, \qquad C^{(1)} = -\zeta_2, \qquad C^{(2)} = 0. \end{aligned} \tag{16}$$

We note that the maximal transcendentality principle [21] holds, which in our case means that if we assign each logarithm and π the transcendentality level 1 and the polylogarithms $\mathrm{Li}_n(x)$ and ζ_n the transcendentality level n, then at the given order of the perturbation theory, the coefficient for the nth pole $1/\epsilon^n$ has the total transcendentality $2l - n$, where l is the number of loops. For a product, the total transcendentality is equal to the sum of the transcendentalities of the factors.

The leading IR behavior of $\mathcal{M}$ can also be computed by considering the Wilson line with one cusp. In this case, the dual description in terms of Wilson loops is known. The same result was obtained in [15] for the operator $\mathcal{V}_X = 2\,\mathrm{Tr}(\Phi_1\overline{\Phi}_1) - \mathrm{Tr}(\Phi_2\overline{\Phi}_2) - \mathrm{Tr}(\Phi_3\overline{\Phi}_3)$ belonging to the energy–momentum tensor supermultiplet.

The $\mathcal{O}_I^{(n)}$, $n = 3$, form factors at the two-loop level. We here consider the results of calculating for the chiral half-BPS operators $\mathcal{O}_I^{(n)}$ defined above. In the second order of the perturbation theory, we have

$$\log \mathcal{M} = \sum_{i=1}^{3} a \left(\frac{s_{i,i+1}}{\mu^2}\right)^{-\epsilon} \left(-\frac{1}{\epsilon^2} + \frac{\zeta_2}{2}\right) + \\ + \sum_{i=1}^{3} a^2 \left(\frac{s_{i,i+1}}{\mu^2}\right)^{-2\epsilon} \left(\frac{\zeta_2}{2\epsilon^2} + \frac{7\zeta_3}{2\epsilon}\right) + \text{fin.part.}$$

As in the preceding case of the form factors of the operators with the conformal dimension two, we can find the first two coefficients in the expansion of the anomalous dimensions:

$$\gamma^{(1)}_{\text{cusp}} = 4, \qquad \gamma^{(2)}_{\text{cusp}} = -8\zeta_2, \qquad G^{(1)} = 0, \qquad G^{(2)} = -7\zeta_3. \tag{17}$$

We note that the values of $\gamma^{(l)}$, as expected, are unchanged and coincide with (16), while those of the collinear anomalous dimension depend on the form factor considered.

We note the highly nontrivial cancellations between the polylogarithms that occurred when calculating $\log\mathcal{M}$. The individual contributions from the scalar integrals have coefficients of the poles consisting of nontrivial combinations of logarithms and polylogarithms of different weight. We see that the IR-factorization property holds for the form factors as for the amplitudes.

The $\mathcal{O}_I^{(n)}$, $n > 3$, form factors at the two-loop level. In the first order of the perturbation theory, similar to the form factors of operators with the conformal dimension two, the answer is given by the "scalar triangle." In the second order, the answer has a form similar to the n=3 case but has an additional term coming from the factored one-loop diagrams. The full set of scalar integrals in the answer is given in Fig. 2.

Expanding over ϵ and calculating the logarithm of the answer, we obtain

$$\log \mathcal{M} = \sum_{i=1}^{n} a \left(\frac{s_{i,i+1}}{\mu^2}\right)^{-\epsilon} \left(-\frac{1}{\epsilon^2} + \frac{\zeta_2}{2}\right) + \\ + \sum_{i=1}^{n} a^2 \left(\frac{s_{i,i+1}}{\mu^2}\right)^{-2\epsilon} \left(\frac{\zeta_2}{2\epsilon^2} + \frac{7\zeta_3}{2\epsilon}\right) + \text{fin.part.}$$

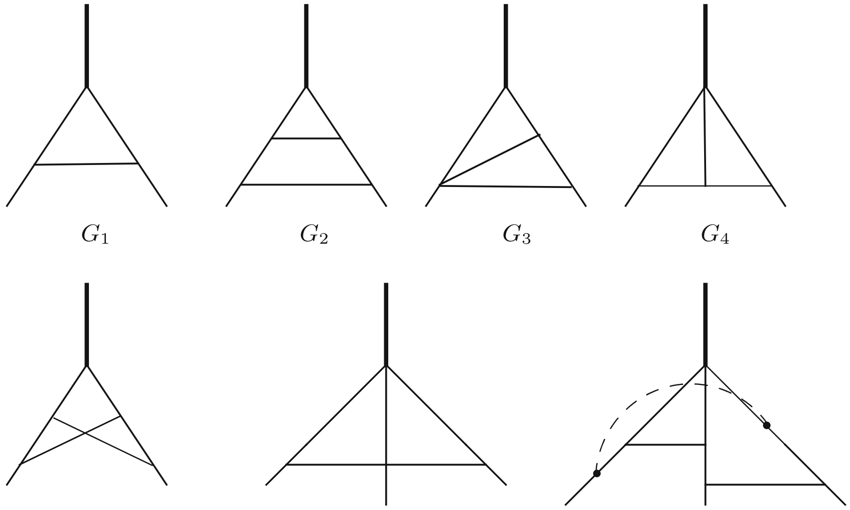

Fig. 2. Scalar integrals in the answer: the arc in G_7 corresponds to the presence of the numerator $(k-p)^2$, the thick black line corresponds to an off-shell leg with the momentum q, and all the other legs are on-shell.

The first two coefficients in the expansion of the anomalous dimensions are

$$\gamma^{(1)}_{\text{cusp}} = 4, \qquad \gamma^{(2)}_{\text{cusp}} = -8\zeta_2, \qquad G^{(1)} = 0, \qquad G^{(2)} = -7\zeta_3.$$

The finite part is

$$\lambda F^{(1)}(s_{12}, \ldots, s_{n1}) + \lambda^2 F^{(2)}(s_{12}, \ldots, s_{n1}) + O(\lambda^3)$$

and is trivial at one loop, $F^{(1)} = 0$, and the two-loop expression $F^{(2)}$ is a complicated combination of polylogarithms and generalized Goncharov polylogarithms of several variables.

We note that the result is still simpler than in the nonsupersymmetric theory and the maximal transcendentality principle still holds [21]. In the case of (1+1)-dimensional kinematics, the expressions for the finite parts of the form factors are essentially simplified [17]. For instance, for the form factor of the operator $\mathcal{O}_I^{(3)}$, the answer is expressed in terms of only logarithms, and the finite part has the form

$$\frac{-\lambda^2}{2880}\left(75\log^4\frac{s_{12}}{s_{13}} + 120\pi^2\log^2\frac{s_{12}}{s_{13}} - 317\pi^4\right).$$

Dual conformal invariance. We note a general property of the integrals in the final result first mentioned in [17]: all the scalar integrals can be obtained from the pseudoconformal integrals in the answer for the amplitudes by a certain limit procedure. This means that the calculation results should be expressed in terms of conformally invariant ratios, which restricts the form of the answer. On the mass shell, generally speaking, this dual conformal symmetry has an anomaly, but this anomaly is completely determined by the cusp anomalous dimension.

As a result, the dual conformal symmetry remains an extremely useful instrument for investigating the properties of amplitudes and the Wilson loops dual to them. Strictly speaking, the integrals that appear when calculating the form factors do not have an exact dual conformal invariance but, as already mentioned, can be obtained from the dually conformal using a certain limit procedure [17], which we describe with

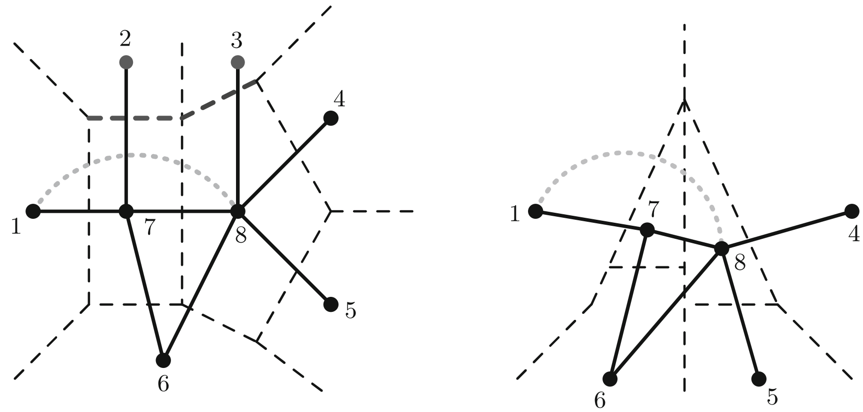

Fig. 3. The two-loop ladder-type triangle diagram and the two-loop box diagram: the arc corresponds to the presence of a numerator.

the example shown in Fig. 3. In the right-hand picture, we show the diagram arising in the form-factor calculation, and in the left-hand picture, we show the "initial" dually conformal diagram encountered when integrating the amplitudes. In the limit as points 2 and 3 go to infinity, the left diagram passes into the right diagram.

IR-finite observables based on form factors. Strictly speaking, the form factors $\langle 0|\mathcal{O}|p_1^{\lambda_1},\ldots,p_n^{\lambda_n}\rangle$ on the mass shell, like the amplitudes, are ill defined in four-dimensional space–time because of the presence of IR divergences, and some IR regulator must therefore be introduced. This leads to the appearance of an additional mass parameter μ in the dimensional regularization, breaking the conformal symmetry. In other words, it may be said that $\langle 0|\mathcal{O}|p_1^{\lambda_1},\ldots,p_n^{\lambda_n}\rangle$ are intermediate objects and the true physical quantities are the IR-safe observables that are constructed from $\langle 0|\mathcal{O}|p_1^{\lambda_1},\ldots,p_n^{\lambda_n}\rangle$ and are free of IR divergences and hence free of the IR regulator. Indeed, for the process $\gamma^* \to$ Jets in QCD, we are usually interested in the total cross section $\sigma_{\mathrm{tot}}(\gamma^* \to \text{Jets})$ or some differential distributions and not in the matrix elements $\langle 0|j_{\mathrm{em}}^{\mathrm{QCD}}|p_1^{\lambda_1},\ldots,p_n^{\lambda_n}\rangle$ themselves. To obtain a physical (final) result, we must include all the processes allowed by the energy–momentum conservation laws in the same order of the perturbation theory.

For instance, we consider the total cross section σ_{tot} for the process $J \to$ all possible from the $\mathcal{N}{=}4$ supermultiplet, where the classical source J is coupled to the $\mathcal{N}{=}4$ supermultiplet by a local gauge-invariant operator $\mathcal{O}$. By the optical theorem,

$$\sigma_{\mathrm{tot}}(s) \sim \frac{1}{s}\,\mathrm{Im}_s\biggl[\int d^Dx\, e^{-iqx}\langle \mathcal{O}(x)\mathcal{O}(0)\rangle\biggr], \qquad q^2 = -s.$$

The two-point correlation function for the operators $\mathcal{O}$ in the conformal theory, in addition to the canonical mass dimension Δ_0, can have an anomalous dimension $\gamma = \gamma(\lambda)$ that depends on the coupling constant,

$$\langle \mathcal{O}(x)\mathcal{O}(0)\rangle \sim \frac{1}{(x^2)^{\Delta_0(1-\epsilon)+\gamma}}.$$

After some calculation, we obtain the total cross section

$$\sigma_{\mathrm{tot}}(s) \sim \frac{1}{\Gamma(\Delta_0+\gamma)\Gamma(\Delta_0+\gamma-1)}\frac{1}{s^{3-\Delta_0-\gamma}}$$

and can study its asymptotic behavior at weak and strong couplings.

In the $\mathcal{N}$=4 SYM theory, as in any conformal field theory, if the operator $\mathcal{O}$ is protected (which means that it does not receive quantum corrections and $\gamma = 0$), then the total cross section behaves as $C/s^{3-\Delta_0}$. From this expression, it might seem that we have a violation of unitarity for processes with operators whose dimension is greater than three, according to the Froissart theorem. The contradiction is resolved because the Froissart theorem is applicable only to renormalizable interactions, i.e., interactions corresponding to operators with a conformal dimension less than or equal to three.

If we consider not σ_{tot} but some differential distribution, then the optical theorem is not so useful, and we must compute the perturbation theory directly. The form factors considered here are "building blocks" for such objects.

4. Discussion

We have summarized our recent achievements in understanding the IR structure of the $\mathcal{N}$=4 SYM theory. As we showed, although the MHV and other helical amplitudes have several attractive aspects, they are not physical objects and depend on the IR regulator. Investigating their properties further is an interesting mathematical problem, but it cannot be considered physically well defined. At the same time, many remarkable properties of the MHV amplitudes are lost for physically well-defined objects.

The form factors are the objects next in complexity after the amplitudes. As we showed, they have many properties similar to those of the scattering amplitudes (the IR divergences factor and so on) although they formally differ from the amplitudes by allowing one leg to go off shell.

The $\mathcal{N}$=4 SYM theory is the first example of a conformal quantum field theory in four-dimensional space–time. There are many indications that it might be integrable in some sense. Integrability in similar cases means the presence of an infinite number of conservation laws, i.e., an infinite number of relations. Possibly, the abovementioned Yangian symmetry is responsible for such a set of relations.

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