Late-time behaviour of the Einstein-Vlasov system with Bianchi I symmetry

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Abstract.
The late-time behaviour of the Einstein-dust system is well understood for homogeneous spacetimes. For the case of Bianchi I we have been able to show that the late-time behaviour of the Einstein-Vlasov system is well approximated by the Einstein-dust system assuming that one is close to the unique stationary solution which is the attractor of the Einstein-dust system.

1. Introduction

The Einstein-Vlasov system with Bianchi symmetry is a system of integro-differential equations on a Lorentzian manifold M which has the structure $M = I \times G$ where I is an interval and G a Lie group. It can be seen as a kind of transport equation, say $Xf = 0$, coupled to other ordinary differential equations. There exists certain limiting process such that the transport equation is transformed into an ordinary differential equation. This happens if the function f becomes a distribution. The resulting system is called Einstein-dust system and the late-time behavior is well understood. It has been studied by the theory of dynamical systems and for almost all Lie groups the ω-limit set is known [1]. The strategy to study the late-time behaviour of the Einstein-Vlasov system consists in trying to show that the late-time behavior is well approximated by the Einstein-dust system. Physically this makes sense since one might expect a decay of the velocity dispersion for expanding cosmological models. The result presented in the following shows that this is true for case of the Lie group $\mathbb{R}^3$ and assuming that one is close to the unique stationary solution which is the attractor of the Einstein-dust system.

2. Einstein-Vlasov system with Bianchi I symmetry

2.1. Einstein equations with Bianchi I symmetry

A Bianchi spacetime is defined to be a spatially homogeneous spacetime whose isometry group possesses a three-dimensional subgroup G that acts simply transitively on the spacelike orbits. They can be classified by the structure constants of the Lie algebra associated to the Lie group. We will only consider the simplest case where the structure constants vanish, i.e. the case of

Bianchi I where the metric has the following form using Gauss coordinates:

$$^4g = -dt^2 + g_{ab}(t)dx^a dx^b.$$
(1)

In terms of coordinate expressions

$$
\begin{aligned}
\rho &= T^{00} \\
j_a &= T^0_a \\
S_{ab} &= T_{ab}
\end{aligned}
$$

where ρ, j_a and $T_{\mu\nu}$ are the energy density, matter current and energy momentum tensor respectively. Using the 3+1 decomposition of the Einstein equations as made in [2] the basic equations are

$$\dot{g}_{ab} = -2k_{ab}$$
(2)

$$\dot{k}_{ab} = Hk_{ab} - 2k_{ac}k^c_b - 8\pi(S_{ab} - \frac{1}{2}g_{ab}\operatorname{tr} S) - 4\pi\rho g_{ab}$$
(3)

$$-k_{ab}k^{ab} + H^2 = 16\pi\rho$$
(4)

$$T_{0a} = 0.$$
(5)

where k_{ab} is the second fundamental form, H its trace and a dot above a letter denotes a derivative with respect to the cosmological (Gaussian) time t. It has been assumed that the cosmological constant vanishes.

2.2. Vlasov equation with Bianchi I symmetry
For the matter model we will take the point of view of kinetic theory. This means that we have a collection of particles (in a cosmological context the particles are galaxies or clusters of galaxies) which are described statistically by a non-negative distribution function $f(x^\alpha, p^\alpha)$ which is the density of particles at a given spacetime point with given four-momentum. We will assume that all the particles have equal mass (one can relax this condition if necessary, see [3]) . We want that our matter model is compatible with our symmetry assumption, so we will also assume that f does not depend on x^a. In addition to that we will assume that there are no collisions between the particles. In this case the distribution function satisfies the Vlasov equation (See (3.38) of [2]):

$$\frac{\partial f}{\partial t} + 2k^a_b p^b \frac{\partial f}{\partial p^a} = 0.$$
(6)

where f is defined on the set determined by the equation

$$-(p^0)^2 + g_{ab}p^a p^b = -m^2$$

called the mass shell. For a given Bianchi I geometry the Vlasov equation can be solved explicitly with the result that if f is expressed in terms of the covariant components p_i then it is independent of time. The non-trivial components of the energy momentum tensor are:

$$\rho = \int f_0(p_i)(m^2 + g^{cd}p_c p_d)^{\frac{1}{2}}(\det g)^{-\frac{1}{2}}dp_1 dp_2 dp_3$$
(7)

$$S_{ab} = \int f_0(p_i)p_a p_b(m^2 + g^{cd}p_c p_d)^{-\frac{1}{2}}(\det g)^{-\frac{1}{2}}dp_1 dp_2 dp_3$$
(8)

For this kind of matter all the energy conditions hold. In particular $\rho \geq \operatorname{tr} S \geq 0$. Our system of equations consists of the equations (2)-(8).

3. Central results

The assumption that the spacetime is close to isotropic is expressed by assuming that the quantity

$$F = \frac{\sigma_{ab}\sigma^{ab}}{H^2}. \tag{9}$$

is small. Here σ_{ab} denotes the trace-free part of the second fundamental form. The quantity F is related to the so called *shear parameter*, which is bounded by the cosmic microwave background radiation and is a dimensionless measure of the anisotropy of the Universe (See chapter 5.2.2 of [1]). The other assumption which is needed is that the spacetime is close to "dust-like". This is expressed by assuming that the absolute value of the momenta of the particles is bounded. We define P as the supremum of the absolute value of the momenta at a given time t:

$$P(t) = \sup\{|p| = (g^{ab}p_a p_b)^{\frac{1}{2}} | f(t,p) \neq 0\} \tag{10}$$

The main result is the following:

Theorem 1. *Consider any C^∞ solution of the Einstein-Vlasov system with Bianchi I-symmetry and with C^∞ initial data. Assume that $F(t_0)$ and $P(t_0)$ are sufficiently small. Then at late times one can make the following estimates:*

$$H(t) = -2t^{-1}(1 + O(t^{-1})) \tag{11}$$
$$P(t) = O(t^{-\frac{2}{3}+\epsilon}) \tag{12}$$
$$F(t) = O(t^{-2}) \tag{13}$$

where ϵ is a small and strictly positive constant. These estimates imply that the spacetime isotropizes and that asymptotically there is a dust-like behaviour. The proof is based on a bootstrap argument. See [4] for the details. Using these estimates one can obtain more detailed information about the behaviour of the metric. In analogy to the Kasner solution one can define the *generalized Kasner exponents* for the non-vacuum case. Let λ_i be the eigenvalues of k_{ij} with respect to g_{ij} we define

$$p_i = \frac{\lambda_i}{H} \tag{14}$$

as the generalized Kasner exponents. Having analyzed carefully the dust case with small data we could conclude that:

Theorem 2. *Consider the same assumptions as in the previous theorem. Then*

$$p_i = \frac{1}{3} + O(t^{-1}) \tag{15}$$

and

$$g_{ab} = t^{+\frac{4}{3}}[\mathcal{G}_{ab} + O(t^{-2})] \tag{16}$$
$$g^{ab} = t^{-\frac{4}{3}}[\mathcal{G}^{ab} + O(t^{-2})] \tag{17}$$

where $\mathcal{G}_{ab}$ and $\mathcal{G}^{ab}$ are independent of t.

4. Conclusions and Outlook

The result can be seen as a generalization of theorem 5.4 of [5]. We obtain the same the result, but a) we also obtain how fast the expressions converge b) we obtain an asymptotic expression for the spatial metric c) we do not assume additional symmetries. However we used a different kind of restriction namely the small data assumptions. For the study of more complicated Lie groups a starting point can be [6] - [9] where the late-time asymptotics are obtained assuming an additional symmetry. Another direction of generalization is the study of the asymptotics towards the initial singularity. In [10] and [11] the case of two fluids has been studied which leads to the non-diagonal case. For the Vlasov case already the non-LRS case has been analyzed in [12]. Already in this case surprising new features like the existence of heteroclinic networks arose. Finally it would be interesting concerning the direction of the initial singularity to show that also in the Vlasov case the off-diagonal components of the metric tend to constants and thus are not important for the dynamics. See [13] for the importance of this behaviour and [14] for consequences in a quantum version.

References

[1] Wainwright J Ellis G F R 1997 *Dynamical Systems in Cosmology* (Cambridge University Press)

[2] Rendall A D 2008 *Partial Differential Equations in General Relativity* (Oxford University Press)

[3] Calogero S and Heinzle J M 2010 Oscillations toward the singularity of LRS Bianchi type IX cosmological models with Vlasov matter *SIAM J. Appl. Dyn. Syst.* 9, 1244-1262

[4] Nungesser E 2010 Isotropization of non-diagonal Bianchi I spacetimes with collisionless matter at late times assuming small data *Class. Quantum Grav.* 27 235025

[5] Rendall A D 1996 The initial singularity in solutions of the Einstein-Vlasov system of Bianchi type I *J. Math Phys.* 37, 438-451.

[6] Rendall A D and Tod K P 1998 Dynamics of spatially homogeneous solutions of the Einstein-Vlasov equations which are locally rotationally symmetric *Class. Quantum Grav.* 16, 1705-1726

[7] Rendall A D and Uggla C 2000 Dynamics of spatially homogeneous locally rotationally symmetric solutions of the Einstein-Vlasov equations *Class. Quantum Grav.* 17, 4697-4713

[8] Rendall A D 2002 Cosmological models and centre manifold theory *Gen. Rel. Grav.* 34, 1277-1294

[9] Calogero S and Heinzle J M 2009 Bianchi Cosmologies with Anisotropic Matter: Locally Rotationally Symmetric Models *Preprint arXiv* 0911.0667v1

[10] Sandin P and Uggla C 2008 Bianchi type I models with two tilted fluids *Class. Quantum Grav.* 25, 225013

[11] Sandin P 2009 Tilted two-fluid Bianchi type I models. *Gen. Rel. Grav.* 41, 2707-2724

[12] Heinzle J M and Uggla C 2006 Dynamics of the spatially homogeneous Bianchi type I Einstein-Vlasov equations *Class. Quantum. Grav.* 23, 3463-3490

[13] Damour T, Henneaux M and Nicolai H 2003 Cosmological Billiards *Class. Quantum Grav.* 20 R145-R200

[14] Kleinschmidt A, Koehn M and Nicolai H 2009 Supersymmetric quantum cosmological billiards *Phys. Rev.* D 80:061701