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A recursive formula for n–point SYM tree amplitudes

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Abstract

We propose a recursive formula for super Yang–Mills color–ordered n–point tree amplitudes based on the cohomology of pure spinor superspace in ten space–time dimensions. The amplitudes are organized into BRST covariant building blocks with diagrammatic interpretation. Manifestly cyclic expressions (no longer than one line each) are explicitly given up to $n = 10$ and higher leg generalizations are straightforward.

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1. Introduction

Elementary particle physics relies on the computation of scattering amplitudes in Yang-Mills theory. Parke and Taylor found compact and simple expressions for maximally helicity violating (MHV) amplitudes in four space-time dimensions [1], which provide an important milestone in discovering hidden structures underlying the S–matrix. Many formal as well as phenomenological advances followed since then, see [2,3] for some reviews.

In this letter we use the framework of the pure spinor formalism [4] to reduce the computation of n–point tree amplitudes in ten-dimensional super-Yang-Mills theory to a recursive cohomology problem in pure spinor superspace. This admits the compact formula (2.1) for the supersymmetric color-ordered n–point scattering amplitude at tree level.

Although the pure spinor framework is initially adapted to ten space-time dimensions, one can still dimensionally reduce the results and extract the physics from any lower dimensional point of view. At any rate, the striking simplicity of our results is exhibited without the need of four–dimensional spinor helicity formalism. Moreover, the simplicity is furnished both for MHV and NMHV helicity configurations in four space–time dimensions.

2. Pure spinor cohomology formula for $A_n(1, 2, \ldots, n)$

The color-ordered tree-level massless super-Yang-Mills amplitudes in ten dimensions will be argued to be determined by the pure spinor superspace cohomology formula[1],

$$\mathcal{A}_n = \langle E_{i_1 \ldots i_{n-1}} V_n \rangle \ , \tag{2.1}$$

where the bosonic superfields of ghost-number two $E_{i_1 \ldots i_m}$ are BRST-closed but not BRST-exact in the momentum phase space of an $n-$point massless amplitude where $s_{i_1 \ldots i_{n-1}} = 0$,

$$Q E_{i_1 \ldots i_p} = 0, \quad E_{i_1 \ldots i_p} = Q M_{i_1 \ldots i_p} \text{ if } s_{i_1 \ldots i_p} \neq 0 \ , \tag{2.2}$$

as will be further explained in the following subsections. The $\langle \ldots \rangle$ bracket denotes a zero mode integration prescription automated in [5] which extracts a certain tensor structure of order $\lambda^3 \theta^5$ from the enclosed superfields [4].

[1] The $n-$point color-ordered formulæ in this letter are all for the ordering $1, 2, \ldots, n$.

2.1. Recursion relations

With the notation where $M_i \equiv V_i$ and assuming $QM_{i_1\ldots i_p} = E_{i_1\ldots i_p}$, $\forall p < n-1$, the superfields $E_{i_1\ldots i_{n-1}}$ can be constructed recursively as

$$E_{i_1\ldots i_{n-1}} = \sum_{p=1}^{n-2} M_{i_1\ldots i_p} M_{i_{p+1}\ldots i_{n-1}} \tag{2.3}$$

in terms of Grassmann odd superfields $M_{i_1\ldots i_p}$. The latter carry $p-1$ inverse powers of Mandelstam invariants $s_{i_1\ldots i_p} = \frac{1}{2}(k_{i_1} + \ldots + k_{i_p})^2$ and can be associated with the collection of Feynman diagrams entering a color ordered $p+1$ point amplitude, see subsection 2.2 for further details. The p sum in (2.3) runs over different partitions of the first $n-1$ legs, so one can interpret (2.1) as a recursive formula for $\mathcal{A}_n$, factorized into $(p+1)$- point and $(n-p)$- point subamplitudes. Apart from a diagrammatic method to construct $M_{i_1\ldots i_p}$, we will give a string-inspired formula in the last section 4.

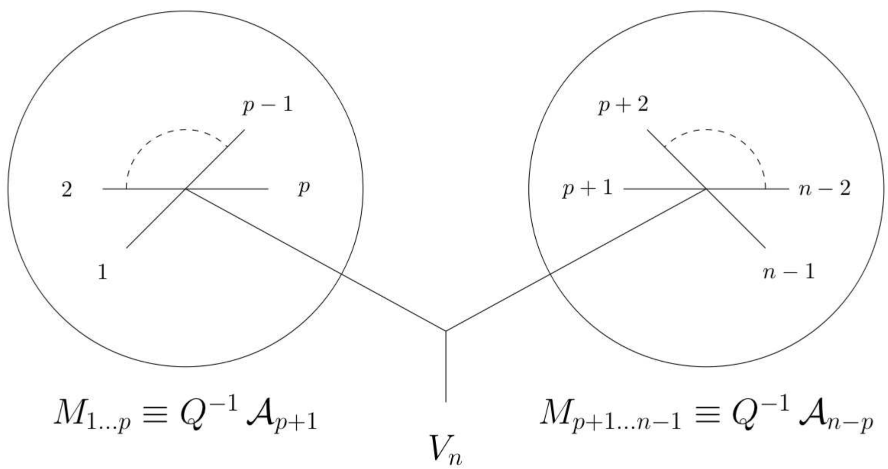

Fig. 1 Cohomology factorization of the $n-$point amplitude

Let us denote the number of kinematic poles configurations in $M_{i_1\ldots i_p}$ or $E_{i_1\ldots i_p}$ by P_{p+1}, then it follows from (2.1) and (2.3) that the number P_n of diagrams with cubic vertices in the color-ordered $n-$point amplitude $\mathcal{A}_n$ can be computed recursively as

$$P_n = \sum_{i=2}^{n-1} P_i P_{n-i+1}, \quad P_2 = P_3 \equiv 1, \qquad n \geq 4. \tag{2.4}$$

The explicit solution to (2.4) agrees with the formula $P_n = 2^{n-2}\frac{(2n-5)!!}{(n-1)!}$ of [6].

2.2. Feynman diagrams and BRST building blocks

In this subsection, we give more details about the fermionic superfields $M_{i_1\ldots i_p}$ and in particular explain their pole structure. They are constructed from ghost number one superfields $T_{j_1\ldots j_p}$ divided by the $p-1$ Mandelstam invariants $s_{j_1 j_2}, s_{j_1 j_2 j_3}, \ldots, s_{j_1\ldots j_p}$ which appear in the BRST variation of $T_{j_1\ldots j_p}$ – this makes sure that each term in $QT_{j_1\ldots j_p}$ cancels one of the poles and different terms conspire to yield an overall BRST closed amplitude. We will define the $T_{j_1\ldots j_p}$ in terms of SYM superfields in the next subsection 2.3; they will turn out to follow naturally from OPE contractions of the SYM vertex operators. The overall pole $M_{i_1\ldots i_{n-1}} \sim s^{-1}_{i_1\ldots i_{n-1}}$ prevents $\mathcal{A}_n$ from being written as $\mathcal{A}_n = \langle Q[M_{i_1\ldots i_{n-1}} V_n]\rangle$ because the kinematics for n massless particles implies that $s_{i_1\ldots i_{n-1}} = 0$. Hence, $\mathcal{A}_n = \langle E_{i_1\ldots i_{n-1}} V_n\rangle$ belongs to the BRST cohomology as required.

Let us give explicit lower order examples $p = 2,3,4,5$ to further specify the $M_{i_1\ldots i_p}$. The $p=2$ case is governed by $QT_{i_1 i_2} = s_{i_1 i_2} V_{i_1} V_{i_2}$ such that $M_{i_1 i_2} := T_{i_1 i_2}/s_{i_1 i_2}$ satisfies $QM_{i_1 i_2} = V_{i_1} V_{i_2} =: E_{i_1 i_2}$. The next examples $p \geq 3$ involve $P_{p+1} = 2, 5, 14, \ldots$ terms according to the color ordered $(p+1)$ point amplitudes $\mathcal{A}_4$, $\mathcal{A}_5$ and $\mathcal{A}_6$:

$$M_{ijk} \equiv \frac{1}{s_{ijk}} \left(\frac{T_{ijk}}{s_{ij}} - \frac{T_{jki}}{s_{jk}} \right)$$

$$M_{ijkl} \equiv \frac{1}{s_{ijkl}} \left(\frac{T_{ijkl}}{s_{ij}s_{ijk}} - \frac{T_{jkil}}{s_{jk}s_{ijk}} - \frac{T_{jkli}}{s_{jk}s_{jkl}} + \frac{T_{klji}}{s_{kl}s_{jkl}} + \frac{T_{lkij} - T_{lkji}}{s_{ij}s_{kl}} \right) \tag{2.5}$$

$$\begin{aligned} M_{ijklm} \equiv \frac{1}{s_{ijklm}} \Big[& \frac{T_{ijklm}}{s_{ij}s_{ijk}s_{ijkl}} - \frac{T_{jkilm}}{s_{jk}s_{ijk}s_{ijkl}} - \frac{T_{jklim}}{s_{jk}s_{jkl}s_{ijkl}} + \frac{T_{kljim}}{s_{kl}s_{jkl}s_{ijkl}} - \frac{T_{jklmi}}{s_{jk}s_{jkl}s_{jklm}} \\ & + \frac{T_{kljmi}}{s_{kl}s_{jkl}s_{jklm}} + \frac{T_{klmji}}{s_{kl}s_{klm}s_{jklm}} - \frac{T_{lmkji}}{s_{lm}s_{klm}s_{jklm}} + \frac{(T_{kljim} - T_{klijm})}{s_{ij}s_{kl}s_{ijkl}} + \frac{(T_{lmjki} - T_{lmkji})}{s_{jk}s_{lm}s_{jklm}} \Big] \\ & + \frac{1}{s_{ijklm}} \Big[\frac{(T_{ijklm} + T_{jikml})}{s_{ij}s_{lm}s_{ijk}} - \frac{(T_{jkilm} + T_{kjiml})}{s_{jk}s_{lm}s_{ijk}} - \frac{(T_{klmij} + T_{lkmji})}{s_{ij}s_{kl}s_{klm}} + \frac{(T_{lmkij} + T_{mlkji})}{s_{ij}s_{lm}s_{klm}} \Big] \end{aligned}$$

We have obtained explicit solutions for the system (2.2) and (2.3) up to $M_{i_1\ldots i_7}$ [7].

Using the BRST variations of T_{ijk} and T_{ijkl},

$$QT_{ijk} = s_{ijk} T_{ij} V_k - s_{ij} (T_{ij} V_k + T_{jk} V_i + T_{ki} V_j)$$

$$\begin{aligned} QT_{ijkl} &= s_{ijkl} T_{ijk} V_l + s_{ijk} \left(T_{ijl} V_k - T_{ijk} V_l + T_{ij} T_{kl} \right) \\ &+ s_{ij} \left(V_i T_{jkl} + T_{ikl} V_j - T_{ijl} V_k + T_{ik} T_{jl} + T_{il} T_{jk} - T_{ij} T_{kl} \right), \end{aligned} \tag{2.6}$$

one can check that QM_{ijk} and QM_{ijkl} indeed reproduce the E_{ijk} and E_{ijkl} which are recursively defined by (2.3). Higher order generalizations of (2.6) are straightforward.

Since each $T_{i_1\ldots i_p}$ requires a specific poles structure dictated by $QT_{i_1\ldots i_p}$, we can interpret it as the endpiece of a color ordered Feynman diagram made of cubic vertices only. The s_{12}, s_{23} and s_{123} poles in M_{123} give rise to the dictionary of Figure 2. According to $P_5 = 5$, there are five diagrams collected in M_{1234} and the last one makes use of the facts that $T_{12[34]} = -T_{34[12]}$ and $QT_{12[34]}$ cancels poles in s_{12}, s_{34} and s_{1234}.

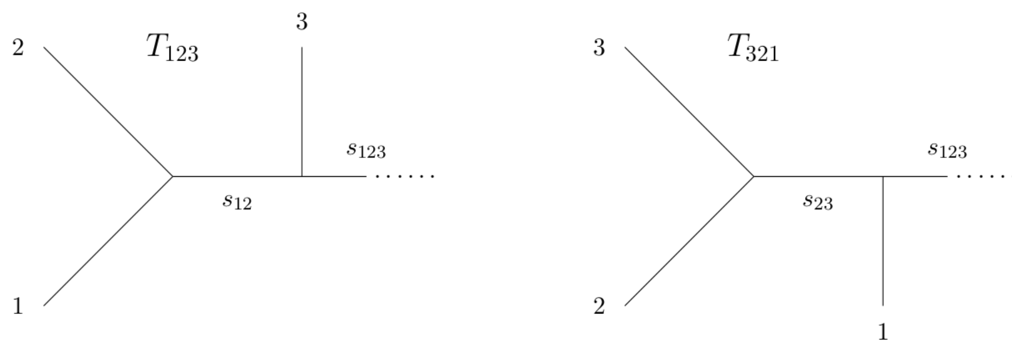

Fig. 2 The M_{123} Feynman diagrams

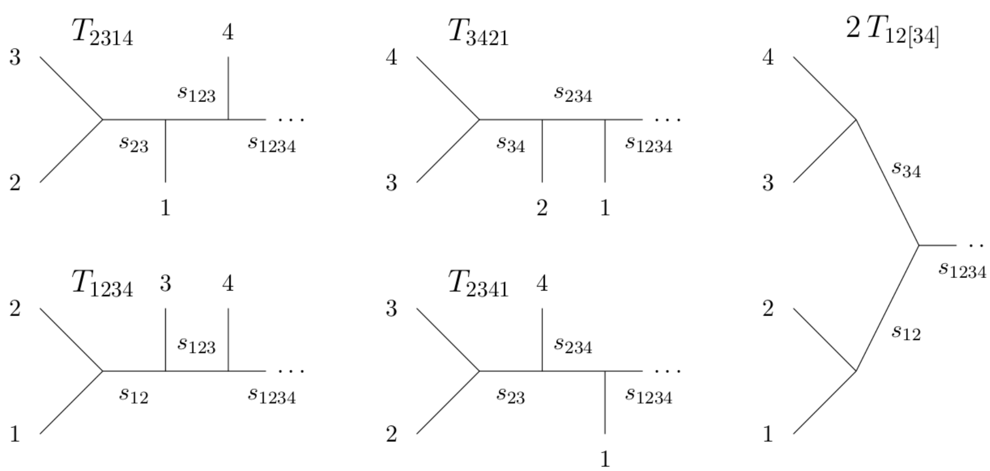

Fig. 3 The M_{1234} Feynman diagrams

For consistency with the diagrammatic interpretation, the $T_{12\ldots p}$ are required to satisfy the symmetry properties present in the corresponding Feynman diagrams. These are

$$T_{ij} = T_{[ij]}, \quad T_{ijk} = T_{[ij]k}, \quad T_{[ijk]} = 0 \tag{2.7}$$

at $p = 2, 3$, in lines with the BRST variations (2.6). The property $T_{12[34]} + T_{34[12]} = 0$ is crucial to preserve the reflection symmetry $(1, 2, 3, 4) \leftrightarrow (4, 3, 2, 1)$ of the fifth diagram in Figure 3. More generally, each $T_{i_1\ldots i_p}$ inherits all the symmetries of $T_{i_1\ldots i_{p-1}}$ in the first $p-1$ labels, so there is one new identity at each rank p (such as $T_{12[34]} + T_{34[12]} = 0$ at $p = 4$) which cannot be inferred from lower order relatives. It can be determined from the symmetries of the diagrams described by $T_{i_1\ldots i_p}$, e.g.

$$T_{ijklm} - T_{ijkml} + T_{lmijk} - T_{lmjik} - T_{lmkij} + T_{lmkji} = 0 \tag{2.8}$$

at $p = 5$. Higher order generalizations of (2.8) will be listed in [7].

2.3. Superfield realization of BRST building blocks

This subsection completes the definition of the $M_{i_1\ldots i_p}$ constituents of $\mathcal{A}_n$ by expressing their building blocks $T_{i_1\ldots i_p}$ in terms of SYM superfields. They are closely related to the OPE residues $L_{2131\ldots p1}$ when $p-1$ integrated vertex operators $U^j(z_j)$ approach their unintegrated counterpart $V^i(z_i)$:

$$\lim_{z_2\to z_1} V^1(z_1)U^2(z_2) \to \frac{L_{21}}{z_{21}}, \quad \lim_{z_p\to z_1} L_{2131\ldots p1}(z_1)U^p(z_p) \to \frac{L_{2131\ldots p1(p+1)1}}{z_{p1}}.$$

Using the explicit form of V^i, U^j and their OPEs we find

$$L_{21} = -A^1_m(\lambda\gamma^m W^2) - V^1(k^1\cdot A^2)$$

$$L_{2131} = -L_{21}((k^1+k^2)\cdot A^3) + (\lambda\gamma^m W^3)\big[A^1_m(k^1\cdot A^2) + A^{1\,n}\mathcal{F}^2_{mn} - (W^1\gamma_m W^2)\big]$$

for two and three legs respectively.

The p-leg residues $L_{2131\ldots p1}$ by themselves do transform BRST covariantly, e.g.

$$QL_{ji} = s_{ij}V_iV_j, \quad QL_{jiki} = s_{ijk}L_{ji}V_k - s_{ij}\big[L_{kj}V_i - L_{ki}V_j + L_{ij}V_k\big],$$

but they do not exhibit any symmetry properties in the labels i, j, k as required for a diagrammatic interpretation. However, many irreducibles of the symmetric group turn out to be BRST exact, e.g. $Q(A_i\cdot A_j) = 2L_{(ij)}$. Only truly BRST cohomological pieces are kept,

$$T_{ij} := L_{[ji]} = L_{ji} - L_{(ji)} = L_{ji} - \frac{1}{2}Q(A_i\cdot A_j).$$

Any higher rank residue $L_{21\dots p1}$ with $p \geq 3$ requires a redefinition in two steps to form the building blocks $T_{12\dots p}$ of $M_{12\dots p}$ as follows: $L_{2131\dots p1} \longrightarrow \tilde{T}_{123\dots p} \longrightarrow T_{123\dots p}$. A first step $\tilde{T}_{123\dots p} = L_{2131\dots p1} + \dots$ removes the BRST trivial parts in $Q\tilde{T}_{123\dots p}$, e.g.

$$\tilde{T}_{ijk} \equiv L_{jiki} + \frac{s_{ij}}{2}\big[(A_j \cdot A_k)V_i - (A_i \cdot A_k)V_j + (A_i \cdot A_j)V_k\big] - \frac{s_{ijk}}{2}(A_i \cdot A_j)V_k$$

$$Q\tilde{T}_{ijk} = s_{ijk}T_{ij}V_k - s_{ij}\big[T_{jk}V_i - T_{ik}V_j + T_{ij}V_k\big]$$

such that the BRST variation of $\tilde{T}_{123\dots p}$ involves $T_{i_1\dots i_{q<p}}$ rather than $L_{i_2 i_1 \dots i_{q<p} i_1}$. But there will be BRST exact components in $\tilde{T}_{123\dots p}$ which still have to be subtracted in a second step. For example, there exist superfields R_{ijk} and O_{ijk} such that [8]

$$QR_{ijk} = 2\tilde{T}_{(ij)k}, \quad QO_{ijk} = -3T_{[ijk]}.$$

The following redefinition yields the hook Young tableau $T_{ijk} = T_{[ij]k}$ with $T_{[ijk]} = 0$

$$T_{ijk} = \tilde{T}_{ijk} - \frac{1}{2}QR_{ijk} + \frac{1}{3}QO_{ijk}$$

suitable to represent the diagrams in M_{ijk}. Similarly, one has to remove $p-1$ BRST trivial irreducibles from $T_{12..p} = \tilde{T}_{12\dots p} + \dots$ where the higher order generalizations of $A_i \cdot A_j$, R_{ijk} and O_{ijk} superfields are related to z_{ij} double poles in the OPE of $U^i(z_i)U^j(z_j)$.

2.4. BRST equivalent expressions for $\mathcal{A}_n$ and cyclic invariance

It follows from (2.3) that $p = n-2$ is the maximum rank of $M_{i_1\dots i_p}$ appearing in the n-point amplitude cohomology formula (2.1). However, these terms are of the form $\langle M_{i_1\dots i_{n-2}} V_{i_{n-1}} V_{i_n}\rangle$ and can be rewritten as $\langle E_{i_1\dots i_{n-2}} M_{i_{n-1} i_n}\rangle$ due to $V_i V_j = E_{ij} = QM_{ij}$ and BRST integration by parts

$$\langle M_{i_1\dots i_p} E_{i_1\dots i_q}\rangle = \langle E_{i_1\dots i_p} M_{i_1\dots i_q}\rangle. \tag{2.9}$$

The decomposition of $E_{i_1\dots i_{n-2}}$ involves at most $M_{i_1\dots i_{n-3}}$, so BRST integration by parts reduces the maximum rank p of $M_{i_1\dots i_p}$ by one. It turns out that the $n-$point cohomology formula (2.1) allows enough BRST integrations by parts as to reduce the maximum rank to $p = [n/2]$. This yields a more economic expression for $\mathcal{A}_n$.

Another benefit of the BRST equivalent $\mathcal{A}_n$ representation in terms of $M_{i_1\dots i_p}$ with $p \leq [n/2]$ lies in the manifest cyclic symmetry. The last leg V_n being singled out in (2.1) obscures the amplitudes' cyclicity. Performing k integrations by parts includes V_n into bigger blocks $M_{i_1\dots i_{k+1}}$ such that the n'th leg appears on the same footing as any other one in the end. We will give examples in the following section 3.

3. $n-$point amplitudes up to $n = 10$

The three-point amplitude [4] is trivially reproduced by (2.1) and (2.3),

$$A_3 = \langle E_{12} V_3 \rangle = \langle V_1 V_2 V_3 \rangle. \tag{3.1}$$

Similarly, (2.1) and (2.3) reproduces the results of [9,10,11] for the four-point amplitude

$$\mathcal{A}_4 = \langle E_{123} V_4 \rangle = \langle V_1 M_{23} V_4 \rangle + \langle M_{12} V_3 V_4 \rangle = \frac{1}{s_{23}} \langle V_1 T_{23} V_4 \rangle + \frac{1}{s_{12}} \langle T_{12} V_3 V_4 \rangle. \tag{3.2}$$

For $n = 5$, the formulæ (2.1) and (2.3) lead to

$$\mathcal{A}_5 = \langle E_{1234} V_5 \rangle = \langle V_1 M_{234} V_5 \rangle + \langle M_{12} M_{34} V_5 \rangle + \langle M_{123} V_4 V_5 \rangle,$$

$$= \frac{\langle T_{123} V_4 V_5 \rangle}{s_1 s_4} - \frac{\langle T_{234} V_1 V_5 \rangle}{s_2 s_5} + \frac{\langle T_{12} T_{34} V_5 \rangle}{s_1 s_3} - \frac{\langle T_{231} V_4 V_5 \rangle}{s_2 s_4} + \frac{\langle T_{342} V_1 V_5 \rangle}{s_3 s_5}. \tag{3.3}$$

As discussed in the previous section, identifying E_{ij} in (3.3) and using (2.9) leads to a manifestly cyclic-invariant form proved in [11]

$$\mathcal{A}_5 = \langle M_{12} V_3 M_{45} \rangle + \text{cyclic}(12345) = \frac{\langle T_{12} V_3 T_{45} \rangle}{s_{12} s_{45}} + \text{cyclic}(12345). \tag{3.4}$$

For $n = 6$ the formula (2.1) reads

$$\mathcal{A}_6 = \langle E_{12345} V_6 \rangle = \langle V_1 M_{2345} V_6 \rangle + \langle M_{12} M_{345} V_6 \rangle + \langle M_{123} M_{45} V_6 \rangle + \langle M_{1234} V_5 V_6 \rangle. \tag{3.5}$$

Integrating the BRST-charge by parts in the first and last terms using (2.9) leads to

$$\mathcal{A}_6 = \langle M_{12} M_{34} M_{56} \rangle + \langle M_{23} M_{45} M_{61} \rangle + \langle M_{123} (M_{45} V_6 + V_4 M_{56}) \rangle$$

$$+ \langle M_{234} (V_5 M_{61} + M_{56} V_1) \rangle + \langle M_{345} (V_6 M_{12} + M_{61} V_2) \rangle,$$

$$= \frac{\langle T_{12} T_{34} T_{56} \rangle}{3 s_{12} s_{34} s_{56}} + \frac{1}{2} \langle \left(\frac{T_{123}}{s_{12} s_{123}} - \frac{T_{231}}{s_{23} s_{123}} \right) \left(\frac{T_{45} V_6}{s_{45}} + \frac{V_4 T_{56}}{s_{56}} \right) \rangle + \text{cyclic}(1 \ldots 6). \tag{3.6}$$

The amplitude (3.6) was first proposed in [11] by using BRST cohomology arguments and proved by the field theory limit of the six-point superstring amplitude in [8]. For $n = 7$,

$$\mathcal{A}_7 = \langle V_1 M_{23456} V_7 \rangle + \langle M_{12} M_{3456} V_7 \rangle + \langle M_{123} M_{456} V_7 \rangle + \langle M_{1234} M_{56} V_7 \rangle + \langle M_{12345} V_6 V_7 \rangle.$$

Identifying $V_i V_j = E_{ij} = Q M_{ij}$ and using (2.9) leads to

$$\mathcal{A}_7 = \langle M_{123} M_{45} M_{67} \rangle + \langle M_{123} M_{456} V_7 \rangle + \langle M_{234} M_{56} M_{71} \rangle + \langle M_{345} M_{67} M_{12} \rangle + \langle M_{456} M_{71} M_{23} \rangle$$

$$+\langle M_{1234}(V_5 M_{67} + M_{56} V_7)\rangle + \langle M_{2345}(V_6 M_{71} + M_{67} V_1)\rangle + \langle M_{3456}(V_7 M_{12} + M_{71} V_2)\rangle,$$

where the generated factors of E_{12345} and E_{23456} have been replaced by M's using the definition (2.3). The maximum rank $M_{i_1\ldots i_4}$ only appear in combination with the BRST-exact superfield $E_{ijk} = V_i M_{jk} + M_{ij} V_k = QM_{ijk}$. Using (2.9) once again leads to a more compact expression with manifest cyclic symmetry,

$$\mathcal{A}_7 = \langle M_{123} M_{45} M_{67}\rangle + \langle V_1 M_{234} M_{567}\rangle + \text{cyclic}(1\ldots7). \tag{3.7}$$

Plugging the solutions (2.5) in (3.7) leads to the Ansatz of [11],

$$\mathcal{A}_7 = \langle V_1 \left(\frac{T_{234}}{s_{23}s_{234}} - \frac{T_{342}}{s_{34}s_{234}}\right)\left(\frac{T_{567}}{s_{56}s_{567}} - \frac{T_{675}}{s_{67}s_{567}}\right)\rangle$$
$$+\langle\left(\frac{T_{123}}{s_{12}s_{123}} - \frac{T_{231}}{s_{23}s_{123}}\right)\frac{T_{45}T_{67}}{s_{45}s_{67}}\rangle + \text{cyclic}(1\ldots7). \tag{3.8}$$

It is easy to check that (3.8) is expanded in terms of 42 kinematic poles.

The procedure to obtain manifestly cyclic symmetric higher-point amplitudes using (2.1) and (2.3) is straightforward and follows the same steps as above. Increasing the number of legs allows further BRST integrations by parts to be performed by identifying and integrating $E_{ij}, E_{ijk}, \ldots$ sucessively at each step, leading to

$$\mathcal{A}_8 = \langle M_{123} M_{456} M_{78}\rangle + \frac{1}{2}\langle M_{1234} E_{5678}\rangle + \text{cyclic}(1\ldots8),$$

$$\mathcal{A}_9 = \frac{1}{3}\langle M_{123} M_{456} M_{789}\rangle + \langle M_{1234}(M_{567} M_{89} + M_{56} M_{789} + M_{5678} V_9)\rangle + \text{cyclic}(1\ldots9),$$

$$\mathcal{A}_{10} = \langle M_{1234}(M_{567} M_{89;10} + M_{5678} M_{9;10})\rangle + \frac{1}{2}\langle M_{12345} E_{6789;10}\rangle + \text{cyclic}(1\ldots10). \tag{3.9}$$

4. Connection to superstring theory

Supersymmetric field theory tree–amplitudes can also be obtained from the low–energy limit of superstring theory where the dimensionless combinations $\alpha' s_{i_1\ldots i_p}$ of Regge slope α' and Mandelstam bilinears are formally sent to zero. Using the pure spinor formalism [4], we will argue in [7] that the full superstring $n-$point amplitude is given by

$$\mathcal{A}_n^{\text{string}}(\alpha') = \prod_{i=2}^{n-2} \int_{z_{i-1}}^{1} \mathrm{d}z_i \prod_{j<k} |z_{jk}|^{-2\alpha' s_{jk}} \sum_{p=1}^{n-2} \frac{\langle T_{12\ldots p}\, T_{n-1,p+1,\ldots,n-2}\, V_n\rangle}{(z_{12}z_{23}\ldots z_{p-1,p})(z_{n-1,p+1}z_{p+1,p+2}\ldots z_{n-3,n-2})}$$

$$+\text{permutations in}\,(2,3,...,n-2) \tag{4.1}$$

in terms of the ubiquitous building blocks $T_{12\ldots p}$, with $z_{jk}=z_j-z_k$. The $\alpha'\to 0$ limit of (4.1) reproduces $\mathcal{A}_n=\sum_{p=1}^{n-2}\langle M_{i_1\ldots i_p}M_{i_{p+1}\ldots i_{n-1}}V_n\rangle$ term by term in the individual p sums. Therefore considering $p=n-2\equiv q$ yields an explicit formula for $M_{i_1\ldots i_p}$:

$$M_{12\ldots q}=\lim_{\alpha'\to 0}\prod_{i=2}^{q}\int_{z_{i-1}}^{1}\mathrm{d}z_i\ \prod_{j<k}|z_{jk}|^{-2\alpha' s_{jk}}\left(\frac{T_{12\ldots q}}{z_{12}z_{23}\ldots z_{q-1,q}}+\text{permutations in}\,(2,3,...,q)\right). \tag{4.2}$$

It has been checked up to $q=6$ that the string inspired computation (4.2) of $M_{12\ldots q}$ is consistent with its construction from the Feynman diagrams in $\mathcal{A}_{q+1}$.

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