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Note on Bonus Relations for $\mathcal{N} = 8$ Supergravity Tree Amplitudes

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Abstract

We study the application of non-trivial relations between gravity tree amplitudes, the bonus relations, to all tree-level amplitudes in $\mathcal{N} = 8$ supergravity. We show that the relations can be used to simplify explicit formulae of supergravity tree amplitudes, by reducing the known form as a sum of $(n-2)!$ permutations obtained by solving on-shell recursion relations, to a new form as a $(n-3)!$-permutation sum. We demonstrate the simplification by explicit calculations of the next-to-maximally helicity violating (NMHV) and next-to-next-to-maximally helicity violating (N^2MHV) amplitudes, and provide a general pattern of bonus coefficients for all tree-level amplitudes.

I. INTRODUCTION

In the past several years there have been enormous progress in unraveling the structure of scattering amplitudes in gauge theory and gravity, such as generalized unitary-cut method at loop level [1], and Britto-Cachazo-Feng-Witten (BCFW) recursion relations at tree level, for Yang-Mills theory [2, 3] and for gravity [4–6]. A particularly important example is the structure of amplitudes in $\mathcal{N} = 4$ super Yang–Mills theory (SYM), which has remarkable simplicities obscured by the usual local formulation and Feynman-diagram calculations. On the other hand, Arkani-Hamed *et al.* have proposed the idea that $\mathcal{N} = 8$ supergravity (SUGRA) may be the quantum field theory with the simplest amplitudes [7], and there is strong evidence for it: recently there have been intensive studies on both the hidden symmetries (e.g. $E_{7(7)}$ symmetry, see [8–11]), and the ultraviolet behavior of the theory (see [12–20] and references therein).

However, we do not need to go beyond the tree level to see the simplicity. As shown in [7], gravity tree amplitudes satisfy non-trivial relations, or "bonus relations", which are absent in SYM color-ordered amplitudes. These bonus relations have been applied to MHV amplitudes in [21] to show the equivalence of various MHV formulae in the literature [4, 22–25], especially to simplify formulae with $(n-2)!$ permutations to those with $(n-3)!$ permutations. The full strength of these relations, however, can only be demonstrated when applied to general, non-MHV amplitudes, and the purpose of the present note is to use bonus relations to simplify explicit formulae of SUGRA tree amplitudes, which are obtained by solving BCFW recursion relations. Before proceeding, let us elaborate on BCFW recursion relations and bonus relations of SUGRA amplitudes.

Supersymmetric BCFW recursion relations [26, 27] hold in both SYM and SUGRA because their amplitudes vanish when two supermomenta are taken to infinity in a complex superdirection [7, 27]. More specifically, under the supersymmetric BCFW shifts of momenta and $SU(\mathcal{N})$ Grassmannian variables,

$$\begin{aligned}\lambda_{\widehat{1}}(z) &= \lambda_1 + z\lambda_n,\\ \widetilde{\lambda}_{\overline{n}}(z) &= \widetilde{\lambda}_n - z\widetilde{\lambda}_1,\\ \eta_{\widehat{n}}(z) &= \eta_n - z\eta_1,\end{aligned} \tag{1.1}$$

SYM and SUGRA amplitudes have at least $1/z$ falloff at large z, thus the contour integral $\oint \frac{dz}{z} M(z)$ can be rewritten as a sum over residues without boundary contributions,

$$M_n = \sum_{L,R} \int d^{4\mathcal{N}}\eta M_L(\widehat{1}, L, \{-\widehat{P}(z_P), \eta\}) \frac{1}{P^2} M_R(\{\widehat{P}(z_P), \eta\}, R, \overline{n}), \tag{1.2}$$

where the poles $z = z_P$ are determined by putting the internal momenta $\widehat{P}(z_P) = \sum_{i\in L} P_i + P_{\widehat{1}}$ on shell. By solving the recursion relations, explicit formulae for up to N^{3}MHV amplitudes, and an algorithm to calculate all tree amplitudes in SUGRA was proposed in [28]. The result can be written as a summation over $(n-2)!$ "ordered gravity subamplitudes" with different permutations of particles $2, \ldots, n-1$. In contrast to SYM color-ordered amplitudes, the SUGRA amplitudes actually have a faster, $1/z^2$, falloff and the contour integral $\oint dz M(z)$ gives the bonus relations,

$$0 = \sum_{L,R} \int d^8\eta M_L(\widehat{1}, L, \{-\widehat{P}(z_P), \eta\}) \frac{z_P}{P^2} M_R(\{\widehat{P}(z_P), \eta\}, R, \overline{n}). \tag{1.3}$$

Similar to the MHV case [21], we shall see that these relations can further simplify the explicit formulae for non-MHV amplitudes by reducing the $(n-2)!$-permutation sum to a new $(n-3)!$-permutation one.

Another important method that has been widely used to calculate gravity tree amplitudes are Kawai-Lewellen-Tye (KLT) relations, first derived in string theory [29] which express (super)gravity tree amplitudes as sums of products of two copies of (super)Yang–Mills amplitudes in the field-theory limit. Recently KLT relations have been proved in gravity [30, 31] and in SUGRA [32] using BCFW recursion relations, without resorting to string theory. While the well-known KLT relations have a form of $(n-3)!$ permutations [33] (see also [31]), in the proof it is natural to use the newly proposed $(n-2)!$ form suitable for BCFW recursion relations [30], and a direct link between these two forms has been derived in [34]. In a related approach, the so-called square relations between gravity and Yang-Mills amplitudes, which can be viewed as a reformulation of KLT relations, have been proposed and proved in [35]. These relations also possess a freedom of going from $(n-2)!$-permutation form to the simpler $(n-3)!$ form, which, similar to the freedom in KLT relations, reflects the Bern-Carrasco-Johansson (BCJ) relations between Yang-Mills amplitudes [35]. For SUGRA amplitudes, the advantage of having solved BCFW relations to some extent will enable us to go beyond this implicit freedom following from BCJ relations, and show the simplification of gravity amplitudes directly in their explicit forms.

The note is organized as following. In section 2 we briefly review tree amplitudes in SUGRA and their bonus relations, especially the simplification of MHV amplitudes when using these relations. Then we apply these relations to some examples beyond MHV amplitudes, including the NMHV and N^{2}MHV amplitudes, and prove these simplified formulae in section 3. The generalization to all tree-level SUGRA amplitudes are presented in section 4.

II. A BRIEF REVIEW OF TREE AMPLITUDES IN SUGRA AND BONUS RELATIONS

A. Tree Amplitudes in SUGRA from BCFW Recursion Relations

By solving Eq. (1.2), all color-ordered SYM tree amplitudes have been obtained and can be written schematically as [36],

$$A_n(1,...,n) = A^{\rm MHV}(1,\ldots,n)\sum_\alpha R_\alpha(1,\ldots,n), \tag{2.1}$$

where

$$A^{\rm MHV}(1,\ldots,n) = \frac{\delta^8(\sum_i \lambda_i\eta_i)}{\langle 12\rangle\langle 23\rangle\cdots\langle n1\rangle} \tag{2.2}$$

is the MHV superamplitudes, and R_α are the so-called dual superconformal invariants, which, for N^{k}MHV amplitudes, are products of k basic invariants of the form,

$$R_{n;a_1b_1;a_2b_2;\ldots;a_rb_r;ab} = \frac{\langle a a-1\rangle\langle b b-1\rangle\ \delta^{(4)}(\langle\xi|x_{b_ra}x_{ab}|\theta_{bb_r}\rangle + \langle\xi|x_{b_rb}x_{ba}|\theta_{ab_r}\rangle)}{x_{ab}^2\langle\xi|x_{b_ra}x_{ab}|b\rangle\langle\xi|x_{b_ra}x_{ab}|b-1\rangle\langle\xi|x_{b_rb}x_{ba}|a\rangle\langle\xi|x_{b_rb}x_{ba}|a-1\rangle}\,, \tag{2.3}$$

where the chiral spinor ξ is given by

$$\langle\xi| = \langle n|x_{na_1}x_{a_1b_1}x_{b_1a_2}x_{a_2b_2}\ldots x_{a_rb_r}\,, \tag{2.4}$$

and dual (super)coordinates are defined as

$$\begin{aligned} x_{ij} &= p_i + p_{i+1} + \cdots + p_{j-1}\,,\\ \theta_{ij} &= \lambda_i\eta_i + \cdots + \lambda_{j-1}\eta_{j-1}\,. \end{aligned} \tag{2.5}$$

There is only one invariant $R = 1$ for MHV case, while we have a sum of $R_{n;a_1b_1}$ with $1 < a_1 < b_1 < n$ for NMHV case. Furthermore, for N^{2}MHV case we have $R_{n;a_1b_1}R^{b_1a_1}_{n;a_1b_1,a_2b_2}$ with $1 < a_1 < a_2 < b_2 \leq b_1 < n$ and $R_{n;a_1b_1}R^{a_1b_1}_{n;a_2b_2}$ with $1 < a_1 < b_1 \leq a_2 < b_2 < n$, where superscripts denote boundary modifications of these invariants [36].

Generally the summation variables α, and boundary modifications, can be represented by a rooted tree diagram [28, 36] (see Fig. 1 and Fig. 2). For N^{k}MHV amplitudes, there are $C_k = \frac{(2k)!}{k!(k+1)!}$ (Catalan number) types of terms labeled by α's corresponding to a path from the root to the k-th level in Fig. 1, and each type can be written as a list of k pairs of labels with a particular order between them, $\alpha \equiv \{n; a_1, b_1; \ldots; a_k, b_k\}$. Not only does the summation over α include all types of terms, but it also sums over all possible $1 < a_i, b_i < n$ in the corresponding order.

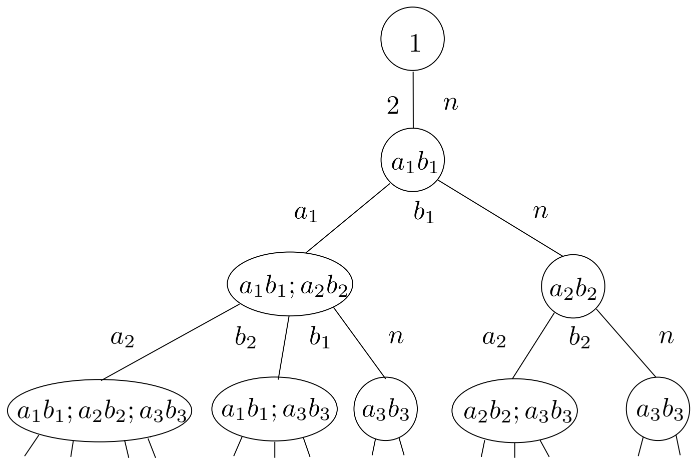

FIG. 1: An rooted tree diagram for tree-level SYM amplitudes. The figure is the same as the tree diagram presented in [28].

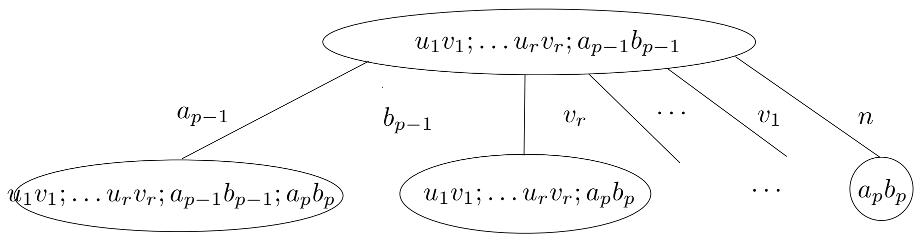

FIG. 2: The rule for going from line $p-1$ to line p (for $p > 1$) in Fig. 1. For every vertex in line $p-1$ of the form given at the top of the diagram, there are $r+2$ vertices in the lower line (line p). The labels in these vertices start with $u_1v_1; \ldots u_rv_r; a_{p-1}b_{p-1}; a_pb_p$ and they get sequentially shorter, with each step to the right removing the pair of labels adjacent to the last pair a_p, b_p until only the last pair is left. The summation limits between each line are also derived from the labels of the vertex above. The left superscripts which appear on the associated R-invariants start with $u_1v_1 \ldots u_rv_rb_{p-1}a_{p-1}$ for the left-most vertex. The next vertex to the right has the superscript $u_1v_1 \ldots u_rv_ra_{p-1}b_{p-1}$, i.e. the same as the first but with the final pair in alphabetical order. The next vertex has the superscript $u_1v_1 \ldots u_rv_r$ and thereafter the pairs are sequentially deleted from the right.

In [28], solving Eq. (1.2) for SUGRA is simplified by using ordered gravity subamplitude $M(1, \ldots, n)$, which satisfy the ordered BCFW recursion relations similar to Yang-Mills theory,

$$M(1, \ldots, n) \equiv \sum_{i=3}^{n-1} \int \frac{d^8\eta}{P^2} M(\widehat{1}, 2, \ldots, i-1, \widehat{P}) M(-\widehat{P}, i, \ldots, n-1, \overline{n}) \,, \tag{2.6}$$

and the sum of $(n-2)!$ permutations of ordered gravity subamplitudes gives the full amplitude,

$$\mathcal{M}_n = \sum_{\mathcal{P}(2,3,\ldots,n-1)} M(1, \ldots, n). \tag{2.7}$$

A solution for $M(1,\ldots,n)$ is obtained in [28],

$$M(1,...,n) = [A^{\rm MHV}(1,\ldots,n)]^2 \sum_\alpha G_\alpha R_\alpha^2(1,\ldots,n), \tag{2.8}$$

where the invariants R_α are exactly the same as those in SYM (including boundary modifications), namely products of basic invariants (2.3), with the same set of summation variables α as given in Fig. 1 and Fig. 2, and the 'dressing factors', G_α, are independent of the Grassmannian variables η_i, and they break dual conformal invariance of the SYM solution. These factors have been calculated explicitly for up to N^{3}MHV amplitudes, for example MHV case,

$$G^{\rm MHV}(1,\ldots,n) = x_{13}^2 \prod_{s=2}^{n-3} \frac{\langle s|x_{s,s+2}x_{s+2,n}|n\rangle}{\langle sn\rangle}\,, \tag{2.9}$$

and there is an algorithm to calculate them in general cases, but we do not need their expressions in this note. In addition, tree-level amplitudes of n-graviton scattering can be obtained from SUGRA superamplitudes (2.7), by choosing fermionic coordinates $\eta = 0$ for positive-helicity gravitons, and integrating over $d^8\eta$ for negative-helicity ones. Details of the solution can be found in [28].

Therefore, SUGRA tree amplitude can be written as a summation of $(n-2)!$ ordered gravity subamplitudes, and each of them has a structure similar to SYM ordered amplitude. In the following we shall use bonus relations to reduce this form to a simpler, $(n-3)!$ form, and first we recall the simplest MHV case.

B. Applying Bonus Relations to MHV Amplitudes

Applying bonus relation to MHV SUGRA tree-level amplitudes was well understood in [21]. From Eq. (2.9), we have the MHV amplitudes as a summation of $(n-2)!$ terms,

$$\mathcal{M}_n^{\rm MHV} = G^{\rm MHV}(1,\ldots n)[A^{\rm MHV}(1,\ldots,n)]^2 + \mathcal{P}(2,3,\ldots,n-1). \tag{2.10}$$

From Fig. 3, we see that there are $(n-2)$ BCFW factorizations and thus the formula can be expressed as,

$$\mathcal{M}_n^{\rm MHV} = M_2 + M_3 + \ldots + M_{n-1}, \tag{2.11}$$

where each M_i is a BCFW term from $\overline{\rm MHV}(\widehat{1}, i, \widehat{P}(z_i)) \times {\rm MHV_{n-1}}$ with $z_i = -\frac{\langle 1i\rangle}{\langle ni\rangle}$. Now since the amplitude has $1/z^2$ fall off, we have a bonus relation which is simple in the MHV case,

$$0 = z_2 M_2 + z_3 M_3 + \ldots + z_{n-1} M_{n-1}. \tag{2.12}$$

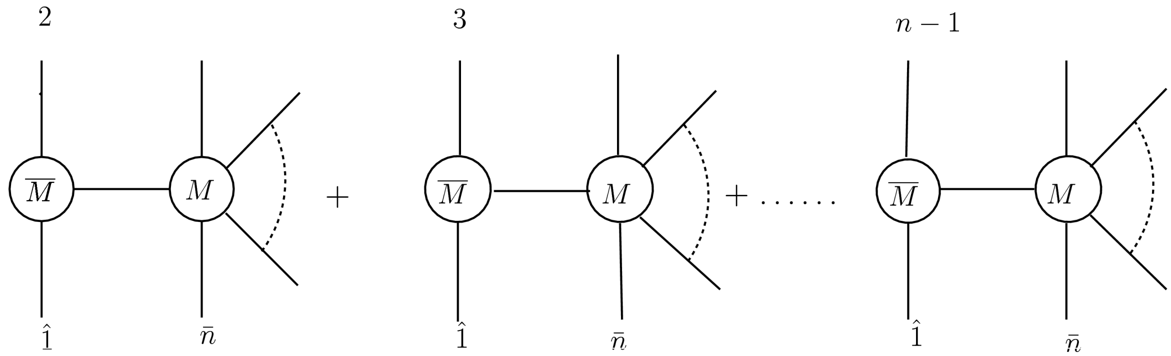

FIG. 3: All factorizations contributing to (2.11) for the MHV amplitude.

Using this relation, we can express the last diagram M_{n-1} in terms of the other $n-3$ diagrams, and a simple manipulation gives us a $(n-3)!$-term formula,

$$\begin{aligned}\mathcal{M}_n^{\mathrm{MHV}} &= B^{\mathrm{MHV}} G^{\mathrm{MHV}}(1,2,\ldots,n)[A^{\mathrm{MHV}}(1,2,\ldots,n)]^2 \\ &+ \mathcal{P}(2,3,\ldots,n-2).\end{aligned} \tag{2.13}$$

where we have defined the MHV bonus coefficient $B^{\mathrm{MHV}} = \frac{\langle 1\ n\rangle\langle n{-}1\ n{-}2\rangle}{\langle 1\ n{-}1\rangle\langle n\ n{-}2\rangle}$. Beyond MHV, we have many more types of BCFW diagrams with complicated structures and the application of bonus relations becomes trickier. In the next section, we shall work out the NMHV and N^{2}MHV cases, and then move on to general amplitudes in section 4.

III. APPLYING BONUS RELATIONS TO NON-MHV GRAVITY TREE AMPLITUDES

A. General Strategy

Before moving on to examples, we first explain the general strategy for applying bonus relations to non-MHV gravity tree amplitudes. For a N^{k}MHV amplitude, inhomogeneous contributions of the form N^{p}MHV $\times$ N^{q}MHV are needed $(p+q+1=k)$[1]. Naively one would like to use "bonus-simplified"[2] lower-point amplitudes for both M_L and M_R in Eq. (1.2), but this is not compatible with the fact that we can only delete one diagram (not two) by applying the bonus relations (1.3), if we want to preserve the structure of ordered BCFW recursion relations.

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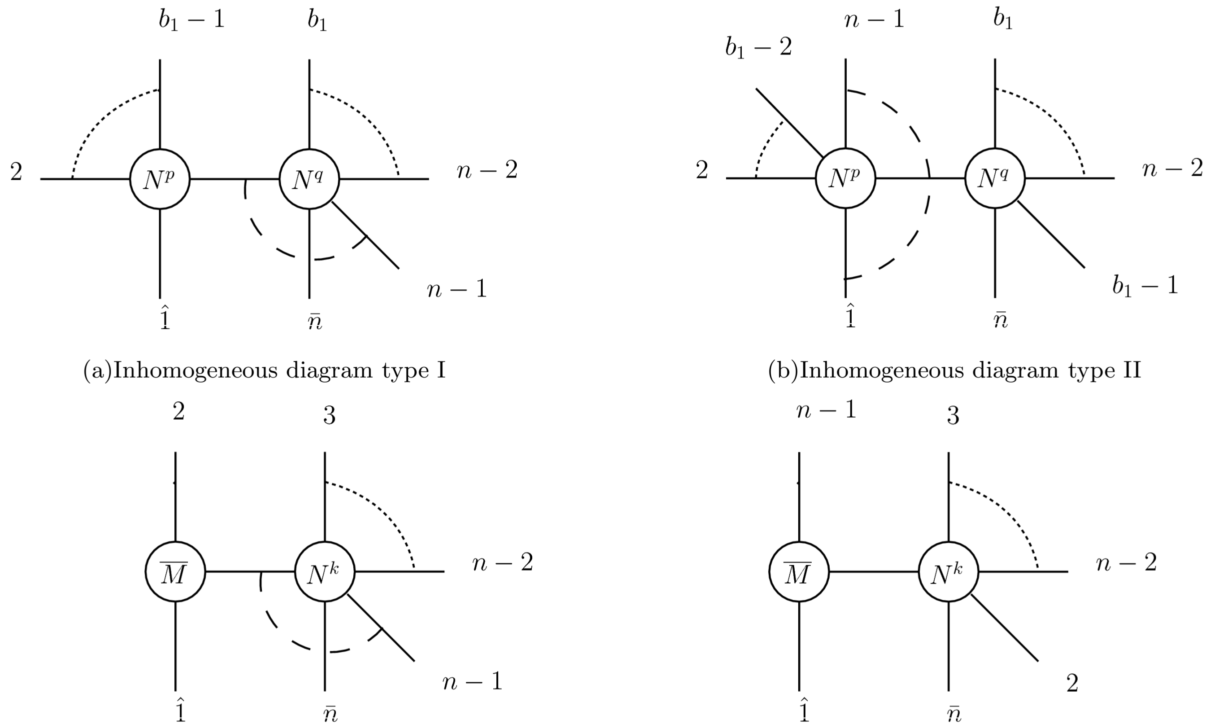
[1] We follow the notations of reference [36] to call the contributions from diagrams of type Fig. 4(a) or Fig. 4(b) as inhomogeneous contributions, while those from Fig. 4(c) as homogeneous ones.

[2] Here "bonus-simplified" means that these lower-point amplitudes used in the BCFW diagrams are simplified by using bonus relations.



(a)Inhomogeneous diagram type I

(b)Inhomogeneous diagram type II

(c)Homogeneous diagram

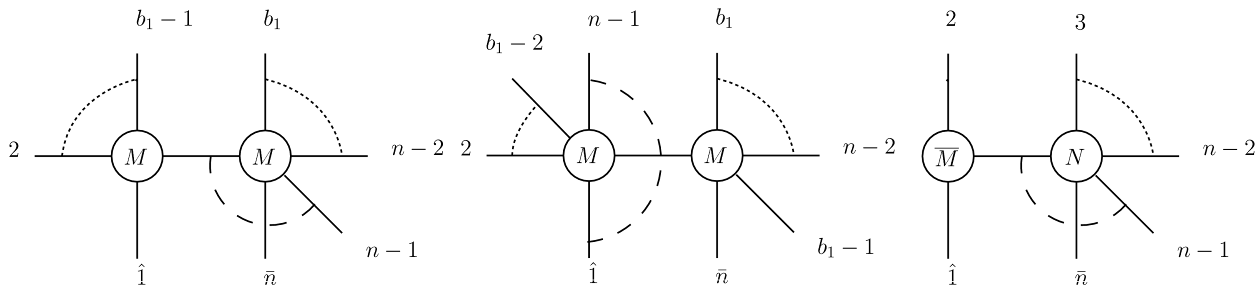
(d)Diagram deleted by bonus relations

FIG. 4: Different types of diagrams for a general N^{k}MHV amplitude, where $k = p + q + 1$. We use a dashed line $----$ connecting three legs to denote a bonus-simplified lower-point amplitude, in which these three legs are kept fixed. For lower-point amplitudes without dashed lines, we use the usual $(n-2)!$ form.

To keep the advantages of the ordered BCFW recursion relations, which are crucial to solve for all tree-level amplitudes, instead we shall apply bonus relations selectively. The idea is illustrated in Fig. 4. Similar to the MHV case, we shall delete Fig. 4(d) by using bonus relations (1.3). To compute the inhomogeneous parts of the amplitudes, we shall use the bonus-simplified amplitude only on one side of a BCFW diagram, namely the lower-point amplitude with the leg $(n-1)$ in it, as indicated in Fig. 4(a) and Fig. 4(b). In this way, the amplitude splits into two types, one type coming from the diagrams of the form as in Fig. 4(a), which has the leg $(n-1)$ adjacent to the leg n and will be called the normal, or type I contributions, and the other one coming from those having the form as in Fig. 4(b), which has the leg $(n-1)$ exchanged with another leg (b_1-1), and will be called the exchanged, or type II contributions,

$$\mathcal{M}_n = \left[A_n^{\mathrm{MHV}}\right]^2 \Big(\sum_{\alpha} B_{\alpha}^{(1,m_1)} G_{\alpha} R_{\alpha}^2 + \sum_{\beta} B_{\beta}^{(2,m_2)} [G_{\beta} R_{\beta}^2 (b_1 - 1 \leftrightarrow n-1)] \Big) + \mathcal{P}(2,3,\ldots,n-2), \tag{3.1}$$

where $(b_1 - 1 \leftrightarrow n-1)$ denotes the exchanges of momenta $(p_{b_1-1} \leftrightarrow p_{n-1})$ as well as the fermionic



(a)Inhomogeneous diagram type I (b)Inhomogeneous diagram type II (c)Homogeneous diagram

FIG. 5: Diagrams for NMHV amplitudes.

coordinates ($\eta_{b_1-1} \leftrightarrow \eta_{n-1}$), and we have used square bracket to indicate that the exchanges act only on the expression inside the bracket. The superscript (i, m_i) in $B_\alpha^{(i,m_i)}$ is used to show the type of this contribution, which will become clear in the examples.

Thus we have seen that, by using bonus relations, any amplitude can be written as a summation of $(n-3)!$ permutations with the coefficients $B_\alpha^{(i,m_i)}$, which will be called bonus coefficients. In this section, we shall calculate all bonus coefficients for NMHV and N^{2}MHV cases, and generalize the pattern observed in these examples to general N^{k}MHV amplitudes in the next section. Once bonus coefficients are calculated, we obtain explicitly all simplified SUGRA tree amplitudes.

B. NMHV Amplitudes

Here we use bonus relations to simplify the $(n-2)!$ form of NMHV amplitudes. First we shall state the general simplified form of NMHV amplitudes, and then prove it by induction. To be concise, we abbreviate the combinations

$$\{n; a_1 b_1\} \equiv G_{n;a_1 b_1} \left[R_{n;a_1 b_1} A^{\text{MHV}}(1, 2, \ldots, n)\right]^2 \tag{3.2}$$

and similar notations will be used in the following sections.

As mentioned above generally, we delete the contributions corresponding to Fig. 4(d) by using the bonus relation (1.3). It is straightforward to compute the inhomogeneous contributions from the two MHV $\times$ MHV diagrams, Fig. 5(a) and Fig. 5(b). Firstly, let us consider the contribution from Fig. 5(a), which corresponds to terms with $a_1 = 2$, and we have

$$M_1 = B^{(1)}_{n;2b_1}\{n; 2b_1\}, \quad \text{with} \;\; 4 \leq b_1 \leq n-1, \tag{3.3}$$

where $B^{(1)}_{n;2b_1}$ are the special cases of the general bonus coefficients $B^{(1)}_{n;a_1 b_1}$. We have used the superscript (1) to indicate that this is the contribution coming from type-I diagram, and similar notations will be used below.

When $b_1 \neq n-1$, the bonus coefficients are given by,

$$B^{(1)}_{n;a_1b_1} = B^{\mathrm{MHV}} \frac{\langle n-1|x_{b_1a_1}x_{b_1n}|n\rangle}{\langle n-1|x_{b_1a_1}x_{a_1n}|n\rangle}. \tag{3.4}$$

Here we note that we can get the above coefficients from the previous ones, namely the bonus coefficients of MHV amplitude, multiplied by the factor $\frac{\langle n-1|x_{b_1a_1}x_{b_1n}|n\rangle}{\langle n-1|x_{b_1a_1}x_{a_1n}|n\rangle}$. It is a general feature of this type of coefficients, for N^{k}MHV case they are given by N^{k-1}MHV coefficients multiplied by the same factor, as we shall see explicitly again in the N^{2}MHV case.

However when $b_1 = n-1$, no bonus relation can be used for the right-hand-side 3-point MHV amplitude in Fig. 5(a), and we find

$$B^{(1)}_{n;a_1n-1} = \frac{\langle 1\ n\rangle}{\langle 1\ n-1\rangle} \frac{\langle n-1|x_{n-1a_1}|n-1]}{\langle n|x_{na_1}|n-1]}. \tag{3.5}$$

For the exchanged diagrams, Fig. 5(b), the contribution can be similarly obtained

$$M_2 = B^{(2)}_{n;2b_1}[\{n;2a_1\}(b_1-1 \leftrightarrow n-1)], \quad \text{with} \ \ 4 \leq b_1 \leq n-1, \tag{3.6}$$

where the bonus coefficients $B^{(2)}_{n;a_1b_1}$ are given by

$$B^{(2)}_{n;a_1b_1} = \frac{\langle 1\ n\rangle}{\langle 1\ n-1\rangle} \frac{\langle n-1\ b_1-2\rangle (x'_{a_1b_1})^2}{\langle n|x_{na_1}x'_{a_1b_1}|b_1-2\rangle}, \tag{3.7}$$

and we have defined $x'_{a_ib_i}$ as,

$$\begin{aligned} x'_{a_ib_i} &\equiv x_{a_ib_i-1} + x_{n-1n} \\ &= x_{a_ib_i}(p_{b_i-1} \leftrightarrow p_{n-1}). \end{aligned} \tag{3.8}$$

All the above calculations do not include the boundary case $a_1 = n-3, b_1 = n-1$, which needs special treatment. This boundary case is special because it recursively reduces to the special 5-point NMHV ($\overline{\mathrm{MHV}}$) amplitude. It does not have the diagram of the type $\overline{\mathrm{MHV}}_3\times$ NMHV, and one has to treat it separately. We apply the bonus relations to this case in the following way: we use Eq. (1.3) to delete the contribution from Fig. 6(a), and compute Fig. 6(b), and we find

$$\mathcal{M}_5 = -\frac{[24][34][51]}{[23][45][41]}\big[\{5;24\}(3\leftrightarrow 4)\big] + \mathcal{P}(2,3). \tag{3.9}$$

By plugging the above 5-point result in Fig. 6(c), we get the boundary term of the 6-point NMHV amplitude

$$M_6^{(\mathrm{boundary})} = \frac{\langle 16\rangle\langle 25\rangle[35][45]x^2_{36}}{\langle 15\rangle[34]\langle 2|1+6|5]\langle 6|1+2|5]}\big[\{6;35\}(4\leftrightarrow 5)\big]. \tag{3.10}$$

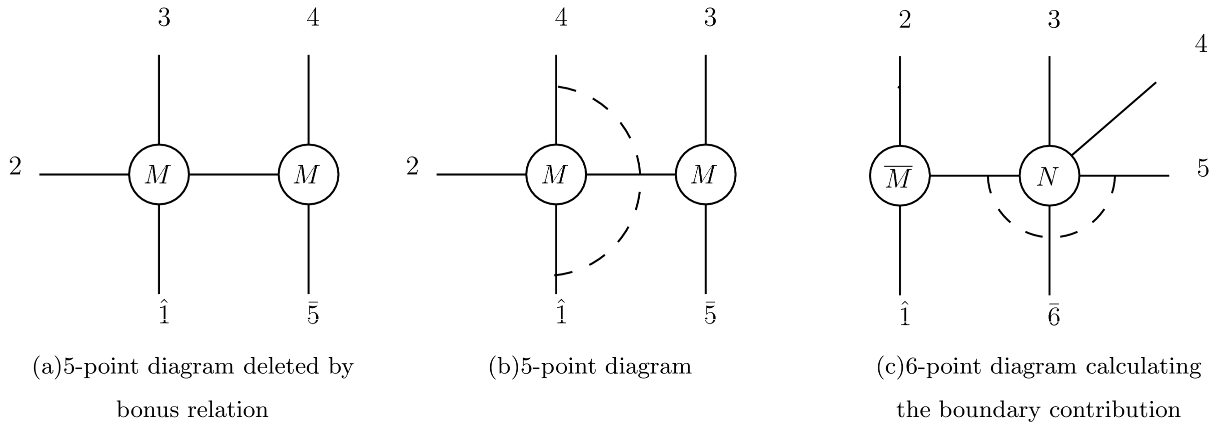


FIG. 6: Diagrams for 5-point NMHV amplitude and the boundary term of 6-point NMHV amplitude. Fig. 6(a) and Fig. 6(b) are used to calculate the bonus-simplified, 5-point, right-hand-side amplitude of Fig. 6(c).

A generic form for the boundary term of the n-point NMHV amplitudes can be obtained as a straightforward generalization of (3.9) and (3.10),

$$M_n^{(\text{boundary})} = B^{(\text{boundary})}_{n;n-3\ n-1}\big[\{n;n-3\ n-1\}(n-2 \leftrightarrow n-1)\big], \tag{3.11}$$

where $B^{(\text{boundary})}_{n;n-3\ n-1}$ is given by,

$$B^{(\text{boundary})}_{n;n-3\ n-1} = \frac{\langle 1n\rangle\langle n-4\ n-1\rangle[n-3\ n-1][n-2\ n-1]x^2_{n-3n}}{\langle 1\ n-1\rangle[n-3\ n-2]\langle n-4|x_{n-3\ n-1}|n-1]\langle n|x_{n-1\ n-3}|n-1]}. \tag{3.12}$$

Putting everything together, we obtain the general formula for NMHV amplitude and as promised, the amplitude indeed can be written as a sum of $(n-3)!$ permutations

$$\begin{aligned}\mathcal{M}_n^{\text{NMHV}} \;=\; &\sum_{a_1=2}^{n-4}\sum_{b_1=a_1+2}^{n-1}\Big(B^{(1)}_{n;a_1b_1}\{n;a_1b_1\} + B^{(2)}_{n;a_1b_1}[\{n;a_1b_1\}(b_1-1 \leftrightarrow n-1)]\Big) + M_n^{(\text{boundary})}\\ &+\mathcal{P}(2,3,\ldots,n-2).\end{aligned} \tag{3.13}$$

1. **Proof by Induction**

Here we shall give an inductive proof for the simplified NMHV formula. For $a_1 = 2$, as we explained above, the formula follows directly from Fig. 5(a) and Fig. 5(b). Therefore we shall focus on the cases when $a_1 \geq 3$, which correspond to the homogeneous contributions from Fig. 5(c). We shall prove that the formula satisfies the BCFW recursion relations.

First note that we have deleted one diagram of the form $M_L^{\text{MHV}}(\hat{1}, n-1, \hat{P}) \times M_R^{\text{MHV}}$ by using bonus relations, this results in a multiplicative prefactor for the overall amplitude, which is given by,

$$(1 - \frac{z_2}{z_{n-1}}) = \frac{\langle 1n\rangle\langle n-1\ 2\rangle}{\langle n2\rangle\langle 1n-1\rangle}. \tag{3.14}$$

Let us consider the bonus coefficient $B^{(1)}_{n;a_1b_1}$, other coefficients $B^{(2)}_{n;a_1b_1}$ and $B^{(\text{boundary})}_{n;n-3\ n-1}$ can be treated similarly. By plugging formula (3.4) into the $(n-1)$-point amplitude $M(-\hat{P},3,4,\ldots,n-1,\bar{n})$ in Fig. 5(c), it is straightforward to check that the second piece of $B^{(1)}_{n;a_1b_1}$, $\frac{\langle n-1|x_{b_1a_1}x_{b_1n}|n\rangle}{\langle n-1|x_{b_1a_1}x_{a_1n}|n\rangle}$, is transformed back to itself under the recursion relations.

For the first piece $B^{\text{MHV}} = \frac{\langle n-1\ n-2\rangle\langle 1\ n\rangle}{\langle n\ n-2\rangle\langle 1\ n-1\rangle}$ of $B^{(1)}_{n;a_1b_1}$, which is the MHV bonus coefficient, the proof is essentially the same as in the MHV case. Taking into account the factor in (3.14) coming from bonus relations, we have

$$\frac{\langle n-1\ n-2\rangle\langle \hat{p}\ n\rangle}{\langle n\ n-2\rangle\langle \hat{p}\ n-1\rangle} \times \frac{\langle 1\ n\rangle\langle n-1\ 2\rangle}{\langle 1\ n-1\rangle\langle n\ 2\rangle} = \frac{\langle n-1\ n-2\rangle\langle 1\ n\rangle}{\langle n\ n-2\rangle\langle 1\ n-1\rangle}. \tag{3.15}$$

Thus the contribution with $B^{(1)}_{n;a_1b_1}$ indeed satisfies the recursion relations.

A final remark is in order. We have used in the proof that $\{n;a_1b_1\}$ satisfy the ordered BCFW recursion relations by themselves.

C. N^{2}MHV amplitudes

In this subsection we consider N^{2}MHV amplitudes as one more example to show the general features of bonus-simplified gravity amplitudes. Similar to NMHV case, let us denote the ordered gravity solutions in the following way

$$\begin{aligned} H^{(1)}_{n;a_1b_1,a_2b_2}\big[R_{n;a_1b_1}R^{b_1a_1}_{n;a_1b_1,a_2b_2}A^{\text{MHV}}(1,2,\ldots,n)\big]^2 &\equiv \quad \{n;a_1b_1,a_2b_2\}_1, \\ H^{(2)}_{n;a_1b_1,a_2b_2}\big[R_{n;a_1b_1}R^{a_1b_1}_{n;a_2b_2}A^{\text{MHV}}(1,2,\ldots,n)\big]^2 &\equiv \quad \{n;a_1b_1,a_2b_2\}_2. \end{aligned}$$

There are four relevant types of diagrams (and a boundary case) which contribute to the general N^{2}MHV amplitudes. The general structure of N^{2}MHV is given in Fig. 7 and the corresponding contributions from each of the four diagrams can be calculated separately.

First we consider the contributions from the diagrams in Fig. 7(b), which are of the form MHV× NMHV. We use bonus-simplified amplitude for the right-hand-side NMHV amplitude and we obtain[3],

$$
\begin{aligned}
M_{\rm I} \;=\; & \sum_{2\le a_1,b_1\le n-1}\;\sum_{b_1\le a_2,b_2<n}\Big(B^{(1,1)}_{n;a_1b_1;a_2b_2}\{n;a_1b_1;a_2b_2\}_2 \\
& + B^{(1,2)}_{n;a_1b_1;a_2b_2}[\{n;a_1b_1;a_2b_2\}_2(b_2-1\leftrightarrow n-1)]\Big) \\
& + \sum_{2\le a_1,b_1\le n-1} B^{(1,\text{boundary})}_{n;a_1b_1;n-3n-1}[\{n;a_1b_1;n-3n-1\}_2(n-2\leftrightarrow n-1)], \qquad (3.16)
\end{aligned}
$$

where in the first sum $a_2\le n-4$ because of the range of summation of the first term in Eq. (3.13). Here the bonus coefficients are given by

$$
\begin{aligned}
B^{(1,1)}_{n;a_1b_1;a_2b_2} &= \frac{\langle 1n\rangle\langle n-1\ n-2\rangle\langle n-1|x_{a_2b_2}x_{b_2n}|n\rangle}{\langle 1n-1\rangle\langle n\ n-2\rangle\langle n-1|x_{a_2b_2}x_{a_2n}|n\rangle}\frac{\langle n-1|x_{a_1b_1}x_{b_1n}|n\rangle}{\langle n-1|x_{a_1b_1}x_{a_1n}|n\rangle} \\
B^{(1,1)}_{n;a_1b_1;a_2b_2} &= \frac{\langle 1n\rangle\langle n-1|x_{n-1a_2}|n-1]}{\langle 1n-1\rangle\langle n|x_{na_2}|n-1]}\frac{\langle n-1|x_{a_1b_1}x_{b_1n}|n\rangle}{\langle n-1|x_{a_1b_1}x_{a_1n}|n\rangle}\quad (b_2=n-1) \\
B^{(1,2)}_{n;a_1b_1;a_2b_2} &= \frac{\langle 1n\rangle\langle n-1\ b_2-2\rangle(x'_{a_2b_2})^2}{\langle 1n-1\rangle\langle n|x_{na_2}x'_{a_2b_2}|b_2-2\rangle}\frac{\langle n-1|x_{a_1b_1}x_{b_1n}|n\rangle}{\langle n-1|x_{a_1b_1}x_{a_1n}|n\rangle} \\
B^{(1,\text{boundary})}_{n;a_1b_1;n-3n-1} &= B^{(\text{boundary})}_{n;n-3\ n-1}\frac{\langle n-1|x_{a_1b_1}x_{b_1n}|n\rangle}{\langle n-1|x_{a_1b_1}x_{a_1n}|n\rangle}, \qquad (3.17)
\end{aligned}
$$

where the last term $B^{(1,\text{boundary})}_{n;a_1b_1;n-3n-1}$ comes from Eq. (3.12). Again the superscripts are used to show the types of the contributions. For instance, in the superscript $(1,1)$ of $B^{(1,1)}_{n;a_1b_1;a_2b_2}$, the first "1" means that it is the type-I contribution, while the second "1" implies that it is a descendant from the NMHV case. A generalization to the N^{k}MHV case will be $B^{(m)}_{n;a_1b_1;\ldots;a_kb_k}$, where m is a string composed of three kinds of labels, "1" "2" and "boundary".

As we have mentioned in the NMHV case, and we want to stress it here again that the bonus coefficients of Fig. 7(b) are simply given as the previous ones, namely the coefficients of NMHV amplitudes, with replacements $(a_1\to a_2, b_1\to b_2)$ and multiplied by the same factor $\frac{\langle n-1|x_{a_1b_1}x_{b_1n}|n\rangle}{\langle n-1|x_{a_1b_1}x_{a_1n}|n\rangle}$.

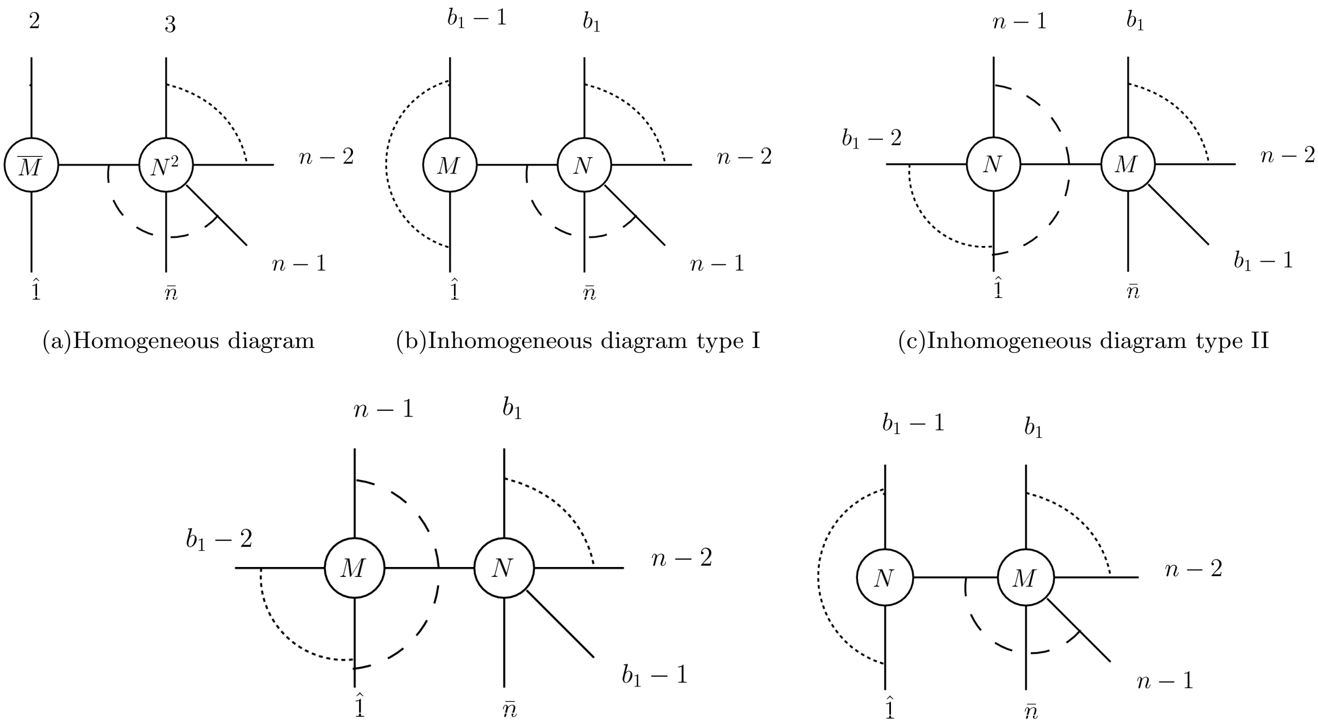
Next, we calculate the contributions from the diagrams in Fig. 7(c) which are of the form NMHV$\times$ MHV and we get

$$
\begin{aligned}
M_{\rm II} = & \sum_{2\le a_1,b_1\le n-1}\;\sum_{a_1\le a_2,b_2<b_1}\Big(B^{(2,1)}_{n;a_1b_1;a_2b_2}\{n;a_1b_1;a_2b_2\}_1(n-1\leftrightarrow b_1-1) \\
& + B^{(2,2)}_{n;a_1b_1;a_2b_2}[\{n;a_1b_1;a_2b_2\}_1(b_2-1\leftrightarrow b_1-1)]\Big) \qquad (3.18) \\
& + \sum_{2\le a_1\le n-3} B^{(2,\text{boundary})}_{n;a_1n-1;n-4n-2}[\{n;a_1n-1;n-3n-1\}_1(n-2\leftrightarrow n-1)].
\end{aligned}
$$

In the above sum we do not include the boundary case $(a_1,b_1,a_2,b_2)=(n-4,n-1,n-4,n-2)$,

---

[3] Here and in the following calculations we have included the corresponding homogeneous terms, for the case we consider the contributions are from Fig. 7(a).



(a)Homogeneous diagram (b)Inhomogeneous diagram type I (c)Inhomogeneous diagram type II

(d)Inhomogeneous diagram type II (e)Inhomogeneous diagram type I

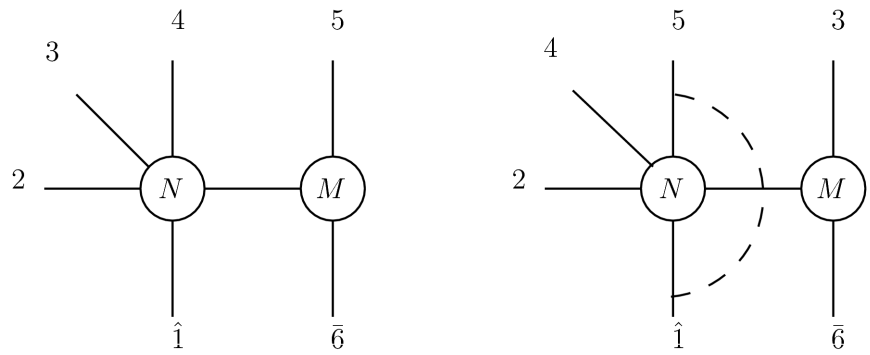
FIG. 7: Diagrams for N^{2}MHV amplitudes.

which we shall study separately. The coefficients are given by

$$
\begin{aligned}
B^{(2,1)}_{n;a_1b_1;a_2b_2} &= \frac{\langle 1n\rangle\langle n-1\ b_1-2\rangle\langle n-1|x_{b_2a_2}x'_{b_2b_1}x'_{a_1b_1}x_{a_1n}|n\rangle(x'_{a_1b_1})^2}{\langle 1n-1\rangle\langle b_1-2|x'_{a_1b_1}x_{a_1n}|n\rangle\langle n-1|x_{b_2a_2}x'_{a_2b_1}x'_{a_1b_1}x_{a_1n}|n\rangle} \\
B^{(2,1)}_{n;a_1b_1;a_2b_2} &= \frac{\langle 1n\rangle\langle n-1|x_{n-1a_2}|n-1](x'_{a_1b_1})^2}{\langle 1n-1\rangle\langle n|x_{na_1}x'_{a_1b_1}x'_{b_1a_2}|n-1]} \quad (b_2=n-2) \\
B^{(2,2)}_{n;a_1b_1;a_2b_2} &= \frac{\langle 1n\rangle\langle n-1\ b_2-2\rangle(x'_{a_2b_2})^2(x'_{a_1b_1})^2}{\langle 1n-1\rangle\langle n|x_{na_1}x'_{a_1b_1}x'_{b_1a_2}x'_{a_2b_2}|b_2-2\rangle} \\
B^{(2,\text{boundary})}_{n;a_1b_1;n-4n-2} &= \frac{\langle 1n\rangle\langle b_1-4\ n-1\rangle[b_1-3\ n-1][b_1-2\ n-1](x'_{b_1-3b_1})^2(x'_{a_1b_1})^2}{\langle 1\ n-1\rangle[b_1-3\ b_1-2]\langle b_1-4|x_{b_1-4\ b_1-1}|n-1]\langle n|x_{na_1}x'_{a_1b_1}x_{b_1-1b_1-3}|n-1]}.
\end{aligned}
\tag{3.19}
$$

By comparing the results with those of NMHV, now we are ready to see the patterns. For this type of diagrams Fig. 7(c), the bonus coefficients can be obtained from the results of NMHV by doing the following replacements on the indices of region momenta x's: $n\to b_1, a_1\to a_2, b_1\to b_2$, and $x\to x'$ when x has the index n with it. Furthermore one should apply the changes on $\langle n|$ and $\langle n-i|$, which correspondingly read $\langle n|\to\langle n|x_{na_1}x'_{a_1b_1}$, and $\langle n-i|$(or $[n-i|$) $\to\langle b_1-i|$(or $[b_1-i|$) for $i>1$. Finally we multiply the obtained answers by a factor $(x'_{a_1b_1})^2$.

The bonus coefficients of the contributions from other diagrams are actually the same as those of the NMHV case. For the sake of completeness, let us write down these contributions: for the



(a)6-point diagram deleted by bonus relations

(b)6-point diagram

FIG. 8: Diagrams for 6-point N^2MHV amplitude.

contribution from Fig. 7(d), we have

$$M_{\mathrm{III}} = \sum_{2\leq a_1,b_1\leq n-1}\ \sum_{b_1\leq a_2,b_2<n} B^{(2)}_{n;a_1b_1;a_2b_2}[\{n;a_1b_1;a_2b_2\}_2(b_1-1\leftrightarrow n-1)], \tag{3.20}$$

where the bonus coefficients $B^{(2)}_{n;a_1b_1;a_2b_2}$ are given by Eq. (3.7); for the other contribution coming from Fig. 7(e), we get

$$M_{\mathrm{IV}} = \sum_{2\leq a_1,b_1\leq n-1}\ \sum_{a_1\leq a_2,b_2<b_1} B^{(1)}_{n;a_1b_1;a_2b_2}\{n;a_1b_1;a_2b_2\}_1, \tag{3.21}$$

and similarly the coefficients are given by Eq. (3.4) and Eq. (3.5).

Again as in the case of Eq. (3.18), this formula does not include the boundary case, $\{n;a_1b_1;a_2b_2\}_1 = \{n;n-4n-1;n-4n-2\}_1$, which should be considered separately, as we shall do below.

Similar to 5-point NMHV amplitude, the 6-point N^2MHV amplitude is special which only receives contributions from diagrams of NMHV $\times$ MHV type and we must treat it separately. We can delete Fig. 8(a) by bonus relations, and the contribution from Fig. 8(b) gives,

$$\mathcal{M}_6 = -\frac{[16][25][45]}{[15][24][56]}[\{6;25,24\}_1(3\leftrightarrow 5)] + \mathcal{P}(2,3,4). \tag{3.22}$$

As the NMHV case (3.11), 6-point N^2MHV amplitude (3.22) can also be similarly generalized, and we obtain the boundary term of the full n-point N^2MHV amplitudes,

$$M^{(\mathrm{boundary})}_n = B^{(\mathrm{boundary})}_{n;n-4\ n-1;n-4\ n-2}[\{n;n-4\ n-1;n-4\ n-2\}_1(n-3\leftrightarrow n-1)], \tag{3.23}$$

where the bonus coefficients are given as

$$B^{(\mathrm{boundary})}_{n;n-4\ n-1;n-4\ n-2} = \frac{\langle 1n\rangle\langle n-5\ n-1\rangle[n-4\ n-1][n-2\ n-1]x^2_{n-4n}}{\langle 1n-1\rangle[n-4\ n-2]\langle n-5|x_{n-4\ n-1}|n-1]\langle n|x_{n-1\ n-4}|n-1]}. \tag{3.24}$$

Therefore we have calculated all the contributions for N^{2}MHV amplitudes and as in the NMHV case, it can also be written as a sum of $(n-3)!$ permutations,

$$\mathcal{M}_n^{\mathrm{N^2MHV}} = M_{\mathrm{I}} + M_{\mathrm{II}} + M_{\mathrm{III}} + M_{\mathrm{IV}} + M_n^{(\mathrm{boundary})} + \mathcal{P}(2,3,\ldots,n-2). \tag{3.25}$$

The result can be proved very similarly by induction as in the NMHV case.

IV. GENERALIZATION TO ALL GRAVITY TREE AMPLITUDES

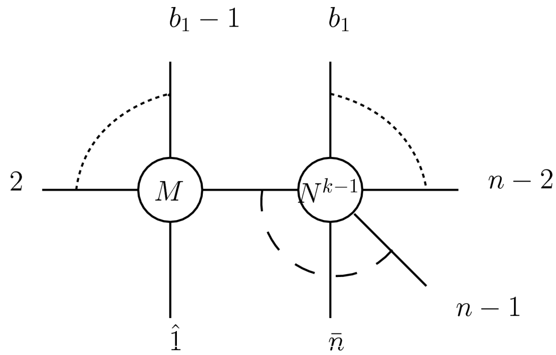
Now we have all the ingredients for generalizing our results and stating the patterns for all tree-level gravity amplitudes. Our way of using bonus relations gives the simplified tree-level N^{k}MHV superamplitude as a sum of $(n-3)!$ permutations, and each of them contains normal and exchanged contributions,

$$\mathcal{M}_n^{\mathrm{N^kMHV}} = \left[A_n^{\mathrm{MHV}}\right]^2 \Big(\sum_\alpha B_\alpha^{(1,m_1)} G_\alpha R_\alpha^2 + \sum_\beta B_\beta^{(2,m_2)} [G_\beta R_\beta^2 (b_1 - 1 \leftrightarrow n-1)] \Big) + \mathcal{P}(2,3,\ldots,n-2). \tag{4.1}$$

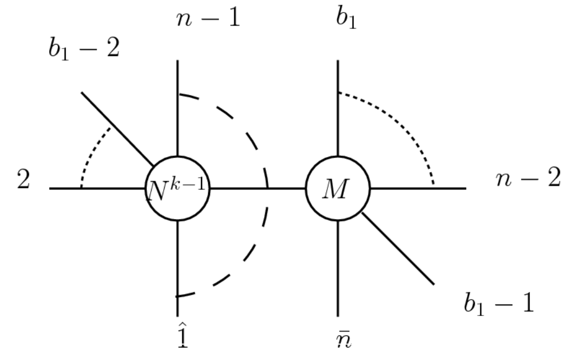
For both the contributions we have k types of terms from k BCFW channels, namely N^{p}MHV $\times$ N^{q}MHV, for $p+q+1=k$ with $0 \le p,q < k$ by reducing the homogeneous term recursively. As we have stressed repeatedly, to respect the ordered structure, we have only used bonus relations on one lower-point amplitude, namely the right-hand-side N^{q}MHV for normal contribution, and the left-hand-side N^{p}MHV for exchanged contribution.

Before presenting all the bonus coefficients for general tree amplitudes, we pause to show by induction that bonus relations roughly reduce the number of terms from $(n-2)!$ in the original solution to $(k+1)(n-3)!$ in the simplified one. To get the previous counting we note that in the N^{p}MHV$\times$ N^{q}MHV channel of the normal contribution, by applying bonus relations to the N^{q}MHV lower-point amplitude we can reduce the number of terms from $(n-2)!/k$ to $(q+1)(n-3)!/k$. Taking into account all channels gives us $(1+2+\ldots+k)(n-3)!/k$ terms, with the same number from the exchanged contribution, thus the simplified form has only $(k+1)(n-3)!$ terms. By parity, one only needs N^{k}MHV amplitudes with $n > 2k+2$ legs and thus the bonus relations can be used to delete at least half of the terms in tree amplitudes. The simplification becomes more significant when $n \gg k$.

Now we generalize the pattern found in the NMHV and N^{2}MHV cases to write down all the bonus coefficients for general tree amplitudes. As we have learned from the examples, once the bonus coefficients of N^{k-1}MHV amplitudes are calculated, then for the N^{k}MHV amplitudes, one only needs to compute two types of new contributions for N^{k}MHV amplitudes, namely the normal



(a)MHV × N^{k-1}MHV



(b)N^{k-1}MHV × MHV

FIG. 9: Two relevant diagrams for computing new bonus coefficients for n-point N^{k}MHV amplitude. The rest of the bonus coefficients can be obtained recursively from the N^{k-1}MHV case.

contribution from MHV × N^{k-1}MHV channel ($q = k-1$) and the exchanged contribution from N^{k-1}MHV × MHV channel ($p = k-1$) (see Fig. 9). All other bonus coefficients $B^{(m)}_{\alpha}$ of N^{p}MHV × N^{q}MHV with $q < k-1$ and $p < k-1$, are the same as those computed previously, namely the results from N^{k-1}MHV amplitudes. Since the summation variables of N^{k}MHV amplitude can be obtained by adding a pair of new labels a_k, b_k to the previous one, α', $\alpha = \{\alpha'; a_k, b_k\}$, the result can be written as

$$B^{(m)}_{\alpha} \;=\; B^{(m)}_{\alpha'}, \tag{4.2}$$

for both normal contributions with $q < k-1$ and exchanged ones with $p < k-1$.

Thus we only need to calculate two new contributions from Fig. 9(a) and Fig. 9(b). It is straightforward to confirm that all the observations we have made for the cases of NMHV and N^{2}MHV can be directly generalized to all tree-level amplitudes. First we shall state the rules and then justify them. Firstly, just like Eq. (3.4) and Eq. (3.17) for NMHV and N^{2}MHV cases, the bonus coefficients of Fig. 9(a), $B^{(1,m_1)}_{\alpha}$, can be similarly obtained by the replacements on the indices of the region momenta x's, $a_i \to a_{i+1}, b_i \to b_{i+1}$, for $B^{(m_1)}_{\alpha'}$ of N^{k-1}MHV amplitudes, then multiplying with a simple common factor of the form $\frac{\langle n-1|x_{a_1b_1}x_{b_1n}|n\rangle}{\langle n-1|x_{a_1b_1}x_{a_1n}|n\rangle}$, which are the same for all tree-level amplitudes,

$$B^{(1,m_1)}_{\alpha} = \frac{\langle n-1|x_{a_1b_1}x_{b_1n}|n\rangle}{\langle n-1|x_{a_1b_1}x_{a_1n}|n\rangle} B^{(m_1)}_{\alpha'}(a_i \to a_{i+1}, b_i \to b_{i+1}). \tag{4.3}$$

Secondly, the bonus coefficients for the new exchanged contributions Fig. 9(b), $B^{(2,m_2)}_{\beta}$, can be obtained by taking $B^{(m_2)}_{\beta'}$ of N^{k-1}MHV amplitudes, and performing the following replacements on the indices of region momenta x's, namely $n \to b_1, a_i \to a_{i+1}, b_i \to b_{i+1}$, and $x \to x'$ when x has index n with it. And for the spinors, we have $\langle n| \to \langle n|x_{na_1}x'_{a_1b_1}$ as well as $|n-i\rangle$(or $|n-i]$) $\to$ $|b_1-i\rangle$(or $|b_1-i]$) for $i>1$. In addition, the obtained answers are further multiplied by a factor $(x'_{a_1b_1})^2$,

$$B_{\beta}^{(2,m_2)} \; = \; (x'_{a_1b_1})^2 B_{\beta'}^{(m_2)}, \tag{4.4}$$

where the arguments of $B_{\beta'}^{(m_2)}$ should be changed under the rules we described above.

All these rules can be understood in a simple way. For the rules of the normal contributions, the common factor is obtained in the following way,

$$(1-\frac{z_i}{z_{n-1}})\frac{\langle n1\rangle}{\langle n-11\rangle} \to (1-\frac{z_i}{z_{n-1}})\frac{\langle n\widehat{P}\rangle}{\langle n-1\widehat{P}\rangle} \to \frac{\langle n-1|x_{a_1b_1}x_{b_1n}|n\rangle}{\langle n-1|x_{a_1b_1}x_{a_1n}|n\rangle}, \tag{4.5}$$

where $(1-\frac{z_i}{z_{n-1}})$ comes from the fact that we delete one diagram using bonus relations, and $\frac{\langle n1\rangle}{\langle n-11\rangle}$ is a factor that always appears in every bonus coefficient.

While for the rules of the exchanged contributions, we find that the factor $(x'_{a_1b_1})^2$ appears because

$$\langle n1\rangle \to \langle \widehat{P}\widehat{1}\rangle \to [\widehat{P}\widehat{1}]\langle \widehat{P}\widehat{1}\rangle \to (x'_{a_1b_1})^2, \tag{4.6}$$

and $\langle n|$ changes in the following way under the recursion relations,

$$\langle n| \to \langle \widehat{P}| \to \langle n1\rangle[1\widehat{P}]\langle \widehat{P}| \to \langle n|x_{na_1}x'_{a_1b_1}. \tag{4.7}$$

Besides, the transformation rule of $x_{n\gamma_i}$ follows as

$$x_{n\gamma_i} \to x_{\widehat{P}\gamma_{i+1}} \to x'_{b_1\gamma_{i+1}}, \tag{4.8}$$

where γ can be a or b and we have used the fact that $p_{\widehat{P}} = p_{b_1} + \cdots + p_{n-2} + p_{b_1-1} + p_{\widehat{n}}$. So in this way, we have a complete understanding of the rules we have proposed.

Finally, as shown in the examples a boundary contribution has to be considered separately because the special case $(k+4)$-point N^{k}MHV amplitude only has diagrams of N^{k-1}MHV $\times$ MHV type. For this special contribution, it is straightforward to obtain a general form,

$$M_n^{(\text{boundary})} \; = \; B_{\beta_0}^{(\text{boundary})}\big[\big(A_n^{\text{MHV}}\big)^2 G_{\beta_0} R^2_{\beta_0}(n-k-1 \leftrightarrow n-1)\big], \tag{4.9}$$

where $\beta_0 = \{n; n-k-2\ n-1; n-k-2\ n-2; \ldots; n-k-2\ n-k\}$, and the coefficients can be written as

$$B_{\beta_0}^{(\text{boundary})} = \frac{\langle 1n\rangle\langle n-k-3\ n-1\rangle[n-k-2\ n-1][n-k\ n-1]x^2_{n-k-2\ n}}{\langle 1n-1\rangle[n-k-2\ n-2]\langle n-k-3|x_{n-k-3\ n-1}|n-1]\langle n|x_{n-1\ n-k-2}|n-1]}. \tag{4.10}$$

Therefore, we have found a set of explicit rules to write down all the bonus coefficients for all tree amplitude in $\mathcal{N}=8$ supergravity.

V. CONCLUSION AND OUTLOOK

In this note, we simplified tree-level amplitudes in $\mathcal{N}=8$ SUGRA, from the BCFW form with a sum of $(n-2)!$ permutations to a new form as a sum of $(n-3)!$ permutations. This is achieved by using the bonus relations, which are relations between tree amplitudes in theories without color ordering. In contrast to the MHV case, a naive use of the bonus relations ruins the structure of the non-MHV ordered tree-level solution, thus we proposed an improved application of the relations, which respects the ordered structure. The key point here is to apply the bonus relations to only one of two lower-point amplitudes in any BCFW diagram, which indeed brings SUGRA amplitudes to a simplified form having a $(n-3)!$-permutation sum with some bonus coefficients. To illustrate the method, we have explicitly calculated simplified amplitudes for the NMHV and N^{2}MHV cases. We have also argued that the pattern generalizes to N^{k}MHV cases, and presented a simple way for writing down the bonus coefficients of all amplitudes, thus one can recursively obtain the simplified form for general SUGRA tree amplitudes.

The simplification is based on an explicit solution from BCFW recursion relations of SUGRA tree amplitudes of [28], which is in spirit similar to but in details different from KLT relations. From a computational point of view, any gravity amplitude obtained from $(n-3)!$ (or the newly proposed $(n-2)!$) form of KLT relations is a sum of $(n-3)!^2$ (or $(n-2)!^2$) terms; at least in the special case of $\mathcal{N}=8$ SUGRA, an explicit solution with only $(n-2)!$ terms was found in [28], which is a significant simplification[4]. Furthermore, in this note we have used the bonus relations to reduce it to a sum with only $(k+1)(n-3)!$ terms. Further simplifications of gravity tree amplitudes are certainly worth investigating.

Apart from the computational advantages, the simplification is also conceptually interesting. The relations between gravity and gauge theories have been reexamined from various perspectives recently [30, 31, 35] (see also [37]). A common feature, of these "gravity"="gauge theory"^{2} methods, is the freedom of rewriting $(n-2)!$ forms of gravity tree amplitudes as $(n-3)!$ forms, essentially by using BCJ relations on the gauge theory side. Our result confirms this freedom at an explicit level by directly using it to simplify SUGRA amplitudes, which also suggests that bonus relations may be regarded as explicit gravity relations induced by Yang-Mills BCJ relations. It may be fruitful to understand the exact connections between our method, general forms of KLT relations, and the square relations. In particular, it would be nice to go beyond SUGRA and see if similar

[4] It would be nice to see if one can derive the explicit $(n-2)!$ form (similarly our simplified $(n-3)!$ form) from $(n-2)!$ (similarly $(n-3)!$) KLT relations. For the simplest MHV case, both have been derived in [34].

simplifications occur generally, given that both BCFW recursion relations and bonus relations are valid in more general gravity theories.

Bonus relations and simplifications we obtained at tree level can also have implications for loop amplitudes. Through the generalized unitarity-cut method, our new form of tree amplitudes can be used in calculations of loop amplitudes. In addition, the square relations have been conjectured to hold at loop level [38], thus we may expect similar simplifications directly for the SUGRA loop amplitudes.

Acknowledgments

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