Mon. Not. R. Astron. Soc. **280,** 1230–1234 (1996)

Space–time modes of relativistic stars

Nils Andersson,[1] Kostas D. Kokkotas[1,2] and Bernard F. Schutz[1]

[1]*Department of Physics and Astronomy, University of Wales College of Cardiff, Cardiff CF2 3YB*
[2]*Department of Physics, Aristotle University of Thessaloniki, Thessaloniki 54006, Greece*

Accepted 1996 January 23. Received 1995 September 21

ABSTRACT

The problem of relativistic stellar pulsations is studied in a somewhat ad hoc approximation that ignores all fluid motions. This Inverse Cowling Approximation (ICA) is motivated by two observations. (1) For highly damped (w-mode) oscillations the fluid plays very little role. (2) If the fluid motion is neglected, the problem for polar oscillation modes becomes similar to that for axial modes. Using the ICA, we find a polar mode spectrum that has all features of the w-mode spectrum of the full problem. Moreover, in the limit of superdense stars, we find the ICA spectrum to be qualitatively similar to that of the axial modes. These results clearly show the importance of general relativity for the pulsation modes of compact stars, and that there are modes whose existence does not depend on motions of the fluid at all, namely pure 'space–time' modes.

Key words: radiation mechanisms: nonthermal – relativity – stars: neutron.

1 INTRODUCTION

The pulsations of relativistic stars have attained increasing interest in the scientific community in the last five years. Recent results have improved our understanding of both this subject and the theory of general relativity itself. Nevertheless, one can easily identify issues that are far from being well understood at the present time.

When Thorne and his colleagues established the theory for relativistic stellar pulsations three decades ago (Thorne & Campolattaro 1967; Thorne 1969), the main results were not very different from the well-known ones of Newtonian theory. Compact objects were found to oscillate at almost exactly the frequencies that Newtonian theory predicted. Relativistic effects only gave rise to a very slow damping of the pulsation. The emission of gravitational radiation implied that the oscillation frequencies were complex with a relatively tiny imaginary part.

The situation recently changed considerably when a new family of oscillation modes for compact stars was found (Kojima 1988; Kokkotas & Schutz 1992). These modes have no relation to the known p- and g-modes in Newtonian theory. Rather, this new family of modes, the existence of which was suggested by a simple model problem (Kokkotas & Schutz 1986), comes from coupling the stellar fluid to the space–time. A characteristic property of the new family is that the fluid motion is very small, and the oscillations are rapidly damped. These modes have been named w-modes (gravitational *w*ave modes), and their existence (together with an additional part of the spectrum) has been verified by Leins, Nollert & Soffel (1993). Moreover, we have recently developed an accurate numerical code that gives reliable results and also reveals the limitations of the specific description of the pulsation problem that has been used in all calculations so far (Andersson, Kokkotas & Schutz 1995).

The oscillations of relativistic stars are often described by a system of four coupled ODEs (Detweiler & Lindblom 1985). Two equations correspond mainly to the fluid pulsations, and the other two to perturbations on the space–time. The two sets of equations couple strongly in the case of highly relativistic stars, but the coupling also depends on the frequency. For slowly damped modes, the equations are, in practice, decoupled in the high- and low-frequency limits. These limits correspond to the fluid p- and g-modes, respectively. The two space–time ODEs then hardly affect the structure of the spectrum at all, and a relativistic generalization of the Cowling approximation (Cowling 1941), in which perturbations of the space–time itself are ignored, yields the correct spectrum (see, e.g., Robe 1988, McDermott, Van Horn & Scholl 1983, Finn 1988 and Lindblom & Splinter 1990).

Inversely, as has been shown by Kokkotas & Schutz (1992), Leins et al. (1993) and Andersson et al. (1995), the fluid hardly pulsates at all in the case of w-modes. A simple way to find the spectrum of such modes might therefore be to omit the equations pertaining to the fluid perturbations altogether – what we will from now on refer to as the Inverse

Cowling Approximation (ICA). In the present work, which is intended as a complement to studies of the full problem, we examine the stellar pulsation problem in this approximation. The major motivation is to clearly show the degree of involvement of general relativity. The ICA should indicate which features of the w-mode spectrum that are related to the fluid motion and which are due to the space–time perturbations. This is an important step towards a better understanding of the nature of the w-modes and the properties of rapidly damped stellar oscillations.

Chandrasekhar & Ferrari (1991) recently showed that axial quasi-normal modes could exist for very compact stars. The possibility of axial modes had previously been discarded, since axial perturbations do not couple to oscillations of the stellar fluid. However, Chandrasekhar & Ferrari realized that 'trapped' modes can occur when the star is so compact that the surface lies inside the peak of the familiar Regge–Wheeler potential barrier (Chandrasekhar 1983). That is, the axial modes can be understood in terms of a potential well (see fig. 1 in Chandrasekhar & Ferrari 1991). Initially, only a few such axial modes were found, and they were all slowly damped. Recent work by Kokkotas (1994) revealed that there are a large number of highly damped axial modes as well. This new set of axial modes is in many ways similar to the polar w-modes.

A good reason for studying the axial modes is that the space–time perturbations do not couple to those of the fluid, and thus the system of equations becomes a very simple one. In fact, the problem is similar to that for a perturbed black hole, although the boundary conditions (now at $r\to 0$) are, of course, different. If we freeze the fluid motion in the case of polar perturbations, we arrive at a similar problem. Using the ICA, we can consequently study the polar pulsations in a way similar to that used for the axial ones. This should provide a better understanding of the relation between the two sets of pulsation modes for relativistic stars.

Recently, two of us have been involved in a study of superdense stars close to the limit of compactness posed by general relativity (Kojima, Andersson & Kokkotas 1995). That study suggested that the origin of the axial modes and the polar w-modes is the same. Both sets are 'space–time' modes that do not depend on the stellar fluid at all for their existence. If that conclusion is correct, one would expect *all* features of the w-modes to be present in the ICA.

2 THE INVERSE COWLING APPROXIMATION

In Regge–Wheeler gauge, the perturbed metric can be written

$$\begin{aligned} ds^2 = & -e^{\nu}(1+r^{\ell}H_0 e^{i\omega t}Y_{\ell m})\,dt^2 \\ & -2i\omega r^{\ell+1}H_1 e^{i\omega t}Y_{\ell m}\,dt\,dr \\ & +e^{\lambda}(1-r^{\ell}H_0 e^{i\omega t}Y_{\ell m})\,dr^2 \\ & +r^2(1-r^{\ell}K e^{i\omega t}Y_{\ell m})(d\theta^2+\sin^2\theta\,d\phi^2), \end{aligned} \tag{1}$$

with H_0, H_1 and K functions of r only. $Y_{\ell m}$ are the standard spherical harmonics, and $M(r)$ acts as an effective mass inside radius r. It is assumed that all perturbations on time as $\exp(i\omega t)$. We also have

$$e^{-\lambda}=1-\frac{2M}{r}. \tag{2}$$

The metric function ν, the pressure p and the mass M follow from the Tolman–Oppenheimer–Volkov equations that determine a stellar equilibrium model. Moreover, Einstein's equations imply the relation (Detweiler & Lindblom 1985)

$$\begin{aligned} &\left[3M+\frac{(\ell-1)(\ell+2)}{2}r-4\pi r^3 p\right]H_0 \\ &=\left[\omega^2 r^3 e^{-\lambda-\nu}-\frac{\ell(\ell+1)}{2}(M+4\pi r^3 p)\right]H_1 \\ &-\left[\omega^2 r^3 e^{-\nu}-\frac{(\ell-1)(\ell+2)}{2}r\right. \\ &\left.-M-4\pi r^3 p+\frac{e^{\lambda}}{r}(M+4\pi r^3 p)^2\right]K. \end{aligned} \tag{3}$$

Hence only two metric functions remain undetermined.

The full pulsation problem consists of four coupled first-order ODEs [equations (8)–(11) of Detweiler & Lindblom (1985)]. These describe the two remaining space–time variables, H_1 and K, as well as two fluid ones, V and W. In this formulation, the ICA corresponds to $V=W=0$, and the perturbation equations for a specific multipole ℓ take the following simple form inside the star:

$$rH_1'=e^{\lambda}(H_0+K)-[\ell+e^{\lambda}+4\pi r^2 e^{\lambda}(p-\rho)]H_1, \tag{4}$$

$$rK'=H_0+\frac{\ell(\ell+1)}{2}H_1-\left[\ell+1-e^{\lambda}\frac{(M+4\pi r^3 p)}{r}\right]K, \tag{5}$$

where a prime denotes a derivtive with respect to r, and equation (3) should be used to replace H_0.

In the exterior the perturbation equations simplify to the well-known Zerilli equation (Fackerell 1971; Chandrasekhar 1983). It is, in fact, possible to transform equations (4) and (5) into a single second-order equation, but we have found the form of this equation to be rather complicated and not numerically convenient. Instead, we approach the above system of equations numerically in the present study. The procedure used is a very simple one. The result of one integration [initiated with the regular solution to equations (4) and (5)] from the centre of the star to the surface is matched to an exterior solution calculated according to the method described by Andersson et al. (1995). The technique used for finding eigenfrequencies here is identical to that discussed in our previous paper.

Before we discuss the numerical results obtained in this way, a few words of caution are in order. We have not defined the ICA in a gauge-invariant way here. Such a definition does not, in fact, seem possible. This does not mean that the idea presented here is without merits, however. The usefulness of an ad hoc approximation such as the ICA is illustrated by the actual results obtained. Even though such results should not be used as indication of new physical phenomena, we will show in the following sections that they provide interesting information. We view the present study as a mathematical experiment that provides information on the relative importance of the space–time variables and the fluid ones.

3 ICA MODES FOR POLYTROPES

We have calculated the quadrupole quasi-normal-mode spectrum in the ICA for the four stellar models of Kokkotas & Schutz (1992). The results are interesting. First of all, we could not find the p- and g-mode spectra. This was certainly expected, since these modes are clearly associated with fluid perturbations which are absent in our approximation (McDermott et al. 1983).

In Fig. 1 we compare the w-mode spectrum to the ICA mode spectrum for model 2 of Kokkotas & Schutz (1992). The similarities between the two spectra are evident. The highly damped part of the ICA spectrum, i.e., the w-modes, is in excellent qualitative agreement with the results of an analysis of the full problem. In fact, the real parts of the frequencies agree surprisingly well. The spacing between consecutive modes is roughly 10 per cent smaller in the ICA than in the full problem. This means that the absolute difference between the real parts for 'corresponding' modes in the ICA and the full problem increases drastically with the frequency. Nevertheless, the relative difference [(Re ω_{n+1} − Re ω_n)/Re ω_{n+1}, where n is an integer labelling the modes] is typically smaller than 10 per cent. As for the damping rates, the ICA imaginary parts are some 20–30 per cent smaller than those for the w-modes of the full problem. As the oscillation frequency increases, this difference decreases and the two spectra approach each other. All characteristic features of the spectrum, such as the existence of a few overdamped modes with small real part (what Leins et al. 1993 referred to as w_{II}-modes) are found where expected.

Even though a discussion of physical effects here must carry a disclaimer because of the gauge-dependency of the ICA, it is interesting to speculate on reasons for why the damping of the ICA modes is slower than that of the w-modes of the full problem. A hand-waving argument is the following. In the full problem the fluid is also 'radiating' gravitational waves with the signature of a w-mode (the perturbations are coupled). The fluid thus acts as a kind of 'gravitational-wave pump', and it seems plausible that the dissipation rate increases. The observed result would follow if the fluid is 'a more efficient radiator of gravitational waves' than the space–time itself. However, does not the evidence from oscillations mainly associated with the fluid (the p-modes) indicate the opposite? The damping of these modes due to gravitational radiation is slow, and the fluid would seem to be a poor radiator. This is certainly true, but it is important to remember that the origin of the fluid modes and the 'space–time' modes we discuss in this paper is quite different. For the fluid modes the space–time perturbations are negligible compared to the fluid ones, and the modes are slowly damped because the coupling between matter and gravitational waves is weak. The situation is different for the w-modes, for which the space–time perturbations play the dominant role. The results in Fig. 1 suggest that the space–time curvature traps gravitational waves more effectively than does the stellar fluid. This seems plausible, but further study of this problem is needed if we are to understand the actual physics involved.

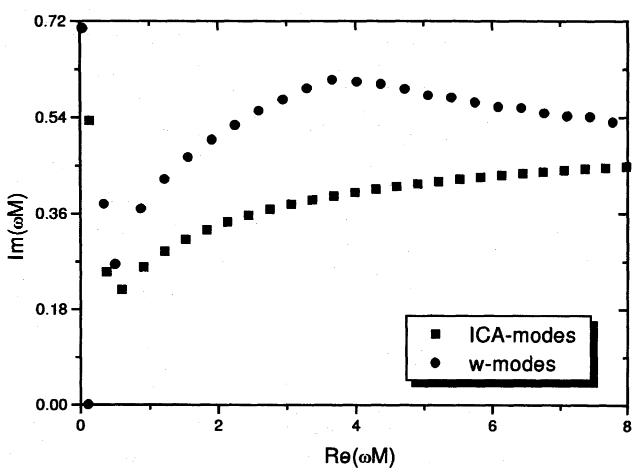

Figure 1. The w-mode spectrum compared to the ICA spectrum for a simple polytropic stellar model. Here, the equation of state is $p = 100\ \mathrm{km}^2\,\rho^2$, and the star has characteristics: $\rho_c = 3 \times 10^{15}\ \mathrm{g\,cm^{-3}}$, $R = 8.861$ km, $M = 1.266\ \mathrm{M}_\odot$, i.e., $2M/R = 0.422$. This corresponds to model 2 of Kokkotas & Schutz (1992).

4 ICA MODES FOR SUPERDENSE STARS

Let us now compare the polar modes obtained in the ICA to the axial modes. To do this, we consider the uniform density model that was studied by Chandrasekhar & Ferrari (1991), Kokkotas (1994) and Kojima et al. (1995). For this model the surface of the star can be inside the peak of the curvature potential. In this way we have a problem that can loosely be characterized as a 'potential with a well inside a barrier', and one would expect modes with small imaginary part (which come mainly from the potential well, i.e., act as quasi-bound states in quantum language) to exist. Furthermore, as was shown by Kokkotas (1994), there are modes with high damping.

Before we proceed to discuss the present results, it is necessary to discuss the discrepancy between previous results for axial modes. In the study of Kokkotas (1994) the damping rate of the modes was found to be roughly half that found by Chandrasekhar & Ferrari (1991). When investigating this issue, we found it to be due to a misprint in equation (19) of Chandrasekhar & Ferrari (1991). If the erroneous equation is used in the numerical calculations, the results of Kokkotas follow. Once the equation is corrected, the numerical results are in good agreement with those of Chandrasekhar & Ferrari. It is, however, important to stress that the results of Kokkotas are qualitatively correct. An infinite spectrum of highly damped modes does, indeed, exist (Kokkotas 1995).

Here we have used the method of Andersson et al. (1995) to approach the exterior problem and verify the qualitative behaviour of the recent axial-mode results of Kokkotas (1994). It is worth noticing at this point that, although the numerical results of Chandrasekhar & Ferrari (1991) are correct, their method of outward numerical integration is very sensitive to the choice of end-point representing 'infinity'. Integration from the surface of the star towards infinity is, in fact, not very reliable in this kind of problem, especially not when one deals with numbers of the order of 10^{-8}, as one must to identify the long-lived modes. Integration towards the surface of the star, as in the method used here, is considerably more stable and accurate.

For these ultracompact models we find that the ICA spectrum is quite similar to the axial-mode spectrum obtained by Kokkotas (1994); see Fig. 2. The ICA modes are generally slower damped than the axial modes throughout the spectrum. This is reminiscent of the result for polar modes of polytropes discussed in the previous section. Our calculations have also unveiled highly damped modes with relatively small real parts in both spectra; see Fig. 2. These modes are, in many ways, similar to the 'new' polar modes identified by Leins et al. (1993) and Andersson et al. (1995). That similar modes exist also for axial perturbations was not known previously, but is evident from our Fig. 2.

The slowest damped modes for very compact stars should be viewed as modes trapped inside the curvature potential barrier (Kojima et al. 1995). The slight difference in damping between the ICA modes and the axial modes in the first part of the spectrum could be due to the difference between the Regge–Wheeler and the Zerilli potential. Although similar, these two potentials are not the same (Chandrasekhar 1983), and a difference in the stellar spectra should be expected at some level.

The rapidly damped modes cannot easily be viewed as trapped modes in this sense. Rather, they are analogous to the w-modes for less compact stellar models discussed in the previous section. In fact, it is worth stressing the considerable qualitative similarity between the rapidly damped axial modes and the polar w-modes (compare Figs 1 and 2). It should also be remembered that the results of Kojima et al. (1995) show that the axial and the polar spectra approach each other as the star becomes increasingly compact. By extending that study to less compact stellar models, one may hope to shed further light on the relationship between the axial modes and the polar w-modes (Andersson, Kojima & Kokkotas 1996). At present, it seems clear that, although axial perturbations do not couple to the stellar fluid (Thorne & Campolattaro 1967), rapidly damped axial modes *should exist also for less compact stellar models*. That is, models for which the surface of the star lies well outside the peak of the Regge–Wheeler potential barrier should support a branch of strongly damped axial modes.

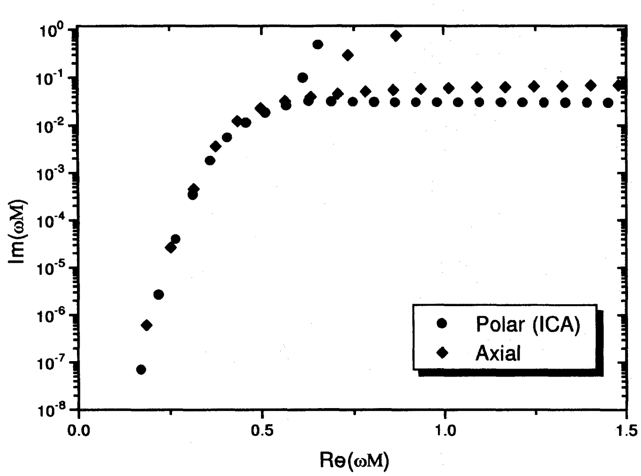

Figure 2. The axial and ICA spectra for a very compact uniform density star. This specific example is a star for which $R/M=2.28$, i.e., $2M/R=0.88$. The modes that correspond to the 'new' polar modes identified by Leins et al. (1993) lie above the general 'string' of modes. That such axial modes exist was not known previously.

5 CONCLUDING DISCUSSION

In this short paper we have presented results for a new approximation relevant to stellar pulsation problems in general relativity. The Inverse Cowling Approximation, which neglects perturbations of the stellar fluid, provides a useful tool for probing the role of the space–time degrees of freedom in their problem. We have compared (1) the spectrum of highly damped modes for polar perturbations (the w-modes) to the corresponding ICA modes for polytropic neutron star models, and (2) the polar ICA modes to the modes for axial perturbations of extremely compact uniform density stars. The results are unequivocal: all essential features of the w-mode spectra are present in the ICA. This is strong support for the idea that the w-modes are 'space–time' modes that do not rely on the motions of the fluid for their existence.

Motivated by the present results, we would argue that this kind of approximation can be used to further improve our present understanding of the stellar pulsation problem. In fact, it works much in the same way as the Cowling approximation does for pulsations mainly associated with the fluid degrees of freedom. In that case, calculations are facilitated by neglecting the perturbations of space–time (McDermott et al. 1983). When relativistic stars are considered, that approximation on its own does not make much sense. It is clear, e.g., from the existence of w-modes, that the gravitational field must be considered as dynamic if all features of the problem are to be accounted for. So, only if the Cowling approximation is complemented with something like the present approximation can one infer the relevant physics.

ACKNOWLEDGMENTS

KDK acknowledges the financial support and the hospitality of the Department of Physics and Astronomy, UWCC. NA acknowledges support from SERC. This work was also supported by an exchange programme from the British Council and the Greek GSRT.

REFERENCES

Andersson N., Kokkotas K. D., Schutz B. F., 1995, MNRAS, 274, 1039
Andersson N., Kojima Y., Kokkotas K. D., 1996, ApJ, in press
Chandrasekhar S., 1983, The Mathematical Theory of Black Holes. Cambridge Univ. Press, Cambridge
Chandrasekhar S., Ferrari V., 1991, Proc. R. Soc. London A, 434, 449
Cowling T. G., 1941, MNRAS, 101, 367
Detweiler S. L., Lindblom L., 1985, ApJ, 292, 12
Fackerell E. D., 1971, ApJ, 166, 197
Finn L. S., 1988, MNRAS, 232, 259
Kojima Y., 1988, Prog. Theor. Phys., 79, 665
Kojima Y., Andersson N., Kokkotas K. D., 1995, Proc. R. Soc. London A, 471, 341

Kokkotas K. D., 1994, MNRAS, 268, 1015
Kokkotas K. D., 1995, MNRAS, 277, 1599
Kokkotas K. D., Schutz B. F., 1986, Gen. Relativ. Gravitation, 18, 913
Kokkotas K. D., Schutz B. F., 1992, MNRAS, 255, 119
Leins M., Nollert H.-P., Soffel M. H., 1993, Phys. Rev. D, 48, 3467
Lindblom L., Splinter R. J., 1990, ApJ, 348, 198
McDermott P. N., Van Horn H. M., Scholl J. F., 1983, ApJ, 268, 837
Robe H., 1968, Ann. d'Ap., 31, 475
Thorne K. S., 1969, ApJ, 158, 1
Thorne K. S., Campolattaro A., 1967, ApJ, 149, 591