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Linear Pulsations and Stability of Differentially Rotating Stellar Models
I. Newtonian Analysis
II. General Relativistic Analysis

BY BERNARD F. SCHUTZ, JR.

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LINEAR PULSATIONS AND STABILITY OF DIFFERENTIALLY ROTATING STELLAR MODELS. I. NEWTONIAN ANALYSIS*

BERNARD F. SCHUTZ, JR.†
California Institute of Technology, Pasadena, California
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ABSTRACT

A systematic method is presented for deriving the Lagrangian governing the evolution of small perturbations of arbitrary flows of a self-gravitating perfect fluid. The method is applied to a differentially rotating stellar model; the result is a Lagrangian equivalent to that of Lynden-Bell and Ostriker. A sufficient condition for stability of rotating stars, derived from this Lagrangian, is simplified greatly by using as trial functions not the three components of the Lagrangian displacement vector ξ, but three scalar functions defined by

$$\rho\xi = \nabla\lambda + \nabla \times (\chi i + \nabla \times \gamma i) \,,$$

where i is an arbitrary vector field. This change of variables saves one from integrating twice over the star to find the effect of the perturbed gravitational field.

I. INTRODUCTION AND SUMMARY

There is usually a very close connection between variational principles and stability criteria. If one has a variational principle that gives the dynamical equations for small perturbations of some equilibrium state, he usually can obtain directly a criterion that tells him whether those perturbations will remain small. In fact, Cotsaftis (1968) has shown that it is in principle always possible to derive at least a sufficient condition for stability from the Lagrangian. The most familiar example of this is the use of the Hamiltonian as a Liapunov function in cases where energy is conserved or dissipated by the perturbations: then positive-definiteness of the Hamiltonian guarantees stability.

In the theory of small pulsations of stellar models made of perfect fluid, the problem of finding a Lagrangian for the pulsational equations has been solved only in the past decade (Chandrasekhar 1964; Chandrasekhar and Lebovitz 1964; Clement 1964; Lynden-Bell and Ostriker 1967; Chandrasekhar and Lebovitz 1968). The Lagrangian for the nonradial pulsations of a nonrotating star was deduced directly from the perturbed equations of motion by Chandrasekhar (1964) and by Chandrasekhar and Lebovitz (1964). Using these same techniques, Lynden-Bell and Ostriker (1967) obtained the Lagrangian for small perturbations of *any* stationary equilibrium configuration of perfect fluid; and they derived from their Lagrangian a sufficient condition for stability, which is essentially that the conserved Hamiltonian be positive-definite. In principle this nearly solves the stability problem, though in practice the criterion is still very difficult to use.

The purpose of this paper is to show that the Lagrangian can also be deduced in a potentially more powerful way from the general perfect-fluid variational principle of Seliger and Whitham (1968); and to show that the resulting stability criterion can be simplified greatly for the purpose of testing realistic models. The method introduced here is potentially more powerful for two reasons. First, it provides a straightforward,

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conceptually simple procedure for deducing the Lagrangian for perturbations of *any* initial flow (not necessarily stationary) with *arbitrary* boundary conditions on the perturbations (boundary conditions have required special considerations in previous work). Second, it is easily generalized to general-relativistic stellar models, where the pulsational equations (cf. Thorne and Campolattaro 1967) are so complicated that they have defied the earlier techniques. In the second paper in this series (Paper II, Schutz 1972), we will apply the method illustrated here to fully relativistic, differentially rotating stellar models, starting from the relativistic version of the Seliger-Whitham variational principle obtained by Schutz (1970) (and obtained independently for special relativity by Schmid 1970a, b). In the present paper we confine ourselves to the Newtonian regime.

The general plan of the paper is as follows. In § II we present the general Lagrangian for the perturbations of *any* motion of a self-gravitating perfect fluid (not restricted to stationary motions). It is the second variation of the Seliger-Whitham Lagrangian. In § III we specialize to the case where the unperturbed flow is a differentially rotating stellar model. We reduce the Lagrangian to a function only of the fluid displacement vector, ξ; and we express the action as an integral over the interior of the star plus an integral over the surface of the star (the surface integral permits the perturbation to obey any boundary condition).

In § IV we write down the sufficient condition for stability, first discovered by Lynden-Bell and Ostriker (1967). We then show that a considerable simplification of the criterion can be effected by dealing not with ξ but with three scalar fields from which ξ can be obtained (in complete generality) by the following construction:

$$\rho\xi = \nabla\lambda + \nabla \times (\chi u_r + \nabla \times \gamma u_r) \,.$$

Finally, in § V we examine the special cases of (i) axially symmetric perturbations of a rotating star (as treated by Chandrasekhar and Lebovitz 1968) and (ii) perturbations of a nonrotating star (treated by Chandrasekhar and Lebovitz 1964). We find that the stability criteria for those cases can also be simplified by using the above expression for ξ. In order to preserve the continuity of the discussion, details of the longer calculations have been placed in appendices.

II. PERTURBATIONS OF AN ARBITRARY FLOW

a) The Velocity-Potential Variational Principle

The starting point for our analysis is the variational principle discovered by Seliger and Whitham (1968). It is by no means the only variational principle for perfect fluids, but it is especially well suited for examining perturbations because it is an Eulerian variational principle. That is, all fluid quantities are expressed in terms of five scalar fields (the velocity potentials ψ, α, β, θ, S); one never needs to deal explicitly with "fluid elements" or "particle paths." Perturbations in the flow come from simple Eulerian perturbations of the velocity potentials, and are much easier to deal with than perturbations in particle paths.

The basis of the variational principle is the representation of the velocity field of the perfect fluid in terms of the five velocity potentials:

$$v = \nabla\psi + \alpha\nabla\beta - S\nabla\theta \,, \tag{1}$$

where S is the entropy per unit mass. The notation follows that of Schutz (1970), with the definition

$$\psi \equiv \phi + \theta S \,, \tag{2}$$

where ϕ was used by Schutz (1970) but will not be used here. It turns out to be more convenient in this paper and especially in Paper II to use the set $(\psi, \alpha, \beta, \theta, S)$, rather

than $(\phi, \alpha, \beta, \theta, S)$. To convert from this notation to that of Seliger and Whitham (1968), make the replacements $\psi \rightarrow \phi$, $\theta \rightarrow -\eta$. (These are changes in *name* only: Seliger and Whitham's ϕ is the *same* as our ψ.)

Each velocity potential obeys a simple "equation of evolution":

$$\frac{\partial \psi}{\partial t} + v \cdot \nabla \psi = -h + TS - \Phi + \tfrac{1}{2} v \cdot v , \tag{3a}$$

$$\frac{\partial \alpha}{\partial t} + v \cdot \nabla \alpha = 0 , \tag{3b}$$

$$\frac{\partial \beta}{\partial t} + v \cdot \nabla \beta = 0 , \tag{3c}$$

$$\frac{\partial S}{\partial t} + v \cdot \nabla S = 0 , \tag{3d}$$

$$\frac{\partial \theta}{\partial t} + v \cdot \nabla \theta = T . \tag{3e}$$

Here T is the temperature; Φ is the gravitational potential,

$$\nabla^2 \Phi = 4\pi G \rho ; \tag{4}$$

and h is the specific enthalpy,

$$h = (E + p)/\rho , \tag{5}$$

where E is the internal thermodynamic energy density, p is the pressure, and ρ is the mass density. The evolution of the velocity potentials fixes the evolution of v through equation (1). In order to make this a well-determined set of equations one must add an equation of state,

$$p = p(h, S) , \tag{6}$$

and the continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho v) = 0 . \tag{7}$$

Equations (3), (4), and (7) constitute seven equations for the seven functions Φ, h, S, ψ, α, β, θ. They are completely equivalent to the Euler equation,

$$\frac{\partial v}{\partial t} + (v \cdot \nabla)v = -\frac{1}{\rho} \nabla p - \nabla \Phi , \tag{8}$$

supplemented by equations (3d), (4), and (7). A rigorous proof of this equivalence has been given by Schutz (1970) for the relativistic version, but it applies equally well here.

Equations (3), (4), and (7) follow from extremizing the action

$$I = \int (\nabla \Phi \cdot \nabla \Phi - 8\pi G p) dt dV , \tag{9}$$

where the integral is over all space and time (dV is an element of volume). The pressure is taken to be a function of h and S through equation (6), and the enthalpy in turn is defined formally as a function of Φ and of the velocity potentials:

$$h = -\Phi - \psi_{,t} - \alpha \beta_{,t} + S \theta_{,t} - \tfrac{1}{2}(\nabla \psi + \alpha \nabla \beta - S \nabla \theta)^2 . \tag{10}$$

Variations in the pressure with respect to the independent variables $(\Phi, \psi, \alpha, \beta, \theta, S)$

are accomplished through the first law of thermodynamics:

$$dp = \rho dh - \rho T dS . \tag{11}$$

The vanishing of the "first variation"

$$\delta I = \int (2\nabla\Phi\cdot\nabla\delta\Phi - 8\pi G\delta p)dtdV$$

$$= \int (2\nabla\Phi\cdot\nabla\delta\Phi - 8\pi G\rho\delta h + 8\pi G\rho T\delta S)dtdV \tag{12}$$

—when δh is expressed in terms of the independent variations $\delta\Phi$, $\delta\psi$, $\delta\alpha$, $\delta\beta$, $\delta\theta$, δS—gives equations (4), (7), and (3b)-(3e). Equation (3a) follows from the rest of equations (3) and equation (10), so it is not an independent Euler-Lagrange equation.

In this paper it is often convenient to use the notation of differential geometry because we wish our expressions to be valid in any curvilinear coordinate system. Thus, we denote partial differentiation by a subscripted comma (as in eq. [10]) and covariant differentiation by a subscripted semicolon. We understand the gradient ∇ to be a covariant derivative. We distinguish contravariant components v^i from covariant components v_i; and we raise and lower indices with the metric tensor g_{ij} [which for spherical polar coordinates is just diag $(1, r^2, r^2 \sin^2 \vartheta)$]. We always integrate over proper volume, $dV = g^{1/2}d^3x$, where $g^{1/2}$—the root of the determinant of the matrix g_{ij}—is the Jacobian of the transformation from Cartesian coordinates to the general curvilinear coordinate system. We are able to integrate by parts because of the identity for any vector A that $\nabla\cdot Ag^{1/2} = A^j_{;j}g^{1/2} = (A^jg^{1/2})_{,j}$.

b) The Second Variation

It is well known that the second variation of a Lagrangian serves itself as a Lagrangian for the small perturbations of whatever state of motion causes the first variation to vanish (cf. Taub 1969 for a recent application to the stability of relativistic stars against radial pulsations). The second variation of equation (9) is just the part of I that is quadratic in the variations $\delta\Phi$, $\delta\psi$, $\delta\alpha$, $\delta\beta$, $\delta\theta$, δS. Thus, starting from equation (12), we find[1]

$$\delta^2 I = \int [2\delta\Phi_{,i}\delta\Phi_{,j}g^{ij} - 8\pi G\delta\rho\delta h - 8\pi G\rho\delta^2 h + 8\pi G\delta(\rho T)\delta S]dtdV . \tag{13}$$

Now, the second variation in h comes from equation (10):

$$\delta^2 h = -2\delta\alpha\delta\beta_{,t} + 2\delta S\delta\theta_{,t} - \delta v\cdot\delta v - 2\delta\alpha v^k\delta\beta_{,k} + 2\delta Sv^k\delta\theta_{,k} . \tag{14}$$

Thus, the Lagrangian density for the perturbations is (dividing eq. [13] by $8\pi G$)

$$L_2 = \frac{1}{4\pi G} g^{ij}\delta\Phi_{,i}\delta\Phi_{,j} - \delta\rho\delta h + \delta(\rho T)\delta S + \rho\delta v\cdot\delta v$$

$$+ 2\rho\delta\alpha(\delta\beta_{,t} + v^k\delta\beta_{,k}) - 2\rho\delta S(\delta\theta_{,t} + v^k\delta\theta_{,k}) . \tag{15}$$

This Lagrangian is perfectly general and makes no assumption about the unperturbed state except that it satisfy the unperturbed velocity-potential equations. In the case of the differentially rotating star, the unperturbed motion is steady, so the coefficients of the quadratic perturbation terms in L_2 will be independent of time; this will enable us to obtain stability criteria.

In using L_2 as the Lagrangian density for the perturbed fluid, we have changed the meaning of $\delta\Phi$, $\delta\psi$, $\delta\alpha$, $\delta\beta$, $\delta\theta$, δS. In the first variation, $\delta\Phi$ was a "virtual" change in the gravitational field. Here, $\delta\Phi$ is the real Eulerian change in Φ produced by the perturbed

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[1] Note that we are looking for second-order changes in functions of the potentials when the potentials are perturbed. By definition, then, the second variation of a potential itself is zero; e.g., $\delta^2\Phi = 0$.

state of the fluid. Extremizing $\int L_2 dVdt$ with respect to *virtual changes in* $\delta\Phi$ gives the perturbed source equation,

$$\nabla^2\delta\Phi = 4\pi G\delta\rho .$$

Similarly, extremizing $\int L_2 dVdt$ with respect to virtual changes in the other perturbations gives the Eulerian perturbations of equations (3b)–(3e) and (7). These equations are completely equivalent to the perturbed Euler equation (eq. [8]), which we write down for future reference:

$$\frac{\partial\delta v}{\partial t} + (\delta v\cdot\nabla)v + (v\cdot\nabla)\delta v = -\frac{1}{\rho}\nabla\delta p + \frac{1}{\rho^2}\delta\rho\nabla p - \nabla\delta\Phi . \qquad (16)$$

c) Discussion

For two reasons the Lagrangian density L_2 is not in a form suitable for a stability analysis.

First, the Lagrangian is degenerate. That is, the momenta $\partial L_2/\partial\delta\Phi_{,t}$, $\partial L_2/\partial\delta\psi_{,t}$, ... are not all independent; in fact, three of them are zero and only one of the remaining three is independent. This is partly a reflection of the fact that not all the six variables are dynamical (cf. Schutz 1971 for further discussion of this point).

Second, the usual criterion for stability is that the perturbations not grow without bound. But even the *unperturbed* potentials ψ, β, and θ grow in time at any given point (cf. eq. [3] or [18]), so we can expect that even for a stable, physically bounded perturbation the perturbations $\delta\psi$, $\delta\beta$, and $\delta\theta$ will grow without limit. This presents no physical difficulty because the potentials themselves are not physically observable. But it presents a mathematical difficulty in that the boundedness of the perturbed velocity potentials is neither necessary nor sufficient for stability.

For these reasons we prefer to express L_2 as a function only of the dynamical variable ξ (the displacement vector of a fluid element).[2] This is accomplished in § III for the case of the differentially rotating star.

It is important to understand that the perturbed action,

$$I_2 = \int L_2 dVdt , \qquad (17)$$

is an integral over *all* space between two arbitrary moments of time. The reason for this is that the Euler-Lagrange equations extremize I_2 only under the condition that the variables $\delta\Phi$, $\delta\psi$, ... be held fixed at the boundary of the region of integration. The only way to ensure that this represents no *physical* constraint on the perturbations is to put the spatial boundary at infinity, where all the perturbations must vanish anyway.[3] (Only $\delta\Phi$ is observable outside the star, and it must approach zero at infinity at least as fast as $1/r^2$. The velocity potentials have no physical significance outside the fluid because ρ and p are zero there, but it is convenient to think of them as existing in the exterior and going smoothly to zero at infinity.) In § III, after we have introduced ξ, we will bring the boundary of the region of integration in to just inside the surface of the star, expressing the contribution from the rest of space as a surface integral at the star's surface. In this manner we will ensure that I_2 be an extremum among all perturbations that obey any physically permissible boundary conditions at the star's surface.

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[2] An alternative procedure is followed in Paper II: We find the (nonconserved) Hamiltonian from L_2 and construct from it a conserved energy density, whose positive-definiteness ensures stability by Liapunov's second theorem (La Salle and Lefschetz 1961). The complexity of the relativistic equations makes that the easier procedure; but in the Newtonian case the procedure we follow here is less difficult and physically more satisfying.

[3] By contrast, requiring the perturbations to vanish at the endpoints in *time* is not a physical restriction: it is a direct carry-over from particle mechanics, where it is the heart of Hamilton's principle. In continuum mechanics one cannot demand as well that the variation vanish at some point in space for all time, for that would be a physical constraint.

III. PERTURBATIONS OF DIFFERENTIALLY ROTATING STARS

a) The Unperturbed Equilibrium

From now on we will consider the Lagrangian, equation (15), only in the context of rotating stars. In this section and the next we make no assumptions about the initial equilibrium except that it be axially symmetric, stationary, and, of course, composed of perfect fluid (no heat flux, no viscosity). In § V we specialize the equilibrium configuration further.

The general stationary axially symmetric flow can be represented by the following set of velocity potentials (r, ϑ, φ are the usual spherical polar coordinates, and t is time):

$$S = \text{arbitrary function of } r \text{ and } \vartheta \text{ ,} \tag{18a}$$

$$\Omega = \text{arbitrary function of } r \text{ and } \vartheta \text{ ,} \tag{18b}$$

$$\alpha = \Omega r^2 \sin^2 \vartheta = \Omega g_{\varphi\varphi} \text{ ,} \tag{18c}$$

$$\beta = \varphi - \Omega t \text{ ,} \tag{18d}$$

$$\theta = Tt \text{ ,} \tag{18e}$$

$$\psi = (-h + TS - \Phi + \tfrac{1}{2} r^2 \sin^2 \vartheta \Omega^2) t \text{ .} \tag{18f}$$

From equation (1) we find

$$v_\varphi = \alpha = \Omega g_{\varphi\varphi} \text{ ,} \tag{19}$$

which means that Ω is the angular velocity,

$$\Omega = v^\varphi = d\varphi/dt \text{ .} \tag{20}$$

Setting v_ϑ and v_r to zero in equation (1) gives the equation of structure

$$\rho^{-1} p_{,j} + \Phi_{,j} - \tfrac{1}{2}\Omega^2 (r^2 \sin^2 \vartheta)_{,j} = 0 \text{ ,} \tag{21a}$$

or

$$\rho^{-1} p_{,j} + \Phi_{,j} - \tfrac{1}{2}\Omega\alpha_{,j} + \tfrac{1}{2}\alpha\Omega_{,j} = 0 \text{ .} \tag{21b}$$

The source equation for Φ, equation (4), has of course the formal solution

$$\Phi(x) = - \int \frac{G\rho(x')}{|x - x'|} dV' \text{ .} \tag{22}$$

Note that although the velocity potentials are conveniently expressed in terms of the spherical polar coordinates, they are scalars and keep the same values in other coordinate systems.

b) Reduction of L_2

We now eliminate the variables $\delta\Phi$, $\delta\psi$, $\delta\alpha$, $\delta\beta$, $\delta\theta$, and δS from L_2 (eq. [15]), replacing them with ξ. The details of the reduction are given in Appendix A. The essential steps are:

i) Solve the perturbed velocity-potential equations for δS, $\delta\alpha$, and $\delta\beta$ in terms of ξ:

$$\delta S = -\xi \cdot \nabla S \text{ ;} \tag{23a}$$

$$\delta\alpha = -\xi \cdot \nabla \alpha \text{ ;} \tag{23b}$$

$$\delta\beta = -\xi \cdot \nabla \beta \text{ .} \tag{23c}$$

ii) Express δv and $\delta \rho$ in terms of ξ:

$$\delta v = \frac{\partial \xi}{\partial t} + (v \cdot \nabla)\xi - (\xi \cdot \nabla)v \; ; \tag{24}$$

$$\delta \rho = -\nabla \cdot (\rho \xi) \; . \tag{25}$$

iii) Formally solve the perturbed source equation for $\delta \Phi$:

$$\delta \Phi(x) = -G \int dV' \rho(x')\xi(x') \cdot \nabla' \frac{1}{|x - x'|} \; . \tag{26}$$

iv) Plug all these expressions into L_2. Perform some integrations by parts so that explicit expressions for $\delta \psi$ and $\delta \theta$ are never needed. Discard all divergences because the integral extends to spatial infinity. Obtain the result

$$L_2 = \frac{1}{4\pi G} \delta \Phi_{,i} \delta \Phi_{,j} g^{ij} - \gamma p (\nabla \cdot \xi)^2 - 2(\nabla \cdot \xi)(\xi \cdot \nabla p)$$

$$- \frac{1}{\rho}(\xi \cdot \nabla \rho)(\xi \cdot \nabla p) + \tfrac{1}{2}\rho(\alpha \Omega_{,j;k} - \Omega \alpha_{,j;k})\xi^j \xi^k$$

$$+ \rho g_{jk} v^l v^m \xi^j_{;l} \xi^k_{;m} + 2\rho g_{jk} \xi^j_{,t} \xi^k_{;l} v^l + \rho g_{jk} \xi^j_{,t} \xi^k_{,t} \; , \tag{27}$$

where $\delta \Phi$ is given by equation (26), γ is the adiabatic index

$$\gamma = \frac{\rho}{p}\left(\frac{\partial p}{\partial \rho}\right)_S \; ,$$

and all quantities except $\delta \Phi$ and ξ have their unperturbed values. This is equivalent to the Lagrangian of Lyden-Bell and Ostriker (1967), specialized to the case of the differentially rotating star.

One ought to wonder if $L_2(\xi)$ is really still the Lagrangian: might not the substitutions of step (iv) fundamentally alter its character? The proof that they don't is, of course, that they don't: it is not hard to show that varying L_2 with respect to ξ gives just the perturbed Euler equation, equation (16), when all perturbed quantities are expressed in terms of ξ.

This is reasonable on general grounds: the action I_2 is an extremum for motions obeying the perturbed versions of equations (3), (4), and (7). If we solve some of these equations for some of the variables in terms of the others and then substitute the solutions back into L_2, then I_2 must still be an extremum for the solution of the rest of the equations. That this is what we have done is evident from equations (23). In the general case, ∇S, $\nabla \alpha$, and $\nabla \beta$ are linearly independent vectors. We have simply re-labeled some of the variables by defining ξ to be a vector whose component on ∇S is $-\delta S$, whose component on $\nabla \alpha$ is $-\delta \alpha$, and whose component on $\nabla \beta$ is $-\delta \beta$. We then eliminated $\delta \theta$, $\delta \psi$, and $\delta \Phi$ in terms of these three components of ξ. The quantity I_2 ought still to be an extremum for whatever δS, $\delta \alpha$, $\delta \beta$ made it an extremum before.

What about uniqueness? It is still possible that our procedure could introduce spurious solutions that extremize the reduced I_2 but not the original. This will in fact happen if one reduces the number of variables in a Lagrangian below the number of true degrees of freedom the system has, because then one has implicitly assumed some relation between one or more degrees of freedom that isn't generally true. As a simple example, consider the free-particle Lagrangian, $\mathcal{L} = \dot{x}^2 + \dot{y}^2$, whose Euler-Lagrange equations have the solution $\dot{x} = $ const., $\dot{y} = $ const. Assume that $\dot{x} = ky$. Substitute this into $\mathcal{L}$: $\mathcal{L} = k^2 y^2 + \dot{y}^2$. The Euler-Lagrange equations still have as one solution

$y = 0$ ($\Rightarrow \dot{x} = \dot{y} = 0$), but they also have the spurious solution $y = \exp (kt\sqrt{2})$. So in general one must exercise care not to infringe on a system's dynamical freedom. In our case we have not introduced spurious solutions: the three components of ξ are the only dynamical variables the pulsating star has.

c) Surface Boundary Conditions: Expressing the Action as an Integral over the Interior of the Star Plus a Surface Integral

One generally prefers to express the action as an integral over the interior of the star, where all the dynamics occurs. Our action $I_2 = \int L_2 dV dt$, with L_2 from equation (27), includes an integral over all of space. The only contribution outside the star is from the term in $\delta\Phi$. We shall see that it can be expressed as a divergence plus a term that is zero outside the star; thus the integral of L_2 outside the star can be expressed as a surface integral evaluated just above the surface of the star.

The star's surface is defined as that place where $p = 0$. For some equations of state this does not imply $\rho = 0$. Outside the surface we must of course have $\rho = 0$, so that ρ may be discontinuous and the terms in L_2 that contain gradients of ρ may be delta-functions at the surface. Therefore, bringing the limit of integration in I_2 to just *inside* the star's surface will bring in a surface integral.

We consider separately the two steps: first bringing the limit in to Σ^+, a surface just outside the star's surface Σ; and second, bringing the limit into Σ^-, a surface just inside Σ.

i) The Integral over the Exterior Region

The only nonzero term in I_2 outside the star comes from $\delta\Phi$. Ignoring for the moment the integral on time, we have

$$\int \nabla \delta\Phi \cdot \nabla \delta\Phi dV = \int [\nabla \cdot (\delta\Phi \nabla \delta\Phi) - \delta\Phi \nabla^2 \delta\Phi] dV$$

$$= -4\pi G \int \delta\Phi \delta\rho dV + \int \nabla \cdot (\delta\Phi \nabla \delta\Phi) dV . \qquad (28)$$

If the region of integration is all space, the second term in the right-hand side vanishes. But if the region of integration is from Σ^+ outward, then the first term is zero and the second term is a surface integral (n is the unit *outward* normal to Σ):

$$\int_{\text{exterior}} \nabla \delta\Phi \cdot \nabla \delta\Phi dV = - \int_{\Sigma^+} \delta\Phi \nabla \delta\Phi \cdot n d\sigma .$$

With this, I_2 becomes

$$I_2 = \int_{\text{out to } \Sigma^+} L_2 dV dt - \frac{1}{4\pi G} \int_{\Sigma^+} \delta\Phi \nabla \delta\Phi \cdot n d\sigma dt . \qquad (29)$$

ii) The Surface Integral

If we integrate the first term on the right-hand side in equation (29) only out to Σ^-, we omit only an infinitesimal volume of space. Only if L_2 has delta-functions at the surface will this region contribute to I_2. As we mentioned previously, a discontinuity in ρ would give such a delta-function. We need not worry about discontinuities in ξ or Ω: we can perfectly well define fields ξ and Ω outside the star that are continuous at its surface. They don't affect I_2 because ρ and p are zero outside. Moreover, there can be no discontinuities in p and $\delta\Phi$ at Σ.

One contribution to the integral of L_2 between Σ^- and Σ^+ might come from the term $\nabla \delta\Phi \cdot \nabla \delta\Phi$. This has no delta-functions, so its net contribution is zero. However, from equation (28) we see that this means

$$4\pi G \int_{\Sigma^-}^{\Sigma^+} \delta\Phi \delta\rho dV = \int_{\Sigma^+} \delta\Phi \nabla \delta\Phi \cdot n d\sigma - \int_{\Sigma^-} \delta\Phi \nabla \delta\Phi \cdot n d\sigma . \qquad (30)$$

If ρ is discontinuous, the term $\delta\rho = -\nabla\cdot(\rho\xi)$ contributes to the left-hand side, and the result is

$$\int_{\Sigma^+}\delta\Phi\nabla\delta\Phi\cdot n d\sigma = \int_{\Sigma^-}\delta\Phi\nabla\delta\Phi\cdot n d\sigma + 4\pi G\int_{\Sigma^-}\delta\Phi\rho\xi\cdot n d\sigma$$

$$= \int_{\Sigma^-}\delta\Phi(\nabla\delta\Phi + 4\pi G\rho\xi)\cdot n d\sigma .\qquad(31)$$

This enables us to move the surface integral in equation (29) from Σ^+ to Σ^-.

The only contribution to the integral of L_2 between Σ^- and Σ^+ comes from the fourth term in equation (27):

$$-\rho^{-1}(\xi\cdot\nabla\rho)(\xi\cdot\nabla p) .$$

Its integral is

$$-\int_{\Sigma^-}^{\Sigma^+}\rho^{-1}(\xi\cdot\nabla\rho)(\xi\cdot\nabla p)dV = \int_{\Sigma^-}(\xi\cdot\nabla p)(\xi\cdot n)d\sigma .\qquad(32)$$

Note that because Σ is a surface of constant pressure, ∇p and n are parallel there. With equations (29), (31), and (32), the action becomes

$$I_2 = \int_{\text{interior}} L_2 dV dt + \int_{\Sigma^-}(\xi\cdot\nabla p)(\xi\cdot n)d\sigma dt - \int_{\Sigma^-}\delta\Phi\left(\rho\xi + \frac{1}{4\pi G}\nabla\delta\Phi\right)\cdot n d\sigma dt ,\qquad(33)$$

where by "interior" we mean the region inside Σ^-.

We should mention that these same surface integrals can be obtained if, instead of integrating L_2 over all space and then bringing the limit of integration in, one always integrates L_2 just over the interior but adds surface terms in order to make Σ a free boundary. This procedure is examined in detail by Courant and Hilbert (1953) under the name "natural boundary conditions." The procedure followed in this section was first suggested to me by Professor Kip Thorne.

IV. STABILITY OF DIFFERENTIALLY ROTATING STARS

a) The Stability Criterion

The Lagrangian density, equation (27), has the form

$$L_2 = \rho\xi_{,t}\cdot\xi_{,t} + \mathcal{Q}[\xi, \xi_{,t}] + \mathcal{C}[\xi, \xi] ,\qquad(34)$$

where $\mathcal{Q}$ and $\mathcal{C}$ are homogeneous quadratic time-independent operators. Moreover, $\mathcal{Q}$ is antisymmetric and $\mathcal{C}$ is symmetric when L_2 is integrated over all space. Note that $\mathcal{C}$ includes all except the last two terms of equation (27). It is easy to show (cf. Kulsrud 1968) that a sufficient condition for stability is (for all ξ bounded everywhere and zero at infinity)

$$-\int_{\text{all space}}\mathcal{C}[\xi, \xi]dV > 0 .\qquad(35)$$

This is sufficient for stability because it guarantees that the "kinetic energy,"

$$K \equiv \int_{\text{all space}}\rho\xi_{,t}\cdot\xi_{,t}dV ,\qquad(36)$$

will remain bounded for all time for all perturbations.

Another way of obtaining the same result is to construct the Hamiltonian density

$$\mathcal{H} = \rho\xi_{,t}\cdot\xi_{,t} - \mathcal{C}[\xi, \xi] .\qquad(37)$$

Because the operator $\mathcal{C}$ is time-independent, the total energy

$$\mathcal{E} \equiv \underset{\text{all space}}{\int} \mathcal{H} dV \, , \tag{38}$$

is constant, so that $\mathcal{H}$ is a Liapunov function whose positive-definiteness guarantees stability. Clearly inequality (35) guarantees the positive-definiteness of $\mathcal{H}$. It is this Liapunov criterion to which we will appeal in Paper II in order to obtain a sufficient condition for the stability of relativistic stars.

For the realistic Newtonian star, inequality (35) is more than just a sufficient condition for stability. According to Lynden-Bell and Ostriker (1967), it is also the condition for secular stability: if friction is introduced, stable modes of pulsation will remain stable if and only if equation (35) is satisfied. It is therefore of great importance to cast the criterion in a form that is easy to test realistic models with. That is the subject of the remainder of this paper. Although the criterion (35) is not new, our way of handling it is.

b) *The Transverse and Longitudinal Parts of* $\rho\xi$

The typical procedure for testing a stellar model for stability is to choose a trial function for ξ, which might have some arbitrary parameters in it, and then to plug it into the operator $\mathcal{C}$ and see if inequality (35) is satisfied for all values of the parameters. This procedure is made very difficult by the term $\nabla\delta\Phi\cdot\nabla\delta\Phi$. In order to find $\delta\Phi$ at any point inside the star one might integrate $\rho\xi$ over the entire star (cf. eq. [26]). This is impractical for all but the simplest stellar models and trial functions.

Fortunately we can overcome this difficulty. The source equation for $\delta\Phi$ is

$$\nabla\cdot(\nabla\delta\Phi) = -4\pi G\nabla\cdot(\rho\xi) \, .$$

This can be integrated to give

$$\nabla\delta\Phi = -4\pi G\mathbf{n}^{L} \, , \tag{39}$$

where $\mathbf{n}^L$ is the longitudinal (curl-free) part of the vector field[4]

$$\mathbf{n} \equiv \begin{cases} \rho\xi & \text{inside the star} \\ 0 & \text{outside the star.} \end{cases} \tag{40}$$

Any piecewise differentiable vector field A that approaches zero at infinity at least as fast as $1/r^2$ can be decomposed into unique longitudinal and transverse parts,

$$A = A^{L} + A^{T} \, , \tag{41a}$$

where (cf. Phillips 1933)

$$A^{L} = \nabla f = \nabla\left[\frac{1}{4\pi}\int A(x')\cdot\nabla'\frac{1}{|x - x'|}dV'\right] \tag{41b}$$

and

$$A^{T} = \nabla \times F = \nabla \times \left[\frac{1}{4\pi}\int A(x') \times \nabla'\frac{1}{|x - x'|}dV'\right] \, . \tag{41c}$$

The function f and the vector F are the unique *continuous* scalar and divergence-free vector potentials of the field A. Note that F is unique only if we demand that it be divergence-free: we can—and later we will—add a gradient to F without changing A^T.

From equation (26) we see that the scalar potential for $\mathbf{n}$ is just $-(4\pi G)^{-1}\delta\Phi$, which proves equation (39). Thus, the gravitational term in $\mathcal{C}$ becomes

$$\frac{1}{4\pi G}\nabla\delta\Phi\cdot\nabla\delta\Phi = 4\pi G\mathbf{n}^{L}\cdot\mathbf{n}^{L} \, . \tag{42}$$

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[4] A good introduction to longitudinal and transverse parts of vector fields can be found in Phillips (1933).

We can achieve a considerable savings of effort in testing a stellar model for stability if instead of choosing a trial function for ξ we choose one for $\mathbf{n}^L$ and one for $\mathbf{n}^T$. The search for a suitable curl-free vector for $\mathbf{n}^L$ and a suitable divergence-free vector for $\mathbf{n}^T$ might still prove difficult, so in the next subsection we will simplify the task even more by introducing three arbitrary scalar functions in place of $\mathbf{n}^T$ and $\mathbf{n}^L$. But first it is convenient to reexpress the stability criterion (35) in terms of $\mathbf{n}$.

Inequality (35) has $\mathcal{C}$ integrated over all space. If we bring the limits of integration in to Σ^-, we pick up the identical surface terms as in equation (33). We can therefore write inequality (35) in the form

$$- \int_{\text{interior}} C[\mathbf{n}, \mathbf{n}] dV - \int_{\Sigma^-} D[\mathbf{n}, \mathbf{n}] \cdot n d\sigma > 0 , \qquad (43)$$

where $C[\mathbf{n}, \mathbf{n}] \equiv \mathcal{C}[\xi, \xi]$, and where (cf. eq. [33])

$$D[\mathbf{n}, \mathbf{n}] = \rho^{-2}(\mathbf{n} \cdot \nabla p)\mathbf{n} - \delta\Phi\mathbf{n}^T . \qquad (44)$$

It is understood in equation (44) that $\delta\Phi$ is $-4\pi G$ times the scalar potential of $\mathbf{n}$.

The operator $C[\mathbf{n}, \mathbf{n}]$ has covariant derivatives of $\mathbf{n}$ in it. When doing calculations one must replace covariant derivatives with ordinary partial derivatives and Christoffel symbols. When one does this in spherical polar coordinates, one finds (now indices j, k run over r, ϑ, φ)

$$C[\mathbf{n}, \mathbf{n}] = 4\pi G g_{jk}\eta^{(L)j}\eta^{(L)k} + \frac{1}{\rho} \Omega^2 g_{jk}\eta^j{}_{,\varphi}\eta^k{}_{,\varphi}$$

$$+ \frac{1}{\rho} \Omega^2 \left(\frac{\alpha}{\Omega}\right)_{,j} (\eta^j\eta^\varphi{}_{,\varphi} - \eta^\varphi\eta^j{}_{,\varphi}) + \frac{1}{2\rho} (\alpha\Omega_{,jk} - \Omega\alpha_{,jk})\eta^j\eta^k$$

$$+ \frac{1}{\rho} \Omega^2[(\eta^r)^2 + 2r \sin 2\vartheta\eta^\vartheta\eta^r + r^2 \cos 2\vartheta(\eta^\vartheta)^2]$$

$$- \frac{\gamma p}{\rho^2} (\nabla \cdot \mathbf{n})^2 + \frac{1}{\rho^3} (\mathbf{n} \cdot \nabla \rho)(\mathbf{n} \cdot S) - \frac{2}{\rho^2} (\nabla \cdot \mathbf{n})(\mathbf{n} \cdot S) . \qquad (45)$$

Here we have defined

$$S \equiv \nabla p - \frac{\gamma p}{\rho} \nabla \rho ,$$

which is the vector Schwarzschild discriminant. For nonrotating stars, $S_r > 0$ is necessary for stability against convection. Components η^r, η^ϑ, η^φ in equation (45) are components of $\mathbf{n}$ on the *unnormalized* coordinate basis vectors e_r, e_ϑ, e_φ.

For future reference it is convenient to write down the entire Lagrangian L_2 from equation (27) in terms of $\mathbf{n}$. It is

$$L_2 = \frac{1}{\rho} g_{jk}\eta^j{}_{,t}\eta^k{}_{,t} + \frac{2}{\rho} \Omega g_{jk}\eta^j{}_{,t}\eta^k{}_{,\varphi} + \frac{1}{\rho} \Omega \left(\frac{\alpha}{\Omega}\right)_{,j} (\eta^j\eta^\varphi{}_{,t} - \eta^\varphi\eta^j{}_{,t}) + C[\mathbf{n}, \mathbf{n}] . \qquad (46)$$

c) *Scalar Potential for $\rho\xi$*

We have seen that it is possible to reduce the number of integrations necessary to test for stability by replacing ξ by $\mathbf{n}$. We now show that it is possible to express $\mathbf{n}$ in terms of three scalars in such a way that the two pieces $\mathbf{n}^L$ and $\mathbf{n}^T$ separate automatically. Then trial functions may be chosen for the scalars without losing the advantage obtained by separating $\mathbf{n}$ into $\mathbf{n}^L$ and $\mathbf{n}^T$.

Our procedure rests on the following theorem: For any vector fields A and i ($i \cdot i \neq 0$) whose Cartesian components are analytic functions of position in the neighborhood of

some point, there exist functions κ, χ, γ also analytic in that neighborhood such that

$$A = \nabla\kappa + \chi i + \nabla \times \gamma i \,. \tag{47}$$

The existence of κ, χ, γ follows from the Cauchy-Kowalewski existence theorem for systems of first-order partial differential equations (cf. Courant and Hilbert 1962). The restriction to analytic functions is probably not important. In practice one can choose i to be analytic almost everywhere. Moreover, the functions κ, χ, γ probably exist for most well-behaved but nonanalytic A as well. Even if they do not exist for some A, it will usually be possible to approximate A as closely as one wishes with analytic functions, except at isolated points. Note that one might need several "patches" to represent A in a finite region.

In the previous subsection we showed that there exist λ and A such that

$$\mathbf{n} = \nabla\lambda + \nabla \times A \,.$$

If we now replace A by equation (47), we obtain

$$\mathbf{n} = \nabla\lambda + \nabla \times (\chi i + \nabla \times \gamma i) \,. \tag{48a}$$

Thus, there always exist λ, χ and γ such that for any analytic, nowhere-zero vector field i

$$\eta^L = \nabla\lambda \,, \tag{48b}$$

$$\eta^T = \nabla \times (\chi i + \nabla \times \gamma i) \,. \tag{48c}$$

We are still free to choose i in any way we might wish. In this paper we will choose $i = e_r$, which is analytic everywhere but at $r = 0$; this will allow our results to assume a convenient form in the nonrotating, spherical case, where the ϑ- and φ-directions are equivalent. One would therefore expect our results to be well-adapted to the study of modes that have analogues in the nonrotating star; they might do less well on other modes. A variant on this is to choose $i = \nabla p/|\nabla p|$ (at the surface, i is the normal), which might do slightly better for isentropic models, where surfaces of ρ and p coincide. On the other hand, for investigations of highly flattened, rapidly rotating models, it might be better to choose $i = e_\varpi$, where ϖ is the radius in cylindrical polar coordinates (ϖ, ϑ, z).

d) Testing for Stability

We define the trial functions a, b, c by

$$\mathbf{n}^L = \nabla a \,, \tag{49a}$$

$$\mathbf{n}^T = \nabla \times A \,, \tag{49b}$$

$$A = -r^2 c e_r + \nabla \times (r^2 b e_r) \,. \tag{49c}$$

Since the star has azimuthal symmetry, we expand

$$a = \sum_{M=0}^{\infty} [a^+{}_M(r, \vartheta, t) \sin M\varphi + a^-{}_M(r, \vartheta, t) \cos M\varphi] \,, \tag{50}$$

and similarly for b and c. Modes corresponding to different M are orthogonal, but plus and minus modes of the same M are mixed by the equations of motion and variational principle. Appendix B contains the details of the reduction of the stability criterion to a condition on $a^\pm$, $b^\pm$, $c^\pm$ for each M. The expressions are very complicated; we will deal only with special cases from now on.

V. SPECIAL CASES

a) Axially Symmetric Perturbations

Axially symmetric perturbations were examined by Lynden-Bell and Ostriker (1967) and in great detail for uniformly rotating stars by Chandrasekhar and Lebovitz (1968). We do not need the restriction to uniform rotation.

Requiring $\mathbf{n}$ to be independent of φ is equivalent to setting $M = 0$ (cf. eq. [B1] of Appendix B). Thus, there is no distinction between plus and minus modes. Representation (B1) for $\mathbf{n}$ becomes

$$\mathbf{n}^L = a_{,r}u_r + a_{,\vartheta}u_\vartheta \tag{51a}$$

$$\mathbf{n}^T = -L^2 bu_r + \frac{1}{r}\frac{\partial}{\partial r}r^2 b_{,\vartheta}u_\vartheta + rc_{,\vartheta}u_\varphi , \tag{51b}$$

where u_r, u_ϑ, and u_φ are *unit* vectors, and where

$$L^2 \equiv \left(\frac{1}{\sin\,\vartheta}\frac{\partial}{\partial\vartheta}\sin\,\vartheta\frac{\partial}{\partial\vartheta} - \frac{M^2}{\sin^2\,\vartheta}\right)$$

is the angular part of the Laplacian.

Notice that the scalar c separates from the other two: its sole function is to determine the φ-component of $\mathbf{n}$. This separation shows up in the equations of motion. The equation for c can be obtained by varying the Lagrangian, equation (46), with respect to η^φ after setting derivatives with respect to φ to zero:

$$-\frac{2}{\rho}r^2\sin^2\,\vartheta\eta^\varphi_{,tt} - \frac{2}{\rho}\Omega\left(\frac{\alpha}{\Omega}\right)_{,j}\eta^j_{,t} = 0 . \tag{52}$$

This is just the equation for the Coriolis acceleration in the azimuthal direction of the displaced fluid element as it is carried around the star. We can integrate this equation:

$$\sin\,\vartheta\eta^\varphi_{,t} = c_{,\vartheta t} = -\frac{\Omega}{r^2\sin^2\,\vartheta}(r^2\sin^2\,\vartheta)_{,j}\eta^j + f(r,\,\vartheta) , \tag{53}$$

where $f(r,\,\vartheta)$ is an arbitrary function that represents an "initial" (i.e., when $\mathbf{n} = 0$) azimuthal velocity perturbation.

Suppose that we take $f \equiv 0$. Then for this restricted class of perturbations we can substitute equation (53) into the Lagrangian density (46), which remains a Lagrangian for η^r and η^ϑ, and in which there are no terms linear in time derivatives of $\mathbf{n}$. From the theorem of Laval, Mercier, and Pellat (1965) we obtain the following necessary and sufficient condition for the stability of the star against our restricted class of perturbations ($f = 0$):

$$\int_{\text{interior}} \left\{\frac{\Omega^2}{\rho r^2\sin^2\,\vartheta}[(r^2\sin^2\,\vartheta)_{,j}\eta^j]^2 - C[\mathbf{n},\,\mathbf{n}]\right\}dV - \int_{\Sigma^-} D[\mathbf{n},\,\mathbf{n}]\cdot\mathbf{n}d\sigma > 0 . \tag{54}$$

Here C and D are the same as in equation (43), reduced to the axially symmetric case.

This condition—as was indicated by Chandrasekhar and Lebovitz (1968)—is only *necessary* for stability against *all* axially symmetric perturbations. However, Lynden-Bell and Ostriker (1967) point out that it is nearly sufficient as well, in the following sense: If all the stellar models that can be obtained from the one we are testing by changing Ω slightly satisfy inequality (54), then the model we are testing is stable against all axially symmetric perturbations. The reason is that a nonzero f in equation (53) means physically that when $\mathbf{n} = 0$ the fluid is given an extra angular velocity of

$f/\rho \sin \vartheta$. If this mode is unstable for some f, then a stellar model differing from the one we are testing by an angular velocity $f/\rho \sin \vartheta$ should be unstable against perturbations with $f = 0$. This argument ignores the effect of the additional angular velocity on the structure (p, ρ) of the equilibrium model, so it is not completely rigorous. Nevertheless, it suggests that inequality (54) ought to be an accurate stability criterion, especially for sequences of models. Note that inequality (43) is still a sufficient condition for stability.

By specializing the calculations of Appendix B to $M = 0$, inequality (54) can also be put in a form that makes testing models easier. This is done in Appendix C.

The special choice of trial function made in § III of Chandrasekhar and Lebovitz (1968) corresponds here to setting $b = 0$. They apparently saw the advantage of using scalars and decomposing **n** into transverse and longitudinal parts, but their trial function with $b = 0$ lacked the generality of our equation (51): its transverse part vanished.

b) The Nonrotating Star

Expressions suitable for analyzing the pulsations of nonrotating stars can be obtained by setting Ω to zero in previous results and expanding a, b, and c in spherical harmonics $Y_L{}^M$. Then the representation (49) of **n** becomes[5]

$$\mathbf{n}^L = \sum_{LM} \left(a_{LM,r} Y_L{}^M \mathbf{u}_r + \frac{1}{r} a_{LM} Y_L{}^M{}_{,\vartheta} \mathbf{u}_\vartheta + \frac{a_{LM}}{r \sin \vartheta} Y_L{}^M{}_{,\varphi} \mathbf{u}_\varphi \right), \qquad (55a)$$

$$\mathbf{n}^T = \sum_{LM} \left\{ b_{LM} Y_L{}^M \mathbf{u}_r + \left[\frac{(r^2 b_{LM})_{,r}}{rL(L+1)} Y_L{}^M{}_{,\vartheta} - \frac{r c_{LM}}{\sin \vartheta} Y_L{}^M{}_{,\varphi} \right] \mathbf{u}_\vartheta \right.$$

$$\left. + \left[\frac{(r^2 b_{LM})_{,r}}{L(L+1)r \sin \vartheta} Y_L{}^M{}_{,\varphi} + r c_{LM} Y_L{}^M{}_{,\vartheta} \right] \mathbf{u}_\varphi \right\} . \qquad (55b)$$

We should note that one can obtain exactly this expression by expanding **n** in Regge-Wheeler (1957) vector spherical harmonics, and then separating $\mathbf{n}^L$ from $\mathbf{n}^T$. That procedure avoids questions of analyticity raised by the theorem proved in § IVc.

Because the underlying star is spherically symmetric, modes belonging to the same L but different M are degenerate, so it suffices to consider the case $M = 0$. Then the action, from equations (46) and (33), becomes

$$I_2 = \int_{\text{interior}} \left[\frac{1}{\rho} \mathbf{n}_{,t} \cdot \mathbf{n}_{,t} + 4\pi \mathbf{n}^L \cdot \mathbf{n}^L - \frac{2}{\rho^2} (\boldsymbol{\nabla} \cdot \mathbf{n})(\mathbf{n} \cdot \mathbf{S}) \right.$$

$$\left. + \frac{1}{\rho^3} (\mathbf{n} \cdot \boldsymbol{\nabla} \rho)(\mathbf{n} \cdot \mathbf{S}) - \frac{p\gamma}{\rho^2} (\boldsymbol{\nabla} \cdot \mathbf{n})^2 \right] dV$$

$$+ \int_{\Sigma^-} \frac{1}{\rho^2} p_{,r}(\eta^r)^2 R^2 \sin \vartheta d\vartheta d\varphi - \int_{\Sigma^-} \delta\Phi \eta^{(T)r} R^2 \sin \vartheta d\vartheta d\varphi , \qquad (56)$$

where R is the radius of the star.

Inspection of I_2 shows that c will enter it only in the $\mathbf{n}_{,t} \cdot \mathbf{n}_{,t}$ term. This is because c generates the "odd parity" (cf. Regge and Wheeler 1957; Thorne and Campolattaro 1967) part of the perturbation, which is a zero-frequency rotational mode. It does not couple to other modes and does not affect the star's stability.

This Lagrangian is equivalent to the variational principle contained in the appendix to Chandrasekhar and Lebovitz (1964). It is interesting that if one varies it with respect

[5] The b in this section is really $L(L + 1)$ times the one in equation (49). Consequently one must set $b = 0$ when $L = 0$.

to a, one gets the divergence of the dynamical equation for $\mathbf{n}$, while if one varies it with respect to b and c, one gets the two independent parts of the curl of that equation. Since a vector field is zero if and only if its divergence and curl are zero, the Euler-Lagrange equations of a, b, and c are equivalent to that of $\mathbf{n}$. Thus, the potentials a, b, and c are also good variables for the variational principle! This presumably also holds for the general variational principle for differentially rotating stars.

The theorem of Laval *et al.* (1965) applies to the Lagrangian for the nonrotating star and gives a necessary and sufficient condition for stability against pulsations of order L:

$$\int_0^R \left\{ -4\pi \left[a_{,r}^2 + \frac{L(L+1)}{r^2} a^2 \right] + \frac{2S}{\rho^2} (a_{,r} + b) \left[\nabla_r^2 a - \frac{L(L+1)}{r^2} a \right] \right.$$

$$- \frac{1}{\rho^3} \rho_{,r} S (a_{,r} + b)^2 + \frac{p\gamma}{\rho^2} \left[\nabla_r^2 a - \frac{L(L+1)}{r^2} a \right]^2 \Bigg\} r^2 dr$$

$$+ \frac{R^2}{\rho(R)^2} p_{,r}(R) [a_{,r}(R) + b(R)]^2 + 4\pi G R^2 a(R) b(R) > 0 , \tag{57}$$

where $S \equiv S_r = p_{,r} - (\gamma p/\rho)\rho_{,r}$ and where we have defined the operator

$$\nabla_r^2 \equiv \frac{1}{r^2} \frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} ,$$

which is the radial part of the Laplacian. The terms evaluated at R are to be evaluated just *inside* the star's surface if there are any discontinuities there.

VI. CONCLUSIONS

We have presented a general method for finding the Lagrangian for arbitrary perturbations of arbitrary flows of a perfect fluid; and we have illustrated the method for the case of differentially rotating stars. It enabled us to reproduce the stability criteria of Lynden-Bell and Ostriker (1967), as well as those obtained by other authors for less general cases.

We also showed that the testing of realistic stellar models with these criteria can be greatly simplified by the introduction of three scalar functions in place of the three components of ξ in such a manner that one need never perform a Green's function integration to determine the perturbed gravitational field. We hope that this will prove to be a useful technique in the future.

In Paper II we will extend these results so far as possible to the general-relativistic case.

APPENDIX A

REDUCTION OF L_2

We wish to transform L_2 from the form

$$L_2 = \frac{1}{4\pi G} g^{ij} \delta\Phi_{,i} \delta\Phi_{,j} - \delta\rho\delta h + \delta(\rho T)\delta S + \rho\delta v \cdot \delta v$$

$$+ 2\rho\delta\alpha(\delta\beta_{,t} + v^k \delta\beta_{,k}) - 2\rho\delta S(\delta\theta_{,t} + v^k \delta\theta_{,k}) \tag{A1}$$

into an expression involving only the unperturbed state of the fluid and ξ, which is defined as the difference between the position of a fluid element in the perturbed state and the position it would have occupied at exactly the same time in the unperturbed flow. As a first step we will express the perturbations themselves in terms of ξ. Then we will substitute them into equation (A1).

a) Expression of the Eulerian Perturbations in Terms of ξ

As mentioned in § IIb, the perturbations are *Eulerian* perturbations, taken at fixed coordinate and time. The vector ξ, on the other hand, is the *Lagrangian* displacement of the fluid. The relations among ξ and the Eulerian perturbations are well known and need not be derived here. One can consult Lynden-Bell and Ostriker (1967) or Lebovitz (1961). The relevant ones are

$$\delta\rho = -\nabla \cdot (\rho\xi) , \tag{A2}$$

$$\delta S = -\xi \cdot \nabla S , \tag{A3}$$

$$\delta p = -\gamma p(\nabla \cdot \xi) - \xi \cdot \nabla p , \tag{A4}$$

and

$$\delta T = \left(\frac{\partial T}{\partial p}\right)_S \delta p + \left(\frac{\partial T}{\partial S}\right)_p \delta S , \tag{A5}$$

supplemented by the Maxwell identity

$$\left(\frac{\partial T}{\partial p}\right)_S = \frac{1}{\rho p \gamma}\left(\frac{\partial p}{\partial S}\right)_\rho = -\frac{1}{\rho^2}\left(\frac{\partial \rho}{\partial S}\right)_p . \tag{A6}$$

In equations (A4) and (A6), γ is the adiabatic index,

$$\gamma = \frac{\rho}{p}\left(\frac{\partial p}{\partial \rho}\right)_S . \tag{A7}$$

Moreover, since $\delta\alpha$ and $\delta\beta$ obey the same equation as δS, we have

$$\delta\alpha = -\xi \cdot \nabla \alpha + (\delta\alpha)_0 , \tag{A8a}$$

$$\delta\beta = -\xi \cdot \nabla \beta + (\delta\beta)_0 . \tag{A8b}$$

Here $(\delta\alpha)_0$ and $(\delta\beta)_0$ are "initial values" of $\delta\alpha$ and $\delta\beta$: their values when $\xi = 0$. They are constants of integration in the following sense:

$$\frac{\partial}{\partial t}(\delta\alpha)_0 + v \cdot \nabla (\delta\alpha)_0 = 0 ,$$

and similarly for $(\delta\beta)_0$. There were no such initial values in equations (A2)–(A5) because we assume that the perturbation is an initial velocity perturbation that does not affect the initial distribution of ρ, p, and S. This does not restrict the generality of our result: changes in the initial perturbed values of ρ, p, and S are equivalent to changes in the unperturbed ρ, p, and S. Instabilities due to such initial conditions will show up in nearby models whose unperturbed ρ, p, and S are the same as those of the original model plus the initial perturbations.

It is not possible to solve explicitly for $\delta\psi$ and $\delta\theta$. We shall need only the equation for $\delta\theta$:

$$\delta\theta_{,t} + v \cdot \nabla \delta\theta + \delta v \cdot \nabla \theta = \delta T , \tag{A9}$$

where the perturbed velocity, δv, is (also from Lynden-Bell and Ostriker [1967])

$$\delta v = \xi_{,t} + (v \cdot \nabla) \xi - (\xi \cdot \nabla) v \tag{A10}$$

$$= \xi_{,t} + \pounds_v \xi . \tag{A11}$$

(Here $\pounds_v$ is the Lie derivative with respect to v.)

With the definition (A10), equation (A2), (A3), (A8a), and (A8b) are equivalent to the perturbed versions of equations (7), (3d), (3b), and (3c), respectively. The last remaining perturbation is $\delta\Phi$, which has the formal solution

$$\delta\Phi = -G \int dV' \rho(x') \xi(x') \cdot \nabla' \frac{1}{|x - x'|} . \tag{A12}$$

b) Expression of L_2 in Terms of ξ

In what follows we will often integrate by parts, using the identity mentioned at the end of § IIa; and we will throw away the resulting divergences, since they become surface integrals at infinity. We will also discard total time derivatives (cf. n. 3).

It is convenient to treat separately the following pieces of L_2 (eq. [A1]):

$$A \equiv 2\rho\delta\alpha(\delta\beta_{,t} + v^k \delta\beta_{,k}) , \tag{A13a}$$

$$B \equiv -2\rho\delta S(\delta\theta_{,t} + v^k \delta\theta_{,k}) , \tag{A13b}$$

$$C \equiv \rho\delta v \cdot \delta v , \tag{A13c}$$

$$D \equiv -\delta\rho\delta h + \delta(\rho T)\delta S . \tag{A13d}$$

i) A. By the perturbed version of equation (3c) we have

$$A = -2\rho\delta\alpha\beta_{,k}\delta v^k .$$

This is the only term in L_2 that explicitly contains $(\delta\alpha)_0$ or $(\delta\beta)_0$. Because the equations derived from L_2 are linear in the perturbations, one should not expect initial values to appear in the Lagrangian. One can in fact show explicitly that

$$A' \equiv -2\rho(\delta\alpha)_0\beta_{,k}\delta v^k$$

is zero to within divergences and time derivatives. The procedure is much the same as that which follows, so we won't go into it explicitly. The remainder of A is

$$A'' = +2\rho(\xi^j\alpha_{,j})\beta_{,k}(\xi^k_{,t} + \pounds_v\xi^k) ,$$

$$= 2\rho[\alpha_{[,j}\beta_{,k]} + \alpha_{(,j}\beta_{,k)}]\xi^j\xi^k_{,t} + 2\rho\alpha_{,j}\beta_{,k}\xi^j\pounds_v\xi^k ,$$

$$= 2\rho\alpha_{[,j}\beta_{,k]}\xi^j\xi^k_{,t} - \rho\frac{\partial}{\partial t}[\alpha_{(,j}\beta_{,k)}]\xi^j\xi^k + 2\rho\alpha_{,j}\beta_{,k}\xi^j\pounds_v\xi^k .$$

This implies

$$A = 2\rho\alpha_{[,j}\beta_{,k]}\xi^j\xi^k_{,t} + \rho\alpha_{(,j}\Omega_{,k)}\xi^j\xi^k + 2\rho\alpha_{,j}\beta_{,k}\xi^j\pounds_v\xi^k . \tag{A14}$$

Here and throughout, brackets around indices denote antisymmetrization, while parentheses denote symmetrization:

$$\alpha_{[,j}\beta_{,k]} \equiv \tfrac{1}{2}\{\alpha_{,j}\beta_{,k} - \alpha_{,k}\beta_{,j}\} ,$$

$$\alpha_{(,j}\beta_{,k)} \equiv \tfrac{1}{2}\{\alpha_{,j}\beta_{,k} + \alpha_{,k}\beta_{,j}\} .$$

ii) B. Equation (A9) converts B to

$$B = -2\rho\delta S\delta T + 2\rho\delta S(\theta_{,k}\delta v^k) \ .$$

The second term can be handled just as A was to give

$$B = -2\rho\delta S\delta T + 2\rho\theta_{[,j}S_{,k]}\xi^j\xi^k_{,\iota} + \rho S_{(,j}T_{,k)}\xi^j\xi^k - 2\rho S_{,j}\theta_{,k}\xi^j\pounds_v\xi^k \ . \quad (A15)$$

iii) $A + B$. Before adding A and B, consider the term

$$2\rho\alpha_{,j}\beta_{,k}\xi^j\pounds_v\xi^k = 2\rho\alpha_{,j}\beta_{,k}\xi^j(\xi^k_{;\iota}v^l - v^k_{;\iota}\xi^l) \ .$$

Manipulations similar to those in (i) convert this to

$$= 2\rho\alpha_{[,j}\beta_{,k]}\xi^j\xi^k_{;\iota}v^l - 2\rho[(\alpha_{(,j}\beta_{,k)})_{;\iota}v^l + 2\alpha_{,j}\beta_{,\iota}v^l_{;k]}\xi^j\xi^k \ ,$$

with a similar expression for the Lie-derivative term in B. Then by adding A and B we get

$$A + B = 2\rho\Omega_{kj}\xi^j(\xi^k_{,\iota} + \xi^k_{;\iota}v^l) - 2\rho\delta S\delta T + \rho(T_{,k}S_{,j} + \Omega_{,k}\alpha_{,j})\xi^k\xi^j$$

$$- \rho[(\alpha_{,j}\beta_{,k} - \theta_{,j}S_{,k})_{;\iota}v^l + 2(\alpha_{,j}\beta_{,\iota} - S_{,j}\theta_{,\iota})v^l_{;k]}\xi^j\xi^k \ , \quad (A16)$$

where we have introduced the vorticity tensor (not to be confused with the angular velocity)

$$\Omega_{kj} \equiv v_{[k,j]} = \alpha_{[,j}\beta_{,k]} + \theta_{[,j}S_{,k]} \ . \quad (A17)$$

Finally, extensive manipulation of the last bracketed term in equation (A16) gives

$$A + B = 2\rho\Omega_{kj}\xi^j(\xi^k_{,\iota} + \xi^k_{;\iota}v^l) - 2\rho\delta S\delta T + \rho T_{,k}S_{,j}\xi^k\xi^j - 2\rho\Omega_{\iota j}v^l_{;k}\xi^j\xi^k \ . \quad (A18)$$

iv) C. From equation (A11) we have

$$\rho\delta v\cdot\delta v = \rho g_{jk}(\xi^j_{,\iota} + \pounds_v\xi^j)(\xi^k_{,\iota} + \pounds_v\xi^k)$$

$$= \rho\xi_{,\iota}\cdot\xi_{,\iota} + 2\rho g_{jk}\xi^j_{,\iota}(\xi^k_{;\iota}v^l - v^k_{;\iota}\xi^l) + \rho g_{jk}(\xi^j_{;\iota}v^l - v^j_{;\iota}\xi^l)(\xi^k_{;m}v^m - v^k_{;m}\xi^m) \ .$$

We treat the last two terms one-by-one:

$$2\rho g_{jk}\xi^j_{,\iota}(\xi^k_{;\iota}v^l - v^k_{;\iota}\xi^l) = 2\rho g_{jk}\xi^k_{,\iota}\xi^k_{;\iota}v^l - 2\rho\Omega_{kj}\xi^j\xi^k_{,\iota} \ ;$$

$$\rho g_{jk}(\xi^j_{;\iota}v^l - v^j_{;\iota}\xi^l)(\xi^k_{;m}v^m - v^k_{;m}\xi^m)$$

$$= \rho g_{jk}v^l v^m\xi^j_{;\iota}\xi^k_{;m} - 2\rho\Omega_{kj}\xi^j\xi^k_{;\iota}v^l + \rho v_{k;j}_{;\iota}v^l\xi^k\xi^j + \rho v_{\iota;j}v^l_{;k}\xi^j\xi^k \ .$$

Assembling terms, we get

$$\rho\delta v\cdot\delta v = \rho\xi_{,\iota}\cdot\xi_{,\iota} - 2\rho\Omega_{kj}\xi^j(\xi^k_{,\iota} + \xi^k_{;\iota}v^l)$$

$$+ 2\rho g_{jk}\xi^j_{,\iota}\xi^k_{;\iota}v^l + \rho g_{jk}v^l v^m\xi^j_{;\iota}\xi^k_{;m} + \rho(v_{k;j}_{;\iota}v^l + v_{\iota;j}v^l_{;k})\xi^j\xi^k \ . \quad (A19)$$

v) Adding C to $A + B$ gives

$$A + B + C = -2\rho\delta S\delta T + \rho T_{,k}S_{,j}\xi^k\xi^j + \rho\xi_{,\iota}\cdot\xi_{,\iota}$$

$$+ 2\rho g_{jk}\xi^j_{,\iota}\xi^k_{;\iota}v^l + \rho g_{jk}v^l v^m\xi^j_{;\iota}\xi^k_{;m} + \rho(v_{k;\iota}v^l)_{;j}\xi^j\xi^k \ . \quad (A20)$$

In spherical polar coordinates, part of the last term becomes

$$v_{k;\iota}v^l = v_{k,\iota}v^l - \Gamma^i_{k\iota}v^l v_i = -\tfrac{1}{2}v^i v^l g_{i\iota,k} = -\tfrac{1}{2}\Omega^2 g_{\varphi\varphi,k}$$

$$= -\tfrac{1}{2}\Omega^2(\alpha/\Omega)_{,k} = \tfrac{1}{2}(\alpha\Omega_{,k} - \Omega\alpha_{,k}) \ .$$

If we differentiate this with respect to j and symmetrize on j and k, we obtain

$$\rho(v_{k;l}v^l)_{,j}\xi^j\xi^k \;=\; \rho[\tfrac{1}{2}\alpha\Omega_{,k;j} - \tfrac{1}{2}\Omega\alpha_{,k;j}]\xi^j\xi^k \,. \tag{A21}$$

vi) D. We add to D two thermodynamic terms from equation (A20) and define

$$E = -\delta\rho\delta h + \delta(\rho T)\delta S - 2\rho\delta S\delta T + \rho T_{,k}S_{,j}\xi^j\xi^k \,,$$

$$= -\frac{1}{\rho}\delta\rho\delta p - \rho\delta T\delta S + \rho T_{,k}S_{,j}\xi^j\xi^k \,. \tag{A22}$$

Upon using equations (A2) through (A7), we find that this reduces to

$$E = -\gamma p(\boldsymbol{\nabla}\cdot\boldsymbol{\xi})^2 - \rho^{-1}(\boldsymbol{\xi}\cdot\boldsymbol{\nabla}\rho)(\boldsymbol{\xi}\cdot\boldsymbol{\nabla}p) - 2(\boldsymbol{\nabla}\cdot\boldsymbol{\xi})(\boldsymbol{\xi}\cdot\boldsymbol{\nabla}p) \,. \tag{A23}$$

vii) The complete Lagrangian is obtained by substituting equations (A20), (A21), and (A23) into equation (A1). Equation (27) is the result.

APPENDIX B

TESTING FOR STABILITY

a) The Stability Criterion in Terms of the Scalars a, b, c

From the definitions of the scalar trial functions, equations (49) and (50), we find (sum on $M \geq 0$ implied)

$$\begin{aligned}
\mathfrak{n}^L = \;&(a^+{}_{M,r}\sin M\varphi + a^-{}_{M,r}\cos M\varphi)u_r \\
&+ (1/r)(a^+{}_{M,\vartheta}\sin M\varphi + a^-{}_{M,\vartheta}\cos M\varphi)u_\vartheta \\
&+ (M/r\sin\vartheta)(a^+{}_M\cos M\varphi - a^-{}_M\sin M\varphi)u_\varphi \,;
\end{aligned} \tag{B1a}$$

$$\begin{aligned}
A = \;&-r^2(c^+{}_M\sin M\varphi + c^-{}_M\cos M\varphi)u_r \\
&+ (rM/\sin\vartheta)(b^+{}_M\cos M\varphi - b^-{}_M\sin M\varphi)u_\vartheta \\
&- r(b^+{}_{M,\vartheta}\sin M\varphi + b^-{}_{M,\vartheta}\cos M\varphi)u_\varphi \,;
\end{aligned} \tag{B1b}$$

$$\begin{aligned}
\mathfrak{n}^T = \;&\left[\left(\frac{M^2}{\sin^2\vartheta}b^+{}_M - \frac{1}{\sin\vartheta}\frac{\partial}{\partial\vartheta}\sin\vartheta b^+{}_{M,\vartheta}\right)\sin M\varphi\right. \\
&\left.+ \left(\frac{M^2}{\sin^2\vartheta}b^-{}_M - \frac{1}{\sin\vartheta}\frac{\partial}{\partial\vartheta}\sin\vartheta b^-{}_{M,\vartheta}\right)\cos M\varphi\right]u_r \\
&+ \left[\left(\frac{1}{r}\frac{\partial}{\partial r}r^2b^+{}_{M,\vartheta} + \frac{rM}{\sin\vartheta}c^-{}_M\right)\sin M\varphi\right. \\
&\left.+ \left(\frac{1}{r}\frac{\partial}{\partial r}r^2b^-{}_{M,\vartheta} - \frac{rM}{\sin\vartheta}c^+{}_M\right)\cos M\varphi\right]u_\vartheta \\
&+ \left[\left(rc^+{}_{M,\vartheta} - \frac{M}{r\sin\vartheta}\frac{\partial}{\partial r}r^2b^-{}_M\right)\sin M\varphi\right. \\
&\left.+ \left(rc^-{}_{M,\vartheta} + \frac{M}{r\sin\vartheta}\frac{\partial}{\partial r}r^2b^+{}_M\right)\cos M\varphi\right]u_\varphi \,.
\end{aligned} \tag{B1c}$$

Here u_r, u_ϑ, u_φ are *unit* vectors.

When these expressions are used in the stability criterion, equation (43), and the integration on φ is performed, modes corresponding to different values of M separate, and we get a separate criterion for each M. In what follows we will accordingly drop the subscript M on the scalars. We will also adopt the notation

$$(a^2 + bc)^{(+-)} \equiv (a^+)^2 + (a^-)^2 + b^+c^+ + b^-c^- , \tag{B2a}$$

$$(ab)^{[+-]} \equiv a^+b^- - a^-b^+ . \tag{B2b}$$

Note that these are *not* the conventional symmetry and antisymmetry symbols: the plus and minus modes are antisymmetrically coupled by the $[+-]$ operation, but they are not coupled at all by the $(+-)$ operation.

A long but straightforward calculation reduces the stability criterion, equations (43), (44), and (45), to this form: A sufficient condition for stability of the modes of order M is that for any a, b, c (with appropriate boundary conditions—see below)

$$- \underset{\text{interior}}{\int\int} C_M r^2 \sin \vartheta\, dr\, d\vartheta - \int_{\Sigma^-} D_M{}^i n_i d\sigma > 0 , \tag{B3}$$

where in both terms *we have already integrated on* φ, and where C_M and $D_M{}^i$ are:

$$
\begin{aligned}
-C_M = \ & \Bigg\{ -4\pi G \Big[a_{,r}{}^2 + \frac{1}{r^2} a_{,\vartheta}{}^2 + \frac{M^2}{r^2 \sin^2 \vartheta} a^2 \Big] \\
& + \frac{2}{\rho} (a_{,r} - L^2 b) \Big[2M^2\Omega^2 \sin \vartheta \Big(a + \frac{\partial}{\partial r} r^2 b \Big) + \mathsf{S}_r \nabla^2 a \Big] \\
& + \frac{2}{\rho} \Big(\frac{1}{r} a_{,\vartheta} + \frac{1}{r} \frac{\partial}{\partial r} r^2 b_{,\vartheta} \Big) \Big[M^2\Omega^2 \sin 2\vartheta \Big(a + \frac{\partial}{\partial r} r^2 b \Big) + \frac{1}{r} \mathsf{S}_\vartheta \nabla^2 a \Big] \\
& + \frac{2}{\rho} 2M^2\Omega^2 r^3 \cos \vartheta c c_{,\vartheta} - \frac{1}{\rho} \frac{\Omega^2 M^4}{r^2 \sin^2 \vartheta} \Big(a + \frac{\partial}{\partial r} r^2 b \Big)^2 + \frac{\gamma p}{\rho^2} (\nabla^2 a)^2 \\
& - \frac{1}{\rho} \Big[\Omega^2 M^2 - 2\Omega\Omega_{,r} r \sin^2 \vartheta + \Omega^2 \cos^2 \vartheta + \frac{1}{\rho^2} \rho_{,r} \mathsf{S}_r \Big] (a_{,r} - L^2 b)^2 \\
& - \frac{1}{\rho r} \Big[-\Omega\Omega_{,r} r \sin 2\vartheta - 2\Omega\Omega_{,\vartheta} \sin^2 \vartheta \\
& \qquad\qquad + \frac{1}{r\rho^2} (\rho_{,\vartheta} \mathsf{S}_r + \rho_{,r} \mathsf{S}_\vartheta) \Big] (a_{,r} - L^2 b) \Big(a_{,\vartheta} + \frac{\partial}{\partial r} r^2 b_{,\vartheta} \Big) \\
& - \frac{1}{\rho} \Big[\Omega^2 M^2 - \Omega\Omega_{,\vartheta} \sin 2\vartheta + \frac{1}{\rho^2 r^2} \rho_{,\vartheta} \mathsf{S}_\vartheta \Big] \Big[\Big(\frac{1}{r} a_{,\vartheta} + \frac{1}{r} \frac{\partial}{\partial r} r^2 b_{,\vartheta} \Big)^2 \\
& \qquad\qquad\qquad\qquad\qquad\qquad + r^2 \frac{M^2}{\sin^2 \vartheta} c^2 \Big] \Bigg\}^{(+-)} \\
& + \Bigg\{ \frac{4}{\rho} M r^2 \Omega^2 \sin^2 \vartheta (a_{,r} - L^2 b) c_{,\vartheta} + \frac{2M}{\rho \sin \vartheta} \mathsf{S}_\vartheta (\nabla^2 a) c \\
& - \frac{\Omega^2 M^2}{\rho} \Big(\frac{M}{r \sin \vartheta} a + r c_{,\vartheta} \Big)^2 \\
& - \frac{rM}{\rho \sin \vartheta} \Big[-\Omega\Omega_{,r} r \sin 2\vartheta - 2\Omega\Omega_{,\vartheta} \sin^2 \vartheta
\end{aligned}
$$

$$+ \frac{1}{r\rho^2} (\rho_{,\vartheta} S_r + \rho_{,r} S_\vartheta)\Big](a_{,r} - L^2 b)c$$

$$- \frac{2M}{\rho \sin \vartheta}\Big[\Omega^2 M^2 - \Omega\Omega_{,\vartheta} \sin 2\vartheta$$

$$+ \frac{1}{r^2\rho^2} \rho_{,\vartheta} S_\vartheta\Big]\Big(a_{,\vartheta} + \frac{\partial}{\partial r} r^2 b_{,\vartheta}\Big) c\Big\}^{[+-]} ; \qquad (B4)$$

$$- D_M{}^r = \Big\{4\pi G a L^2 b - \frac{1}{\rho^2} p_{,r}(a_{,r} - L^2 b)^2$$

$$- \frac{1}{r\rho^2} p_{,\vartheta}(a_{,r} - L^2 b)\Big(\frac{1}{r} a_{,\vartheta} + \frac{1}{r}\frac{\partial}{\partial r} r^2 b_{,\vartheta}\Big)\Big\}^{(+-)}$$

$$+ \Big\{- \frac{M}{\rho^2 \sin \vartheta} p_{,\vartheta}(a_{,r} - L^2 b)c\Big\}^{[+-]} ; \qquad (B5)$$

and

$$- D_M{}^\vartheta = \Big\{-4\pi G \frac{1}{r^2} a \frac{\partial}{\partial r} r^2 b_{,\vartheta} - \frac{1}{r\rho^2} p_{,r}(a_{,r} - L^2 b)\Big(\frac{1}{r} a_{,\vartheta} + \frac{1}{r}\frac{\partial}{\partial r} r^2 b_{,\vartheta}\Big)$$

$$- \frac{1}{r^2\rho^2} p_{,\vartheta}\Big(\frac{1}{r} a_{,\vartheta} + \frac{1}{r}\frac{\partial}{\partial r} r^2 b_{,\vartheta}\Big)^2 - \frac{M^2}{\rho^2 \sin^2 \vartheta} p_{,\vartheta}c^2\Big\}^{(+-)}$$

$$+ \Big\{-4\pi G \frac{M}{\sin \vartheta} ac - \frac{M}{\rho^2 \sin \vartheta} p_{,r}(a_{,r} - L^2 b)c$$

$$- \frac{2M}{r\rho^2 \sin \vartheta} p_{,\vartheta}\Big(\frac{1}{r} a_{,\vartheta} + \frac{1}{r}\frac{\partial}{\partial r} r^2 b_{,\vartheta}\Big) c\Big\}^{[+-]} ; \qquad (B6)$$

and where we have used the notation

$$L^2 \equiv \frac{1}{\sin \vartheta} \frac{\partial}{\partial \vartheta} \sin \vartheta \frac{\partial}{\partial \vartheta} + \frac{1}{\sin^2 \vartheta} \frac{\partial^2}{\partial \varphi^2} = \frac{1}{\sin \vartheta} \frac{\partial}{\partial \vartheta} \sin \vartheta \frac{\partial}{\partial \vartheta} - \frac{M^2}{\sin^2 \vartheta} ;$$

$$S_r = p_{,r} - \frac{\gamma p}{\rho} \rho_{,r} ; \qquad S_\vartheta = p_{,\vartheta} - \frac{\gamma p}{\rho} \rho_{,\vartheta} .$$

While this expression is complicated, it should be reasonably adaptable to computer calculations.

b) *Boundary Conditions on a, b, c*

Though we have not restricted the perturbation ξ to have any particular value at the star's surface, there are nevertheless some weak boundary conditions on a, b, and c that arise from the vanishing of $\mathbf{n}$ (and ξ) at the star's center and from the vanishing of $\mathbf{n}$ (but not ξ) at the star's surface (Σ^-) if ρ vanishes there.

The demand that $\mathbf{n}$ vanish at the star's center requires that $\mathbf{n}^L = -\mathbf{n}^T$, but *not* that each vanish separately. This implies

$$\left.\begin{aligned} a^+{}_{,r} &= L^2 b^+ , \\[2mm] a^+{}_{,\vartheta} &= - \frac{\partial}{\partial r} r^2 b^+{}_{,\vartheta} - \frac{r^2 M}{\sin \vartheta} c^- , \\[2mm] a^+ &= - \frac{\partial}{\partial r} r^2 b^+ - \frac{r^2 \sin \vartheta}{M} c^-{}_{,\vartheta} , \end{aligned}\right\} \quad \text{at the star's center} \qquad (B7)$$

plus the conjugate equations (plus and minus interchanged).

At the surface of the star (actually at Σ^-, where the surface integral is evaluated) we demand only that a, b, and c be finite with finite derivatives, except if $\rho = 0$ there. Then again we must have $\mathbf{n}^L = -\mathbf{n}^T + O(\rho)$; that is, $\mathbf{n}$ must vanish at least as fast as ρ near Σ^-. So the same equations (B7) must hold at the surface, to order ρ. (This is also true, of course, anywhere else that ρ vanishes.)

c) Eigenfrequencies of Stable Modes

If condition (B3) is satisfied, the star is stable. In that case the eigenfrequencies of oscillation are the stationary values of the roots of the following quadratic expression (cf. Lynden-Bell and Ostriker 1967):

$$\omega^2 V_1 + \omega V_2 + V_3 = 0, \tag{B8}$$

with

$$V_1 = \frac{1}{\pi} \int_{\text{interior}} \frac{1}{\rho} \mathbf{n} \cdot \mathbf{n} r^2 \sin \vartheta \, dr \, d\vartheta \, d\varphi$$

$$= \int\int_{\text{interior}} \frac{1}{\rho} r^2 \sin \vartheta \, dr \, d\vartheta \left\{ (a_{,r} - L^2 b)^2 + \left(\frac{1}{r} a_{,\vartheta} + \frac{1}{r} \frac{\partial}{\partial r} r^2 b_{,\vartheta} \right)^2 \right.$$

$$\left. + \frac{r^2 M^2}{\sin^2 \vartheta} c^2 + \frac{M^2}{r^2 \sin^2 \vartheta} \left(a + \frac{\partial}{\partial r} r^2 b \right)^2 \right\}^{(+-)}$$

$$+ \int\int_{\text{interior}} \frac{1}{\rho} r^2 \sin \vartheta \, dr \, d\vartheta \left\{ \frac{rM}{\sin \vartheta} (a_{,r} - L^2 b) c + \left(\frac{M}{r \sin \vartheta} a + r c_{,\vartheta} \right)^2 \right\}^{(+-)}; \tag{B9}$$

$$V_2 = \frac{1}{\pi} \int_{\text{interior}} \frac{2\Omega}{\rho} \left[g_{jk} \eta^j \eta^k_{,\varphi} + \left(\frac{\alpha}{\Omega} \right)_{,j} \eta^j \eta^\varphi \right] r^2 \sin \vartheta \, dr \, d\vartheta \, d\varphi$$

$$= \int\int_{\text{interior}} \frac{2\Omega}{\rho} r^2 \sin \vartheta \, dr \, d\vartheta \left\{ \frac{2M^2}{\sin \vartheta} \left(a_{,\vartheta} + \frac{\partial}{\partial r} r^2 b_{,\vartheta} \right) c + 2r \sin \vartheta (a_{,r} - L^2 b) c_{,\vartheta} \right.$$

$$\left. + 2 \cos \vartheta \left(a_{,\vartheta} + \frac{\partial}{\partial r} r^2 b_{,\vartheta} \right) c_{,\vartheta} - \frac{2M^2 \cos \vartheta}{\sin^2 \vartheta} \left(a + \frac{\partial}{\partial r} r^2 b \right) c \right\}^{(+-)}$$

$$+ \int\int_{\text{interior}} \frac{2\Omega}{\rho} r^2 \sin \vartheta \, dr \, d\vartheta \left\{ (a_{,r} - L^2 b)^2 + \frac{1}{r^2} \left(a_{,\vartheta} + \frac{\partial}{\partial r} r^2 b_{,\vartheta} \right)^2 \right.$$

$$- \frac{M^2}{r^2 \sin^2 \vartheta} \left(a + \frac{\partial}{\partial r} r^2 b \right)^2 + r^2 c_{,\vartheta}^2 - \frac{2}{r} (a_{,r} - L^2 b) \left(a + \frac{\partial}{\partial r} r^2 b \right)$$

$$\left. - \frac{\cos \vartheta}{r^2 \sin \vartheta} \left(a_{,\vartheta} + \frac{\partial}{\partial r} r^2 b_{,\vartheta} \right) \left(a + \frac{\partial}{\partial r} r^2 b \right) + \frac{r^2 \cos \vartheta}{\sin \vartheta} c_{,\vartheta} c \right\}^{(+-)}; \tag{B10}$$

and

$$V_3 = \int\int_{\text{interior}} C_M r^2 \sin \vartheta \, dr \, d\vartheta + \int_{\Sigma^-} D_M{}^i n_i d\sigma, \tag{B11}$$

where C_M and $D_M{}^i$ are given by equations (B4), (B5), and (B6). Thus the trial functions permit estimation of eigenfrequencies for the stable case. Unfortunately one cannot estimate e-folding times for the unstable modes in this manner (see Lynden-Bell and Ostriker 1967).

APPENDIX C

STABILITY OF AXIALLY SYMMETRIC PERTURBATIONS

The necessary condition for stability of axially symmetric perturbations obtained by Chandrasekhar and Lebovitz (1968) and by Lynden-Bell and Ostriker (1967) can be expressed in terms of scalars with the method of Appendix B.

The condition is (eq. [54])

$$\int_{\text{interior}} \left\{ \frac{\Omega^2}{\rho r^2 \sin^2 \vartheta} [(r^2 \sin^2 \vartheta)_{,j} \eta^j]^2 - C[\mathbf{n}, \mathbf{n}] \right\} dV - \int_{\Sigma^-} D[\mathbf{n}, \mathbf{n}] \cdot \mathbf{n} d\sigma > 0 . \quad \text{(C1)}$$

We can obtain C and D from Appendix B by setting $M = 0$ in equations (B4), (B5), and (B6). We can then expand the first term in inequality (C1) in terms of a, b, and c, and add it to C. The result is that a necessary condition for stability of a differentially rotating star against axially symmetric perturbations is that, for all a, b, c satisfying the boundary conditions described in Appendix B,

$$- \iint_{\text{interior}} C_A r^2 \sin \vartheta dr d\vartheta - \int_{\Sigma^-} D_A{}^i n_i d\sigma > 0, \quad \text{(C2)}$$

where

$$-C_A = -4\pi G \left(a_{,r}{}^2 + \frac{1}{r^2} a_{,\vartheta}{}^2 \right) + \frac{2S_r}{\rho} (a_{,r} - L^2 b) \nabla^2 a$$

$$+ \frac{2S_\vartheta}{\rho r^2} \left(a_{,\vartheta} + \frac{\partial}{\partial r} r^2 b_{,\vartheta} \right) \nabla^2 a + \frac{\gamma p}{\rho^2} (\nabla^2 a)^2$$

$$- \frac{1}{\rho} \left[-2\Omega\Omega_{,r} r \sin^2 \vartheta + \Omega^2 (1 + 3 \sin^2 \vartheta) + \frac{1}{\rho^2} \rho_{,r} S_r \right] (a_{,r} - L^2 b)^2$$

$$- \frac{1}{\rho r} (-\Omega\Omega_{,r} r \sin 2\vartheta - 2\Omega\Omega_{,\vartheta} \sin^2 \vartheta$$

$$+ 4\Omega^2 \sin 2\vartheta) \left(a_{,\vartheta} + \frac{\partial}{\partial r} r^2 b_{,\vartheta} \right) (a_{,r} - L^2 b)$$

$$- \frac{1}{\rho r^2} \left(-\Omega\Omega_{,\vartheta} \sin 2\vartheta + 4\Omega^2 \cos^2 \vartheta + \frac{1}{\rho^2 r^2} \rho_{,\vartheta} S_\vartheta \right) \left(a_{,\vartheta} + \frac{\partial}{\partial r} r^2 b_{,\vartheta} \right)^2 ; \quad \text{(C3)}$$

$$-D_A{}^r = 4\pi G a L^2 b - \frac{1}{\rho^2} p_{,r} (a_{,r} - L^2 b)^2 - \frac{1}{\rho^2 r^2} p_{,\vartheta} (a_{,r} - L^2 b) \left(a_{,\vartheta} + \frac{\partial}{\partial r} r^2 b_{,\vartheta} \right) ; \quad \text{(C4)}$$

and

$$-D_A{}^\vartheta = - \frac{4\pi G}{r^2} a \frac{\partial}{\partial r} r^2 b_{,\vartheta} - \frac{1}{\rho^2 r^2} p_{,r} (a_{,r} - L^2 b) \left(a_{,\vartheta} + \frac{\partial}{\partial r} r^2 b_{,\vartheta} \right)$$

$$- \frac{1}{\rho^2 r^2} p_{,\vartheta} \left(\frac{1}{r} a_{,\vartheta} + \frac{1}{r} \frac{\partial}{\partial r} r^2 b_{,\vartheta} \right)^2 . \quad \text{(C5)}$$

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