Junction conditions across perturbed contact discontinuities

Bernard F. Schutz *Department of Applied Mathematics and Astronomy, University College, PO Box 78, Cardiff CF1 1XL*

Summary. Near an oscillating 'contact discontinuity' — a discontinuity which particles do not cross — the appropriate junction condition is that the Lagrangian change in the pressure be continuous. This corrects earlier claims that an inequivalent condition, continuity of the Eulerian pressure change, should be used when the usual linearized equations for the Eulerian perturbation are employed.

There has recently been some interest in the oscillations of stars which contain surfaces of discontinuity, such as white dwarfs with crystalline cores and liquid envelopes (Hansen & Van Horn 1979) or neutron stars with solid crusts. If the oscillation has a short enough period, the surface of discontinuity may be idealized as moving with the particles of which the star is made. Such an advected surface characterizes a 'contact discontinuity', in the terminology of Ledoux & Walraven (1958). These authors recommend that the linear perturbation equations should be solved near the discontinuity by applying the 'junction condition' that the Eulerian change in the pressure should be continuous. This condition can, however, give incorrect results, essentially because the Eulerian form of the linear perturbation equations is not valid near the discontinuity. The correct junction condition is that the Lagrangian change in the pressure be continuous.

The difficulty near the discontinuity is illustrated schematically in Fig. 1, in which it is assumed that the density is discontinuous but the pressure is, of course, always continuous. Because the density is discontinuous, the pressure gradient is discontinuous. If we *define* the Eulerian change in the pressure, $\delta p(x)$, to be the difference between $p_0(x)$ and $p_1(x)$, then δp is continuous. But $\delta \nabla p$ need not be 'small': between x_d and x'_d, it is much larger than elsewhere. The Eulerian perturbation of the equations of motion does not, therefore, involve uniformly 'small' terms, and will not be valid *if linearized*.

The Lagrangian equations suffer no such problem. Since particles do not cross the discontinuity, the Lagrangian changes in all quantities remain small, as will be clear from inspection of Fig. 1. For example, the linearized perturbation equations in the adiabatic case can be expressed entirely in terms of the displacement vector ξ (Lynden-Bell & Ostriker 1967), so that the correct junction conditions are that (i) the component of ξ
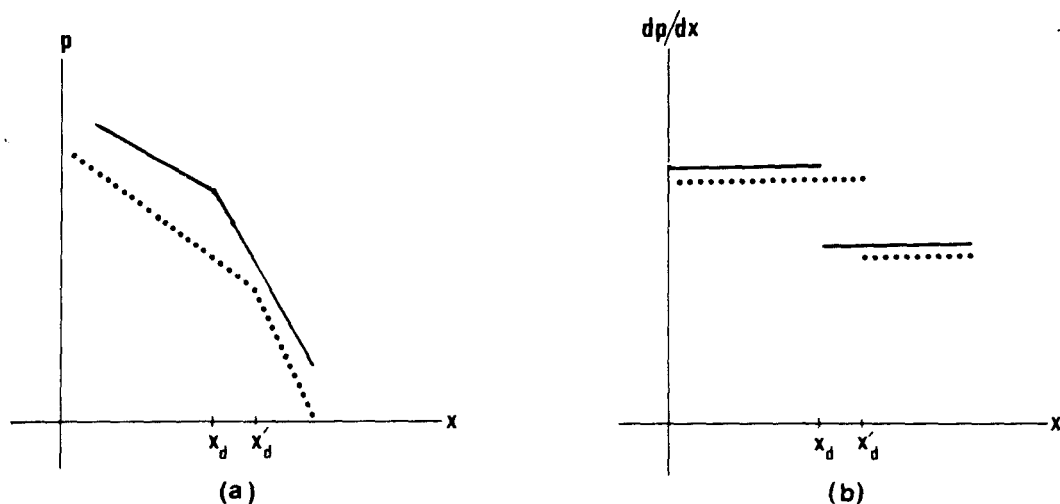
Figure 1. If the density changes discontinuously, then the pressure gradient will be discontinuous as well. Such a case is illustrated for a star before (solid line) and after (dotted line) the perturbation is applied. The pressure appears in (a), its gradient in (b). While Lagrangian changes remain small near the discontinuity, Eulerian changes do not.

normal to the surface of discontinuity be continuous, and (ii) $\nabla \cdot \xi$ be continuous. The second condition ensures that the Lagrangian change in the pressure, Δp, is likewise continuous. (The calculation of Hansen & Van Horn (1979) referred to earlier is Lagrangian, so it correctly incorporates these conditions.) But note that the true Eulerian change δp defined above is no longer related to Δp by the usual equation

$$\Delta p = \delta p + \xi \cdot \nabla p, \tag{1}$$

because ∇p is discontinuous. When ξ moves a particle across the unperturbed position of the discontinuity, the term $\xi \cdot \nabla p (x)$ is not a good approximation to the change in the unperturbed pressure field from x to $x + \xi$.

It is possible to define a false 'Eulerian' change $\delta' p$ in terms of Δp by equation (1), and similarly to define $\delta' \rho$, $\delta' v$, $\delta' s$, etc. by the analogues of equation (1) as given in Lynden–Bell & Ostriker (1967) or, somewhat differently, in Friedman & Schutz (1978). These false changes do satisfy the ordinary linearized Eulerian version of the equations of motion, but there is no physical reason to demand $\delta' p$ be continuous across the surface of discontinuity In fact from equation (1) we learn that

$$[\Delta p] = 0 \Rightarrow [\delta' p] = - [\xi \cdot \nabla p] \tag{2}$$

where $[f]$ represents the jump in f across the surface. It follows that the usual linearized Eulerian perturbation equations can be used, but only be appyling the junction condition (2), not by demanding continuity of $\delta' p$.

Acknowledgments

References

Friedman, J. L. & Schutz, B. F., 1978. *Astrophys. J.*, **221**, 937–957.
Hansen, C. J. & Van Horn, H. M., 1979. *Astrophys. J.*, **233**, 253–258.
Ledoux, P. & Walraven, Th., 1958. *Handbuch der Physik* vol. 51, p. 455, Springer-Verlag, Berlin.
Lynden-Bell, D. & Ostriker, J. P., 1967. *Mon. Not. R. astr. Soc.*, **136**, 293.