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Modular realizations of hyperbolic Weyl groups

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Abstract

We study the recently discovered isomorphisms between hyperbolic Weyl groups and modular groups over integer domains in normed division algebras. We show how to realize the group action via fractional linear transformations on generalized upper half-planes over the division algebras, focusing on the cases involving quaternions and octonions. For these we construct automorphic forms, whose explicit expressions depend crucially on the underlying arithmetic properties of the integer domains. Another main new result is the explicit octavian realization of $W^+(\mathrm{E}_{10})$, which contains as a special case a new realization of $W^+(\mathrm{E}_8)$ in terms of unit octavians and their automorphism group.

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1 Introduction

Following the work of Feingold and Frenkel [1] it was realized in [2] that very generally there are isomorphisms between the Weyl groups of hyperbolic Kac–Moody algebras and (finite extensions or quotients of) modular groups over integer domains in the division algebras $\mathbb{A}$ of real, complex, quaternionic or octonionic numbers; $\mathbb{A} = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$. The simplest case is associated with the rational integers $\mathbb{Z} \subset \mathbb{R}$: the modular group $\mathrm{PSL}(2,\mathbb{Z})$ is isomorphic to the even part of the Weyl group of the canonical hyperbolic extension of the Lie algebra $\mathfrak{sl}(2,\mathbb{R})$ [1]. This is but the first example of a rich and interesting family of novel isomorphisms that rely on the existence of integers within the division algebras of higher dimensions [2], culminating in the relation $W^+(\mathrm{E}_{10}) \cong \mathrm{PSL}(2,\mathsf{O})$ between the even Weyl group of E_{10} (the hyperbolic extension of the exceptional Lie algebra E_8) and the integer octonions $\mathsf{O} \subset \mathbb{O}$.

These isomorphisms are interesting for several reasons. The underlying arithmetic structure of the various integer domains could help in understanding the structure of the still elusive hyperbolic Kac–Moody algebras, especially for the "maximally extended" hyperbolic algebra E_{10} which is expected to possess very special properties.[1] This is certainly a tantalizing possibility but we will not pursue it further in this work. Instead, we focus on the modular theory associated with the novel modular groups that appear in the isomorphisms, and the associated "generalized upper half planes". More specifically, we will discuss the quaternionic and the octonionic cases,

[1]For instance, the arithmetic structure of the modular group $\mathrm{PSL}(2,\mathsf{O})$ may impose more stringent constraints on the characters and root multiplicities of E_{10} than the corresponding modular groups do for the lower rank hyperbolic algebras. E_8 and its hyperbolic extension E_{10} are also special because they are the only eligible algebras with (Euclidean or Lorentzian) even self-dual root lattices.

corresponding to the hyperbolic extensions of the split real $D_4 \equiv \mathfrak{so}(4,4)$ and E_8 Lie algebras. In the quaternionic case in particular, we give a detailed description of the "generalized upper half plane" of dimension five on which the modular group acts naturally.

Our main motivation derives from potential applications of this modular theory in quantum gravity and M-theory. As shown recently, the generalized upper half planes are the configuration spaces for certain models of (anisotropic) mini-superspace quantum gravity, and a subclass of the automorphic forms associated with the modular groups studied here appear as solutions to the Wheeler–DeWitt equation in the cosmological billiards limit [3–5], a version of mini-superspace quantum gravity which we refer to as "arithmetic quantum gravity". Automorphic functions (particularly Eisenstein series) also feature prominently in recent studies of non-perturbative effects in string and M-theory, see [6,7] for early work and [8–10] for recent progress concerning the split exceptional groups $G = E_7$ and E_8, with corresponding arithmetic groups $E_7(\mathbb{Z})$ and $E_8(\mathbb{Z})$.[2] There exist far reaching conjectures concerning the infinite-dimensional extensions E_9, E_{10} and E_{11}, but it is less clear how to make sense of (or even *define*) their discrete subgroups (see, however, [11]). Nevertheless, our results can be viewed as a first step towards extending these ideas to the infinite-dimensional duality group E_{10}, since $\mathrm{PSL}(2,\mathsf{O})$, being the even Weyl group, is expected to be contained in any hypothetical arithmetic subgroup of E_{10}. A possible link with the work of [3–5] is provided by a conjecture of [13] according to which the solution of the Wheeler–DeWitt equation in M-theory is a vastly generalized automorphic form with respect to $E_{10}(\mathbb{Z})$. Our results on $D_4 \equiv \mathfrak{so}(4,4)$ can also be viewed from a string theory perspective, since $SO(4,4)$ can be realized as the symmetry of type I supergravity in ten dimensions without any vector multiplets on a four-torus. It is also the continuous version of T-duality of type II strings.

Within the general theory of automorphic forms for groups of real rank one, the main new feature that distinguishes our construction is the link with the division algebras $\mathbb{A}$ and the integer domains $\mathcal{O} \subset \mathbb{A}$ (which always contain $\mathbb{Z}$ as a "real" subset). Moreover, the arithmetic modular groups considered here are all identified with the even subgroups of certain hyperbolic Weyl groups. That is, each of the modular groups considered here is isomorphic to the *even* Weyl group of a canonical hyperbolic extension $\mathfrak{g}^{++}$ of an associated simple finite-dimensional split Lie algebra $\mathfrak{g}$,

$$\Gamma \equiv W^+_{\mathrm{hyp}} \equiv W^+(\mathfrak{g}^{++}). \tag{1.1}$$

[2] In some cases, the relevant "automorphic forms" are not eigenfunctions of the Laplacian but have source terms [12].

More specifically, the groups Γ are discrete subgroups of the groups $\mathrm{SO}(1, n+1)$ acting on the (Lorentzian) root space of $\mathfrak{g}^{++}$, but with additional restrictions implying special (and as yet mostly unexplored) arithmetic properties. The standard Poincaré series construction then implies a simple general expression for an automorphic function on the generalized upper half plane as a sum over images of the group action as

$$f(u, v) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} (\gamma \cdot v)^s, \tag{1.2}$$

where $u \in \mathbb{A}$ and $v \in \mathbb{R}_{>0}$ parametrize the upper half plane. (This expression is convergent for $\mathrm{Re}(s)$ sufficiently large and can be analytically continued to most complex values of s by means of a suitable functional relation.) The subgroup Γ_∞ appearing in this sum is conventionally defined as the subgroup of the modular group Γ leaving invariant the cusp at infinity; here this group becomes identified with the even Weyl group of the affine subalgebra $\mathfrak{g}^+ \subset \mathfrak{g}^{++}$,

$$\Gamma_\infty \equiv W^+_{\mathrm{aff}} \equiv W^+(\mathfrak{g}^+) \subset \Gamma. \tag{1.3}$$

As we will explain, the cusp at infinity in the generalized upper half plane here corresponds to the affine null root of the affine algebra $\mathfrak{g}^+$, in accord with the fact that W_{aff} can be defined as the subgroup of W_{hyp} stabilizing the affine null root [14]. At the same time W^+_{aff} is a *maximal parabolic subgroup* of $\Gamma = W^+_{\mathrm{hyp}}$. Because $W_{\mathrm{aff}} = W_{\mathrm{fin}} \ltimes \mathcal{T}$, where $W_{\mathrm{fin}} \equiv W(\mathfrak{g})$ and $\mathcal{T}$ is the abelian group of affine translations, the minimal parabolic subgroup is $\mathcal{T} \subset \Gamma_\infty$.

We briefly position our work relative to the existing mathematical literature that we are aware of. Generalized upper half planes have appeared for example in [15] where they are defined as cosets $\mathrm{GL}(n, \mathbb{R})/\mathrm{O}(n) \times \mathrm{GL}(1, \mathbb{R})$, on which the discrete groups $\mathrm{GL}(n, \mathbb{Z})$ act as generalized modular groups. This definition fits with our definition only for $n = 2$ (corresponding to $\mathbb{A} = \mathbb{R}$). More generally, one can define generalized upper half planes as quotient spaces $G/K(G)$ where G is a non-compact group and $K(G)$ its compact subgroup, and then consider the action of some arithmetic subgroup $G_{\mathbb{Z}} \subset G$ on this space. Our discrete groups Γ are subgroups of $\mathrm{SO}(1, n+1; \mathbb{R})$ and live on the well-known symmetric space $\mathrm{SO}(1, n+1)/\mathrm{SO}(n+1)$ [16]. Therefore our analysis is in principle part of the general theory of reductive groups of real rank one (see for example [17–22]). What makes it stand out and interesting for us — apart from the fascinating potential applications in fundamental physics — is the precise nature of the discrete subgroups Γ, which are different from the arithmetic groups $G_{\mathbb{Z}}$ usually considered in this context,[3] and the link to integers in normed division algebras and to hyperbolic Weyl groups, which has not been exposed in the existing literature to the best of our knowledge. For example, it would obscure the underlying arithmetic structure, if one tried to describe our final expression (6.11) in terms of a complicated lattice sum in $\mathbb{R}^{2n}$. It is not even clear to us how to derive such a sum without the knowledge of the integral structure.

We emphasize that it is not our intention here to give a complete account of the theory of automorphic forms for the modular groups we study (after all, even for $\Gamma = \mathrm{SL}(2,\mathbb{Z})$ this is a subject which easily fills a whole book [20, 21]). Rather, we regard the present work merely as a first step towards a theory that remains to be fully developed in future work. In particular, since we are using a direct and specific construction in terms of Poincaré series, we have not explored the adelic approach to modular forms (see for example [23]) although we anticipate that this could yield interesting results in this context.

This paper is structured as follows. In Section 2, we review the construction of normed division algebras and introduce the generalized upper half planes that we will utilize in the remainder of the text. Section 3 is devoted to the isomorphism between modular groups and hyperbolic Weyl groups in general; Section 4 then deals in detail with the case of quaternions and $\mathrm{D}_4 \equiv \mathfrak{so}(4,4)$, whereas in Section 5, we present the modular group $\mathrm{PSL}(2,\mathsf{O})$ over the non-associative integer octonions O that appears for E_{10}, the hyperbolic extension of E_8. In particular, it gives a novel realization of the even (and finite) Weyl group $W^+(\mathrm{E}_8)$ in terms of the 240 octavian units and the finite automorphism group $\mathrm{G}_2(2)$ of the octavians. Automorphic forms for all these groups are defined and analysed in detail in Section 6, with special emphasis on the most interesting case $\mathbb{A} = \mathbb{O}$. These sections form the heart of this paper and contain the bulk of our new results. Several appendices contain some additional results and technical details that we use in the main text.

2 Division algebras and upper half planes

2.1 Cayley–Dickson doubling

The division algebras of complex numbers, quaternions and octonions can be defined recursively from the real numbers by a procedure called *Cayley–Dickson doubling* [24, 25]. Let $\mathbb{A}$ be a real finite-dimensional algebra

[3] These are obtained by choosing a metric to define the matrix group $\mathrm{SO}(1, n+1)$ and then restricting the matrix entries to (rational) integers.

with a conjugation, i.e., a vector space involution $x \mapsto \bar{x}$ such that $\overline{ab} = \bar{b}\,\bar{a}$ for any a, $b \in \mathbb{A}$. In the Cayley–Dickson doubling, one introduces a vector space $i\mathbb{A}$, isomorphic to $\mathbb{A}$, where i is a new imaginary unit, and considers the direct sum $\mathbb{A} \oplus i\mathbb{A}$ of these two vector spaces. The product of two elements in $\mathbb{A} \oplus i\mathbb{A}$ is defined as [25]

$$(a + ib)(c + id) := (ac - d\bar{b}) + i(cb + \bar{a}d), \tag{2.1}$$

and conjugation as

$$\overline{a + ib} := \bar{a} - ib \qquad (\text{with } ai = i\bar{a}). \tag{2.2}$$

Starting from $\mathbb{A} = \mathbb{R}$ (with the identity map as conjugation) and successively applying the Cayley–Dickson doubling one thus obtains an infinite sequence of so-called *Cayley–Dickson algebras*. The first doubled algebra is of course the familiar algebra $\mathbb{C}$ of complex numbers. The following two algebras in the sequence are $\mathbb{H}$ and $\mathbb{O}$, consisting of quaternions and octonions, respectively. Studying these algebras one finds that $\mathbb{H}$ is associative but not commutative, whereas $\mathbb{O}$ is neither associative nor commutative. Nevertheless, $\mathbb{O}$ is an *alternative* algebra, which means that any subalgebra generated by two elements is associative.

Any Cayley–Dickson algebra admits a positive-definite inner product

$$(a, b) := \tfrac{1}{2}(a\bar{b} + b\bar{a}), \tag{2.3}$$

with the associated norm $|a|^2 \equiv (a, a) = a\bar{a}$. As a consequence of (2.1) and (2.2) the norm of the "doubled" expression $a + ib \in \mathbb{A} \oplus i\mathbb{A}$ is

$$|a + ib|^2 = |a|^2 + |b|^2. \tag{2.4}$$

From the doubling formulas (2.1) and (2.2), it follows:

$$\begin{aligned} |(a + ib)(c + id)|^2 &= |ac - d\bar{b}|^2 + |cb + \bar{a}d|^2 \\ &= (|a|^2 + |b|^2)(|c|^2 + |d|^2) + 2\,\mathrm{Re}\Big((\bar{d}a)(cb) - (b\bar{d})(ac)\Big), \end{aligned} \tag{2.5}$$

where $\mathrm{Re}\, a := \frac{1}{2}(a + \bar{a})$. Because for $a, b, c \in \mathbb{A}$ (for all $\mathbb{A}$)

$$\mathrm{Re}\, a(bc) = \mathrm{Re}\,(ab)c, \quad \mathrm{Re}\, ab = \mathrm{Re}\, ba, \tag{2.6}$$

we obtain

$$|(a + ib)(c + id)|^2 = |a + ib|^2 |c + id|^2 + 2\,\mathrm{Re}\Big(\bar{d}\,\{a, c, b\}\Big), \tag{2.7}$$

where the associator $\{a,b,c\} := a(bc) - (ab)c$ vanishes for associative algebras. The fact that $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$ are associative thus implies that the composition property

$$|ab| = |a||b| \tag{2.8}$$

holds for their doubled algebras $\mathbb{C}, \mathbb{H}$ and $\mathbb{O}$. Algebras with this property are called *normed division algebras*. It follows from (2.8) that if $ab = 0$ then a or b must be zero, so these algebras have no zero divisors. Hurwitz' theorem (see [24,25]) states that $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ and $\mathbb{O}$ are the only (real and finite-dimensional) normed division algebras with unity. For instance, since the octonions are non-associative, their Cayley–Dickson double does not satisfy the norm composition property (2.8). This is a real 16-dimensional algebra, often called the *sedenions*. It is of course non-associative (since it contains the octonions) but also non-alternative, and does possess zero divisors.

2.2 Generalized upper half planes

For every normed division algebra $\mathbb{A}$ we define the *generalized upper half plane*

$$\mathcal{H} \equiv \mathcal{H}(\mathbb{A}) := \{z = u + iv \quad \text{with } u \in \mathbb{A} \text{ and real } v > 0\}\,. \tag{2.9}$$

By i we will always denote the new imaginary unit not contained in $\mathbb{A}$, while $u \equiv u^0 + \sum u^i e_i$ with the imaginary units e_i in $\mathbb{A}$. Hence $\mathcal{H}(\mathbb{A})$ is contained in a hyperplane in the Cayley–Dickson double $\mathbb{A} \oplus i\mathbb{A}$, and of (real) dimension $\dim_{\mathbb{R}} \mathcal{H}(\mathbb{A}) = (\dim_{\mathbb{R}} \mathbb{A}) + 1$. By complex conjugation (2.2) we get $\bar{z} = \bar{u} - iv$, and thus $\bar{z}$ parametrizes the corresponding "lower half plane" $\overline{\mathcal{H}}$. From (2.7), we see that the composition property

$$|zz'| = |z||z'| \quad z, z' \in \mathcal{H}(\mathbb{A}) \tag{2.10}$$

continues to hold for *all* normed division algebras $\mathbb{A} = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$. In particular, it still holds for $\mathbb{A} = \mathbb{O}$, even though in this case z is a sedenion, because the associator in (2.7) still vanishes as long as v remains real. Likewise, the alternative laws

$$(aa)z = a(az), \quad (az)a = a(za), \quad (za)a = z(aa) \tag{2.11}$$

remain valid for $a \in \mathbb{O}$ and $z \in \mathcal{H}(\mathbb{O})$ by Cayley–Dickson doubling (2.1). This will be important below when we define modular transformations.

The line element in $\mathcal{H}(\mathbb{A})$ is

$$ds^2 = \frac{|du|^2 + dv^2}{v^2} \tag{2.12}$$

for $u \in \mathbb{A}\,(= \mathbb{R}, \mathbb{C}, \mathbb{H} \text{ or } \mathbb{O})$ and $v > 0$, with $|du|^2 \equiv d\bar{u}\,du$. Using v to parametrize the geodesic, the geodesic equation is (with $u' \equiv du/dv$)

$$\frac{d}{dv}\left(\frac{1}{v\sqrt{1+|u'|^2}}\frac{du}{dv}\right) = 0. \tag{2.13}$$

It is then straightforward to see that geodesics are given by straight lines parallel to the "imaginary" ($= v$) axis, or by half circles starting at $u_1 \in \mathbb{A}$ and ending at $u_2 \in \mathbb{A}$ on the boundary $v = 0$ of $\mathcal{H}(\mathbb{A})$. The equation of this geodesic half circle for $z = u + iv \in \mathcal{H}(\mathbb{A})$ reads

$$(u - u_1)(\bar{u} - \bar{u}_2) + v^2 = 0. \tag{2.14}$$

The length of any geodesic segment connecting two points $z_1, z_2 \in \mathcal{H}(\mathbb{A})$ is

$$d(z_1, z_2) = \log \frac{|z_1 - z_2^*| + |z_1 - z_2|}{|z_1 - z_2^*| - |z_1 - z_2|} \tag{2.15}$$

or, equivalently,

$$d(z_1, z_2) = 2\,\text{artanh}\,\frac{|z_1 - z_2|}{|z_1 - z_2^*|} = \text{arcosh}\left(1 + \frac{|z_1 - z_2|^2}{v_1 v_2}\right), \tag{2.16}$$

where we defined

$$z = u + iv \Rightarrow z^* := u - iv \in \overline{\mathcal{H}(\mathbb{A})}. \tag{2.17}$$

Formula (2.16) generalizes the familiar formula from complex analysis to all division algebras. Likewise, the volume element on $\mathcal{H}(\mathbb{A})$ is given by

$$d\text{vol}(z) := \frac{d^n u\, dv}{v^{n+1}} \tag{2.18}$$

and the Laplace–Beltrami operator reads[4]

$$\triangle_{\text{LB}} = v^{n+1}\frac{\partial}{\partial v}\left(v^{1-n}\frac{\partial}{\partial v}\right) + v^2\frac{\partial}{\partial u}\frac{\partial}{\partial \bar{u}}, \tag{2.19}$$

with $n = 1,\, 2,\, 4$ and 8 for $\mathbb{A} = \mathbb{R},\, \mathbb{C},\, \mathbb{H}$ and $\mathbb{O}$, respectively.

[4]Note that $\frac{\partial}{\partial u}\frac{\partial}{\partial \bar{u}}|u|^2 \equiv \sum_{i=0}^{n-1}\frac{\partial^2}{\partial u_i^2}|u|^2 = 2n$ in our conventions.

As far as the geometry is concerned, the "generalized upper half planes" $\mathcal{H}(\mathbb{A})$ are special examples of the general coset spaces

$$\mathcal{H}_n = \frac{\mathrm{SO}(1, n+1)}{\mathrm{SO}(n+1)} \tag{2.20}$$

for $n = 1, 2, 4, 8$. These are all hyperbolic spaces of constant negative curvature, which can be embedded as unit hyperboloids

$$\mathcal{H}_n = \left\{ x^\mu \in \mathbb{R}^{1,n+1} \,\middle|\, x^\mu x_\mu = -x^+ x^- + x\cdot x = -1\,;\, x^\pm > 0 \right\} \tag{2.21}$$

into the forward lightcone of $(n+2)$-dimensional Minkowski space $\mathbb{R}^{1,n+1}$, with $x \in \mathbb{R}^n$. These hyperboloids are isometric to the upper half planes introduced above by means of the mapping

$$x^- = \frac{1}{v}, \quad x^+ = v + \frac{|u|^2}{v}, \quad x = \frac{u}{v}. \tag{2.22}$$

where the last equation identifies the components $x^j = u^j/v$ in the Euclidean subspace $\mathbb{R}^n$. The line element (2.12) is the pull-back of the Minkowskian line element $ds^2 = -dx^+ dx^- + dx\cdot dx$. Similarly, it is straightforward to check that

$$\int d\mathrm{vol}(z) \big(\cdots \big) = \int d^n x \, dx^+ \, dx^- \, \delta\big(x^+ x^- - \bar{x}x - 1 \big) \big(\cdots \big) \tag{2.23}$$

and to derive the Laplace–Beltrami operator (2.19) from the Klein–Gordon operator in $\mathbb{R}^{1,n+1}$. Consequently, the geodesic length (2.16), the volume element (2.18) and the Laplace–Beltrami operator (2.19) are left invariant under the isometry group $\mathrm{SO}(1, n+1)$ of the embedding space (and the unit hyperboloid). Because the even Weyl groups (or modular groups) to be considered below are all discrete subgroups of $\mathrm{SO}(1, n+1)$, these geometric objects are *a fortiori* invariant under these discrete groups as well. The main new feature here distinguishing the cases $n = 1, 2, 4, 8$ from the general case (2.21) is the link with the division algebras and their algebraic structure, which is evident in particular in the form of the modular transformations (3.17) below.

For completeness, we discuss the Green function on $\mathcal{H}(\mathbb{A})$ in Appendix C and (periodic) geodesics in Appendix D for the quaternionic case.

3 Weyl groups as modular groups

As vector spaces with a positive-definite inner product, the normed division algebras $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$ can be identified with the root spaces of finite-dimensional Kac–Moody algebras of rank 1, 2, 4, 8, respectively. The root lattices are then identified with lattices in these algebras. In several cases of interest, the lattices close under multiplication and thus endow the root lattice with the structure of a (possibly non-associative) ring [2].

The fundamental Weyl reflection with respect to a simple root a is defined as a reflection in the hyperplane orthogonal to a,

$$x \mapsto x - 2\frac{(a,\, x)}{(a,\, a)}a. \tag{3.1}$$

When we identify the roots with elements in the division algebra $\mathbb{A}$ this can be written

$$x \mapsto x - |a|^{-2}(a\bar{x} + x\bar{a})a = x - |a|^{-2}(a\bar{x}a + x|a|^2) = -\frac{a\bar{x}a}{|a|^2}, \tag{3.2}$$

where we have used the definition of the inner product, and the alternativity of $\mathbb{A}$. For simply-laced algebras (the only case we will consider in this paper), we normalize the simple roots to unit norm. In particular, if ε_i denote the so normalized simple roots of the finite algebra, then the associated finite Weyl group W_{fin} is generated by the fundamental Weyl reflections

$$x \mapsto w_i(x) = -\varepsilon_i \bar{x} \varepsilon_i, \tag{3.3}$$

for $x \in \mathbb{A}$ with i ranging over the rank of the algebra. We will always choose the orientation of the simple roots such that the highest root θ of the algebra is equal to the real unit, i.e., $\theta = 1$.

3.1 Hyperbolic overextensions

To any simple finite-dimensional Lie algebra (finite Kac–Moody algebra) of rank r one can associate an infinite-dimensional (indefinite) Kac–Moody algebra of rank $r+2$ as follows. One first constructs the non-twisted affine extension, thereby increasing the rank by one. Then one adds an additional node with a single line to the affine node in the Dynkin diagram. The resulting algebras are often called "over-extended" [26,27] and in many cases turn out to be hyperbolic Kac–Moody algebras.[5] The Cartan–Killing metric on the Cartan subalgebra is always Lorentzian. We will number the affine and over-extended nodes by 0 and −1, respectively. Denoting the finite-dimensional split real algebra by $\mathfrak{g}$, the associated over-extension will be denoted by $\mathfrak{g}^{++}$ and we use the index $I = -1, 0, 1, \ldots, r$ to denote the simple roots of $\mathfrak{g}^{++}$. The affine extension will be denoted by $\mathfrak{g}^{+}$.

The infinite Weyl groups associated with these overextended algebras were studied in [2]. The Weyl group acts on the Lorentzian vector space $\mathbb{A} \oplus \mathbb{R}^{1,1}$, *alias* the Minkowski space $\mathbb{R}^{1,n+1}$, consisting of real linear combinations of the simple roots. We can identify this vector space with the Jordan algebra[6] $H_2(\mathbb{A})$ of Hermitian matrices

$$X := \begin{pmatrix} x^+ & x \\ \bar{x} & x^- \end{pmatrix} = X^\dagger \tag{3.4}$$

where $x^\pm \in \mathbb{R}$ and $x \in \mathbb{A}$. This vector space is Lorentzian with respect to the norm

$$||X||^2 := -\det X = -x^+x^- + \bar{x}x \tag{3.5}$$

and the associated bilinear form

$$(X,\, Y) = \tfrac{1}{2}\left(||X+Y||^2 - ||X||^2 - ||Y||^2\right). \tag{3.6}$$

This definition of the norm differs from the one in [2] by a factor of 1/2. We also define the simple roots by

$$\alpha_{-1} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \alpha_0 = \begin{pmatrix} -1 & -1 \\ -1 & 0 \end{pmatrix}, \quad \alpha_i = \begin{pmatrix} 0 & \varepsilon_i \\ \bar{\varepsilon}_i & 0 \end{pmatrix}, \tag{3.7}$$

where $|\varepsilon_i| = 1$, so that they have unit length, $||\alpha_I||^2 = 1$, instead of length 2, which is the standard normalization for a simply laced algebra. The reason for this is that we want the norm to coincide with the standard norm in the division algebra $\mathbb{A}$, as soon as we restrict the root space to the finite subalgebra.

[5]But not always: For instance, the finite algebra A_8 extends to $A_8^{++} \equiv AE_{10}$ which is indefinite, but not hyperbolic.

[6]A Jordan algebra is a commutative (but possibly non-associative) algebra where any two elements X and Y satisfy $X^2 \circ (Y \circ X) = (X^2 \circ Y) \circ X$. This identity holds for $H_2(\mathbb{A})$ with $\circ$ being the symmetrized matrix product. However, we will here only consider the Jordan algebras $H_2(\mathbb{A})$ as vector spaces, without using the Jordan algebra property.

The fundamental Weyl reflections with respect to the simple roots (3.7) are given by [2]

$$w_I : X \mapsto M_I \bar{X} M_I^\dagger, \tag{3.8}$$

where

$$M_{-1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad M_0 = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}, \quad M_i = \begin{pmatrix} \varepsilon_i & 0 \\ 0 & -\bar{\varepsilon}_i \end{pmatrix}, \tag{3.9}$$

and we denote the thus generated hyperbolic Weyl group by $W \equiv W_{\text{hyp}}$.

Since Weyl transformations preserve the norm, we can consider their action on elements of fixed norm, and in particular on those elements that lie on the unit hyperboloid $||X||^2 = -1$ inside the forward light-cone of the Lorentzian space. As we showed in the previous section, this unit hyperboloid is isometric to the generalized upper half plane $\mathcal{H}(\mathbb{A})$ via the isometric embedding (2.22). The lightcone $||X||^2 = 0$ in $H_2(\mathbb{A})$ then corresponds to (a double cover of) the boundary $\partial\mathcal{H}(\mathbb{A})$, consisting of the subspace $\mathbb{A}$ (that is, $v = 0$) and the single point $z = i\infty$. The latter point is called "cusp at infinity", and plays a special role because it is associated with the affine null root

$$\delta = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}. \tag{3.10}$$

To see this more explicitly we note that the lines with constant u parallel to the imaginary axis in $\mathcal{H}(\mathbb{A})$ are obtained via (2.22) by projection of the null line

$$x^- = 1, \quad x^+ = t + |u|^2, \quad x = u \tag{3.11}$$

onto the unit hyperboloid inside the forward lightcone of $\mathbb{R}^{1,n+1}$. The tangent vector of this null line is then identified with (3.10) again via (2.22), and the boundary point on the unit hyperboloid corresponding to the cusp $z = i\infty$ is reached for $t \to \infty$. Choosing any other null direction inside the forward lightcone one can reach all other points in the boundary, that is $\mathbb{A}$ with $v = 0$.

Because the Weyl transformations generated by the repeated action (3.8) preserve the norm $||X||^2$, they leave invariant the unit hyperboloid. As shown in [3, 4] the projection of the fundamental reflections of the infinite

Weyl group W onto the unit hyperboloid, and hence the generalized upper half plane, induce the following modular action on $z \in \mathcal{H}(\mathbb{A})$

$$w_{-1}(z) = \frac{1}{\bar{z}}, \quad w_0(z) = -\bar{z} + 1, \quad w_i(z) = -\varepsilon_i \bar{z} \varepsilon_i, \tag{3.12}$$

where ε_i are the (unit) simple roots of the underlying finite algebra as above. Furthermore, $1/\bar{z} = z/|z|^2$.

3.2 Even Weyl group

We are here mainly interested in the *even* part of the Weyl group, which is realized by "holomorphic" transformations (see (3.17) below) and which we wish to interpret as a generalized modular group. The even Weyl group $W^+ \subset W$ consists of the words of even length in W. As the generators of the even Weyl group we take the transformations

$$s_I := w_I w_\theta \tag{3.13}$$

with $I = -1, 0$ or $I = i$, where w_θ is the reflection on the highest root $\theta = 1$. These generators act as

$$s_I : X \mapsto S_I X S_I^\dagger, \tag{3.14}$$

where

$$S_{-1} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad S_0 = \begin{pmatrix} -1 & -1 \\ 0 & -1 \end{pmatrix}, \quad S_i = \begin{pmatrix} \varepsilon_i & 0 \\ 0 & \bar{\varepsilon}_i \end{pmatrix}. \tag{3.15}$$

On the upper half plane it follows from (3.3) (with $\varepsilon = \theta = 1$) that

$$w_\theta(z) = -\bar{z}, \tag{3.16}$$

and the generators (3.13) act as

$$\boxed{s_{-1}(z) = -\frac{1}{z}, \quad s_0(z) = z + 1, \quad s_i(z) = \varepsilon_i z \varepsilon_i.} \tag{3.17}$$

These transformations are "holomorphic" on the upper half plane and generalize the well-known expression for modular transformations of the two-dimensional upper half plane $\mathcal{H}(\mathbb{R})$ under $\mathrm{PSL}(2, \mathbb{Z})$.

In [2] $\tilde{s}_0 = w_{-1} w_0$ and $\tilde{s}_i = w_{-1} w_i$ were used as generating elements. Our choice here is more convenient since it refers to the universal element $\theta = 1$, but unlike the one in [2], the generating set (3.13) is not minimal. For $\mathbb{A} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ this redundancy is expressed in the extra relations

$$s_1 = 1 \quad \text{for } \mathbb{A} = \mathbb{R}, \quad s_1 s_2 = 1 \quad \text{for } \mathbb{A} = \mathbb{C},$$
$$s_1 s_3 s_4 = 1 \quad \text{for } \mathbb{A} = \mathbb{H} \tag{3.18}$$

with appropriate numbering of the roots, and always in the basis, where $\theta = 1$. For the octonions the relevant relation is more tricky because of non-associativity.

Quite generally, the even Weyl groups of the over-extended algebras considered in [2] are of the form

$$W^+ \equiv W^+_{\text{hyp}} \cong \text{PSL}(2, \mathcal{O}), \tag{3.19}$$

where $\mathcal{O} \subset \mathbb{A}$ is a suitable set of algebraic integers in $\mathbb{A}$. The roots of the associated finite Kac–Moody algebra then correspond to the units of $\mathcal{O}$ (a unit being an invertible element of the ring). In some cases, one has to consider finite extensions of this group or finite quotients (depending on whether the algebra is not simply laced or has diagram automorphisms); a detailed analysis of all cases can be found in [2]. The even Weyl group W^+ is a normal subgroup of index two in the full Weyl group and for many purposes it is sufficient to study it. In the following sections, we will focus on two distinguished cases, one related to D_4 and the quaternions (where $\mathcal{O} = \mathsf{H}$, the ring of "Hurwitz numbers") and one related to E_8 and the octonions (where $\mathcal{O} = \mathsf{O}$, the "octavians").

The Weyl transformations (3.12) define a fundamental domain $\mathcal{F}_0 \subset \mathcal{H}(\mathbb{A})$, which is the image of the fundamental Weyl chamber in $H_2(\mathbb{A})$ under the projection (2.22). Likewise, the *even* Weyl transformations (3.17) define a fundamental domain $\mathcal{F}$, which contains two copies of $\mathcal{F}_0$, and a "projection" K of $\mathcal{F}$ onto $\mathbb{A}$; see figure 1 below for an illustration. As we already explained, the cusp $z = i\infty$ is the image of the boundary point of the unit hyperboloid which is reached by following and projecting any null ray inside the forward lightcone along the affine null root δ.

From (2.18) it follows that

$$\int_{\mathcal{F}} d\text{vol}(z) < \infty, \tag{3.20}$$

if K does not extend beyond the unit sphere in the $\mathbb{A}$-plane. It may touch the unit sphere at isolated points since the improper integral will still be convergent and finite. This happens among the over-extended algebras in

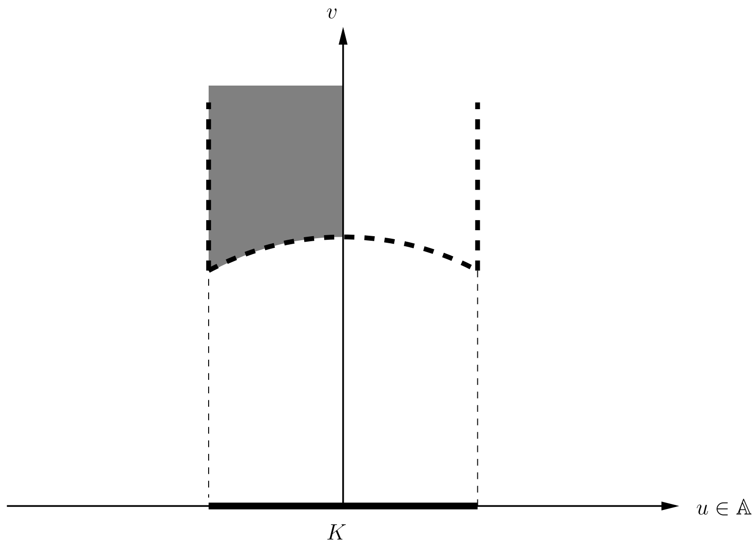


Figure 1: *Sketch of the fundamental domains $\mathcal{F}_0$ (in grey) and $\mathcal{F}$ (within heavy dashed lines). The shape of $\mathcal{F}$ is always that of a "skyscraper" over a compact domain K contained in the unit ball of $\mathbb{A}$. This "skyscraper", whose bottom has been cut off, extends to infinite height in v.*

particular for the algebras A_7^{++}, B_8^{++} and D_8^{++}.[7] The finite volume of the fundamental domain is in one-to-one correspondence with the hyperbolicity of the underlying Kac–Moody algebra [27].

For all $\mathcal{H}(\mathbb{A})$, the geodesic length (2.16) is invariant under the continuous isometry group $\mathrm{SO}(1, n+1)$ and therefore also under the generating elements of the Weyl group (3.12), that is under the full Weyl group, as can also be verified explicitly for the generating Weyl reflections (3.12). For instance, for the inversion we get

$$w_{-1} : |z_1 - z_2| \to |\bar{z}_2^{-1} - \bar{z}_1^{-1}| = (|z_1||z_2|)^{-1}|z_1 - z_2| \tag{3.21}$$

using alternativity. By iteration it then follows that

$$d(z_1, z_2) = d(w(z_1), w(z_2)) \tag{3.22}$$

for all elements w of the Weyl group $w \in W_{\mathrm{hyp}}$. Similarly, for all $w \in W_{\mathrm{hyp}}$ the hyperbolic volume element is left invariant:

$$d\mathrm{vol}(z) = d\mathrm{vol}(w(z)). \tag{3.23}$$

[7]See [28] for further discussions of the fundamental domain of hyperbolic Kac–Moody algebras and explicit volume computations.

Finally, the Laplace–Beltrami operator is also invariant in the following sense. Setting $z' := w(z)$ and denoting by $\triangle'_{\rm LB}$ the Laplace–Beltrami operator with respect to the primed coordinates z', we have *for any function* $f(z)$ and for all $w \in W_{\rm hyp}$

$$\triangle'_{\rm LB} f(z') = \triangle_{\rm LB} f(w(z)). \tag{3.24}$$

3.3 Affine Weyl group

The hyperbolic Weyl group has a natural subgroup associated with the affine subalgebra, corresponding to the embedding $\mathfrak{g}^+ \subset \mathfrak{g}^{++}$. We denote this subgroup by $W_{\rm aff} \equiv W(\mathfrak{g}^+)$. It is known [29] that it has the structure

$$W_{\rm aff} = W_{\rm fin} \ltimes \mathcal{O}, \tag{3.25}$$

where $\mathcal{O}$ is the integer domain corresponding to the root lattice of the finite-dimensional algebra $\mathfrak{g}$. In (3.25), $\mathcal{O}$ is to be thought of as a free abelian group $\mathcal{T}$ of translations on the finite root lattice on which the finite Weyl group $W_{\rm fin}$ acts. Written in matrix form the translations take the form

$$T_y = \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix}, \tag{3.26}$$

and obey $T_x T_y = T_{x+y}$ for all $x, y \in \mathcal{O}$. In the associative cases, one can write the finite Weyl transformations as matrices as

$$M_{a,b} = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \tag{3.27}$$

for units $a, b \in \mathcal{O}$, possibly with restrictions depending on $\mathfrak{g}$.[8] It is easy to see that the transformations leaving invariant the null root (3.10) are exactly the T_x and the $M_{a,b}$. This is to be expected as the affine Weyl group can be defined as the stabilizer of the affine null root [14].

4 D_4 and its hyperbolic extension

Now we specialize to $\mathbb{A} = \mathbb{H}$ in which case the relevant ring of algebraic integers is $\mathcal{O} = \mathsf{H}$, the "Hurwitz numbers", to be defined below. The results of this section generalize many well-known results for the usual upper half

[8]We will encounter an explicit example of such restrictions in the case $\mathfrak{g} = D_4$ below.

plane ($\mathbb{A} = \mathbb{R}$). Since they contain similar results for $\mathbb{A} = \mathbb{C}$ we will not specially consider this case (which has been dealt with in the literature [30]). Our main task will be to find closed formulas for arbitrary elements of the group $\mathrm{PSL}(2, \mathsf{H})$ and its modular realization on $\mathcal{H}(\mathbb{H})$. In comparison to the commutative cases $\mathbb{A} = \mathbb{R}, \mathbb{C}$, this case presents new features because of the non-commutativity of the matrix entries. However, a direct generalization to $\mathbb{A} = \mathbb{O}$ is not possible because of the non-associativity of the matrix entries; this case will be treated separately in Section 5.

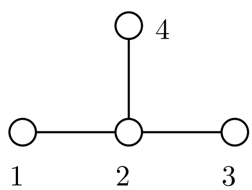
4.1 D_4 and the integer quaternions

The quaternions are obtained by applying the Cayley–Dickson doubling procedure described in Section 2.1 to $\mathbb{A} = \mathbb{C}$. An arbitrary quaternion $x \in \mathbb{H}$ is written as

$$x = x_0 + x_1 e_1 + x_2 e_5 + x_3 e_6, \tag{4.1}$$

where x_0, x_1, x_2, $x_3 \in \mathbb{R}$, and we have the usual rules of quaternionic multiplication $e_1 e_5 = -e_5 e_1 = e_6$, etc. for the imaginary quaternionic units e_1, e_5 and e_6.[9] The norm is $|x|^2 \equiv x\bar{x} = x_0^2 + x_1^2 + x_2^2 + x_3^2$.

We are here interested in quaternionic integers. The *Hurwitz integers* (or just *Hurwitz numbers*) H are those quaternions (4.1) for which the coefficients x_0, x_1, x_2, x_3 are either all integers or all half-integers. They constitute a lattice $\mathsf{H} \subset \mathbb{H}$ and form a non-commutative ring of integers (actually, a "maximal order") inside $\mathbb{H}$. The key feature is that H at the same time can be identified with the root lattice of the algebra $\mathrm{D}_4 \equiv \mathfrak{so}(4,4)$. Indeed, the simple roots of D_4, labeled according to the Dynkin diagram



[9]The imaginary units are conventionally designated as i, j, k but here we prefer to use $e_1 \equiv i$, $e_5 \equiv j$ and $e_6 \equiv k$ because we want to reserve the letter i for the new imaginary unit used in the Cayley–Dickson doubling. For the doubling $\mathbb{H} = \mathbb{C} \oplus i\mathbb{C}$ we thus identify $i \equiv e_5$ and $x = (x_0 + x_1 e_1) + i(x_2 - x_3 e_1)$. Our notation conforms with the one used in Section 5 for the octonions, such that the obvious D_4 roots inside E_8 are simply obtained from (4.2) by multiplication with e_7.

can be identified with the following set of *Hurwitz units*:

$$\varepsilon_1 = e_1, \quad \varepsilon_2 = \frac{1}{2}(1 - e_1 - e_5 - e_6), \quad \varepsilon_3 = e_5, \quad \varepsilon_4 = e_6. \tag{4.2}$$

The D_4 root lattice is spanned by integer linear combinations of these simple roots, and with the above choice of basis, the highest root $\theta = \varepsilon_1 + 2\varepsilon_2 + \varepsilon_3 + \varepsilon_4$ is indeed equal to 1. The new feature is that such combinations also close under multiplication, thereby endowing the D_4 root lattice with the structure of a non-commutative ring. The 24 roots of D_4 are then identified with the 24 *units* in H. Comparison with (3.17) shows that we now have the extra relation $s_1 s_3 s_4 = 1$ for the generating set (3.13), see (3.18).

The elements in $W^+(D_4)$ are the words of even length where the letters are the fundamental Weyl reflections. For example, those of length two act as

$$x \mapsto \varepsilon_i(\overline{\varepsilon_j \bar{x} \varepsilon_j})\varepsilon_i = \varepsilon_i(\bar{\varepsilon}_j x \bar{\varepsilon}_j)\varepsilon_i = (\varepsilon_i \bar{\varepsilon}_j) x (\bar{\varepsilon}_j \varepsilon_i). \tag{4.3}$$

It follows [2, 25] that any element in $W^+(D_4)$ has the form of a combined left and right multiplication,

$$x \mapsto axb, \tag{4.4}$$

where a and b are (in general different) Hurwitz units, subject to the constraint that the product ab is an element of the quaternionic group

$$ab \in \mathcal{Q} \equiv \{\pm 1, \pm e_1, \pm e_5, \pm e_6\}. \tag{4.5}$$

The restriction to the quaternionic group arises from triality of the D_4 root system as the outer triality automorphisms are not elements of the Weyl group. In [2] this result was generalized to the hyperbolic extension $W(D_4^{++})$ in a way that we will describe next.

4.2 The even hyperbolic Weyl group $W^+(D_4^{++})$

Consider transformations acting on $X \in H_2(\mathbb{H})$ by

$$X \to s(X) = SXS^\dagger \tag{4.6}$$

with

$$S = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad a, b, c, d \in \mathbb{H}. \tag{4.7}$$

(Of course, these formulas trivially specialize to $\mathbb{A} = \mathbb{R}, \mathbb{C}$.) Such transformations preserve the norm of X if and only if

$$\det(SS^\dagger) = 1. \tag{4.8}$$

This is true for all the normed division algebras up to and including the quaternions.[10] The associated continuous group of matrices will be denoted by

$$\mathrm{SL}(2, \mathbb{H}) = \left\{ S = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \,:\, a, b, c, d \in \mathbb{H}, \quad \det(SS^\dagger) = 1 \right\}. \tag{4.9}$$

Its projective version

$$\mathrm{PSL}(2, \mathbb{H}) := \mathrm{SL}(2, \mathbb{H}) / \{\mathbf{1}, -\mathbf{1}\} \tag{4.10}$$

is isomorphic to $\mathrm{SO}_0(1,5)$, see [24, 31]. Let us note the explicit expressions

$$\begin{aligned} \det(SS^\dagger) &= |a|^2|d|^2 + |b|^2|c|^2 - 2\,\mathrm{Re}\big(a\bar{c}d\bar{b}\big) \\ &= |ad - bc|^2 - 2\,\mathrm{Re}\big(a[\bar{c}, d]\bar{b}\big) \end{aligned} \tag{4.11}$$

and the explicit form of the (left and right) inverse matrix for $S \in SL(2, \mathbb{H})$

$$S^{-1} = \begin{pmatrix} |d|^2\bar{a} - \bar{c}d\bar{b} & |b|^2\bar{c} - \bar{a}b\bar{d} \\ |c|^2\bar{b} - \bar{d}c\bar{a} & |a|^2\bar{d} - \bar{b}a\bar{c} \end{pmatrix}. \tag{4.12}$$

The discrete groups $\mathrm{SL}(2, \mathsf{H}) \subset \mathrm{SL}(2, \mathbb{H})$ and $\mathrm{PSL}(2, \mathsf{H}) \subset \mathrm{PSL}(2, \mathbb{H})$ are obtained from the above groups by restricting the entries a, b, c, d to be Hurwitz integers. From (4.11) it then follows that in order for S to be an element of $\mathrm{SL}(2, \mathsf{H})$ or $\mathrm{PSL}(2, \mathsf{H})$, the pairs (a, b) and (c, d) must each be *left coprime*, and the pairs (a, c) and (b, d) must each be *right coprime* (i.e., share no common left or right factor $g \in \mathsf{H}$ with $|g| > 1$). In fact, these conditions are not only necessary, but also sufficient (for some choice of the remaining two entries), as we show in Section 5.5.

[10] For real and complex numbers the determinant can be factorized to give $|\det(S)| = 1$. Making use of commutativity then allows the reduction to $\det(S) = 1$ in the projective version.

Straightforward computation using (3.4) and (4.6) gives the transformations of the various components of X

$$
\begin{aligned}
s(x^+) &= \bar{a}a\,x^+ + \bar{b}b\,x^- + ax\bar{b} + b\bar{x}\bar{a},\\
s(x) &= a\bar{c}\,x^+ + b\bar{d}\,x^- + ax\bar{d} + b\bar{x}\bar{c},\\
s(x^-) &= \bar{c}c\,x^+ + \bar{d}d\,x^- + cx\bar{d} + d\bar{x}\bar{c}.
\end{aligned}
\tag{4.13}
$$

This formula is, of course, also valid for $\mathbb{A} = \mathbb{R}, \mathbb{C}$. Likewise, it holds for general quaternionic matrices S as long as these are invertible.

The Weyl groups of the hyperbolic over-extensions of rank four algebras are discrete subgroups of $\mathrm{PSL}(2, \mathbb{H})$ (see [2] for details). For the hyperbolic over-extension D_4^{++} of D_4 one finds

$$
W^+(\mathrm{D}_4^{++}) \cong \mathrm{PSL}^{(0)}(2, \mathsf{H}) := \Big\{ S \in \mathrm{PSL}(2, \mathsf{H}) \,|\, ad - bc \equiv 1 \bmod \mathsf{C} \Big\}, \tag{4.14}
$$

where C is the (two-sided) commutator ideal $\mathsf{C} = \mathsf{H}[\mathsf{H}, \mathsf{H}]\mathsf{H}$ introduced in [2]. In other words, the Weyl group of D_4^{++} is not simply $\mathrm{PSL}(2, \mathsf{H})$: the elements S in $\mathrm{PSL}^{(0)}(2, \mathsf{H})$ must satisfy the additional constraint $ad - bc \equiv 1$ modulo C, which is an index 4 sublattice in H. As shown in [2], $\mathrm{PSL}^{(0)}(2, \mathsf{H})$ is an index 3 subgroup of $\mathrm{PSL}(2, \mathsf{H})$: if one extends the Weyl group by outer automorphisms related to the diagram automorphisms ("triality") one obtains all of $\mathrm{PSL}(2, \mathsf{H})$, corresponding to all even symmetries of the D_4^{++} root lattice written in quaternionic coordinates.

Below, we will encounter two important subgroups of $\mathrm{PSL}^{(0)}(2, \mathsf{H})$. One is the (even) affine subgroup $W^+_{\text{aff}} \subset W^+(\mathrm{D}_4^{++})$, which is the semi-direct product $W_{\text{fin}} \ltimes \mathcal{T}$ of the finite Weyl group $W_{\text{fin}} = W^+(\mathrm{D}_4)$ and the abelian group $\mathcal{T}$ of affine translations. This is a maximal parabolic subgroup of $W^+(\mathrm{D}_4^{++})$ whose elements in the quaternionic matrix representation are given by

$$
S = \begin{pmatrix} a & u \\ 0 & b \end{pmatrix}, \tag{4.15}
$$

where a, b are unit quaternions with $ab \in \mathcal{Q}$ and $u \in \mathsf{H}$ is an arbitrary Hurwitz integer. The other important subgroup is the translation subgroup $\mathcal{T}$ itself, consisting of the matrices

$$
T_u = \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}, \tag{4.16}
$$

where again $u \in \mathsf{H}$ is an arbitrary Hurwitz number. The following little lemma will be useful in Section 6.2.

Lemma 4.1. *The left coprime pairs* $(c,d) \in \mathsf{H}^2$ *parametrize the coset spaces* $W^+(\mathrm{D}_4^+)\backslash W^+(\mathrm{D}_4^{++}) \equiv W^+_{\mathrm{aff}}\backslash W^+_{\mathrm{hyp}}$. *The equivalence classes* $(c,d) \sim (ec, ed)$ *for a unit* $e \in \mathsf{H}$ *uniquely parametrize the cosets.*

Proof. Suppose that the left coprime lower entries (c,d) are given and that a, b and $\tilde{a}, \tilde{b}$ are two different pairs of Hurwitz numbers completing c, d to two different matrices S and $\tilde{S}$ in $\mathrm{PSL}^{(0)}(2,\mathsf{H})$. Then an easy computation using (4.12) shows that

$$\tilde{S}S^{-1} = \begin{pmatrix} q & * \\ 0 & 1 \end{pmatrix}, \tag{4.17}$$

where $q \in \mathcal{Q}$ is in the quaternionic group (4.5). Hence all such matrices are related by an upper triangular element of $\mathrm{PSL}^{(0)}(2,\mathsf{H})$ of type (4.15), that is, an element of W^+_{aff}. Since (4.15) also allows for left multiplication of the pair (c,d) by a Hurwitz unit we arrive at the claim. □

4.3 Action of $\mathbf{PSL^{(0)}(2,H)}$ on $\mathcal{H}(\mathbb{H})$

We now wish to interpret the even Weyl group $W^+(\mathrm{D}_4^{++}) \equiv \mathrm{PSL}^{(0)}(2,\mathsf{H})$ as a modular group Γ acting on the quaternionic upper half plane $\mathcal{H}(\mathbb{H})$. To exhibit the nonlinear "modular" action of Γ we map the forward unit hyperboloid $||X||^2 = -1$ in $\mathbb{R}^{1,5}$ to the upper half plane $\mathcal{H}(\mathbb{H})$ by means of the projection (2.22). Accordingly, we consider (2.9), but now specialize to $u \in \mathbb{H}$. Using the formulas of the previous section, in particular (2.22) and (4.13), we obtain

$$v' = \frac{v}{D}, \quad u' = \frac{1}{D}\Big[(au+b)(\bar{u}\bar{c}+\bar{d}) + a\bar{c}\,v^2\Big] \tag{4.18}$$

with $z' = u' + iv' \equiv s(z) \equiv s(u+iv)$, where

$$D \equiv D(S,u,v) := |cu+d|^2 + |c|^2v^2 = |cz+d|^2 \tag{4.19}$$

and (2.4) has been used. Observe that now $cz + d \in \mathbb{O}$. The transformations (4.18) can be combined into a single formula

$$z' = \frac{(au+b)(\bar{u}\bar{c}+\bar{d}) + a\bar{c}\,v^2 + iv}{|cz+d|^2}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}^{(0)}(2,\mathsf{H}), \tag{4.20}$$

which is our expression for the most general quaternionic modular transformation. Here z' is manifestly in the upper half plane $\mathcal{H}(\mathbb{H})$, and also reduces to the standard formula $z' = (az+b)(cz+d)^{-1}$ for the commutative cases $u \in \mathbb{R}$ (or $\mathbb{C}$) and $a,b,c,d \in \mathbb{R}$ (or $\mathbb{C}$). However, (4.20) fails for $\mathbb{A} = \mathbb{O}$ because of non-associativity. One can check explicitly that the transformations (4.18) and the identifications (2.22) are consistent with (4.13).

As a special case we have the modular transformations (4.15) corresponding to the even affine Weyl group. The induced modular action of the most general element $\gamma \in \Gamma_\infty \equiv W^+_{\text{aff}} \equiv W^+_{\text{fin}} \ltimes \mathsf{H}$ is

$$\gamma : z \to \gamma(z) = aub + y\bar{b} + iv \tag{4.21}$$

with units a, b such that $ab \in \mathcal{Q}$ (cf. (4.5)) and $y \in \mathsf{H}$ an arbitrary Hurwitz integer. This transformation corresponds to a finite (even) Weyl transformation followed by a constant shift along the root lattice H. In particular, it follows that such transformations leave invariant the "cusp" $z = i\infty$ in $\mathcal{H}(\mathbb{H})$ — in analogy with the action of the usual shift matrix T for the real modular group PSL(2, $\mathbb{Z}$) (in that case, W^+_{fin} is trivial). As we explained, this is in accord with the fact that the affine Weyl group W_{aff} is the subgroup of W_{hyp} leaving invariant the affine null root δ [14].

5 E_8 and its hyperbolic extension

We now turn to the largest exceptional algebra E_8 and its affine and hyperbolic extensions $\mathrm{E}_9 \equiv \mathrm{E}_8^+$ and $\mathrm{E}_{10} \equiv \mathrm{E}_8^{++}$. When extending the results of the previous section to $\mathbb{A} = \mathbb{O}$ a main obstacle is the non-associativity of the octonions. We have the abstract isomorphism [2]

$$W^+(\mathrm{E}_{10}) \cong \mathrm{PSL}(2, \mathsf{O}) \tag{5.1}$$

and we will exhibit a presentation of this group in terms of cosets with respect to the cusp stabilizing affine Weyl group $W^+(E_9)$; a presentation that will prove particularly useful when constructing non-holomorphic automorphic forms in Section 6. We emphasize that realizing part of this group in terms of 2×2 matrices with integer octonionic ("octavian") entries is certainly not sufficient, because such matrices violate associativity and thus cannot by themselves define any group. Nevertheless, the generating transformations (3.17) for $\Gamma \equiv \mathrm{PSL}(2, \mathsf{O})$ are still valid in the form given there: by the alternativity of the octonions no parentheses need to be specified for nested products involving only two different octonions. On our way to describing $W^+(\mathrm{E}_{10})$ we will also obtain formula (5.23) below that gives a rather explicit expression for the action of an E_8 Weyl group element in terms of unit octavians and the $\mathrm{G}_2(2)$ automorphisms of the octavians.

5.1 E_8 and the integer octonions

As a basis for the octonion algebra one usually takes the real number 1 and seven "imaginary" units e_1, e_2, …, e_7 that anticommute and square to -1. One then chooses three of them to be in a quaternionic subalgebra. For instance, taking (e_1, e_5, e_6) as a basic quaternionic triplet as in Section 4, the octonion multiplication has the following properties:

$$e_i e_j = e_k \Rightarrow e_{i+1} e_{j+1} = e_{k+1}, \quad e_{2i} e_{2j} = e_{2k}, \tag{5.2}$$

where the indices are counted mod 7. Using these rules, the whole multiplication table can be derived from only one equation, for example $e_1 e_5 = e_6$.

The root lattice of E_8 can be identified with the non-commutative and non-associative ring of *octavian integers*, or simply *octavians* O (for their construction, see [25]). Within this ring, the 240 roots of E_8 are then identified with the set of *unit octavians*, which are the 240 invertible elements in O. More specifically, the octavian units can be divided into

$$\begin{array}{ll} \text{2 real numbers:} & \pm 1, \\ \text{112 Brandt numbers:} & \tfrac{1}{2}(\pm 1 \pm e_i \pm e_j \pm e_k), \\ \text{126 imaginary numbers:} & \tfrac{1}{2}(\pm e_m \pm e_n \pm e_p \pm e_q), \quad \pm e_r, \end{array} \tag{5.3}$$

where

$$\begin{array}{rcccccccc} ijk &=& 124, & 137, & 156, & 236, & 257, & 345, & 467, \\ mnpq &=& 3567, & 2456, & 2347, & 1457, & 1346, & 1267, & 1235, \end{array} \tag{5.4}$$

and $r = 1, 2, \ldots, 7$.[11] The lattice O of octavians is the set of all integral linear combinations of these unit octavians.

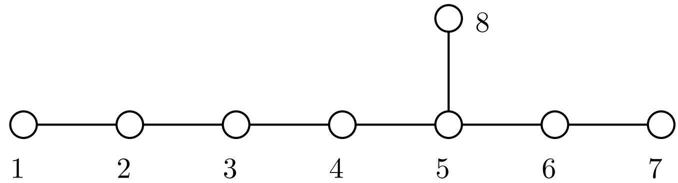
Recall that the automorphism group $\mathrm{Aut}\,\mathbb{O}$ of the octonionic multiplication table (5.2) is the exceptional group G_2 [24]. For the ring O we have the analogous discrete result [25]

$$\mathrm{Aut}\,\mathsf{O} = G_2(2), \quad |G_2(2)| = 12\,096, \tag{5.5}$$

where $G_2(2)$ is the finite (but not simple) group G_2 over the field $\mathbb{F}_2$ (a Chevalley group).

[11] Imaginary units and Brandt numbers are called "eyes" and "arms" in [25].

We choose the simple roots of E_8, according to the Dynkin diagram



as the following unit octavians,

$$\begin{aligned}
\varepsilon_1 &= \tfrac{1}{2}(1 - e_1 - e_5 - e_6), \quad \varepsilon_2 = e_1, \\
\varepsilon_3 &= \tfrac{1}{2}(-e_1 - e_2 + e_6 + e_7), \quad \varepsilon_4 = e_2, \\
\varepsilon_5 &= \tfrac{1}{2}(-e_2 - e_3 - e_4 - e_7), \quad \varepsilon_6 = e_3, \\
\varepsilon_7 &= \tfrac{1}{2}(-e_3 + e_5 - e_6 + e_7), \quad \varepsilon_8 = e_4.
\end{aligned} \tag{5.6}$$

The identification (5.6) of the simple roots of E_8 as unit octavians is not unique. The main advantage of the choice of basis here (similar to [33] but different from the one in [2]) is that the 126 roots of the E_7 subalgebra are given by *imaginary* octavians (defined to obey $e^2 = -1$), while the 112 roots corresponding to the two **56** representations of E_7 in the decomposition of the adjoint **248** of E_8 are given by *Brandt numbers*, and the highest root is $\theta = 1$ in accord with our general conventions. By definition, a Brandt number a obeys $a^3 = \pm 1$. As explained in [25, 34], for such numbers (and only for them) the map $x \to axa^{-1}$ is an automorphism of the octonions, see also Appendix A.

Note that only the leftmost node of the Dynkin diagram corresponds to a Brandt number, while all others correspond to imaginary unit octavians. Also, the root basis for the obvious D_4 subalgebra inside E_8 is simply obtained from the one of Section 4 by multiplying $\varepsilon_4, \varepsilon_5, \varepsilon_6$ and ε_8 with e_7.

Using the scalar product (2.3) it is straightforward to reproduce from (5.6) the E_8 Cartan matrix via

$$A_{ij} = 2(\varepsilon_i, \varepsilon_j). \tag{5.7}$$

We saw in Section 4 that any element in the finite Weyl group $W^+(D_4)$ can be written as a bi-multiplication

$$x \mapsto axb, \tag{5.8}$$

where a and b are Hurwitz units. But the number of different such transformations ($= 12 \times 24$) exceeds the order of $W^+(\mathrm{D}_4)$ (which is $= 96$). This is why, in order to obtain only elements in $W^+(\mathrm{D}_4)$, we needed to impose the extra condition $ab \in \mathcal{Q}$. For $W^+(\mathrm{E}_8)$ we would like to generalize the expression (5.8) to unit *octavians* a and b. Because of the non-associativity, we have to place parentheses, for instance by taking

$$x \mapsto (ax)b. \tag{5.9}$$

Since the Dynkin diagram of E_8 does not have any symmetries (unlike the one of D_4) all such transformations do belong to $W^+(\mathrm{E}_8)$. But counting these expressions, we find that their number (modulo sign changes $(a, b) \leftrightarrow (-a, -b)$) now is *smaller* than the order of the even Weyl group, which is

$$|W^+(\mathrm{E}_8)| = 120 \times 240 \times 12\,096. \tag{5.10}$$

The extra factor 12 096 is exactly the order of the automorphism group of the octavians. This observation, made in [2] (see also [35]), suggests that the expression (5.9) should be generalized to

$$x \mapsto (a\varphi(x))b, \tag{5.11}$$

where φ is an arbitrary automorphism of the octavians. However, it was noted in [2] that the number of *independent* such expressions is in fact smaller than $120 \times 240 \times 12\,096$, since the transformation (5.9) is already an automorphism if a is a Brandt number (that is, $a^3 = \pm 1$) and $b = \pm\bar{a}$ (see above and Appendix A). In Section 5.3, we will explain how the formula (5.11) can be modified in these cases, so that it indeed expresses all even Weyl transformations of E_8. As a first step we consider the E_7 subalgebra obtained by deleting the leftmost node in the Dynkin diagram and thus corresponding to imaginary octavians.

5.2 The even Weyl group of E_7

In this subsection we use a_B to denote bimultiplication from left and right with the same octavian a:

$$a_B : \mathsf{O} \to \mathsf{O}, \quad x \mapsto axa. \tag{5.12}$$

We will show that $W^+(\mathrm{E}_7)$ is the set of all transformations

$$g_B h_B \varphi : \mathsf{O} \to \mathsf{O}, \quad x \mapsto g(h\,\varphi(x)h)g, \tag{5.13}$$

where g and h are imaginary unit octavians and φ is an automorphism of the octavians. It follows from (3.3), and the identification (5.6) of the simple E_7 roots as imaginary unit octavians, that the generators of $W^+(E_7)$ have this form. To show that this holds for all elements in $W^+(E_7)$ we thus have to show that the composition of two such transformations $g_{1B}h_{1B}\varphi_1$ and $g_{2B}h_{2B}\varphi_2$ again can be written in the form (5.13). From the definition of an automorphism we have

$$\begin{aligned}(g_{1B}h_{1B}\varphi_1)(g_{2B}h_{2B}\varphi_2) &= g_{1B}h_{1B}\,\varphi_1(g_2)_B\,\varphi_1(h_2)_B\,\varphi_1\varphi_2\\ &= (e_Bf_Bf_Be_B)g_{1B}h_{1B}\,\varphi_1(g_2)_B\,\varphi_1(h_2)_B\varphi_1\varphi_2, \quad (5.14)\end{aligned}$$

where we in the last step have inserted the identity map in form of $e_Bf_Bf_Be_B$ for two arbitrary imaginary unit octavians e and f. Any unit octavian can be written as a product of two imaginary unit octavians, so we can choose e and f in (5.14) such that

$$fe = \pm((\varphi_1(h_2)\,\varphi_1(g_2))h_1)g_1, \quad (5.15)$$

or equivalently,

$$((((fe)g_1)h_1)\,\varphi_1(g_2))\,\varphi_1(h_2) = \pm 1. \quad (5.16)$$

Then, according to Corollary A.4, the map

$$\varphi_3 \equiv f_Be_Bg_{1B}h_{1B}\,\varphi_1(g_2)_B\,\varphi_1(h_2)_B \quad (5.17)$$

is an automorphism. Inserting this into (5.14) we obtain

$$\begin{aligned}(g_{1B}h_{2B}\varphi_1)(g_{2B}h_{2B}\varphi_2) &= (e_Bf_Bf_Be_B)g_{1B}h_{1B}\,\varphi_1(g_2)_B\,\varphi_1(h_2)_B\varphi_1\varphi_2\\ &= e_Bf_B(f_Be_Bg_{1B}h_{1B}\,\varphi_1(g_2)_B\,\varphi_1(h_2)_B)\varphi_1\varphi_2\\ &= e_Bf_B(\varphi_3\varphi_2\varphi_1). \quad (5.18)\end{aligned}$$

Since φ_3, φ_1 and φ_2 all are automorphisms, their product $\varphi_3\varphi_2\varphi_1$ is an automorphism as well.

Conversely, any transformation (5.13) is an isometry of the root system. Since the group of isometries of the root system of a finite algebra is the semidirect product of the Weyl group and the symmetry group of the Dynkin diagram (which is trivial for E_7), it follows that any transformation (5.13) belongs to the Weyl group. Furthermore, it belongs to the *even* Weyl group since it does not involve the conjugate $\bar{x}$.

We have thus proven that $W^+(\mathrm{E}_7)$ is the set of all expressions (5.13). But there is some redundancy in this set. If $g_1h_1 = g_2h_2$, then the transformation

$$\varphi_3 = h_2g_2g_1h_1 \tag{5.19}$$

is an automorphism according to Corollary A.4, and we have

$$g_1h_1\varphi_1 = g_2h_2\varphi_2, \tag{5.20}$$

where $\varphi_2 = \varphi_3\varphi_1$. Thus only the product $\pm gh$ matters modulo automorphisms φ, and this product can be any unit octavian. Put differently, any element of $W^+(\mathrm{E}_7)$ is characterized by an automorphism in $G_2(2)$ and a unit octavian a which must then be factorized as gh with imaginary units g and h, unless a is itself imaginary. Accordingly, the order of $W^+(\mathrm{E}_7)$ is $120 \times 12\,096$.

5.3 From $W(\mathrm{E}_7)$ to $W(\mathrm{E}_8)$

Having understood the Weyl group of E_7 the step to E_8 is more conceptual than technical — it is an application of the orbit-stabilizer theorem, which is valid for any group $\mathcal{G}$ acting on a set $\mathcal{X}$. If $\mathcal{H} \subset \mathcal{G}$ is the subgroup stabilizing an element $x \in \mathcal{X}$, the theorem says that, for any $g \in \mathcal{G}$, the coset $g\mathcal{H}$ is the set of all elements in $\mathcal{G}$ that take x to gx.

The Weyl group of E_7 is the subgroup of $W(\mathrm{E}_8)$ that stabilizes the highest root of E_8. Applying the orbit-stabilizer theorem, it is enough to find one transformation w in $W(\mathrm{E}_8)$ for each element x in the orbit $W(\mathrm{E}_8)\,\theta$, such that $w(\theta) = x$. Then the set of *all* such transformations is the coset $w\,W(\mathrm{E}_7)$. The root system of E_8 is one single orbit under $W(\mathrm{E}_8)$ and the highest root θ is 1 in our identification with the octavians. Thus we need a transformation in $W(\mathrm{E}_8)$ for each unit octavian b, such that $\theta = 1$ is mapped to b. But this is easy to find, we just take the transformation to be right (or left) multiplication by b. It follows if we linearize the composition property (2.8), or alternatively by (2.3) together with the first of the Moufang identities (A.1), that

$$(xb,\, yb) = |b|^2(x,\, y), \tag{5.21}$$

so right multiplication by b is an isometry if b has unit length. Like for E_7 any isometry of the root system is an element of the Weyl group, and right multiplications of unit octavians belong to the even Weyl group $W^+(\mathrm{E}_8)$ since they do not involve conjugation.

Thus any element of $W^+(\mathrm{E}_8)$ can be written as

$$x \mapsto (f(e\,\varphi(x)\,e)f)b, \tag{5.22}$$

where e and f are imaginary unit octavians, b an arbitrary unit octavian, and φ an automorphism of the octavians. Using the third of the Moufang identities (A.1), we can write this as

$$\begin{aligned} x \mapsto\ & (f(e\,\varphi(x)\,e)f)b \\ &= f((e\,\varphi(x)\,e)(fb)) \\ &= f(e\,(\varphi(x)\,(e(fb)))) \\ &= f(e\,(\varphi(x)\,d)), \end{aligned} \tag{5.23}$$

where $d = e(fb)$. This differs from the formula (5.11) suggested in [2], only by the factorization of a into two imaginary unit octavians e and f. In fact, the factors e and f do not need to be imaginary but can also be *real*, that is ± 1. Then if a in (5.11) is imaginary, the factorization is trivial; we can just take $e = a$ and $f = 1$, and (5.23) reduces to (5.11). Only when a is a Brandt number we need to replace (5.11) by (5.23), where we can choose any factorization of a into two imaginary unit octavians e and f.

When we consider E_8 as a subalgebra of $\mathrm{E}_{10} = \mathrm{E}_8^{++}$, the Weyl group $W(\mathrm{E}_8)$ acts in a non-trivial way on the simple root α_0 (and trivially on α_{-1}). We can thus extend the action of $W(\mathrm{E}_8)$ to the whole root space of E_{10}. The Weyl transformation (5.23) can then be written as[12]

$$\begin{pmatrix} x^+ & x \\ \bar{x} & x^- \end{pmatrix} \mapsto \begin{pmatrix} x^+ & f(e\,(\varphi(x)\,d)) \\ ((\bar{d}\,\varphi(\bar{x}))\bar{e})\bar{f} & x^- \end{pmatrix} \tag{5.24}$$

or, in more compact form,

$$X \mapsto F(E(D^\dagger \varphi(X) D)E^\dagger)F^\dagger, \tag{5.25}$$

where

$$D = \begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix}, \quad E = \begin{pmatrix} e & 0 \\ 0 & 1 \end{pmatrix}, \quad F = \begin{pmatrix} f & 0 \\ 0 & 1 \end{pmatrix}. \tag{5.26}$$

[12]The fact that the formula "$X \mapsto SXS^\dagger$" requires refinement by placing parentheses inside the matrix elements was anticipated in [2].

5.4 From $W(\mathrm{E}_8)$ to $W(\mathrm{E}_9)$

As mentioned in Section 3.3 (see also [29]), to any point x on the root lattice of the finite Kac–Moody algebra E_8 we can associate a *translation* t_x that acts on the root space of E_{10} as

$$t_y : X \mapsto X + 2(X,\delta)y + 2(X,\, y)\delta - 2|y|^2(X,\,\delta)\delta. \tag{5.27}$$

where δ is the null root (3.10). Then $t_x t_y = t_{x+y}$ for any x and y, and all translations form a free group of translations $\mathcal{T} \equiv \mathsf{O}$ of rank eight. This group is a normal subgroup of the Weyl group of E_9, which can then be written as a semidirect product

$$W(\mathrm{E}_9) = W(\mathrm{E}_8) \ltimes \mathcal{T} \equiv W(\mathrm{E}_8) \ltimes \mathsf{O}. \tag{5.28}$$

Furthermore, the translations belong to the even Weyl group, so we have

$$W^+(\mathrm{E}_9) = W^+(\mathrm{E}_8) \ltimes \mathcal{T}. \tag{5.29}$$

When we identify the root space of E_{10} with the Jordan algebra $H_2(\mathbb{O})$, we can write the action of t_y as

$$t_y : X \mapsto T_y\, X\, T_y^\dagger, \tag{5.30}$$

where

$$T_y = \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix}. \tag{5.31}$$

For such matrices we indeed need not worry about non-associativity because

$$\begin{aligned}(T_yX)T_y{}^\dagger = T_y(XT_y{}^\dagger) &= \begin{pmatrix} x^+ + y\bar{x} + x\bar{y} + yx^-\bar{y} & x + yx^- \\ \bar{x} + x^-\bar{y} & x^- \end{pmatrix} \\ &= \begin{pmatrix} x^+ + 2(x,\, y) + x^-|y|^2 & x + x^-y \\ \bar{x} + x^-\bar{y} & x^- \end{pmatrix} = t_y(X),\end{aligned} \tag{5.32}$$

where δ is as in (3.10). It follows that any element in $W^+(\mathrm{E}_9)$ can be written in the form (5.25), but now with

$$D = \begin{pmatrix} 1 & 0 \\ c & d \end{pmatrix}, \tag{5.33}$$

where c is an arbitrary octavian, so that $D^\dagger$ is an upper triangular matrix. Since at least one of the diagonal entries in D is equal to 1 the expression (5.25) is still well defined.

5.5 From $W(\mathrm{E}_9)$ to $W(\mathrm{E}_{10})$

The tool for describing $W(\mathrm{E}_{10})$ in terms of $W(\mathrm{E}_9)$ is the orbit-stabilizer theorem, just like in going from $W(\mathrm{E}_7)$ to $W(\mathrm{E}_8)$. We use the fact that $W^+(\mathrm{E}_9)$ stabilizes the null root $-\delta$ of (3.10). Therefore the left cosets $W^+(\mathrm{E}_{10})/W^+(\mathrm{E}_9)$ are in bijection with the orbit of $-\delta$ under $W^+(E_{10})$. In this section we will see that the orbit of $-\delta$ under $W^+(E_{10})$ in turn can be parametrized by pairs of right coprime octavians, if we define coprimality for octavians in an appropriate way based on the Euclidean algorithm. (This is not the only possible generalization of the notion of coprimality when going from the Hurwitz numbers to the non-associative octavians — see below). We first recall the validity of the (right) Euclidean algorithm for octavians (see e.g., [25] for a proof).

Theorem 5.1. *Let a and c be two octavians. Then, for some integer $n \geq 0$, there exist octavians q_1, q_2, …, q_n, q_{n+1} and r_0, r_1, …, r_n, such that*

$$\begin{aligned} a &= q_1 c - r_1, \\ c &= q_2 r_1 - r_2, \\ r_1 &= q_3 r_2 - r_3, \\ &\cdots \\ r_{n-2} &= q_n r_{n-1} - r_n, \\ r_{n-1} &= q_{n+1} r_n \end{aligned} \tag{5.34}$$

and $|r_1| > \cdots > |r_n| > 0$.[13]

We now define a and c to be right coprime if (for some choice of possible n and $q_1, q_2 \ldots, q_{n+1}$), the last non-vanishing remainder is a unit, $|r_n| = 1$. For Hurwitz numbers this definition is equivalent to the one that we gave in Section 4.2: two Hurwitz numbers are left (right) coprime if and only if they share no common left (right) factor $g \in \mathsf{H}$ with $|g| > 1$. For octavians the two definitions are no longer equivalent, due to non-associativity. One can construct counterexamples of a and c such that they are right coprime but

[13] In this way of writing the Euclidean algorithm, we have changed the sign of the remainders r_i compared to other authors. This choice of sign proves to be more convenient in the analysis to follow.

still have a non-trivial common right divisor.[14] We have not been able to decide on the converse statement, i.e., if having only trivial common right divisor implies being right coprime in the sense of the Euclidean algorithm. Left coprimality is defined analogously as we will spell out below.

With the above definition of coprimality of octavians we have the following lemma.

Lemma 5.2. *Let $\mathcal{O}_{-\delta}$ be the orbit of $-\delta$ under the action of $W^+(\mathrm{E}_{10})$. Then*

$$\mathcal{O}_{-\delta} = \left\{ \begin{pmatrix} |a|^2 & a\bar{c} \\ c\bar{a} & |c|^2 \end{pmatrix} : a \text{ and } c \text{ are right coprime octavians} \right\}. \tag{5.35}$$

Proof. Let $\mathcal{S}$ be the right hand side of (5.35). By examining the action of the generators (3.15) of $W^+(\mathrm{E}_{10})$ one finds that $\mathcal{S}$ is preserved under the action of $W^+(\mathrm{E}_{10})$. Since $-\delta$ itself is contained in $\mathcal{S}$ we conclude that $\mathcal{O}_{-\delta}$ is contained in $\mathcal{S}$.

To prove the other inclusion, we need to find a $W^+(\mathrm{E}_{10})$ element $w_{a,c}$ associated to any pair of right coprime octavians a and c such that $w_{a,c}(-\delta)$ equals the matrix on the right hand side of (5.35). Since a and c are right coprime, there exist octavians $q_1, q_2, \ldots, q_{n+1}$ such that the remainders $r_1, r_2, \ldots, r_n$, defined by (5.34), satisfy $|r_1| > \cdots > |r_n| > 0$ and $|r_n| = 1$. We now consider the $W^+(\mathrm{E}_{10})$ element

$$w_{a,c} = t_{q_1} \circ s_{-1} \circ \cdots \circ t_{q_{n+1}} \circ s_{-1} \circ u_{r_n}, \tag{5.36}$$

where t_q has been defined in (5.30), s_{-1} is as in (3.15) and u_{r_n} is given by

$$u_{r_n}(X) = U_{r_n} X U_{r_n}^\dagger \quad \text{with} \quad U_{r_n} = \begin{pmatrix} r_n & 0 \\ 0 & \bar{r}_n \end{pmatrix}. \tag{5.37}$$

Thus u_{r_n} is an element of $W^+(\mathrm{E}_8) \subset W^+(\mathrm{E}_{10})$ if and only if $|r_n| = 1$. The notation $w_{a,c}$ is not completely accurate as the element also depends on the choice of Euclidean algorithm decomposition, but all (presently) important

[14] Consider for example $a = e_1 + e_2$, $c = e_1 + e_3$. Then one can write $a = q_1 c - r_1$ with

$$q_1 = \tfrac{1}{2}\left(2 + e_1 + e_4 - e_5 - e_7\right), \quad r_1 = \tfrac{1}{2}\left(-1 + e_1 + e_3 + e_7\right),$$

so they are right co-prime in the sense of (5.34). At the same time, they have the common right divisor $g = 1 + e_1$.

features of $w_{a,c}$ depend only on a and c. We calculate $w_{a,c}(-\delta)$ by induction. One first verifies that

$$(t_{q_{i+1}} \circ s_{-1}) \begin{pmatrix} |r_i|^2 & r_i\bar{r}_{i+1} \\ r_{i+1}\bar{r}_i & |r_{i+1}|^2 \end{pmatrix} = \begin{pmatrix} |r_{i-1}|^2 & r_{i-1}\bar{r}_i \\ r_i\bar{r}_{i-1} & |r_i|^2 \end{pmatrix}, \tag{5.38}$$

for all $i = 0, \ldots, n$, if we let r_{-1}, r_0 and r_{n+1} be equal to a, c and 0, respectively. Since trivially

$$u_{r_n}(-\delta) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} |r_n|^2 & r_n\bar{r}_{n+1} \\ r_{n+1}\bar{r}_n & |r_{n+1}|^2 \end{pmatrix}, \tag{5.39}$$

we conclude that

$$w_{a,c}(-\delta) = \begin{pmatrix} |r_{-1}|^2 & r_{-1}\bar{r}_0 \\ r_0\bar{r}_{-1} & |r_0|^2 \end{pmatrix} = \begin{pmatrix} |a|^2 & a\bar{c} \\ c\bar{a} & |c|^2 \end{pmatrix}. \tag{5.40}$$

This shows that all elements of $\mathcal{S}$ are contained in $\mathcal{O}_{-\delta}$, completing the proof of the lemma. □

By the orbit-stabilizer theorem this lemma implies that $W^+(E_{10})$ can be written as

$$W^+(\mathrm{E}_{10}) = \bigcup_{a,c\in\mathsf{O} \text{ right coprime}} w_{a,c}W^+(\mathrm{E}_9). \tag{5.41}$$

By repeating the same arguments leading to (5.41) but using a right action with the inverse elements we find that $W^+(E_{10})$ is not only the union of left cosets (5.41) but also the union of the right cosets

$$W^+(\mathrm{E}_{10}) = \bigcup_{c,d\in\mathsf{O} \text{ left coprime}} W^+(\mathrm{E}_9)\tilde{w}_{c,d}, \tag{5.42}$$

where we sum now over *left* coprime octavians and

$$\tilde{w}_{c,d} = u_{\bar{r}_n} \circ s_{-1} \circ t_{q_{n+1}} \circ \cdots \circ s_{-1} \circ t_{q_1}. \tag{5.43}$$

These octavians $q_1, \dots q_{n+1}$ and r_n appearing here are the elements in the left Euclidean algorithm

$$\begin{aligned} d &= cq_1 - r_1, \\ c &= r_1 q_2 - r_2, \\ r_1 &= r_2 q_3 - r_3, \\ &\dots \\ r_{n-2} &= r_{n-1} q_n - r_n, \\ r_{n-1} &= r_n q_{n+1}. \end{aligned} \tag{5.44}$$

Neither of the unions in (5.41) or (5.42) is disjoint, as is evident from the proof of the lemma, where the precise value of r_n did not enter in the calculation. Therefore, all d and c that are related by only changing the unit r_n give rise to the same image and therefore to the same coset. In Appendix B, we show that conversely, if two pairs (c, d) and (c', d') give rise to the same coset, then they must be related in this way. Hence, the union can be made disjoint by identifying those pairs of octavians that differ only by changing r_n. In the associative case this is precisely the content of Lemma 4.1.

The reason why have included u_{r_n} in the definition of $w_{a,c}$, although it is an element of $W^+(\mathrm{E}_8) \subset W^+(\mathrm{E}_9)$ and thus acts trivially on $-\delta$, comes from restricting to the associative case. For D_4 and the associated Hurwitz numbers, one can write $w_{a,c}$ in matrix form as

$$w_{a,c} : X \mapsto S_{a,c} X S_{a,c}^\dagger, \tag{5.45}$$

where $S_{a,c}$ is the $\mathrm{PSL}(2, \mathsf{H})$ element

$$S_{a,c} = T_{q_1} S_{-1} \cdots T_{q_{n+1}} S_{-1} U_{r_n}. \tag{5.46}$$

This matrix will then have the form

$$S_{a,c} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \tag{5.47}$$

where b and d depend on the precise choice of Euclidean decomposition. The construction of $S_{a,c}$ shows that the left coprimality condition on (a, c) is not only necessary for the existence of such a matrix in $\mathrm{PSL}(2, \mathsf{H})$, but also sufficient, as we claimed in Section 4.2. The same is true for the left coprimality condition on (b, d), and for the right coprimality conditions on (a, b) and (c, d).

6 Automorphic functions on $\mathcal{H}(\mathbb{A})$

In this section, we construct Maass wave forms and Poincaré series on $\mathcal{H}(\mathbb{A})$ invariant under the Weyl groups studied in the preceding sections. We will mostly concentrate on the case $\mathbb{A} = \mathbb{O}$ because it is the most interesting, and because the extension (or rather, specialization) to the other division algebras is straightforward. The resulting expressions are simple, reminiscent of known expressions for $\mathrm{PSL}(2,\mathbb{Z})$ (see for example [15]) and reflect the arithmetic structure of the integer domains associated with the underlying Kac–Moody algebra. Our analysis can be seen as a first step towards developing a more general theory of automorphic functions and Maass wave forms of these Weyl groups.

6.1 Maass wave forms

The definitions of this subsection apply to all the normed division algebras (including $\mathbb{A} = \mathbb{O}$). Following [15, 21] we define the scalar product between two functions $f, g : \mathcal{H}(\mathbb{A}) \to \mathbb{C}$ by means of the invariant measure (2.18)

$$(f, g) := \int_{\mathcal{F}} \overline{f(z)}\, g(z)\, d\mathrm{vol}(z). \tag{6.1}$$

A *Maass wave form of type s* with respect to the modular group Γ acting on $\mathcal{H}(\mathbb{A})$ is then defined to be a non-zero complex function $f \in \mathrm{L}^2(\mathcal{F}, \mathbb{C})$ on $\mathcal{F} \subset \mathcal{H}(\mathbb{A})$ obeying

- $f(z) = f(\gamma(z))$ for $\gamma \in \Gamma$,
- $\triangle_{\mathrm{LB}} f(z) = -s(n-s) f(z)$,

where $n = \dim_{\mathbb{R}} \mathbb{A}$ as always, and $s \in \mathbb{C}$. If we also demand

- $\int_K f(u + iv)\, d^n u = 0$ for all $v > 0$,

where K is the "projection" of $\mathcal{F}$ onto $\mathbb{A}$, cf. figure 1, we would obtain so-called "cusp forms", i.e., automorphic functions that vanish at the cusp at infinity. However, it will be sufficient for our purposes here to require the functions to be Fourier expandable around $v = \infty$.

Among the Maass wave forms one can introduce a further distinction according to whether they are even or odd. Because $\Gamma \equiv W^+_{\mathrm{hyp}}$ is an index 2 subgroup in the full hyperbolic Weyl group W_{hyp} we have two possibilities for extending the group action, namely

$$f(w(z)) = \begin{cases} f(z) \\ \det(w)\, f(z) \end{cases}, \quad w \in W_{\mathrm{hyp}}. \tag{6.2}$$

Taking the special reflection $w_\theta \in W_{\mathrm{hyp}}$ as a reference (see (3.16)), we see that these definitions respectively are equivalent to

$$f(-\bar{z}) = \begin{cases} +f(z), \\ -f(z). \end{cases} \tag{6.3}$$

Maass wave forms obeying the first (second) condition are referred to as *even (odd)* Maass wave forms. Employing the generating Weyl reflections (3.12) it is straightforward to see that odd Maass wave forms vanish on $\partial\mathcal{F}_0$, hence obey Dirichlet boundary conditions on $\mathcal{F}_0$ (that is, on all faces of the fundamental Weyl chamber), while even Maass wave forms have vanishing normal derivatives on $\partial\mathcal{F}_0$, hence obey Neumann boundary conditions. If one interprets f as a quantum mechanical wave function, both possibilities are compatible with the rules of quantum mechanics because only scalar products of wave functions (for which the sign factors cancel) are observable [3–5].

A main goal of the theory is to analyse the spectral decomposition of the Laplace–Beltrami operator on $\mathrm{L}^2(\mathcal{F}, \mathbb{C})$ [15, 21]. As it turns out, for the discrete eigenvalues the associated eigenfunctions must be determined numerically (see e.g., [30] for results on $\mathbb{A} = \mathbb{C}$). For odd Maass wave forms, the spectrum is purely discrete, and one can establish the bound [3, 4][15]

$$-\triangle_{\mathrm{LB}} \geq \frac{n^2}{4} \tag{6.4}$$

generalizing an argument of [21]. For even $f(z)$ one has in addition to the discrete spectrum a continuous spectrum that can be constructed by means of Eisenstein series and extends from $n^2/4$ over the positive real axis (see below).

[15] In fact, for $\mathrm{SL}(2, \mathbb{Z})$ Maass wave forms a better bound is $E \geq 3\pi^2/2$ (see [15], p. 71), and we expect such improved lower bounds also to exist for the other normed division algebras $\mathbb{A}$.

6.2 Poincaré series

We recall that the Laplacian on hyperbolic space $\mathcal{H}(\mathbb{A})$ of dimension $n+1$ is given by (2.19), again with $n = \dim_{\mathbb{R}} \mathbb{A}$. For $I_s(z) = v^s$ or $I_{n-s}(z) = v^{n-s}$ with $s \in \mathbb{C}$, we compute

$$-\triangle_{\mathrm{LB}}\, I_s(z) = E_s\, I_s(z) = s(n-s)\, I_s(z). \tag{6.5}$$

Note that for real values of s we have $E_s > 0$ if and only if $0 < s < n$.

We are here interested in functions that are both eigenfunctions of the Laplacian $\triangle_{\mathrm{LB}}$ *and* automorphic (but not necessarily square integrable). To turn the function I_s into an automorphic form, we have to make it symmetric under modular transformation by “averaging” over the modular group $\Gamma \equiv W^+_{\mathrm{hyp}}$. Abstractly, this is achieved by defining the *Poincaré series* (or restricted Eisenstein series)

$$\mathcal{P}_s(z) := \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} I_s(\gamma(z)). \tag{6.6}$$

Whenever this sum converges it is an eigenfunction of the Laplace–Beltrami operator by virtue of the invariance property (3.24), which implies that every term in the sum is separately an eigenfunction with the same eigenvalue. The restriction to summing over cosets of the stabilizer of $I_s(z)$ ensures that the sum is well-defined. Furthermore $I_s(w_\theta(z)) = I_s(-\bar{z}) = +I_s(z)$, so we conclude that $\mathcal{P}_s(z)$ satisfies Neumann boundary conditions on $\mathcal{F}_0$.

We emphasize again that the above definition, though standard [15, 21], is special inasmuch all our “modular groups” Γ are hyperbolic Weyl groups, with the corresponding affine Weyl subgroups Γ_∞ as the stabilizer groups of the cusp at infinity. The relation of the hyperbolic Weyl group to integer domains in $\mathbb{A}$ will bring the arithmetic structure to the fore in the final expressions for $\mathcal{P}_s$. We will perform the analysis for $\Gamma = W^+(\mathrm{E}_{10})$ since for example the quaternionic case $W^+(\mathrm{D}_4^{++})$ is contained in it by specialization.

In order to evaluate the sum in (6.6) we employ the description (5.42) of the right cosets $\Gamma_\infty \backslash \Gamma$. Consider therefore two left coprime octavians c and d with corresponding left Euclidean decomposition (5.44) and associated left coset representative

$$\tilde{w}_{c,d} = u_{\bar{r}_n} \circ s_{-1} \circ t_{q_{n+1}} \circ \cdots \circ s_{-1} \circ t_{q_1}. \tag{6.7}$$

as in (5.43).

Lemma 6.1. *Let (z_i) be a sequence of elements in $\mathcal{H}(\mathbb{A})$ defined recursively by $z_0 = u_0 + iv_0 := z$ and $z_{k+1} = u_{k+1} + iv_{k+1} := (s_{-1} \circ t_{q_{k+1}})(z_k) = -(z_k + q_{k+1})^{-1}$. Then*

$$v_{n+1} = \frac{v_i}{|r_i z_i + r_{i-1}|^2} \tag{6.8}$$

for all $i = n+1$ down to $i = 0$. (Here, we let $r_{n+1} = 0$, $r_0 = c$ and $r_{-1} = d$ as before.)

Proof. For $i = n+1$ the claim is true since $r_{n+1} = 0$ and $|r_n| = 1$. The induction step consists of

$$\begin{aligned} v_{n+1} &= \frac{v_i}{|r_i z_i + r_{i-1}|^2} = \frac{v_{i-1}}{|r_i z_i + r_{i-1}|^2 |z_{i-1} + q_i|^2} \\ &= \frac{v_i}{|-r_i(z_{i-1} + q_i)^{-1} + r_{i-1}|^2 |z_{i-1} + q_i|^2} \\ &= \frac{v_{i-1}}{|r_{i-1} z_{i-1} + r_{i-1} q_i - r_i|^2} \\ &= \frac{v_{i-1}}{|r_{i-1} z_{i-1} + r_{i-2}|^2}. \end{aligned} \tag{6.9}$$

In one step one requires that $r_{i-1} = [r_{i-1}(z_{i-1} + q_i)](z_{i-1} + q_i)^{-1}$. Remarkably, this step remains valid even for $\mathcal{H}(\mathbb{O})$: though such a relation is not true generally over the sedenions due to lack of alternativity, it holds in the present case since the imaginary part of $z_{i-1} + q_i$ is i multiplied with a real number only. For the same reason one can also use multiplicativity of the norm. □

Since the imaginary part of z_{n+1} is not changed by the final u_{r_n} we obtain

$$I_s\left(\tilde{w}_{c,d}(z)\right) = v_{n+1}^s = \frac{v_0^s}{|r_0 z_0 + r_{-1}|^{2s}} = \frac{v^s}{|cz + d|^{2s}}. \tag{6.10}$$

The fact that r_n does not change the result of this computation indicates that one can choose the unit r_n freely. As we stated in Section 5.5, if and only if c' and d' are defined by the same q_i as c and d but with r'_n instead of r_n the corresponding $\tilde{w}_{c',d'}$ will be in the same coset as $\tilde{w}_{c,d}$ and hence $I_s(\tilde{w}_{c',d'}(z)) = I_s(\tilde{w}_{c,d}(z))$. Therefore we can undo the overcounting associated with the non-disjointness of (5.42) by dividing by the number of choices for r_n, that is, the number of units of the integer domain in $\mathbb{A}$.

Finally, we obtain the following explicit expression for the Poincaré series

$$\mathcal{P}_s(z) = \frac{1}{N} \sum_{c,d\in\mathcal{O} \text{ left coprime}} \frac{v^s}{|cz+d|^{2s}}, \tag{6.11}$$

where N is the number of units, which is 240 for the octavians and 24 for the Hurwitz numbers. This expression crucially uses the number theoretic properties of the integer domain $\mathcal{O} \subset \mathbb{A}$ underlying the hyperbolic Weyl group.

One can also define the *unrestricted* Eisenstein series

$$\mathcal{E}_s(z) := \sum_{(c,d)\in\mathcal{O}^2\setminus\{(0,0)\}} \frac{v^s}{|cz+d|^{2s}}. \tag{6.12}$$

This Eisenstein series is related to (6.11) by

$$\mathcal{E}_s(z) = \zeta_{\mathcal{O}}(s)\mathcal{P}_s(z), \tag{6.13}$$

where

$$\zeta_{\mathcal{O}}(s) = \sum_{0\neq a\in\mathcal{O}} |a|^{-s} = \sum_{n\in\mathbb{N}} \frac{\sigma_{\mathcal{O}}(n)}{n^s} \tag{6.14}$$

is the (Dedekind) zeta function associated with the appropriate integers and $\sigma_{\mathcal{O}}(n)$ counts the number of roots of (squared) length $2n$ on the root lattice (which is always a multiple of the number of units). This zeta function is also the correct factor of proportionality in the non-associative case as can be seen by considering any pair (c, d). The last vanishing remainder r_n is not necessarily a unit but has well-defined norm $|r_n|^2$. Then one can again construct all pairs (c', d') which yield the same summand in (6.12) by letting r_n range over all octavians of given norm $|r_n|^2$. This produces exactly the zeta function (encoding the same information as the E_8 lattice theta function).

We expect that with an appropriately defined completed zeta function $\xi_{\mathcal{O}}(s)$ the following functional relation holds:[16]

$$\xi_{\mathcal{O}}(s)\mathcal{P}_s(z) = \xi_{\mathcal{O}}(n-s)\mathcal{P}_{n-s}(z), \tag{6.15}$$

[16]For example, the completed Riemann zeta function is

$$\xi(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)$$

and satisfies the functional relation $\xi(s) = \xi(1-s)$ which can be used to define the Riemann zeta function by analytic continuation outside its domain of convergence $\mathrm{Re}(s) > 1$. There exist appropriate generalizations for other algebraic integers.

where n is the dimension of the division algebra and this functional relation should be related to the ones studied in the real rank one case in [19].

We now turn to the Fourier expansion of the Poincaré series. Because the sum (6.6) is periodic under integer shifts $u \to u + o$ for $o \in \mathcal{O}$, it can be expanded into a Fourier series. Since it is an eigenfunction of the Laplacian, the Fourier coefficients have to also satisfy differential equations and we obtain

$$\mathcal{P}_s(z) = v^s + a(s)v^{n-s} + v^{n/2} \sum_{\mu \in \mathcal{O}^* \setminus \{0\}} a_\mu K_{s-n/2}(2\pi|\mu|v)\mathrm{e}^{2\pi i \mu(u)}, \quad (6.16)$$

where $\mathcal{O}^* \subset \mathbb{A}$ is the lattice dual to the lattice of integers $\mathcal{O} \subset \mathbb{A}$ relevant to the hyperbolic Weyl group and $K_\nu(v)$ is the solution to the Bessel equation

$$v^2 \frac{d^2 K_\nu}{dv^2} + v \frac{dK_\nu}{dv} + (v^2 - \nu^2)\, K_\nu = 0, \quad (6.17)$$

which vanishes exponentially for large v. The coefficients a_μ in (6.16) are further constrained by the even part of the finite Weyl group which acts as a set of (generalized) rotations on $\mathcal{O}$, so that $a_\mu = a_{s(\mu)}$ for all $s \in W_{\text{fin}}^+$. Further constraints come from a Hecke algebra (if it can be suitably defined) and will render the coefficients a_μ multiplicative over $\mathbb{A} = \mathbb{R}, \mathbb{C}$. The first two terms in (6.16) correspond to the so-called constant terms. For (6.15) to hold one needs $a(s) = \xi_{\mathcal{O}}(n-s)/\xi_{\mathcal{O}}(s)$. The remaining terms fall off exponentially as $v \to \infty$. In string theory applications, v is associated with the string dilaton and hence the first two terms correspond to perturbative effects whereas the exponentially suppressed terms are non-perturbative in the string coupling.

By standard arguments one can show that (6.11) and (6.12) are convergent for $\mathrm{Re}(s) > n/2$. However, these functions are never square integrable with respect to (6.1). For both $\mathrm{Re}\, s > n/2$ and $\mathrm{Re}\, s < n/2$, the functions $\mathcal{E}_s(z)$ are not normalizable with respect to the invariant measure (2.18) when integrated over the fundamental domain as in (6.1). This is due to the divergence of the v-integral for $v \to \infty$ because of the "constant terms" v^s and v^{n-s}, both of which appear with non-vanishing coefficients in (6.16), whereas the "tail" involving Bessel functions decays exponentially for large v. However, for the special values on the "critical line" ($r \in \mathbb{R}$)

$$s = \frac{n}{2} + ir \quad (6.18)$$

we obtain "almost" normalizable states with eigenvalues

$$E = \frac{n^2}{4} + r^2 \geq \frac{n^2}{4}. \tag{6.19}$$

Notwithstanding a rigorous proof, this can be roughly seen as follows. Exploiting the expansion (6.16) the leading terms behave like v^s or v^{n-s}, with the subleading terms being integrable in v. Substituting these "dangerous" terms into (6.1) and neglecting finite contributions, we get (with $\mathrm{Re}\, s = n/2$)

$$\int_{\mathcal{F}} d\mathrm{vol}(z)\, \overline{\mathcal{E}_s(z)} \mathcal{E}_{s'}(z) \sim \int_K d^n u \int_{\sqrt{1-|u|^2}}^{\infty} \frac{dv}{v} \exp\big[i(\pm r' \pm r) \ln v\big]. \tag{6.20}$$

The integral over u is finite because K is compact. Changing variables to $\xi = \ln v$ and extending the range of integration to $v = 0$, we get

$$\int_{-\infty}^{\infty} d\xi \, \exp\big[i(\pm r' \pm r)\xi\big] = 2\pi\delta(r' \pm r). \tag{6.21}$$

Hence, for these values of s, the Eisenstein functions are δ-function normalizable up to finite corrections. As a result, there is a continuous part of the spectrum

$$\mathrm{spec}_c(-\triangle_{\mathrm{LB}}) = \left[\frac{n^2}{4}, \infty\right) \tag{6.22}$$

for *even* Maass wave forms. In addition there are discrete eigenvalues for even Maass wave forms, which are embedded in the continuum (but which must be determined numerically). The spectrum of odd Maass wave forms is purely discrete.

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Appendix A Automorphisms of octonions

In this appendix, we will prove a theorem about the automorphisms of the octonions that we use in Section 5.2. First we list the Moufang identities that we also use

$$(ax)(ya) = a(xy)a, \quad ((xa)y)a) = x(aya), \quad a(x(ay)) = (axa)y. \tag{A.1}$$

These identities follow from the alternative laws (2.11). (For a proof, see [36].) Another useful identity that follows from the alternative laws and the Moufang identities is

$$(a^2xa)(a^{-1}ya) = a^2(xy)a. \tag{A.2}$$

Indeed, we have

$$\begin{aligned}(a^2xa)(a^{-1}ya) &= (a(ax)a)(a^{-1}ya) \\ &= a((axa)(a^{-1}y))a \\ &= a(a(xy))a = a^2(xy)a.\end{aligned} \tag{A.3}$$

Now we can prove the following lemma.

Lemma A.1. *For any integer $n \leq 1$ and any z, a_1, a_2, $\dots$, $a_n \in \mathbb{O}$ we have*

$$\begin{aligned}&a_1(a_2(\cdots(a_n z a_n^{-1})\cdots)a_2^{-1})a_1^{-1} \\ &\quad = a_1^{-2}(a_2^{-2}\cdots(a_n^{-2}(b_n z)a_n^{-1})\cdots a_2^{-1})a_1^{-1},\end{aligned} \tag{A.4}$$

where

$$b_n = a_n^2(a_{n-1}^2\cdots(a_2^2 a_1^3 a_2)\cdots a_{n-1})a_n. \tag{A.5}$$

Proof. We prove this by induction over n. The case $n = 1$ follows easily by alternativity. Suppose now that the identity (A.4) holds for some integer

$n \geq 1$. Then we have

$$a_1(a_2 \cdots (a_n(a_{n+1} z a_{n+1}^{-1}) a_n^{-1}) \cdots a_2^{-1}) a_1^{-1}$$
$$= a_1^{-2}(a_2^{-2} \cdots (a_n^{-2}(b_n(a_{n+1} z a_{n+1}^{-1})) a_n^{-1}) \cdots a_2^{-1}) a_1^{-1}, \qquad \text{(A.6)}$$

so we only have to show that

$$b_n(a_{n+1} z a_{n+1}^{-1}) = a_{n+1}^{-2}(b_{n+1} z) a_{n+1}^{-1}. \qquad \text{(A.7)}$$

To do this we use that

$$b_n = a_{n+1}^{-2} b_{n+1} a_{n+1}^{-1}, \qquad \text{(A.8)}$$

so that the left-hand side of (A.7) becomes

$$b_n(a_{n+1} z a_{n+1}^{-1}) = (a_{n+1}^{-2} b_{n+1} a_{n+1}^{-1})(a_{n+1} z a_{n+1}^{-1}). \qquad \text{(A.9)}$$

and then (A.7) follows from (A.3). □

Lemma A.2. *For any integer $n \leq 1$ and any x, y, a_1, a_2, …, $a_n \in \mathbb{O}$ we have*

$$(a_1(\cdots (a_n x a_n^{-1}) \cdots) a_1^{-1})(a_1(\cdots (a_n y a_n^{-1}) \cdots) a_1^{-1})$$
$$= a_1^{-2}(a_2^{-2} \cdots (a_n^{-2}((b_n x) y) a_n^{-1}) \cdots a_2^{-1}) a_1^{-1}, \qquad \text{(A.10)}$$

where

$$b_n = a_n^2(a_{n-1}^2 \cdots (a_2^2 a_1^3 a_2) \cdots a_{n-1}) a_n. \qquad \text{(A.11)}$$

Proof. By Lemma A.1 we have

$$(a_1(\cdots (a_n x a_n^{-1}) \cdots) a_1^{-1})(a_1(\cdots (a_n y a_n^{-1}) \cdots) a_1^{-1})$$
$$= (a_1^{-2} \cdots (a_n^{-2}(b_n x) a_n^{-1}) \cdots a_1^{-1})(a_1(\cdots (a_n y a_n^{-1}) \cdots) a_1^{-1}) \qquad \text{(A.12)}$$

and then (A.10) follows by using (A.3) successively. □

No we can prove the main result of this appendix.

Theorem A.3. *A transformation*

$$\varphi : x \mapsto a_1(a_2(\cdots (a_n x a_n^{-1}) \cdots) a_2^{-1}) a_1^{-1} \qquad \text{(A.13)}$$

of the octonions, where a_1, …, $a_n \neq 0$, is an automorphism if and only if

$$b_n = a_n^2(a_{n-1}^2(\cdots (a_2^2 a_1^3 a_2) \cdots a_{n-1}) a_n \in \mathbb{R}. \qquad \text{(A.14)}$$

Proof. Setting $xy = z$, we have in Lemma A.1 and Lemma A.2 obtained expressions for $\varphi(xy)$ and $\varphi(x)\varphi(y)$, respectively. The difference between these two expressions is

$$\varphi(xy) - \varphi(x)\varphi(y) = a_1^{-2}(a_2^{-2} \cdots (a_n^{-2}\{b_n,\, x,\, y\}a_n^{-1}) \cdots a_2^{-1})a_1^{-1}. \quad \text{(A.15)}$$

Since there are no zero divisors, this difference is zero if and only if the associator $\{b_n,\, x,\, y\} = b_n(xy) - (b_n x)y$ vanishes. Since this must happen for all x and y, the necessary and sufficient condition is $b_n \in \mathbb{R}$. □

The theorem simplifies considerably if we take $a_1, \ldots, a_n$ to be unit octonions, and furthermore imaginary.

Corollary A.4. *A transformation*

$$\varphi : x \mapsto a_1(a_2(\cdots(a_n x a_n)\cdots)a_2)a_1 \quad \text{(A.16)}$$

of the octonions, where $a_1, \ldots, a_n$ are imaginary unit octonions or real unit octonions, is an automorphism if and only if

$$(((a_1a_2)a_3)\cdots a_n) = \pm 1. \quad \text{(A.17)}$$

Proof. This follows directly from Theorem A.3 if we use that $a^{-1} = \pm a$ and $a^2 = a^{-2} = \pm 1$ for any imaginary unit octonion or real unit octonion a. □

We use this corollary in Section 5.2 when we identify the imaginary unit octavians with the simple roots of E_7, and the corresponding bimultiplications with generators of $W^+(\mathrm{E}_7)$.

Appendix B 2n-dimensional representation of W^+_{hyp}

The representation (3.15) of the even hyperbolic Weyl group as conjugation on Hermitian (2×2)-matrices over $\mathbb{A}$ is an $(n+2)$-dimensional representation, corresponding to the action on the Cartan subalgebra. Alternatively, it is the action in the vector (fundamental) representation of $\mathrm{SO}(1, n+1)$ in which W^+_{hyp} is embedded as a discrete subgroup. The isomorphism $W^+_{\mathrm{hyp}} \cong \mathrm{PSL}(2, \mathcal{O})$ for integers $\mathcal{O} \subset \mathbb{A}$ suggests also a natural action on 2-component vectors of elements in $\mathbb{A}$. Such a representation corresponds to the spinor representation of $\mathrm{SO}(1, n+1)$.

Concretely, we consider the *right* action of $\mathrm{PSL}(2,\mathcal{O})$ on the $2n$-dimensional space of two components *rows*

$$\{(a_1,a_2)\,:\,a_1,a_2\in\mathbb{A}\}\,/\sim, \tag{B.1}$$

where $\sim$ denotes the equivalence relation associated with the "P" in $\mathrm{PSL}(2,\mathcal{O})$. Except for the complex case it corresponds to $(a_1,a_2)\sim(-a_1,-a_2)$. The action of $\mathrm{PSL}(2,\mathcal{O})$ is by right multiplication of the generators (3.15).

In order to check that this defines a representation of W^+_{hyp} one has to verify the defining relations. We do this in a non-associative example for the product S_iS_j of two generators such that $(\varepsilon_i\varepsilon_j)^2=\pm1$. From the definition one gets

$$(a_1,a_2)\cdot(S_iS_j)^2=\big((((a_1\varepsilon_i)\varepsilon_j)\varepsilon_i)\varepsilon_j,(((a_2\bar\varepsilon_i)\bar\varepsilon_j)\bar\varepsilon_i)\bar\varepsilon_j\big). \tag{B.2}$$

Now, using the Moufang identities one calculates

$$\begin{aligned}(((a_1\varepsilon_i)\varepsilon_j)\varepsilon_i)\varepsilon_j&=(a_1\varepsilon_i)(\varepsilon_j\varepsilon_i\varepsilon_j)=(a_1(\bar\varepsilon_j\bar\varepsilon_i\bar\varepsilon_j))(\varepsilon_j\varepsilon_i\varepsilon_j\varepsilon_i\varepsilon_j\varepsilon_i\varepsilon_j)\\&=\pm(a_1(\bar\varepsilon_j\bar\varepsilon_i\bar\varepsilon_j)(\varepsilon_j\varepsilon_i\varepsilon_j)=\pm a_1,\end{aligned} \tag{B.3}$$

where alternativity has been used in a number of places. Hence, we arrive at $(S_iS_j)^2=\mathrm{id}\in\mathrm{PSL}(2,\mathcal{O})$ as required. The calculations in the other cases are similar.

One advantage of this representation is that one can easily exhibit the invariant content of elements like $\tilde w_{c,d}$ in (5.43). A calculation shows that

$$(0,1)\cdot\tilde w_{c,d}=(c,d). \tag{B.4}$$

In the associative case this is clear since $\tilde w_{c,d}$ is then given by a matrix with bottom row (c,d). In the non-associative case one cannot write a single matrix for $\tilde w_{c,d}$ but the above computation shows that it still behaves as a matrix with bottom row (c,d) when it acts on $(0,1)$.[17] This insight allows us to derive the overcounting factor N appearing in (6.11). Suppose that (c',d') is a pair of left coprime octavians that gives rise to the same coset as

[17] In this representation, the stabilizer of $(0,1)$ is only the translation group $\mathcal{T}$ and not the full (even) affine Weyl group $\Gamma_\infty\equiv W^+_{\mathrm{aff}}=W^+_{\mathrm{fin}}\ltimes\mathcal{T}$. Therefore, we can "resolve" the presence of u_{r_n} in (5.43).

(c,d), that is,

$$W^+(\mathrm{E}_9)\tilde{w}_{c,d} = W^+(\mathrm{E}_9)\tilde{w}_{c',d'}. \tag{B.5}$$

Since $\Gamma_\infty = W^+_{\mathrm{aff}}$ is generated (non-minimally) by the translations t_y of (5.31) and the rotations u_ε of (5.37) this means that $\tilde{w}_{c',d'} = w\tilde{w}_{c,d}$, where w is a product of such translations and rotations. Acting on $(0,1)$ we get

$$(c', d') = (0,1)\cdot(w\tilde{w}_{c,d}). \tag{B.6}$$

The rotations in w will change $(0,1)$ to $(0,\varepsilon)$ where ε is a unit, and the translations act trivially on such rows.

We conclude that c' and d' are determined by the left Euclidean algorithm with the same q_i as for (c,d) but with $r_n' = \varepsilon r_n$ instead of r_n. Therefore all pairs that represent the same coset must be related in such a way. One can show that for all choices of ε one obtains distinct pairs, and therefore the overcounting in (6.11) is equal to the number of units in $\mathcal{O}$.

Appendix C Green functions on $\mathcal{H}(\mathbb{A})$

For $z_1, z_2 \in \mathcal{H}(\mathbb{A})$ define (cf. (2.16))

$$\lambda(z_1,z_2) := \frac{|u_1-u_2|^2+(v_1-v_2)^2}{4v_1v_2} \equiv \frac{1}{2}\sinh^2\frac{d(z_1,z_2)}{2}. \tag{C.1}$$

Acting with the Laplace–Beltrami operator (2.19) on the first argument of $\lambda(z_1,z_2)$ we get

$$\triangle_{\mathrm{LB}}\,\lambda(z_1,z_2) = (n+1)\left(\frac{1}{2}+\lambda(z_1,z_2)\right) \tag{C.2}$$

and thus for $\xi\in\mathbb{R}$ and $\lambda\equiv\lambda(z_1,z_2)$

$$\begin{aligned}\triangle_{\mathrm{LB}}\big[\xi+\lambda\big]^{-s} &= s\big[\xi+\lambda\big]^{-s-2}\\ &\quad\times\left[(s+1)(\lambda+\lambda^2)-(n+1)\left(\frac{1}{2}+\lambda\right)(\xi+\lambda)\right].\end{aligned} \tag{C.3}$$

Following [21], we define the Green function (or the "propagator") as

$$G_s(\lambda(z,w)) = \int_0^1 \left[\xi(1-\xi)\right]^{s-\frac{n+1}{2}}\left[\xi+\lambda(z,w)\right]^{-s}d\xi, \tag{C.4}$$

for z, $w \in \mathcal{H}(\mathbb{A})$. This integral converges for $s > \frac{n-1}{2}$ and $\lambda(z,w) > 0$. We will assume $n > 1$ from now on, as the case $n = 1$ is treated in much detail in [21].

For non-coincident points $z \neq w$, application of the operator $\triangle_{\mathrm{LB}} + s(n-s)$ gives an integrand proportional to

$$s \frac{d}{d\xi} \left[\Big(\xi(1-\xi)\Big)^{s+\frac{1-n}{2}} \big(\xi + \lambda(z,w)\big)^{-s-1} \right]. \tag{C.5}$$

Therefore, the boundary terms, and hence the integral vanish for $s > \frac{n-1}{2}$. This shows that, for $z \neq w$, we have

$$\Big[\triangle_{\mathrm{LB}} + s(n-s)\Big] G_s\big(\lambda(z,w)\big) = 0. \tag{C.6}$$

For coincident arguments the integrand is singular, and we must reason more carefully. To determine the behavior for small λ we split the integral as

$$\int_0^1 = \int_0^A + \int_A^B + \int_B^1 \tag{C.7}$$

with suitable $0 < A < B < 1$. Since for $\lambda \geq 0$ and $0 < \xi < 1$

$$\left| \frac{\xi(1-\xi)}{\xi + \lambda} \right| < 1 \tag{C.8}$$

the first integral is bounded above by

$$\int_0^A \frac{d\xi}{(\xi+\lambda)^{\frac{n+1}{2}}} = A\lambda^{-\frac{n+1}{2}} + \mathcal{O}\big(A^2 \lambda^{-\frac{n+3}{2}}\big) \tag{C.9}$$

Choosing $A = \lambda^{\frac{n+1}{2}}$, we see that this part of the integral is $\mathcal{O}(1)$ for small λ. For the middle integral we have

$$\int_A^B \frac{d\xi}{(\xi+\lambda)^{\frac{n+1}{2}}} = \frac{2}{n-1} \left[\frac{1}{(\lambda+A)^{\frac{n-1}{2}}} - \frac{1}{(\lambda+B)^{\frac{n-1}{2}}} \right]. \tag{C.10}$$

so that with (say) $B = \frac{1}{2}$ we get

$$\int_{\lambda^{(n+1)/2}}^{1/2} \frac{d\xi}{(\xi+\lambda)^{\frac{n+1}{2}}} = \frac{2}{n-1} \lambda^{-\frac{n-1}{2}} + \mathcal{O}(1) \quad \text{for } \lambda \to 0. \tag{C.11}$$

The remaining integral is again bounded as

$$\int_{1/2}^{1} (1-\xi)^{s-\frac{n+1}{2}} d\xi = \mathcal{O}(1) \tag{C.12}$$

for $s > \frac{n-1}{2}$. From (C.11) we see that the Green function behaves as $\sim \lambda^{-(n-1)/2}$ for small λ, and therefore (C.6) must be amended to

$$\Big[\triangle_{\text{LB}} + s(n-s)\Big] G_s\big(\lambda(z,w)\big) = \frac{4\pi^{\frac{n+1}{2}}}{(n-1)\Gamma\left(\frac{n+1}{2}\right)}\, \delta^{(n+1)}(z,w) \tag{C.13}$$

where we have used the standard formula for the volume of the $(n+1)$-dimensional unit sphere. The automorphic Green function is then defined as

$$\mathbb{G}_s(z/w) := \sum_{\gamma\in\Gamma} G_s(z,\gamma(w)) \tag{C.14}$$

in analogy with the case $n = 1$ [21].

Appendix D Geodesics in $\mathcal{H}(\mathbb{H})$ and periodic orbits

Geodesics in $\mathcal{H}(\mathbb{H})$ are given by half-circles and straight lines. For the usual complex upper half plane it is known (see e.g., [32]) that we can associate to each hyperbolic element[18] $M \in \mathrm{PSL}(2,\mathbb{R})$ a geodesic that is mapped onto itself by the action of M (the same is true for all matrices conjugate to M). Let us consider the "imaginary axis" in $\mathcal{H}(\mathbb{H})$, parametrized by $z(t) = it$ with $0 < t < \infty$. For this geodesic, the hyperbolic motion leaving it invariant is obviously given by $\gamma_t = \mathrm{diag}\,(t^{1/2}, t^{-1/2})$, whereby i is mapped to $\gamma_t(i) = \gamma_t(z(1)) = it$. It is straightforward to see that all geodesics in $\mathcal{H}(\mathbb{H})$ are $\mathrm{PSL}(2,\mathbb{H})$ images of one another. First, straight line geodesics centered at $u \neq 0$ are trivially obtained by acting on $z(t) = it$ with the shift matrix

$$T_u = \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}. \tag{D.1}$$

For arbitrary $u_1 \neq u_2 \in \mathbb{H}$ the circular geodesic with endpoints u_1 and u_2 can (for example) be obtained by acting on $z(t)$ with the matrix

$$C_{u_1,u_2} = \frac{1}{\sqrt{|u_1 - u_2|}} \begin{pmatrix} u_2 & u_1 \\ 1 & 1 \end{pmatrix}. \tag{D.2}$$

[18] Recall that a hyperbolic element $M \in \mathrm{SL}(2,\mathbb{R})$ has reciprocal real eigenvalues or, equivalently, satisfies $|\mathrm{Tr}\,\mathrm{M}| > 2$.

Use of (4.20) results in the explicit parametrization of the geodesic half-circle

$$z'(t) = \frac{u_1 + u_2\, t^2 + it\, |u_1 - u_2|}{1 + t^2} \qquad (0 < t < \infty)\ . \tag{D.3}$$

The hyperbolic motions leaving invariant this geodesic half-circle are now simply given by (for all t)

$$M_t = S \begin{pmatrix} t^{1/2} & 0 \\ 0 & t^{-1/2} \end{pmatrix} S^{-1}, \tag{D.4}$$

where S is either T_u or C_{u_1,u_2}. The formula expressing M_t explicitly in terms of t, u_1, u_2 follows directly from (D.2) and (4.12) but is not very illuminating. However, it implies the inequality

$$\mathrm{Re}\,\big(\mathrm{Tr}\, M_t\big) = t^{1/2} + t^{-1/2} \geq 2. \tag{D.5}$$

Note that the real part of the trace is cyclic even over quaternionic matrices and hence furnishes an invariant for matrices up to conjugation. In other words, we may adopt (D.5) as the quaternionic generalization of the usual condition of hyperbolicity for an $\mathrm{SL}(2,\mathbb{R})$ matrix.

Consider now the case when $M \equiv M_{t_0}$ happens to be integral for some value $t_0 > 1$, i.e., $M_{t_0} \in \mathrm{PSL}^{(0)}(2,\mathsf{H})$. Then the associated geodesic gives rise to a *periodic orbit in the fundamental domain* $\mathcal{F}$ of the modular group $\mathrm{PSL}^{(0)}(2,\mathsf{H})$. This is most easily seen by following the geodesic circle successively through the images of the fundamental domain $\mathcal{F}$ until we reach the image of $\mathcal{F}$ under the special transformation M, where we connect up again to the original geodesic curve intersecting $\mathcal{F}$. (The integrality condition $M \in \mathrm{PSL}^{(0)}(2, \mathsf{H})$ may be viewed as the analog of the rationality condition for periodic orbits on tori.) The length ℓ_p of this periodic orbit is easily calculated by mapping it back to the imaginary axis, whence

$$\ell_p = \int_1^{t_0} \frac{dv}{v} = \log t_0 \tag{D.6}$$

by the invariance of the geodesic length under the modular group. With (D.5) we recover the quaternionic analog of the well-known formula for

$\mathrm{PSL}(2,\mathbb{Z})$ [32]

$$2\cosh\frac{\ell_p}{2} = \left|\mathrm{Re}\,(\mathrm{Tr}\,M)\right|. \tag{D.7}$$

The limiting case $\ell_p \to \infty$ corresponds to the infinite geodesic along the imaginary axis. Consequently, the number of periodic orbits in the fundamental domain $\mathcal{F}$ increases exponentially with the geodesic length of the orbit, as it does for $\mathrm{PSL}(2,\mathbb{Z})$ (see, e.g., [32]).

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