ON THE CLASSICAL GEOMETRY OF EMBEDDED SURFACES IN TERMS OF POISSON BRACKETS

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Abstract. We consider surfaces embedded in a Riemannian manifold of arbitrary dimension and prove that many aspects of their differential geometry can be expressed in terms of a Poisson algebraic structure on the space of smooth functions of the surface. In particular, we find algebraic formulas for Weingarten's equations, the complex structure and the Gaussian curvature.

1. Introduction

Given a manifold Σ, it is interesting to study in what ways information about the geometry of Σ can be extracted as algebraic properties of the algebra of smooth functions $C^\infty(\Sigma)$. In case Σ is a Poisson manifold, this algebra has a second (apart from the commutative multiplication of functions) bilinear (non-associative) algebra structure realized as the Poisson bracket. The bracket is compatible with the commutative multiplication via Leibniz rule, thus carrying the basic properties of a derivation.

On a surface Σ, with local coordinates u^1 and u^2, one may define

$$\{f,h\} = \frac{1}{\sqrt{g}}\left(\frac{\partial f}{\partial u^1}\frac{\partial h}{\partial u^2} - \frac{\partial h}{\partial u^1}\frac{\partial f}{\partial u^2}\right),$$

where g is the determinant of the metric tensor, and one can readily check that $\big(C^\infty(\Sigma), \{\cdot,\cdot\}\big)$ is a Poisson algebra. Having only this very particular combination of derivatives at hand, it seems at first unlikely that one can encode geometric information of Σ in Poisson algebraic expressions. Surprisingly, it turns out that many differential geometric quantities can be computed in a completely algebraic way, as stated in Theorem 3.5 and in Theorem 4.1. For instance, the Gaussian curvature of a surface embedded in $\mathbb{R}^3$ can be written as

$$K = -\frac{1}{2}\sum_{i,j=1}^{3}\{x^i,n^j\}\{x^j,n^i\},$$

where $x^i(u^1,u^2)$ are the embedding coordinates and $n^i(u^1,u^2)$ are the components of a unit normal vector at each point of Σ.

Let us also mention that our initial motivation for studying this problem came from matrix regularizations of Membrane Theory. Classical solutions in Membrane Theory are 3-manifolds with vanishing mean curvature in $\mathbb{R}^{1,d}$. Considering one of the coordinates to be time, the problem can also be formulated in a dynamical way as surfaces sweeping out volumes of vanishing mean curvature. In this context, a regularization was introduced replacing the infinite dimensional function algebra on the surface by an algebra of $N\times N$ matrices [Hop82]. If we let $T^{(N)}$ be a linear map from smooth functions to hermitian $N \times N$ matrices, the regularization is required to fulfill

$$\lim_{N\to\infty} \left|\left|T^{(N)}(f)T^{(N)}(g) - T^{(N)}(fg)\right|\right| = 0,$$
$$\lim_{N\to\infty} \left|\left|N[T^{(N)}(f), T^{(N)}(h)] - iT^{(N)}(\{f,h\})\right|\right| = 0,$$

where $||\cdot||$ denotes the operator norm, and therefore it is natural to regularize the system by replacing (commutative) multiplication of functions by (non-commutative) multiplication of matrices and Poisson brackets of functions by commutators of matrices.

Although we may very well consider $T^{(N)}(\frac{\partial f}{\partial u^1})$, its relation to $T^{(N)}(f)$ is in general not simple. However, the particular combination of derivatives found in $T^{(N)}(\{f,h\})$ is easily expressed in terms of a commutator of $T^{(N)}(f)$ and $T^{(N)}(h)$. In the context of Membrane Theory, it is desirable to have geometrical quantities in a form that can easily be regularized, which is the case for any expression constructed out of multiplications and Poisson brackets.

2. Preliminaries

To introduce the relevant notations, we shall recall some basic facts about submanifolds, in particular Gauss' and Weingarten's equations (see e.g. [KN96a, KN96b] for details). Let Σ be a two dimensional manifold embedded in a Riemannian manifold M with $\dim M = 2 + p \equiv m$. Local coordinates on M will be denoted by $x^1,\ldots,x^m$, local coordinates on Σ by u^1,u^2, and we regard $x^1,\ldots,x^m$ as being functions of u^1,u^2 providing the embedding of Σ in M. The metric tensor on M is denoted by $\bar{g}_{ij}$ and the induced metric on Σ by g_{ab}; indices i,j,k,l,n run from 1 to m, indices a,b,c,d,p,q run from 1 to 2 and indices A,B,C,D run from 1 to p. Furthermore, the covariant derivative and the Christoffel symbols in M will be denoted by $\bar{\nabla}$ and $\bar{\Gamma}^i_{jk}$ respectively.

The tangent space $T\Sigma$ is regarded as a subspace of the tangent space TM and at each point of Σ one can choose $e_a = (\partial_a x^i)\partial_i$ as basis vectors in $T\Sigma$, and in this basis we define $g_{ab} = \bar{g}(e_a,e_b)$. Moreover, we choose a set of normal vectors N_A, for $A = 1,\ldots,p$, such that $\bar{g}(N_A,N_B) = \delta_{AB}$ and $\bar{g}(N_A,e_a) = 0$.

The formulas of Gauss and Weingarten express the relation between the covariant derivative in M and the covariant derivative in Σ (denoted by ∇) as

$$\bar{\nabla}_X Y = \nabla_X Y + \alpha(X,Y) \tag{2.1}$$
$$\bar{\nabla}_X N_A = -W_A(X) + D_X N_A \tag{2.2}$$

where $X,Y \in T\Sigma$ and $\nabla_X Y$, $W_A(X) \in T\Sigma$ and $\alpha(X,Y)$, $D_X N_A \in T\Sigma^\perp$. By expanding $\alpha(X,Y)$ in the basis $\{N_1,\ldots,N_p\}$ one can write (2.1) as

$$\bar{\nabla}_X Y = \nabla_X Y + \sum_{A=1}^{p} h_A(X,Y)N_A, \tag{2.3}$$

and we set $h_{A,ab} = h_A(e_a,e_b)$. It follows from the above equations that following relations exist

$$h_{A,ab} = -\bar{g}\big(e_a, \bar{\nabla}_b N_A\big) \tag{2.4}$$
$$(W_A)^a_b = g^{ac}h_{A,cb}, \tag{2.5}$$

where g^{ab} denotes the inverse of g_{ab}. From Gauss' formula (2.1) one obtains an expression for the curvature R of Σ in terms of the curvature $\bar{R}$ of M

$$\begin{aligned} g\big(R(X,Y)Z,W\big) = \bar{g}\big(\bar{R}(X,Y)Z,W\big) - \bar{g}\big(\alpha(X,Z),\alpha(Y,W)\big) \\ + \bar{g}\big(\alpha(Y,Z),\alpha(X,W)\big), \end{aligned} \tag{2.6}$$

where $X,Y,Z,W \in T\Sigma$. The Gaussian curvature K of Σ can be computed as the sectional curvature (of Σ) in the plane spanned by e_1 and e_2, i.e.

$$K = \frac{g\big(R(e_1,e_2)e_2,e_1\big)}{g(e_1,e_1)g(e_2,e_2) - g(e_1,e_2)^2} \tag{2.7}$$

which, by using (2.6), yields

$$K = \frac{1}{g}\bar{g}\big(\bar{R}(e_1,e_2)e_2,e_1\big) + \sum_{A=1}^{p} \frac{\det(h_{A,ab})}{g} \tag{2.8}$$

where $g = \det(g_{ab})$. We also recall the mean curvature vector, defined as

$$H = \frac{1}{2}\sum_{A=1}^{p} \big(\operatorname{tr} W_A\big) N_A. \tag{2.9}$$

3. Poisson algebraic formulation

In this section we will prove that one can express many aspects of the differential geometry of an embedded surface in terms of a Poisson algebra structure introduced on $C^\infty(\Sigma)$. Namely, let $\rho : \Sigma \to \mathbb{R}$ be an arbitrary non-vanishing density, i.e. an object transforming as

$$\tilde{\rho}(v) = \left|\frac{\partial(u^1,u^2)}{\partial(v^1,v^2)}\right| \rho\big(u(v)\big) \tag{3.1}$$

under general coordinate transformations, and define

$$\{f,h\} = \frac{1}{\rho}\varepsilon^{ab}\big(\partial_a f\big)\big(\partial_b h\big) \tag{3.2}$$

for all $f,h \in C^\infty(\Sigma)$, where ε^{ab} is antisymmetric with $\varepsilon^{12} = 1$. It is straightforward to check that $\big(C^\infty(\Sigma), \{\cdot,\cdot\}\big)$ is a Poisson algebra.

The above Poisson bracket also arises from the choice of a volume form on Σ. Namely, since any 2-form is closed on a two dimensional manifold, a volume form ω is also a symplectic form. For any smooth function f, a symplectic form defines a vector field $X_f \in T\Sigma$ associated with f through the relation

$$\omega(X_f, Y) = df(Y)$$

for all $Y \in T\Sigma$. Furthermore, since ω is closed, one defines a Poisson bracket by setting

$$\{f,h\} = \omega(X_f, X_h),$$

which, in local coordinates where $\omega = \rho(u^1,u^2)du^1 \wedge du^2$, coincides with (3.2).

Let $x^1(u^1,u^2),\ldots,x^m(u^1,u^2)$ be the embedding of Σ, and let $n_A^1,\ldots,n_A^m$ denote the components of the normal vector N_A. We introduce the following tensors

$$\mathcal{P}^{ij} = \{x^i, x^j\} \tag{3.3}$$

$$\mathcal{S}_A^{ij} = \frac{1}{\rho}\varepsilon^{ab}\big(\partial_a x^i\big)\big(\bar{\nabla}_b N_A\big)^j = \{x^i, n_A^j\} + \{x^i, x^k\}\bar{\Gamma}^j_{kl} n_A^l, \tag{3.4}$$

and by lowering one of the indices one can regard $\mathcal{P}$ and $\mathcal{S}_A$ as maps $TM \to TM$. Our convention is to lower the second index, i.e

$$\begin{aligned}\mathcal{P}(X) &= \mathcal{P}^{ik}\bar{g}_{kj}X^j\partial_i\\ \mathcal{S}_A(X) &= \mathcal{S}_A^{ik}\bar{g}_{kj}X^j\partial_i\\ \mathcal{S}_A^T(X) &= \bar{g}_{ik}\mathcal{S}_A^{kj}X^i\partial_j.\end{aligned}$$

Out of these objects, we define two compound maps $TM \to TM$ as

$$\mathcal{A}_A(X) = -\mathcal{P}\mathcal{S}_A^T(X) \tag{3.5}$$

$$\mathcal{B}_A(X) = \mathcal{P}\mathcal{S}_A(X), \tag{3.6}$$

whose components in the coordinate basis are

$$(\mathcal{A}_A)^i_k = \{x^i, x^j\}\,\bar{g}_{jj'}\{n_A^{j'}, x^{k'}\}\bar{g}_{k'k} + \{x^i, x^j\}\bar{g}_{jj'}\bar{\Gamma}^{j'}_{ll'}n_A^{l'}\{x^l, x^{k'}\}\bar{g}_{k'k} \tag{3.7}$$

$$(\mathcal{B}_A)^i_k = \{x^i, x^j\}\bar{g}_{jj'}\{x^{j'}, n_A^{k'}\}\bar{g}_{k'k} + \{x^i, x^j\}\bar{g}_{jj'}\{x^{j'}, x^l\}\bar{\Gamma}^{k'}_{ll'}n_A^{l'}\bar{g}_{k'k}. \tag{3.8}$$

Let us now investigate some properties of the maps defined above.

Proposition 3.1. *For $X \in TM$ it holds that*

$$\begin{aligned}\mathcal{S}_A(X) &= -\frac{1}{\rho}\bar{g}\big(X, \bar{\nabla}_a N_A\big)\varepsilon^{ab}e_b\\ \mathcal{P}(X) &= -\frac{1}{\rho}\bar{g}\big(X, e_a\big)\varepsilon^{ab}e_b\\ \mathcal{P}^2(X) &= -\frac{g}{\rho^2}\bar{g}\big(X, e_a\big)g^{ab}e_b.\end{aligned}$$

In particular, $\mathcal{P}(X), \mathcal{S}_A(X) \in T\Sigma$, and for $Y \in T\Sigma$ one obtains $\mathcal{P}^2(Y) = -(g/\rho^2)Y$.

Proof. Let us provide a proof for the statements concerning $\mathcal{P}$. The statement about $\mathcal{S}_A$ is proven in an analogous way.

Any $X \in TM$ is written in coordinates as $X^i\partial_i$. Applying $\mathcal{P}$ to this vector yields

$$\begin{aligned}\mathcal{P}(X) &= \frac{1}{\rho}\varepsilon^{ab}\big(\partial_a x^i\big)\big(\partial_b x^j\big)\bar{g}_{jk}X^k\partial_i = \frac{1}{\rho}\bar{g}\big(X, e_b\big)\varepsilon^{ab}\big(\partial_a x^i\big)\partial_i\\ &= -\frac{1}{\rho}\bar{g}\big(X, e_a\big)\varepsilon^{ab}e_b.\end{aligned}$$

Applying $\mathcal{P}$ once more gives

$$\mathcal{P}^2(X) = -\frac{1}{\rho}\bar{g}\big(\mathcal{P}(X), e_a\big)\varepsilon^{ab}e_b = \frac{1}{\rho^2}\varepsilon^{ab}\varepsilon^{pq}g_{qa}\bar{g}\big(X, e_p\big)e_b,$$

and using the fact that $gg^{bp} = \varepsilon^{ab}\varepsilon^{qp}g_{qa}$ one obtains

$$\mathcal{P}^2(X) = -\frac{g}{\rho^2}\bar{g}(X, e_p)g^{pb}e_b,$$

which proves the statement. □

Since $\mathcal{P}$ and $\mathcal{S}_A$ can be restricted to $T\Sigma$, one may consider components both of the type $\mathcal{P}^i_j$ and of the type $\mathcal{P}^a_b$. Therefore, there are two possible ways of defining a trace, namely

$$\begin{aligned}\operatorname{Tr}\mathcal{P} &= \mathcal{P}^i_i,\\ \operatorname{tr}\mathcal{P} &= \mathcal{P}^a_a.\end{aligned}$$

Proposition 3.2. *With W_A denoting the Weingarten maps it holds that*

$$\operatorname{Tr}\mathcal{B}_A = \operatorname{tr}\mathcal{B}_A = \operatorname{Tr}\mathcal{A}_A = \operatorname{tr}\mathcal{A}_A = \frac{g}{\rho^2}\operatorname{tr}W_A$$
$$\operatorname{Tr}\mathcal{B}_A^2 = \operatorname{tr}\mathcal{B}_A^2 = \operatorname{Tr}\mathcal{A}_A^2 = \operatorname{tr}\mathcal{A}_A^2 = \frac{g}{\rho^2}\operatorname{tr}W_A^2.$$

It turns out that the map $\mathcal{B}_A$, when restricted to $T\Sigma$, is actually proportional to the Weingarten map.

Proposition 3.3. *For $X \in TM$ it holds that*

$$\mathcal{B}_A(X) = -\frac{g}{\rho^2}\bar{g}\big(X, \bar{\nabla}_a N_A\big)g^{ab}e_b,$$

and, in particular, if $Y \in T\Sigma$ then

$$\mathcal{B}_A(Y) = \frac{g}{\rho^2}W_A(Y).$$

Proof. Using the results in Proposition 3.1 one computes

$$\mathcal{B}_A(X) = \mathcal{P}\Big(-\frac{1}{\rho}\bar{g}(X, \bar{\nabla}_a N_A)\varepsilon^{ab}e_b\Big) = \frac{1}{\rho^2}\varepsilon^{ab}\varepsilon^{pq}g_{bp}\bar{g}\big(X, \bar{\nabla}_a N_A\big)e_q$$
$$= -\frac{g}{\rho^2}\bar{g}\big(X, \bar{\nabla}_a N_A\big)g^{aq}e_q.$$

Let us take $Y \in T\Sigma$ and write $Y = Y^c e_c$

$$\mathcal{B}_A(Y) = -\frac{g}{\rho^2}Y^c\bar{g}(e_c, \bar{\nabla}_a N_A)g^{ab}e_b = \frac{g}{\rho^2}Y^c h_{A,ca}g^{ab}e_b$$
$$= \frac{g}{\rho^2}Y^c(W_A)^b_c e_b = \frac{g}{\rho^2}W_A(Y). \qquad \square$$

The trace of the squares of $\mathcal{P}$ and $\mathcal{S}_A$ also contain geometric information, as shown in the next Proposition.

Proposition 3.4. *For the maps $\mathcal{P}$ and $\mathcal{S}_A$ it holds that*

$$\operatorname{Tr}\mathcal{S}_A^2 = \operatorname{tr}\mathcal{S}_A^2 = -\frac{2}{\rho^2}\det(h_{A,ab}) \tag{3.9}$$

$$\operatorname{Tr}\mathcal{P}^2 = \operatorname{tr}\mathcal{P}^2 = -2\frac{g}{\rho^2}. \tag{3.10}$$

Proof. We provide proofs for the statements involving tr. It is easy to see that the components of $\mathcal{S}_A$, when restricted to $T\Sigma$, are

$$(\mathcal{S}_A)^a_b = -\frac{1}{\rho}\varepsilon^{ac}h_{A,cb},$$

which implies that

$$\operatorname{tr}\mathcal{S}_A^2 = (\mathcal{S}_A)^a_c(\mathcal{S}_A)^c_a = \frac{1}{\rho^2}\varepsilon^{ap}h_{A,pc}\varepsilon^{cq}h_{A,qa} = -\frac{2}{\rho^2}\det(h_A).$$

Similarly, one finds that $\mathcal{P}^a_b = -g_{bc}\varepsilon^{ca}/\rho$, which implies that

$$\operatorname{tr}\mathcal{P}^2 = \mathcal{P}^a_b\mathcal{P}^b_a = \frac{1}{\rho^2}\varepsilon^{ca}\varepsilon^{pb}g_{bc}g_{ap} = -2\frac{g}{\rho^2}. \qquad \square$$

From (2.8) it now follows that one can compute the Gaussian curvature in terms of traces of $\mathcal{S}_A^2$. We collect this result, together with the expression for the mean curvature vector, in the following theorem.

Theorem 3.5. *Let K denote the Gaussian curvature and let H denote the mean curvature vector of Σ. Then*

$$K = \frac{1}{g}\bar{g}\big(\bar{R}(e_1,e_2)e_2,e_1\big) - \frac{\rho^2}{2g}\sum_{A=1}^{p}\operatorname{Tr}\mathcal{S}_A^2, \tag{3.11}$$

$$H = \frac{\rho^2}{2g}\sum_{A=1}^{p}\big(\operatorname{Tr}\mathcal{B}_A\big)N_A. \tag{3.12}$$

Note that when $M=\mathbb{R}^m$ the above expressions become

$$K = -\frac{\rho^2}{2g}\sum_{A=1}^{p}\sum_{i,j=1}^{m}\{x^i,n_A^j\}\{x^j,n_A^i\} \tag{3.13}$$

$$H = \frac{\rho^2}{2g}\sum_{A=1}^{p}\sum_{i,j,k=1}^{m}\{x^i,x^j\}\{x^j,n_A^i\}n_A^k\partial_k. \tag{3.14}$$

Coming back to Weingarten's formula for the covariant derivative of a normal vector

$$\bar{\nabla}_X N_A = -W_A(X) + D_X N_A,$$

we have shown that the Weingarten maps W_A can be written in terms of $\mathcal{B}_A$. One may ask the question if there is a relation also between $D_X N_A$ and $\mathcal{B}_A$? Surprisingly, there is such a relation. Namely, let

$$(D_X)_{AB} = \bar{g}\big(N_A, D_X N_B\big) \tag{3.15}$$

denote the components of the covariant derivative (in the direction of $X\in T\Sigma$) in the normal space with respect to the basis $N_1,\ldots,N_p$. Then one has the following result.

Proposition 3.6. *Let D denote the covariant derivative in the normal space. Then for all $X\in T\Sigma$ it holds that*

$$\bar{g}\big(\mathcal{B}_A(N_B),X\big) = \frac{g}{\rho^2}(D_X)_{AB},$$

where $(D_X)_{AB}$ denotes the components of D_X relative to the basis $N_1,\ldots,N_p$.

Proof. For a vector $X=X^a e_a$, it follows from Weingarten's formula (2.2) that

$$(D_X)_{BA} = \bar{g}\big(N_B,\bar{\nabla}_X N_A\big) = X^a\bar{g}(N_B,\bar{\nabla}_a N_A).$$

On the other hand, with the formula from Proposition 3.3, one computes

$$\begin{aligned}\bar{g}\big(\mathcal{B}_A(N_B),X\big) &= -\frac{g}{\rho^2}\bar{g}\big(N_B,\bar{\nabla}_a N_A\big)g^{ab}g_{bc}X^c = -\frac{g}{\rho^2}\bar{g}\big(N_B,\bar{\nabla}_a N_A\big)X^a\\ &= -\frac{g}{\rho^2}(D_X)_{BA} = \frac{g}{\rho^2}(D_X)_{AB}.\end{aligned}$$

The last equality is due to the fact that D is a covariant derivative, which implies that $0 = D_X\bar{g}(N_A,N_B) = \bar{g}(D_X N_A,N_B) + \bar{g}(N_A,D_X N_B)$. □

Hence, it is possible to state Weingarten's formula entirely in terms of $\mathcal{B}_A$.

Theorem 3.7. *Let $N_1,\ldots,N_p$ be an orthonormal basis of the normal space. For all $X\in T\Sigma$ it holds that*

$$\frac{g}{\rho^2}\bar{\nabla}_X N_A = -\mathcal{B}_A(X) + \sum_{B=1}^{p}\bar{g}\big(\mathcal{B}_A(N_B),X\big)N_B, \tag{3.16}$$

for $A = 1, \dots, p$.

Let us also note that since $\alpha(X,Y) = X^a g_{ac}(W_A)^c_b Y^b$, we can also rewrite Gauss' formula as

$$\nabla_X Y = \bar{\nabla}_X Y - \frac{\rho^2}{g}\sum_{A=1}^{p}\bar{g}\big(\mathcal{B}_A(X), Y\big)N_A. \tag{3.17}$$

In the particular case when $\rho = \sqrt{g}$ all formulas simplify, and the most important ones become

$$K = \frac{1}{g}\bar{g}\big(\bar{R}(e_1,e_2)e_2,e_1\big) - \frac{1}{2}\sum_{A=1}^{p}\operatorname{Tr}\mathcal{S}_A^2,$$

$$H = \frac{1}{2}\sum_{A=1}^{p}\big(\operatorname{Tr}\mathcal{B}_A\big)N_A,$$

$$\bar{\nabla}_X N_A = -\mathcal{B}_A(X) + \sum_{B=1}^{p}\bar{g}\big(\mathcal{B}_A(N_B), X\big)N_B$$

$$\nabla_X Y = \bar{\nabla}_X Y - \sum_{A=1}^{p}\bar{g}\big(\mathcal{B}_A(X), Y\big)N_A.$$

Note that if the ambient manifold M is pseudo-Riemannian, corresponding formulas can be worked out when the induced metric on Σ is non-degenerate.

4. Complex structures and the construction of normal vectors

To every Riemannian metric on Σ one can associate an almost complex structure $\mathcal{J}$ through the formula

$$\mathcal{J}(X) = \frac{1}{\sqrt{g}}\varepsilon^{ac}g_{cb}X^b e_a,$$

and since on a two dimensional manifold any almost complex structure is integrable, $\mathcal{J}$ is a complex structure on Σ. It follows immediately from Proposition 3.1 that this complex structure can be expressed in terms of $\mathcal{P}$.

Theorem 4.1. *Defining* $\mathcal{J}_M(X) = (\rho/\sqrt{g})\mathcal{P}(X)$ *for all* $X \in TM$ *yields the following results:*

(1) $\mathcal{J}_M(X) = \mathcal{J}(X)$ *for all* $X \in T\Sigma$,
(2) $-\mathcal{J}_M^2(X) = g^{ab}\bar{g}(X, e_a)e_b$ *is the projection of* $X \in TM$ *onto* $T\Sigma$.

It is a standard result that the metric on Σ is hermitian with respect to the complex structure $\mathcal{J}$ and that the Kähler form

$$\Omega(X,Y) = g\big(X, \mathcal{J}(Y)\big), \tag{4.1}$$

induces the Poisson bracket

$$\{f,h\}_\Omega = \frac{1}{\sqrt{g}}\varepsilon^{ab}\big(\partial_a f\big)\big(\partial_b h\big) = \frac{\rho}{\sqrt{g}}\{f,h\}.$$

Note that when choosing $\rho = \sqrt{g}$ (which implies $\mathcal{J}_M = \mathcal{P}$), the Poisson bracket induced from Ω coincides with the one defined in (3.2).

Theorem 4.1 also provides an expression for the projection operator in terms of the Poisson bracket of the embedding coordinates only, with no reference to the normal vectors. Therefore, one can in principle construct p orthonormal normal vectors from $x^1, \dots, x^m$ as follows: Choose an arbitrary frame $Y_1, \dots, Y_m$ of TM and set

$$\tilde{Y}_k = Y_k + \mathcal{J}_M^2(Y_k), \tag{4.2}$$

which implies that $\tilde{Y}_k \in T\Sigma^\perp$. Then apply the Gram-Schmidt orthonormalization procedure to $\tilde{Y}_1, \dots, \tilde{Y}_m$ to obtain p orthonormal vectors in $T\Sigma^\perp$.

Another way of constructing normal vectors is obtained from the following result:

Proposition 4.2. *For any choice of $j_1, \dots, j_{p-1} \in \{1, \dots, m\}$ the vector*

$$Z_{j_1 \cdots j_{p-1}} = \frac{\rho}{2\sqrt{g(p-1)!}} \bar{g}^{ij} \varepsilon_{jklj_1 \cdots j_{p-1}} \{x^k, x^l\} \partial_i, \tag{4.3}$$

where $\varepsilon_{i_1 \cdots i_m}$ is the Levi-Civita tensor, is normal to $T\Sigma$, i.e. $\bar{g}(Z_{j_1 \cdots j_{p-1}}, e_a) = 0$ for $a = 1, 2$.

Proof. The proof consists of a straightforward computation:

$$\begin{aligned}\frac{\sqrt{g(p-1)!}}{\rho} \bar{g}(e_a, Z_{j_1 \cdots j_{p-1}}) &= \frac{1}{2} \bar{g}_{im} (\partial_a x^m) \bar{g}^{ij} \varepsilon_{jklj_1 \cdots j_{p-1}} \{x^k, x^l\} \\ &= \frac{1}{2\rho} \varepsilon_{jklj_1 \cdots j_{p-1}} \varepsilon^{bc} \big(\partial_a x^j\big)\big(\partial_b x^k\big)\big(\partial_c x^l\big) = 0,\end{aligned}$$

since a, b, c can only take on two different values and $\big(\partial_a x^j\big)\big(\partial_b x^k\big)\big(\partial_c x^l\big)$ is contracted with $\varepsilon_{jklj_1 \cdots j_{p-1}}$, which is antisymmetric in j, k and l. □

In general, $Z_{j_1 \cdots j_{p-1}}$ defines $p(p+1)(p+2)/6$ normal vectors, and one can again apply the Gram-Schmidt orthonormalization procedure to extract p orthonormal vectors in $T\Sigma^\perp$. However, it turns out that one can use $Z_{j_1 \cdots j_{p-1}}$ to construct another set of normal vectors, avoiding explicit use of the Gram-Schmidt procedure, as follows: Let I, J, L denote multi-indices of length $p-1$, i.e. $I = i_1 \cdots i_{p-1}$, and writing $Z_{i_1 \cdots i_{p-1}} \equiv Z_I$ we introduce

$$\mathcal{Z}_I^J = \bar{g}_{ij} Z_I^i Z^{Jj}. \tag{4.4}$$

Considered as a matrix over multi-indices, $\mathcal{Z}$ is symmetric and we let E_I, μ_I denote orthonormal eigenvectors and their corresponding eigenvalues. Using the eigenvectors to define

$$\hat{N}_I = E_I^J Z_J, \tag{4.5}$$

one finds that $\bar{g}(\hat{N}_I, \hat{N}_J) = \mu_I \delta_{IJ}$, i.e. the vectors are orthogonal.

Lemma 4.3. *For $\mathcal{Z}_I^J$ defined as above, it holds that*

$$\sum_L \mathcal{Z}_I^L \mathcal{Z}_L^J = \mathcal{Z}_I^J \tag{4.6}$$

$$\sum_I \mathcal{Z}_I^I = p. \tag{4.7}$$

Proof. Both (4.6) and (4.7) can easily be proven once one has the following result

$$Z_K^i Z^{jK} = \bar{g}^{ij} + \frac{\rho^2}{g} \big(\mathcal{P}^2\big)^{ij}. \tag{4.8}$$

The above formula is proven using that

$$\begin{aligned}\varepsilon_{ijkK}\varepsilon^{lmnK} &= (p-1)!\big(\delta^{[l}_{[i}\delta^{m}_{j}\delta^{n]}_{k]}\big)\\ &= (p-1)!\Big[\delta^l_i\delta^m_j\delta^n_k - \delta^l_i\delta^m_k\delta^n_j - \delta^l_j\delta^m_i\delta^n_k + \delta^l_j\delta^m_k\delta^n_i - \delta^l_k\delta^m_j\delta^n_i + \delta^l_k\delta^m_i\delta^n_j\Big],\end{aligned}$$

which gives

$$Z^i_K Z^{jK} = \frac{\rho^2}{g}\Big[-\frac{1}{2}\big(\operatorname{Tr}\mathcal{P}^2\big)\bar{g}^{ij} + \big(\mathcal{P}^2\big)^{ij}\Big] = \bar{g}^{ij} + \frac{\rho^2}{g}\big(\mathcal{P}^2\big)^{ij}.$$

Formula (4.7) is now immediate, and to obtain (4.6) one simply notes that since $Z_J \in T\Sigma^\perp$ and $\mathcal{P}^2$ projects onto $T\Sigma$, it holds that $\mathcal{P}^2(Z_J) = 0$. □

From Lemma 4.3 it follows that an eigenvalue of $\mathcal{Z}$ is either 0 or 1, which implies that $\hat{N}_I = 0$ or $\bar{g}(\hat{N}_I, \hat{N}_I) = 1$, and that the number of non-zero vectors is $\operatorname{Tr}\mathcal{Z} = \mathcal{Z}^I_I = p$. Hence, the p non-zero vectors among $\hat{N}_I$ constitute an orthonormal basis of $T\Sigma^\perp$, and it follows that one can replace any sum over two normal vectors N_A by a contraction of $\hat{N}_I$ with $\hat{N}^I$. As an example, let us work out explicit formulas for the Gaussian curvature and the mean curvature vector in the case when $M = \mathbb{R}^m$. For that, it is convenient to have the following lemma at hand.

Lemma 4.4. *Defining*

$$\mathcal{S}^{ij}(X) = \frac{1}{\rho}\varepsilon^{ab}\big(\partial_a x^i\big)\big(\bar{\nabla}_b X\big)$$

for $X \in TM$ it holds that

$$\operatorname{Tr}\mathcal{S}(fN)\mathcal{S}(hN') \equiv \mathcal{S}^i_j(fN)\mathcal{S}^j_i(hN') = fh\operatorname{Tr}\mathcal{S}(N)\mathcal{S}(N')$$

for all $N, N' \in T\Sigma^\perp$ and $f, h \in C^\infty(M)$.

Proof. Using $\bar{\nabla}_b fN = (\partial_b f)N + f\bar{\nabla}_b N$ one obtains

$$\begin{aligned}\mathcal{S}^i_j(fN)\mathcal{S}^j_i(hN') &= fh\mathcal{S}^i_j(N)\mathcal{S}^j_i(N')\\ &\quad + \frac{1}{\rho^2}\varepsilon^{ab}\varepsilon^{pq}\big(\partial_a x^i\big)\big(\partial_b f\big)N^j\bar{g}_{jk}\big(\partial_p x^k\big)h\big(\bar{\nabla}_q N'\big)^l\bar{g}_{li}\\ &\quad + \frac{1}{\rho^2}\varepsilon^{ab}\varepsilon^{pq}\big(\partial_a x^i\big)f\big(\bar{\nabla}_b N\big)^j\bar{g}_{jk}\big(\partial_p x^k\big)\big(\partial_q h\big)N'^l\bar{g}_{li}\\ &\quad + \frac{1}{\rho^2}\varepsilon^{ab}\varepsilon^{pq}\big(\partial_a x^i\big)\big(\partial_b f\big)N^j\bar{g}_{jk}\big(\partial_p x^k\big)\big(\partial_q h\big)N'^l\bar{g}_{li}\\ &= fh\mathcal{S}^i_j(N)\mathcal{S}^j_i(N')\end{aligned}$$

as $N^j\bar{g}_{jk}\big(\partial_p x^k\big) = N'^l\bar{g}_{li}\big(\partial_a x^i\big) = 0$ since $N, N' \in T\Sigma^\perp$. □

Proposition 4.5. *If $M = \mathbb{R}^m$ then*

$$K = -\frac{\rho^4}{8g^2(p-1)!}\sum \varepsilon_{jklI}\varepsilon_{imnI}\{x^i,\{x^k,x^l\}\}\{x^j,\{x^m,x^n\}\} \tag{4.9}$$

$$H = \frac{\rho^4}{8g^2(p-1)!}\sum \varepsilon_{iklI}\varepsilon_{k'mnI}\{x^i,x^j\}\{x^j,\{x^k,x^l\}\}\{x^m,x^n\}\partial_{k'}. \tag{4.10}$$

Proof. When $M = \mathbb{R}^m$ we one use equation (3.13) to write

$$K = -\frac{\rho^2}{2g}\sum_{A=1}^{p}\sum_{i,j=1}^{m}\{x^i, n_A^j\}\{x^j, n_A^i\} = -\frac{\rho^2}{2g}\sum_{I}\sum_{i,j=1}^{m}\{x^i, \hat{N}_I^j\}\{x^j, \hat{N}_I^i\}$$

since exactly p of the vectors $\hat{N}_I$ are non-zero and orthonormal. Inserting the definition of $\hat{N}_I$ into the above expression and using Lemma 4.4, together with the orthonormality of the eigenvectors E_I, yields

$$\begin{aligned} K &= -\frac{\rho^2}{2g}\sum_{I,J,L}\sum_{i,j=1}^{m}\{x^i, E_I^J Z_J^j\}\{x^j, E_I^L Z_L^i\} \\ &= -\frac{\rho^2}{2g}\sum_{I,J,L}E_I^J E_I^L\sum_{i,j=1}^{m}\{x^i, Z_J^j\}\{x^j, Z_L^i\} \\ &= -\frac{\rho^2}{2g}\sum_{J}\sum_{i,j=1}^{m}\{x^i, Z_J^j\}\{x^j, Z_J^i\} \end{aligned}$$

which, by inserting the definition of Z_I and again using Lemma 4.4, becomes

$$K = -\frac{\rho^4}{8g^2(p-1)!}\sum \varepsilon_{jklI}\varepsilon_{imnI}\{x^i, \{x^j, x^k\}\}\{x^j, \{x^m, x^n\}\}.$$

The formula for H can be proven analogously. □

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