Future asymptotics of tilted Bianchi type II cosmologies

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Abstract

In this paper we study the future asymptotics of spatially homogeneous Bianchi type II cosmologies with a tilted perfect fluid with a linear equation of state. By means of Hamiltonian methods we first find a monotone function for a special tilted case, which subsequently allows us to construct a new set of monotone functions for the general tilted type II cosmologies. In the context of a new partially gauge invariant dynamical system, this then leads to a proof for a theorem that for the first time gives a complete description of the future asymptotic states of the general tilted Bianchi type II models. The generality of our arguments suggests how one can produce monotone functions that are useful for determining the asymptotics of other tilted perfect fluid cosmologies, as well as for other sources.

1   Introduction

Spatially homogeneous anisotropic perfect fluid models have during the last decades been successfully studied using a dynamical systems approach. The book [1] summarizes most of the presently known results about the so-called non-tilted perfect fluid cosmologies, while the more general 'tilted' perfect fluid models have been primarily investigated more recently [2]—[14].

In all of the papers investigating tilted models, the analysis has rested on standard techniques from dynamical systems theory. Most of the results concern the identification of fixed points and a subsequent linear stability analysis of these points. In order to get a grip on the global aspects of the solutions, an effective tool is the use of monotone functions. However, such monotone functions are hard to find, and in most of the previous works on tilted models the monotone functions were obtained by brute force, trial and error, and luck. It would therefore be desirable to have a more systematic method to seek and find such monotone functions.

For non-tilted spatially homogeneous perfect fluid models, virtually all known results crucially rely on the existence of conserved quantities and monotone functions. These have turned out to be connected to the existence of certain symmetries, intimately associated with conservation laws such as the preservation of the number of particles in a fluid element, and the so-called scale-automorphism group [15]. Although not necessary, the symmetries and associated structures were, to a large extent, found by means of Hamiltonian techniques, see ch. 10 in [1] and [15]. One aim of this paper is to illustrate that one can fruitfully use similar methods that previously have been applied to non-tilted models to tilted ones. To do so, we will consider an example—the tilted Bianchi type II models.

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The tilted perfect fluid Bianchi type II models have been analyzed before as a dynamical system in [5]. In that paper, as well as in the present one, the perfect fluid was assumed to obey a linear equation of state characterized by $\tilde{p}/\tilde{\rho} = w = \text{const}$, where $\tilde{p}$ and $\tilde{\rho}$ are the pressure and energy density with respect to the rest frame of the fluid, respectively; special cases of interest are dust, $w = 0$, radiation, $w = \frac{1}{3}$, a stiff fluid, $w = 1$, while a cosmological constant Λ can formally be regarded as a perfect fluid with $w = -1$, however, in this paper we will consider the range $-1 < w < 1$.

In [5] the fixed points of the system were found and their linear stability properties were studied. It was observed that the future stability of the fixed points depended on the equation of state parameter w, but that a future stable fixed point existed for all w in the range $-1 < w < 1$ under consideration. On the basis of the linear analysis and numerical computations it was conjectured that the future linearly stable fixed points were global attractors in the full state space. This conjecture was corroborated by means of monotone functions for some ranges of w. However, the complete picture was not fully drawn, mainly because of the lack of a set of sufficiently restrictive monotone functions. In this paper we will use methods that previously have been applied to the non-tilted models and find a set of new monotone functions that are sufficiently restrictive to determine the future asymptotics of all tilted solutions, thus filling in the missing gaps in the conclusions drawn in [5].

The outline of the paper is as follows. In Section 2 we give a new partially gauge invariant dynamical system for the general tilted Bianchi type II models, derived in Appendix A. Then, based on the structure of monotone functions obtained for more special models by means of Hamiltonian methods, presented in Appendix B, we obtain the new monotone functions in Section 3, which leads to a new theorem that describes the future asymptotic states of all tilted Bianchi type II cosmologies. Finally, in Section 4 we conclude with a discussion about the structure of the monotone functions, why they exist, and why one can hope to expect similar structures in other models.

2 Dynamical systems description of tilted Bianchi type II cosmologies

In [16] the so-called conformally Hubble-normalized orthonormal frame equations are given in full generality. These are in turn specialized to the spatially homogeneous Bianchi case in Appendix A and then to the presently studied Bianchi type II models with a general tilted perfect fluid with a linear equation of state. Then a subsequent set of new variables, invariant under frame rotations in the 23-plane (see Appendix A), yield the following state vector and dynamical system.

State vector:
$$\mathbf{S} = (\Sigma_+, \bar{\Sigma}, \tilde{\Sigma}^2, \check{\Sigma}^2, \Omega_{\mathrm{k}}, v^2), \tag{1}$$

where we treat $\tilde{\Sigma}^2$, $\check{\Sigma}^2$, and v^2 as variables, but where we have refrained from giving them new names.

Evolution equations:

$$\Sigma'_+ = -(2-q)\Sigma_+ - 3\check{\Sigma}^2 + 4\Omega_{\mathrm{k}} + \tfrac{1}{2}(1+w)G_+^{-1}v^2\Omega, \tag{2a}$$

$$\bar{\Sigma}' = -(2-q)\bar{\Sigma} - 2\sqrt{3}\tilde{\Sigma}^2 + \sqrt{3}\check{\Sigma}^2 + \tfrac{\sqrt{3}}{2}(1+w)G_+^{-1}v^2\Omega, \tag{2b}$$

$$(\tilde{\Sigma}^2)' = -2(2-q-2\sqrt{3}\bar{\Sigma})\tilde{\Sigma}^2, \tag{2c}$$

$$(\check{\Sigma}^2)' = -2[2-q-3\Sigma_+ + \sqrt{3}\bar{\Sigma}]\check{\Sigma}^2, \tag{2d}$$

$$\Omega'_{\mathrm{k}} = 2(q-4\Sigma_+)\Omega_{\mathrm{k}}, \tag{2e}$$

$$(v^2)' = 2G_-^{-1}(1-v^2)[3w-1-\Sigma_+ - \sqrt{3}\bar{\Sigma}]v^2. \tag{2f}$$

Constraint equation:
$$f(\mathbf{S}) = 4\check{\Sigma}^2\Omega_{\mathrm{k}} - (1+w)^2 G_+^{-2}v^2\Omega^2 = 0. \tag{2g}$$

In the above equations Ω is given by the Gauss constraint

$$\Omega = 1 - \Sigma^2 - \Omega_{\mathrm{k}}, \tag{3}$$

where

$$\Sigma^2 = \Sigma_+^2 + \bar{\Sigma}^2 + \tilde{\Sigma}^2 + \check{\Sigma}^2 \,. \tag{4}$$

The condition $\Omega \geq 0$ in combination with (3) yields $0 \leq \Sigma_+^2 + \bar{\Sigma}^2 + \tilde{\Sigma}^2 + \check{\Sigma}^2 + \Omega_{\mathrm{k}} \leq 1$. The deceleration parameter q is given by

$$q = 2\Sigma^2 + \tfrac{1}{2}G_+^{-1}[1 + 3w + (1 - w)v^2]\Omega \,, \tag{5}$$

while

$$G_\pm = 1 \pm wv^2 \,; \tag{6}$$

finally, $'$ denotes differentiation with respect to a dimensionless time parameter τ, determined by $d\tau = H\,dt$, where t is the clock time along the congruence normal to the spatially homogeneous hypersurfaces.

We now give a brief description of the invariant subsets and fixed points of the system (2), which is analogous to the analysis given by Hewitt et al. [5]. Note that although our system is invariant under frame rotations in the 23-plane it is not invariant under all rotations. Hence there exist multiple representations of solutions, for further comments on this, see [5].

	Name	Restrictions	Dimension	Interior/Boundary
i)	Non-tilted non-vacuum Bianchi type II	$v^2 = \check{\Sigma}^2 = 0$	4	Boundary
ii)	Non-tilted non-vacuum Bianchi type I	$v^2 = \Omega_{\mathrm{k}} = 0$	4	Boundary
iii)	Vacuum Bianchi type II	$\check{\Sigma}^2 = \Omega = 0$	4	Boundary
iv)	Vacuum Bianchi type I (Kasner)	$\Omega_{\mathrm{k}} = \Omega = 0$	4	Boundary
v)	Extreme tilt	$v^2 = 1$	4	Boundary
vi)	Orthogonally transitive Bianchi type II	$\tilde{\Sigma}^2 = 0$	4	Interior

Table 1: Invariant sets of the state space. The last column indicates if a subset is part of the boundary of the general tilted Bianchi type II models or if it is an interior subset.

Fixed points:

i) The flat Friedmann solution, F: $-1 < w < 1$,

$$\Sigma_+ = \bar{\Sigma} = \tilde{\Sigma}^2 = \check{\Sigma}^2 = \Omega_{\mathrm{k}} = v^2 = 0 \,.$$

ii) The Collins-Stewart solution, CS: $-\tfrac{1}{3} < w < 1$,

$$\Sigma_+ = \frac{1}{8}(3w + 1)\,, \quad \bar{\Sigma} = \tilde{\Sigma}^2 = \check{\Sigma}^2 = v^2 = 0\,, \quad \Omega_{\mathrm{k}} = \frac{3}{64}(3w + 1)(1 - w)\,.$$

iii) Hewitt's solution, H: $\tfrac{3}{7} < w < 1$,

$$\Sigma_+ = \frac{1}{8}(3w + 1)\,, \quad \bar{\Sigma} = \frac{\sqrt{3}}{8}(7w - 3)\,, \quad \tilde{\Sigma}^2 = 0\,, \quad \check{\Sigma}^2 = \frac{3(1 - w)(11w + 1)(7w - 3)}{16(17w - 1)}\,,$$

$$\Omega_{\mathrm{k}} = \frac{3(1 - w)(5w + 1)(3w - 1)}{4(17w - 1)}\,, \quad v^2 = \frac{(3w - 1)(7w - 3)}{(11w + 1)(5w + 1)}\,.$$

iv) Hewitt et al.'s 1-parameter set of solutions,[1] HL: $w = \tfrac{5}{9}$, $0 < b = \mathrm{const} < 1$,

$$\Sigma_+ = \frac{1}{3}\,, \quad \bar{\Sigma} = \frac{1}{3\sqrt{3}}\,, \quad \tilde{\Sigma}^2 = \frac{4}{27}\,b\,, \quad \check{\Sigma}^2 = \frac{4(4b + 1)(8 - 3b)}{513}\,,$$

$$\Omega_{\mathrm{k}} = \frac{(2b + 1)(17 - 8b)}{171}\,, \quad v^2 = \frac{3(4b + 1)(2b + 1)}{(17 - 8b)(8 - 3b)}\,.$$

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[1]There is a misprint in [5] where the square root on the b in the Σ_1 expression of the line of fixed points, HL, has disappeared.

v) Hewitt et al.'s extreme tilted point, Het: $-1 < w < 1$,

$$\Sigma_+ = \frac{1}{3}, \quad \bar{\Sigma} = \frac{1}{3\sqrt{3}}, \quad \tilde{\Sigma}^2 = \frac{4}{27}, \quad \check{\Sigma}^2 = \frac{100}{513}, \quad \Omega_{\mathrm{k}} = \frac{3}{19}, \quad v^2 = 1.$$

The system also admits the following fixed point sets: *the Kasner circle* K°, for which $v^2 = 0$, *the Kasner lines* $\mathrm{KL}_{\pm}$, for which $v^2 = $ const, and *the extremely tilted Kasner circle* Ket°, for which $v^2 = 1$. These subsets reside on the Bianchi type I vacuum boundary, i.e., $\Sigma^2 = 1$, with $\tilde{\Sigma} = \check{\Sigma} = 0$, see [5], however, since these fixed points do not play a prominent role in this paper we refrain from giving them explicitly.

Remark. Below we will refer to the relevant fixed point values for Σ_+ and $\bar{\Sigma}$ by Σ_{+0} and $\bar{\Sigma}_0$, respectively.

3   Future asymptotes in tilted Bianchi type II cosmology

In what follows certain monotone functions will play a crucial role. Based on our results in Appendix A and B, we hence begin by deriving them.

3.1   Monotone functions

There are several auxiliary equations that are useful in the context of monotonic functions, see Appendix A:

Auxiliary equations:

$$\Omega' = [2q - (1 + 3w) + (1 + w)(3w - 1 - \Sigma_+ - \sqrt{3}\bar{\Sigma})G_+^{-1}v^2]\Omega, \tag{7a}$$

$$Q' = -[2(1 - q) + \Sigma_+ + \sqrt{3}\bar{\Sigma}]Q, \tag{7b}$$

$$\Psi' = [2q - (1 + 3w)]\Psi, \tag{7c}$$

where

$$Q = (1 + w)G_+ - 1\, v\, \Omega, \qquad \Psi = \Gamma^{-(1-w)}\, G_+^{-1}\, \Omega. \tag{8}$$

Since

$$2q - (1 + 3w) = 4\Sigma^2 - (1 + 3w)(1 - \Omega) + (1 - 3w)(1 + w)G_+^{-1}v^2\Omega \geq 0, \quad \text{if} -1 < w \leq -1/3, \tag{9}$$

as follows from (47), Ψ is a monotonically increasing function when $-1 < w \leq -1/3$; henceforth we denote Ψ in this interval of the equation of state by M_{F}.

Before we continue, let us introduce some notation:

$$\phi_* = 1 - \Sigma_{+0}\Sigma_+ - \bar{\Sigma}_0\bar{\Sigma}, \tag{10a}$$

$$\varphi_* = [\Sigma_{+0}(\Sigma_+ - \Sigma_{+0}) + \bar{\Sigma}_0(\bar{\Sigma} - \bar{\Sigma}_0)]^2, \tag{10b}$$

$$\bar{\varphi}_* = [\bar{\Sigma}_0(\Sigma_+ - \Sigma_{+0}) - \Sigma_{+0}(\bar{\Sigma} - \bar{\Sigma}_0)]^2, \tag{10c}$$

where the subscript $*$ henceforth denotes a specific fixed point, while Σ_{+0} and $\bar{\Sigma}_0$ are the associated fixed point values for Σ_+ and $\bar{\Sigma}$, respectively. In the following it is important that $\phi_* > 0$, which can be seen as follows:

$$\phi_* = \tfrac{1}{2}\left[1 - \Sigma_{+0}^2 - \bar{\Sigma}_0^2 + 1 - \Sigma_+^2 - \bar{\Sigma}^2 + (\Sigma_+ - \Sigma_{+0})^2 + (\bar{\Sigma} - \bar{\Sigma}_0)^2\right] > \tfrac{1}{2}\left[1 - \Sigma_{+0}^2 - \bar{\Sigma}_0^2\right] > 0, \tag{11}$$

where we have used the Gauss constraint (3) and $\Omega > 0$, $\Omega_{\mathrm{k}} > 0$.

For non-tilted perfect fluid models

$$M_{\mathrm{CS}:\mathrm{v}^2=0} = \phi_{\mathrm{CS}}^{-2}\,\Omega_k^m\,\Omega^{1-m} = \frac{\Omega_k^m\,\Omega^{1-m}}{(1 - \Sigma_{+0}\Sigma_+)^2}; \quad m = \frac{3(1 - w)\Sigma_{+0}}{8(1 - \Sigma_{+0}^2)}, \quad \Sigma_{+0} = \frac{1}{8}(1 + 3w), \quad \bar{\Sigma}_0 = 0, \tag{12}$$

is a monotonically increasing function. However, it is of interest to generalize this function by replacing Ω with Ψ, which is equal to Ω in the non-tilted case, i.e.,

$$M_{\mathrm{CS}} = \phi_{\mathrm{CS}}^{-2}\,\Omega_k^m\,\Psi^{1-m}, \tag{13}$$

which leads to the following time derivative in the present fully tilted state space:

$$(\ln M_{\mathrm{CS}})' = 3\phi_{\mathrm{CS}}^{-1}\left[(1-w)\left(\frac{\varphi_{\mathrm{CS}}}{(1-\Sigma_{+0}^2)\Sigma_{+0}^2} + \frac{\bar{\varphi}_{\mathrm{CS}}}{\Sigma_{+0}^2} + \tilde{\Sigma}^2\right) + \frac{1}{8}(3-7w)(2\check{\Sigma}^2 + (1+w)G_+^{-1}v^2\Omega)\right], \quad (14)$$

and hence M_{CS} is monotonically increasing when $-1/3 < w \le 3/7$.

In Appendix B we derive a monotonic function for the tilted orthogonally transitive case $\tilde{\Sigma}^2 = 0$ which can be written as

$$M_{\mathrm{H}} = \phi_{\mathrm{H}}^{-(3+13w)}(\check{\Sigma}^2)^{\frac{7w-3}{2}}\Omega_{\mathrm{k}}^{3w-1}\Psi^4. \quad (15)$$

It sometimes turns out to be the case that a monotone function for a given state space is also monotone in a more general state space in which the original is embedded in, at least for a limited range of the equation of state parameter, see e.g. [10, 15]. We hence compute the time derivative for M_{H} in the full tilted case; this gives us

$$(\ln M_{\mathrm{H}})' = \phi_{\mathrm{H}}^{-1}\left[\frac{49(16\varphi_{\mathrm{H}} + 3(1-w)(3+13w)\bar{\varphi}_{\mathrm{H}})}{8 + \frac{3}{8}(91w-31)(7w-3)} + \frac{3}{4}(5-9w)(13w+3)\tilde{\Sigma}^2\right], \quad (16)$$

and hence M_{H} is monotonically increasing when $3/7 < w \le 5/9$.

The above monotonic functions all have the form

$$M_* = \phi_*^{-\beta}(\tilde{\Sigma}^2)^{\alpha_1}(\check{\Sigma}^2)^{\alpha_2}\Omega_{\mathrm{k}}^{\alpha_3}(\Psi)^{\alpha_4}, \quad (17)$$

where $\beta = 2(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4)$, and $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \beta \ge 0$. For the individual cases we have

$$M_{\mathrm{F}}: \quad \alpha_1 = \alpha_2 = \alpha_3 = 0, \ \alpha_4 = 1, \ \beta = 2, \quad (18)$$

$$M_{\mathrm{CS}}: \quad \alpha_1 = \alpha_2 = 0, \ \alpha_3 = m, \ \alpha_4 = 1-m, \ \beta = 2, \quad (19)$$

$$M_{\mathrm{H}}: \quad \alpha_1 = 0, \ \alpha_2 = \tfrac{1}{2}(7w-3), \ \alpha_3 = 3w-1, \ \alpha_4 = 4, \ \beta = 3+13w. \quad (20)$$

Let us assume the form (17) in order to find a monotonic function for the range $5/9 < w < 1$. We obtain

$$M_{\mathrm{Het}} = \phi_{\mathrm{Het}}^{-46}(\tilde{\Sigma}^2)^4(\check{\Sigma}^2)^{10}\Omega_{\mathrm{k}}^9, \quad (21)$$

and hence $\alpha_1 = 4, \ \alpha_2 = 10, \ \alpha_3 = 9, \ \alpha_4 = 0, \ \beta = 46$, which leads to

$$(\ln M_{\mathrm{Het}})' = \phi_{\mathrm{Het}}^{-1}\left[243\varphi_{\mathrm{Het}} + 207\bar{\varphi}_{\mathrm{Het}} + \frac{23(9w-5)}{3}G_+^{-1}(1-v^2)\Omega\right], \quad (22)$$

and thus M_{Het} is monotonically increasing when $5/9 < w < 1$.

3.2 Future asymptotic limits

Let us denote the invariant set for the general tilted Bianchi type case, for which $(1-v^2)\,v^2\,\tilde{\Sigma}^2\,\check{\Sigma}^2\,\Omega\,\Omega_{\mathrm{k}} \ne 0$, by $\mathcal{S}_{\mathrm{Gen}}$, while we denote the orthogonally transitive case, for which $(1-v^2)\,v^2\,\check{\Sigma}^2\,\Omega\,\Omega_{\mathrm{k}} \ne 0$, by $\mathcal{S}_{\mathrm{OT}}$.

A local analysis, as in [5], reveals that the state space **S** have fixed points as *local sinks* according to Table 2, which leads to the following bifurcation diagram:

$$\mathrm{F} \xrightarrow{w=-1/3} \mathrm{CS} \xrightarrow{w=3/7} \mathrm{H} \xrightarrow{w=5/9,\,b=0} \mathrm{HL} \xrightarrow{w=5/9,\,b=1} \mathrm{Het}. \quad (23)$$

It was conjectured in [5] that the above local results hold globally modulo a set of solutions of measure zero. Using the above monotone functions one can show that this is indeed the case, and also identify all the exceptional solutions. In the following theorem we show that *all* solutions that belong to $\mathcal{S}_{\mathrm{Gen}}$ end up asymptotically at the above sinks:

Theorem 3.1. *For all* $\mathbf{x} \in \mathcal{S}_{\mathrm{Gen}}$

$$\omega(\mathbf{x}) = \begin{cases} \mathrm{F} & -1 < w \le -1/3 \\ \mathrm{CS} & -1/3 < w \le 3/7 \\ \mathrm{H} & 3/7 < w < 5/9 \\ \mathrm{HL} & w = 5/9 \\ \mathrm{Het} & 5/9 < w < 1 \end{cases}.$$

Range of w	Sink
$-1 < w \le -1/3$	F
$-1/3 < w \le 3/7$	CS
$3/7 < w < 5/9$	H
$w = 5/9$	HL
$5/9 < w < 1$	Het

Table 2: Sinks for $\mathcal{S}_{\text{Gen}}$.

The proof makes use of the *Monotonicity Principle* [1, 17], which gives information about the global asymptotic behavior of solutions of a dynamical system. It is stated as follows: Let ϕ_τ be a flow on $\mathbb{R}^n$ with $\mathbf{X}$ being an invariant set. Furthermore, let M be a $\mathcal{C}^1$ function $M : \mathbf{X} \to \mathbb{R}$. Then if M is increasing on orbits, then for all $x \in X$

$$\omega(x) \subseteq \{ \mathbf{s} \in \overline{X} \backslash X \mid \lim_{\mathbf{y} \to \mathbf{s}} M(\mathbf{y}) \ne \inf_X M \}. \tag{24}$$

Proof. We make use that $\mathcal{S}_{\text{Gen}}$ is a relatively compact set and hence that every orbit in $\mathcal{S}_{\text{Gen}}$ has an ω-limit point in $\overline{\mathcal{S}}_{\text{Gen}}$, moreover, the ω-limit set of every orbit in $\mathcal{S}_{\text{Gen}}$ must be an invariant set. In every case $\inf_{\mathcal{S}_{\text{Gen}}} M_* = 0$, and hence we only have to investigate the set where $(\ln M_*)' = 0$. It turns out that in all cases the invariant set associated with $(\ln M_*)' = 0$ is precisely the pertinent fixed point(s), which thus is the ω-limit of every orbit in $\mathcal{S}_{\text{Gen}}$. Hence the proof, and the situations, is virtually identical to that of the non-tilted Bianchi type II perfect fluid case given in [5] on p. 151. For the cases $-1 < w \le -1/3$, $-1/3 < w \le 3/7$, $3/7 < w < 5/9$ and $w = 5/9$, $5/9 < w < 1$, one uses M_F, M_{CS}, M_H, and M_{Het}, respectively. $\qquad\square$

Remark. It follows that F, CS, H, Het attract a 4-parameter set of solutions, as does the line HL, however, in this case it follows from the reduction theorem, see e.g. [18], that each point on the line attracts a 3-parameter set since the line is transversally hyperbolic.

Corollary 3.2. *For all* $\mathbf{x} \in \mathcal{S}_{\text{OT}}$

$$\omega(\mathbf{x}) = \begin{cases} F & -1 < w \le -1/3 \\ CS & -1/3 < w \le 3/7 \\ H & 3/7 < w < 1 \end{cases}.$$

Proof. This follows immediately from the previous proof, in combination with noticing the form for $(\ln M_H)'$ when $\tilde{\Sigma}^2 = 0$. $\qquad\square$

We hence have established that the local bifurcation diagram

$$F \xrightarrow{w=-1/3} CS \xrightarrow{w=3/7} H \tag{25}$$

from [5] reflects the global features of the solution space of $\mathcal{S}_{\text{OT}}$.

4 Discussion

In this paper we have found that there exists a collection of monotonically increasing functions that completely determine and describe the future asymptotics of tilted Bianchi type II models. Furthermore, they all take the form

$$M_* = \phi_*^{-\beta} (\tilde{\Sigma}^2)^{\alpha_1} (\check{\Sigma}^2)^{\alpha_2} \Omega_{\text{k}}^{\alpha_3} (\Psi)^{\alpha_4}, \tag{26}$$

where $\beta = 2(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4)$, and α_1, α_2, α_3, α_4, $\beta \ge 0$. Moreover, the time derivatives of the monotone functions M_{CS}, M_H, M_{Het}, with nonzero $\Sigma_{+0}^2 + \overline{\Sigma}_0^2$ (and hence non-zero $\varphi_* + \bar{\varphi}_*$) all take the form

$$(\ln M_*)' = \phi_*^{-1}(A_* \varphi_* + B_* \bar{\varphi}_* + \text{Inv}_*), \tag{27}$$

where A_* and B_* are constants, and Inv_* a function that vanishes on one of the invariant sets in Table 1. Outside these invariant sets the function Inv_* is positive only in a limited range of values for w. In the M_{CS} case $\mathrm{Inv}_{\mathrm{CS}}$ is zero for the non-tilted subset for which $v^2 = \check{\Sigma}^2 = 0$; in the M_{H} case $\mathrm{Inv}_{\mathrm{H}}$ is zero for the orthogonally transitive subset for which $\tilde{\Sigma}^2 = 0$; finally, in the M_{Het} case $\mathrm{Inv}_{\mathrm{Het}}$ is zero for the extreme tilt subset for which $v^2 = 1$. Clearly it is easier to find these monotone functions on these subsets first and then extend them to the larger state space, as we did in order to find the key monotone function M_{H} in Appendix B, moreover, we also first computed M_{Het} for the extreme tilt subset $v^2 = 1$, although we did not use Hamiltonian methods in this case.[2]

The importance of the hierarchical structure of Bianchi cosmology, where we have systems with boundaries on boundaries, have been emphasized before, see e.g. [19, 20, 21] and references therein. Here we see yet another context for this observation, which suggests that one should first try to find monotone functions for subsets and then attack the case one is really interested in. Hence one should first identify subsets for a given state space and write them on the form $Z_A = 0$ and then, if there exists a locally future stable fixed point on a given subset that admits subsets $Z_a = 0$, attempt to find monotone functions of the form

$$M_* = \phi_*^{-\beta} \prod_a Z_a^{\alpha_a} . \tag{28}$$

Is there a deeper reason for why monotone functions like this should exist? The analysis of the non-tilted case in [15] suggests that the existence of these monotone functions are related to the scale-automorphism group. In the tilted case this group can be viewed as consisting of an off-diagonal special automorphism group and a diagonal scale-automorphism group. The off-diagonal special automorphism give rise to conserved momenta, if the underlying symmetry is not broken by source terms, and hence also to monotone functions, in a similar way as for the non-tilted models, see [15]. We have not discussed such monotone functions here since we did not need them for the future asymptotics, however, they could be of help for the much more difficult past asymptotic behavior. However, the off-diagonal automorphisms also have other dynamical consequences. It is because of the off-diagonal automorphisms we only had diagonal shear degrees of freedom in ϕ_* in the present case, and it is because of this one would expect the present analysis to also be of relevance for other tilted models, and for other sources. Moreover, in the present case we have encountered a hierarchy of source subsets that is typical. It is the increasing complexity of the source that breaks the vacuum symmetry group associated with the scale-automorphism group, creating a hierarchy of monotone functions associated with different source subsets. This is also what one encounters in the non-tilted case [15], however, with an increasingly complex source this phenomenon seems to be even more pronounced! Hence a systematic attempt on the tilted models, or other sources, strongly suggests a deeper investigation into the dynamical consequences for the scale-automorphism group for the various relevant subsets. This is quite ambitious task and we have therefore refrained from doing it here; instead we have made use of the structures one can expect to arise from such an analysis without deriving all the details that completely determines all the monotone functions from the scale-automorphism group (for a hint of how this can be done, see the complete analysis of the class A non-tilted models from a Hamiltonian perspective in [15]) in order to see if this is likely to be a fruitful project. The answer seems to be yes.

Acknowledgments

A   Derivation of the dynamical system

In spatially homogeneous cosmology the space-time is foliated by a geodesically parallel family of spatially homogeneous slices with a timelike unit normal vector n^a, see, e.g., [1, 22, 23] and references therein.

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[2]If one wants to use Hamiltonian methods to deal with extreme tilt, one first has to observe that these models have the same equations as those associated with a source that takes the form of a perfect fluid with a radiation equation of state $w = 1/3$, but with a null vector field replacing the timelike 4-velocity.

A.1 Perfect fluids

Splitting the total stress-energy tensor T_{ab} with respect to n^a yields:

$$T_{ab} = \rho\, n_a\, n_b + 2q_{(a}\, n_{b)} + p\, h_{ab} + \pi_{ab}\,, \tag{29a}$$

$$\rho = n^a\, n^b\, T_{ab}\,, \qquad q_a = -h_a{}^b\, n^c\, T_{bc}\,, \qquad p = \tfrac{1}{3}\, h^{ab}\, T_{ab}\,, \qquad \pi_{ab} = h_{\langle a}{}^c\, h_{b\rangle}{}^d\, T_{cd}\,, \tag{29b}$$

where $h_{ab} = n_a n_b + g_{ab}$ and $A_{\langle ab\rangle} = h_a{}^c h_b{}^d A_{cd} - \tfrac{1}{3} h_{ab} h^{cd} A_{cd}$; ρ, p is the total energy density and total effective pressure, respectively, measured in the rest space of n^a. In this paper we consider a perfect fluid, which yields the stress-energy tensor:

$$T^{ab} = (\tilde{\rho} + \tilde{p})\tilde{u}^a\tilde{u}^b + \tilde{p}g^{ab}\,, \tag{30}$$

where $\tilde{\rho}$ and $\tilde{p}$ are the energy density and pressure, respectively, in the rest frame of the fluid, while $\tilde{u}^a$ is its 4-velocity; throughout we assume that $\tilde{\rho} \geq 0$. Making a 3+1 split with respect to n^a, leads to

$$\tilde{u}^a = \Gamma(n^a + v^a)\,; \qquad n_a v^a = 0\,, \qquad \Gamma = (1 - v^2)^{-1/2}\,, \tag{31}$$

where v^a is the three-velocity of the fluid, also known as the tilt vector; this gives

$$\tilde{\rho} = \Gamma^{-2}\, G_+^{-1}\, \rho\,, \quad q^a = (1+w)G_+^{-1}\, \rho\, v^a\,, \quad p = w\rho + \tfrac{1}{3}(1 - 3w)q_a v^a\,, \quad \pi_{ab} = q_{\langle a} v_{b\rangle}\,, \tag{32}$$

where $G_\pm = 1 \pm w\, v^2$, $w = \tilde{p}/\tilde{\rho}$.

A.2 Orthonormal frame equations

In Bianchi cosmology the metric can be written as

$$^4\mathbf{g} = -N^2(x^0)\, dx^0 \otimes dx^0 + g_{\hat{i}\hat{j}}(x^0)\, (\boldsymbol{\omega}^{\hat{i}} + N^{\hat{i}}\, dx^0) \otimes (\boldsymbol{\omega}^{\hat{j}} + N^{\hat{j}}\, dx^0) \qquad (\hat{i}, \hat{j} = 1, 2, 3)\,, \tag{33}$$

where $\{\boldsymbol{\omega}^{\hat{i}}\}$ is a left-invariant[3] co-frame on G dual to a left-invariant spatial frame $\{\boldsymbol{e}_{\hat{i}}\}$. This frame is a basis of the Lie algebra with structure constants $C^{\hat{i}}{}_{\hat{j}\hat{k}}$, i.e.,

$$[\boldsymbol{e}_{\hat{i}}, \boldsymbol{e}_{\hat{j}}] = C^{\hat{k}}{}_{\hat{i}\hat{j}}\, \boldsymbol{e}_{\hat{k}} = (\epsilon_{\hat{i}\hat{j}\hat{m}}\, n^{\hat{m}\hat{k}} + 2a_{[\hat{i}}\, \delta_{\hat{j}]}{}^{\hat{k}})\, \boldsymbol{e}_{\hat{k}} \qquad \text{or, equivalently,} \qquad d\boldsymbol{\omega}^{\hat{i}} = -\tfrac{1}{2} C^{\hat{i}}{}_{\hat{j}\hat{k}}\, \boldsymbol{\omega}^{\hat{j}} \wedge \boldsymbol{\omega}^{\hat{k}}\,, \tag{34}$$

where we have decomposed the structure constants $C^{\hat{i}}{}_{\hat{j}\hat{k}}$ as [24]:

$$C^{\hat{i}}{}_{\hat{j}\hat{k}} = \epsilon_{\hat{j}\hat{k}\hat{m}}\, n^{\hat{m}\hat{i}} + a_{\hat{n}}\, \delta^{\hat{n}\hat{i}}_{\hat{j}\hat{k}}\,, \qquad a_{\hat{i}} = \tfrac{1}{2} C^{\hat{j}}{}_{\hat{i}\hat{j}}\,, \tag{35}$$

where $a_{\hat{i}}\, a_{\hat{j}} = \tfrac{1}{2} h \epsilon_{\hat{i}\hat{k}\hat{m}} \epsilon_{\hat{j}\hat{p}\hat{n}}\, n^{\hat{k}\hat{p}} n^{\hat{m}\hat{n}}$, $a^2 = a_{\hat{i}}\, a^{\hat{i}} = \tfrac{1}{2} h[(n^{\hat{i}}{}_{\hat{i}})^2 - n^{\hat{i}}{}_{\hat{j}}\, n^{\hat{j}}{}_{\hat{i}}]$, where h is a constant group invariant that is unaffected by frame choices. The Bianchi models are divided into two main classes: The class A models for which $a_{\hat{i}} = 0$, and the class B models for which $a_{\hat{i}} \neq 0$.

For simplicity we will set the shift vector to zero, i.e., $N^{\hat{i}} = 0$. This leads to the orthonormal frame $^4\mathbf{g} = -(\boldsymbol{\omega}^0)^2 + (\boldsymbol{\omega}^1)^2 + (\boldsymbol{\omega}^2)^2 + (\boldsymbol{\omega}^3)^2$, and a frame $\{\boldsymbol{e}_\alpha\}$ that is dual to $\{\boldsymbol{\omega}^\alpha\}$, $\alpha = 1, 2, 3$,

$$\boldsymbol{e}_0 = N^{-1} \frac{\partial}{\partial x^0}\,, \qquad \boldsymbol{e}_\alpha = e_\alpha{}^{\hat{i}}(x^0)\, \hat{\boldsymbol{e}}_{\hat{i}} = e_\alpha{}^{\hat{i}}(x^0)\, e_{\hat{i}}{}^i \frac{\partial}{\partial x^i} \quad \text{where} \quad \delta^{\alpha\beta} e_\alpha{}^{\hat{i}}(x^0)\, e_\beta{}^{\hat{j}}(x^0) = g^{\hat{i}\hat{j}}(x^0)\,, \tag{36}$$

where $g^{\hat{i}\hat{j}}(x^0)$ is the left-invariant contravariant spatial metric associated with $g_{\hat{i}\hat{j}}(x^0)$ $(e_{\hat{i}}{}^i = e_{\hat{i}}{}^i(x^1, x^2, x^3))$.

Since the unit normal to the spatial symmetry surfaces $\mathbf{n} = \boldsymbol{e}_0$ by definition is hypersurface forming, and since it is the tangent to a geodesic congruence due to spatial homogeneity, we obtain

$$[\boldsymbol{e}_0, \boldsymbol{e}_\alpha] = C^\beta{}_{0\alpha}\, \boldsymbol{e}_\beta = f_\alpha{}^\beta\, \boldsymbol{e}_\beta = -[H\, \delta_\alpha{}^\beta + \sigma_\alpha{}^\beta + \epsilon_\alpha{}^\beta{}_\gamma\, \Omega^\gamma]\boldsymbol{e}_\beta\,, \tag{37a}$$

$$[\boldsymbol{e}_\alpha, \boldsymbol{e}_\beta] = C^\gamma{}_{\alpha\beta}\, \boldsymbol{e}_\gamma = [2a_{[\alpha}\, \delta_{\beta]}{}^\gamma + \epsilon_{\alpha\beta\delta}\, n^{\delta\gamma}]\, \boldsymbol{e}_\gamma\,, \tag{37b}$$

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[3]See [1] p. 38 on the meaning of group invariant frames and their relation to orthonormal frames.

where H is the Hubble variable; $\sigma_{\alpha\beta}$ is the shear associated with $\mathbf{n}$; Ω^α is the Fermi rotation which describes how the spatial triad rotates with respect to a gyroscopically fixed so-called Fermi frame.[4] The relations (36) and (37) yield

$$H = \tfrac{1}{3}\left(g^{-\frac{1}{2}}\right) \boldsymbol{e}_0 \left(g^{\frac{1}{2}}\right) = -\tfrac{1}{3} e^\alpha{}_{\hat{i}}\, \boldsymbol{e}_0 \left(e_\alpha{}^{\hat{i}}\right), \qquad \sigma_{\alpha\beta} = -e^\gamma{}_{\hat{i}}\, \delta_{\gamma\langle\alpha}\, \boldsymbol{e}_0 \left(e_{\beta\rangle}{}^{\hat{i}}\right), \tag{38a}$$

$$\Omega^\alpha = \tfrac{1}{2}\epsilon^\alpha{}_\beta{}^\gamma\, e^\beta{}_{\hat{i}}\, \boldsymbol{e}_0 \left(e_\gamma{}^{\hat{i}}\right), \qquad\qquad n^{\alpha\beta} = g^{-\frac{1}{2}} e^\alpha{}_{\hat{i}}\, e^\beta{}_{\hat{j}}\, n^{\hat{i}\hat{j}}, \qquad a_\alpha = e_\alpha{}^{\hat{i}}\, a_{\hat{i}}, \tag{38b}$$

where g is the determinant of the spatial metric $g_{\hat{i}\hat{j}}$, and $e^\alpha{}_{\hat{i}}$ is the inverse of $e_\alpha{}^{\hat{i}}$, i.e.,

$$g = \det(g_{\hat{i}\hat{j}}) = (\det(e^\alpha{}_{\hat{i}}))^2 = (\det(e_\alpha{}^{\hat{i}}))^{-2}, \qquad e^\alpha{}_{\hat{i}}\, e_\beta{}^{\hat{i}} = \delta^\alpha{}_\beta. \tag{39}$$

The matter conservation equation $\boldsymbol{\nabla}_a T^{ab} = 0$ for a perfect fluid with a linear equation of state yield

$$(\ln \rho)^{\cdot} = (1+w) G_+^{-1}[-3H + f_{\alpha\beta}\, v^\alpha v^\beta + 2 a_\alpha v^\alpha], \tag{40a}$$

$$\dot{v} = G_-^{-1}\left(1-v^2\right) \left[3wH + f_{\alpha\beta}\, c^\alpha c^\beta - 2w\, a_\beta\, c^\beta\, v\right] v, \tag{40b}$$

$$\dot{c}_\alpha = [\delta_\alpha{}^\beta - c_\alpha c^\beta][f_\beta{}^\gamma\, c_\gamma - v(a_\beta + \epsilon_{\beta\gamma\delta}\, n^{\delta\zeta}\, c_\zeta\, c^\gamma)], \tag{40c}$$

where $a_\beta + \epsilon_{\beta\gamma\delta}\, n^{\delta\zeta}\, c_\zeta\, c^\gamma = C^\zeta{}_{\beta\gamma}\, c_\zeta\, c^\gamma$, and where instead of the 3-velocity v_α we have found it convenient to introduce $v = \sqrt{v_\alpha v^\alpha} \geq 0$ and the unit vector $c_\alpha = v_\alpha/v$ as variables. To obtain a more compact notation, we also introduced $\dot{f} = (f)^{\cdot} = \boldsymbol{e}_0 f = N^{-1} df/dx^0 = df/dt$, where t is the clock time associated with the normal congruence of the spatial symmetry surfaces (i.e. $N = 1$ for this parameterization).

It is of interest to consider the particle number density $\tilde{n}$ and the chemical potential $\tilde{\mu}$, which, for a linear equation of state, can be defined as (see [23, 26] and references therein)

$$\tilde{\rho} = \tilde{n}^{1+w}, \qquad \tilde{\mu} = (1+w)\, \tilde{n}^w. \tag{41}$$

Defining

$$l = \tilde{n}\, g^{\frac{1}{2}}\, \Gamma, \tag{42}$$

yields the evolution equation $(\ln l)^{\cdot} = 2 a_\alpha v^\alpha = 2(a_\alpha c^\alpha)\, v$, and hence l is a constant of the motion whenever $a_\alpha v^\alpha = 0$, e.g. for the class A perfect fluid models. Another quantity of interest is Taub's spatial circulation 1-form [27, 28] $t_a = \tilde{\mu}\, \tilde{u}_a$, whose spatial components can be written as $t_\alpha = \tilde{\mu}\, \Gamma\, v_\alpha = \tilde{\mu}\, \Gamma\, v\, c_\alpha$, with the norm $\tilde{\mu}\, \Gamma\, v$, which satisfies $(\ln \tilde{\mu}\, \Gamma\, v)^{\cdot} = f_{\alpha\beta}\, c^\alpha\, c^\beta$, which, together with (40c) yields $\dot{t}_\alpha = \tilde{\mu}\, \Gamma\, v\, (f_{\alpha\beta} - v\, C^\gamma{}_{\alpha\beta}\, c_\gamma)\, c^\beta$. This in turn leads to that $t_{\hat{i}} = e^\alpha{}_{\hat{i}}\, t_\alpha$, where $\boldsymbol{e}_0\, (e^\alpha{}_{\hat{i}}) = -f_\beta{}^\alpha\, e^\beta{}_{\hat{i}}$, obeys the equation

$$\dot{t}_{\hat{i}} = -(\tilde{\mu}\, \Gamma)^{-1}\, C^{\hat{k}}{}_{\hat{i}\hat{j}}\, t_{\hat{k}}\, t^{\hat{j}}. \tag{43}$$

A.3   The Hubble-normalized dynamical systems approach

In the conformal Hubble normalized approach one factors out the Hubble variable H by means of a conformal transformation which yields dimensionless quantities [29, 16]. In the spatially homogeneous case this amounts to the following:

$$(\Sigma_{\alpha\beta}, R^\alpha, N^{\alpha\beta}, A_\alpha) = \frac{1}{H}(\sigma_{\alpha\beta}, \Omega^\alpha, n^{\alpha\beta}, a_\alpha), \qquad (\Omega, P, Q_\alpha, \Pi_{\alpha\beta}) = \frac{1}{3H^2}(\rho, p, q_\alpha, \pi_{\alpha\beta}), \tag{44}$$

where we have chosen to normalize the stress-energy quantities with $3H^2$ rather than H^2 in order to conform with the usual definition of Ω; in the perfect fluid case this leads to that

$$Q^\alpha = (1+w)G_+^{-1}\, v\, \Omega\, c^\alpha, \quad Q = (1+w)G_+^{-1}\, v\, \Omega, \quad P = w\Omega + \tfrac{1}{3}(1-3w)v\, Q, \quad \Pi_{\alpha\beta} = v\, Q\, c_{\langle\alpha}c_{\beta\rangle}. \tag{45}$$

In addition to this we choose a new dimensionless time variable τ by means of the lapse choice $N = H^{-1}$. Since H is the only variable with dimension, its evolution equation decouples from the remaining equations for dimensional reasons:

$$H' = -(1+q)H; \qquad q = 2\Sigma^2 + \tfrac{1}{2}(\Omega + 3P), \qquad \Sigma^2 = \tfrac{1}{6}\Sigma_{\alpha\beta}\Sigma^{\alpha\beta}, \tag{46}$$

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[4]The sign in the definition of Ω^α is the same as in [25, 16], but opposite of that in [1].

where a prime denotes $d/d\tau$ and where q is the deceleration parameter, obtained by means of one of Einstein's equations $G_{ab} = T_{ab}$ —the Raychaudhuri equation (we use units $c = 1$ and $8\pi G = 1$, where c is the speed of light and G is Newton's gravitational constant); in the perfect fluid case we obtain

$$q = 2\Sigma^2 + \tfrac{1}{2}(\Omega + 3P) = 2\Sigma^2 + \tfrac{1}{2}G_+^{-1}[1 + 3w + (1 - w)v^2]\Omega \,. \tag{47}$$

The remaining Einstein field equations together with the Jacobi identities yield the following set of equations, which we divide into evolution equations and constraints:

Evolution equations

$$\Sigma'_{\alpha\beta} = -(2 - q)\Sigma_{\alpha\beta} - 2\epsilon^{\gamma\delta}{}_{\langle\alpha}\,\Sigma_{\beta\rangle\gamma}\,R_\delta - {}^3\mathcal{R}_{\langle\alpha\beta\rangle} + 3\Pi_{\alpha\beta} \,, \tag{48a}$$

$$(N^{\alpha\beta})' = (3q\delta_\gamma{}^{(\alpha} - 2F_\gamma{}^{(\alpha)})N^{\beta)\gamma} \,, \tag{48b}$$

$$A'_\alpha = F_\alpha{}^\beta\,A_\beta \,, \tag{48c}$$

$$\Omega' = (2q - 1)\,\Omega - 3P + 2A_\alpha\,Q^\alpha - \Sigma_{\alpha\beta}\Pi^{\alpha\beta} \,, \tag{48d}$$

$$v' = -G_-^{-1}\,(1 - v^2)\,\left[1 - 3w + \Sigma_{\alpha\beta}\,c^\alpha c^\beta + 2w\,(A_\beta\,c^\beta)\,v\right]v \,, \tag{48e}$$

$$c'_\alpha = [\delta_\alpha{}^\beta - c_\alpha c^\beta][F_\beta{}^\gamma\,c_\gamma - v(A_\beta + \epsilon_\beta{}^\gamma{}_\delta\,N^{\delta\zeta}\,c_\zeta\,c_\gamma)] \,. \tag{48f}$$

Constraint equations

$$0 = 1 - \Sigma^2 - \Omega_{\rm k} - \Omega \,, \tag{49a}$$

$$0 = (3\delta_\alpha{}^\gamma\,A_\beta + \epsilon_\alpha{}^{\delta\gamma}\,N_{\delta\beta})\,\Sigma^\beta{}_\gamma - 3Q_\alpha \,, \tag{49b}$$

$$0 = A_\beta\,N^\beta{}_\alpha \,, \tag{49c}$$

where $\Sigma^2 = \tfrac{1}{6}\Sigma_{\alpha\beta}\Sigma^{\alpha\beta}$, $A^2 = A_\alpha A^\alpha$, $F_\alpha{}^\beta = q\,\delta_\alpha{}^\beta - \Sigma_\alpha{}^\beta - \epsilon_\alpha{}^\beta{}_\gamma\,R^\gamma$, and

$$^3\mathcal{R}_{\langle\alpha\beta\rangle} = \mathcal{B}_{\langle\alpha\beta\rangle} - 2\epsilon^{\gamma\delta}{}_{\langle\alpha}\,N_{\beta\rangle\gamma}\,A_\delta \,; \quad \Omega_{\rm k} = \tfrac{1}{12}\mathcal{B}^\alpha{}_\alpha + A^2 \,; \quad \mathcal{B}_{\alpha\beta} = 2N_{\alpha\gamma}\,N^\gamma{}_\beta - N^\gamma{}_\gamma\,N_{\alpha\beta} \,. \tag{50}$$

The Gauss constraint (49a) and the Codazzi constraint (49b) will figure prominently throughout.

It is of interest to note that

$$Q' = -[2 - q - F_{\alpha\beta}\,c^\alpha\,c^\beta + 2(A_\alpha\,c^\alpha)\,v]\,Q \,. \tag{51}$$

Another quantity of interest, intimately connected with l, is defined via $\tilde{n}\,\Gamma = g^{-1/2}\,l$; raising the r.h.s. with $1 + w$ and normalizing with H yields the quantity[5]

$$\Psi = \Gamma^{-(1-w)}\,G_+^{-1}\,\Omega \,, \qquad (\ln\Psi)' = 2q - (1 + 3w) + 2(1 + w)(A_\alpha\,c^\alpha)\,v \,. \tag{52}$$

A.4   Bianchi type II

For the Bianchi type II models we have $A_\alpha = 0$, and in addition we can choose a spatial frame $\boldsymbol{e}_\alpha$ to be an eigenframe of the matrix $N_{\alpha\beta}$, with $N_{11} \neq 0$, while otherwise $N_{\alpha\beta} = 0$. This leads to that eq. (48b) yields $R_2 = \Sigma_{31}$, $R_3 = -\Sigma_{12}$, and that the Codazzi constraint (49b) implies

$$v_1 = 0 = c_1 \,. \tag{53}$$

Inserting the conditions of eq. (53) into eq. (48f) gives the following relation:

$$0 = \Sigma_{12}\,c_2 + \Sigma_{31}\,c_3 = -R_3\,c_2 + R_2\,c_3 \qquad \Leftrightarrow \qquad \epsilon_{AB}\,R^A\,c^B = 0 \,, \tag{54}$$

where $A, B = 2, 3$ and ϵ_{AB} is the two-dimensional permutation that has $\epsilon_{23} = 1$ (hence $c_A\,c^A = 1$). It follows from $\epsilon_{AB}\,R^A\,c^B = 0$ that $R_A \propto c_A \propto Q_A$, where the last relation holds when $\Omega \neq 0$; hence Σ_{12} and Σ_{31} are linearly dependent and can be replaced by a single variable.

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[5]This quantity has appeared before in the literature, e.g. in [5], where $\Gamma^{-(1-w)}\,G_+^{-1}$ has been denoted by β.

We have the freedom to rotate in the 23-plane, which is expressed in the field equations as the freedom to choose R_1. To obtain a set of variables that are invariant under such rotations we introduce the following new shear variables

$$\Sigma_+ = \tfrac{1}{2}\Sigma_A{}^A = -\tfrac{1}{2}\Sigma_{11}\,, \qquad\qquad \bar\Sigma = \tfrac{1}{\sqrt{3}}(\Sigma_{AB} - \Sigma_+\delta_{AB})(c^A c^B - \tfrac{1}{2}\delta^{AB})\,, \tag{55a}$$

$$\tilde\Sigma = \tfrac{1}{\sqrt{3}}(\Sigma_{AB} - \Sigma_+\delta_{AB})\epsilon^B{}_C(c^A c^C - \tfrac{1}{2}\delta^{AC})\,, \qquad \check\Sigma^2 = \tfrac{1}{3}\left(\Sigma_{12}^2 + \Sigma_{31}^2\right)\,. \tag{55b}$$

Hewitt et al. [5] make use of the freedom to rotate in the 23-plane to set $c_2 = 0$, which yields that $c_3 = 1$ and $R_2 = 0 = \Sigma_{31}$. This leads to a correspondence between the variables $\Sigma_-, \Sigma_1, \Sigma_3$ in [5] to our variables, when setting $c_2 = 0$, according to $\bar\Sigma = -\Sigma_-$, $\check\Sigma^2 = \Sigma_3^2$, $\tilde\Sigma^2 = \Sigma_1^2$.

Because of the existence of discrete symmetries one can simplify the analysis, e.g. the eigenvalue analysis, by introducing the following state vector $\mathbf{S} = (\Sigma_+, \bar\Sigma, \tilde\Sigma^2, \check\Sigma^2, \Omega_k, v^2)$, where $\Omega_k = N_{11}^2/12$, and where Ω can be obtained in terms of $\mathbf{S}$ via the Gauss constraint (49a). This leads to the dynamical system given in Section 2.

B   Hamiltonian considerations and derivation of monotone functions

B.1   Hamiltonian considerations

The scalar Hamiltonian is given by

$$\tilde{\mathcal{H}} = 2N\, g^{\frac{1}{2}} n^a\, n^b (G_{ab} - T_{ab}) = 2\tilde N\, g\, n^a\, n^b (G_{ab} - T_{ab}) = \tilde N \mathcal{H} = \tilde N (T + U_{\mathrm g} + U_{\mathrm f})\,, \tag{56}$$

where we have defined $\tilde N = Ng^{-1/2}$, and where $(T, U_{\mathrm g}, U_{\mathrm f}) = 6\,g\,H^2(-1+\Sigma^2, \Omega_k, \Omega)$. By means of (32), (41), and (42), this leads to that

$$U_{\mathrm f} = 2g\rho = 2l^{1+w}\, g^{(1-w)/2}\, \Gamma^{1-w} G_+\,, \tag{57}$$

while T and $U_{\mathrm g}$ depends on the model and the metric representation.

In [30] it was shown that the tilted orthogonally transitive Bianchi models exhibit a so-called timelike homothetic Jacobi symmetry. It was later realized that such symmetries are related to the existence of monotonic functions (Uggla in ch. 10 in [1], and [15]). Unfortunately the analysis in [30] is quite cumbersome and we will therefore make a new derivation of the 'homothetic structure' and from this derive a monotone function. To do so, we need to connect the spatially homogeneous frame in (33) with the orthonormal frame. This is done in two steps: (i) diagonalization by means of the off-diagonal special automorphism group, (ii) normalization by means of diagonal scaling. We hence write $e_\alpha{}^{\hat i}$ in the transformation (36), i.e., $\boldsymbol{e}_\alpha = e_\alpha{}^{\hat i}(x^0)\,\hat{\boldsymbol{e}}_{\hat i}$, as $e_\alpha{}^{\hat i} = (D^{-1})_\alpha{}^{\tilde j}(S^{-1})_{\tilde j}{}^{\hat i}$, or $e^\alpha{}_{\hat i} = D^\alpha{}_{\tilde j} S^{\tilde j}{}_{\hat i}$, where $S^{\tilde j}{}_{\hat i}$, since it is assumed to be a special automorphism transformation, leaves $a_{\hat i}$ and $n^{\hat i \hat j}$ unaffected. In addition we define the new metric variables $\beta^1, \beta^2, \beta^3$ via the matrix $D^\alpha{}_{\tilde j}$ so that

$$(D^{-1})_\alpha{}^{\tilde j} = \begin{pmatrix} \exp(-\beta^1) & 0 & 0 \\ 0 & \exp(-\beta^2) & 0 \\ 0 & 0 & \exp(-\beta^3) \end{pmatrix}\,, \tag{58}$$

where $\beta^\alpha = \beta^\alpha(x^0)$; hence $g^{1/2} = \exp(\beta^1 + \beta^2 + \beta^3)$.

To obtain the Hamiltonian for the tilted orthogonally transitive Bianchi models we choose a spatially homogeneous frame so that the line element can be written as (33) with all structure constants being zero except $n^{\hat 1 \hat 1} = \hat n_1$. In the orthogonally transitive case we can specify the spatially homogeneous frame so that $g_{\hat i \hat j}$ in (33) have one off-diagonal component, $g_{\hat 1 \hat 2}$, and so that the perfect fluid velocity has a single non-zero component, $v_{\hat 3}$. Hence we follow [30] and write[6]

$$S^\alpha{}_{\tilde j} = \begin{pmatrix} 1 & -\sqrt{2}\hat n_1 \theta^3(x^0) & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\,. \tag{59}$$

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[6]There is a typographical error in eq. (2.61) in [30]; the exponent should be -1 and not $-1/2$ of $(\hat m^{(3)})$ in the expression for s_3.

It follows from (38b) that

$$n^{11} = \exp(\beta^1 - \beta^2 - \beta^3)\hat{n}_1 \,. \tag{60}$$

Due to that θ_3 is associated with the off-diagonal special automorphism group, the associated momentum, which is proportional to σ_{12}, is conserved [30]. Furthermore, note that (43) yields the constant of the motion $t_{\hat{3}} = \hat{t}_3 = \text{const}$. This is consistent with the Codazzi constraint, also known as the Hamiltonian momentum constraint, which linearly relates these two constants to each other. The constants $\hat{n}_1$, l, $\hat{t}_3$ allows us to write $T + U_{\rm g} + U_{\rm f}$ in (56) so that

$$\mathcal{H} = T + U_{\rm g} + U_{\rm f} = T_{\rm d} + U_{\rm c} + U_{\rm g} + U_{\rm f} \,, \tag{61}$$

where $U_{\rm c}$ is the so-called centrifugal potential, which is proportional to the Σ_{12}^2 term in Σ^2, where, furthermore, $U_{\rm c}$, $U_{\rm g}$, and $U_{\rm f}$ are all expressible in terms of β^α, and no other time dependent quantities. Expressing $\tilde{N}T_{\rm d}$ in terms of the $\dot{\beta}^\alpha$ or the associated momenta π_α, see [15] and also [31], by means of (38) and the transformations in this subsection, leads to

$$\tilde{N}T_{\rm d} = 2\tilde{N}^{-1}\,\mathcal{G}_{\gamma\delta}\dot{\beta}^\gamma\dot{\beta}^\delta = \tfrac{1}{4}\tilde{N}\,\mathcal{G}^{\gamma\delta}\pi_\gamma\pi_\delta \,, \tag{62}$$

where $\mathcal{G}_{\gamma\delta}$ is known as the minisuperspace metric for the diagonal degrees of freedom, which, together with its inverse $\mathcal{G}^{\alpha\beta}$ is given by

$$\mathcal{G}_{\alpha\beta} = \begin{pmatrix} 0 & -1 & -1 \\ -1 & 0 & -1 \\ -1 & -1 & 0 \end{pmatrix}; \qquad \mathcal{G}^{\alpha\beta} = \tfrac{1}{2}\begin{pmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix}. \tag{63}$$

The centrifugal potential $U_{\rm c} = 2\,g\,H^2\,\Sigma_{12}^2 = 2\,g\,\sigma_{12}^2$ re-expressed via the Codazzi constraint in terms of $\hat{t}_3$ yields

$$U_{\rm c} \propto \exp[-2(\beta^1 - \beta^2)] \,. \tag{64}$$

By means of $U_{\rm g} = 6\,g\,H^2\,\Omega_{\rm k}$ and (60) we find that

$$U_{\rm g} \propto \exp(4\beta^1) \,. \tag{65}$$

We can write $\tilde{u}^a\,\tilde{u}_a = -1$ as $0 = 1 - \Gamma^2 + \tilde{\mu}^{-2}\,t_\alpha\,t^\alpha$, and by defining

$$F = (1+w)^{-2}\,l^{-2w}\,g^w\,t_\alpha\,t^\alpha = (1+w)^{-2}\,l^{-2w}\,g^w\,g^{\hat{i}\hat{j}}\,t_{\hat{i}}\,t_{\hat{j}} \,, \tag{66}$$

we obtain

$$0 = 1 - \Gamma^2 + F\,\Gamma^{2w} \,, \tag{67}$$

which allows one to, in general, implicitly express Γ^2 in terms of F, i.e., $\Gamma^2 = \Gamma^2(F)$. In the present case we find that $F = (1+w)^{-2}\,l^{-2w}\,g^w\,g^{\hat{3}\hat{3}}\,t_{\hat{3}}\,t_{\hat{3}}$, which yields

$$F = (1+w)^{-2}\,l^{-2w}\,\hat{t}_3^2\,\exp[w(\beta^1 + \beta^2) - (1-w)\beta^3] \,. \tag{68}$$

It follows that

$$U_{\rm f} \propto \exp[(1-w)(\beta^1 + \beta^2 + \beta^3)]\phi. \tag{69}$$

where $\phi := \Gamma^{1-w}G_+$ is a function of the particular combination $w(\beta^1 + \beta^2) - (1-w)\beta^3$ only.

B.2   Derivation of monotone functions

Based on the Hamiltonian for the diagonal degrees of freedom for the orthogonally transitive type II case we in this subsection derive a monotonic function that is of key importance for understanding the dynamics. In ch. 10 in [1] and in [15] it is shown that monotone functions are associated with 'homothetic' symmetries of the potential, i.e., we require that there exists a vector $\mathbf{c} = c^\alpha\partial_{\beta^\alpha}$ such that $\mathbf{c}U = \mathbf{c}(U_{\rm c} + U_{\rm g} + U_{\rm f}) = r\,U$, where r is a constant. For this to be possible we require (i) that $\mathbf{c}\phi = 0$, and hence that

$$\mathbf{c}(w(\beta^1 + \beta^2) - (1-w)\beta^3) = w(c^1 + c^2) - (1-w)c^3 = 0 \,, \tag{70}$$

and (ii) $\mathbf{c}\,U_c = r\,U_c$, $\mathbf{c}\,U_g = r\,U_g$, and $\mathbf{c}\,U_f = r\,U_g$, which due to the condition (i) yields $\mathbf{c}\exp[(1-w)(\beta^1 + \beta^2 + \beta^3)] = r\exp[(1-w)(\beta^1 + \beta^2 + \beta^3)]$. This leads to

$$-2(c^1 - c^2) = r\,, \qquad 4c^1 = r\,, \qquad (1-w)(c^1 + c^2 + c^3) = r\,, \tag{71}$$

which yields $c^2 = 3c^1$, $c^3 = 4c^1 w/(1-w)$; w.l.g we can choose $c^1 = 1 - w$, which gives

$$(c_1, c_2, c_3) = (1 - w, 3(1-w), 4w)\,, \qquad r = 4(1-w)\,. \tag{72}$$

The causal character of this vector with respect to the metric $\mathcal{G}_{\alpha\beta}$ is crucial [15]; we obtain

$$\mathcal{G}_{\gamma\delta}c^\gamma\,c^\delta = -2(1-w)(3+13w)\,, \tag{73}$$

which is timelike if $-3/13 < w < 1$. The above properties of the potential U implies that the model satisfies the criteria given in [15] for admitting a monotonic function given by[7]

$$M \propto (c^\alpha\,\pi_\alpha)^2 \exp\left[-r\,\frac{\mathcal{G}_{\gamma\delta}c^\gamma\,\beta^\delta}{\mathcal{G}_{\gamma\delta}c^\gamma\,c^\delta}\right]\,. \tag{74}$$

We now have to express M in the dynamical systems variables $\mathbf{S}$. We do so in two steps by first using $\Sigma_{\alpha\alpha} = \Sigma_\alpha$, and then in the second step we go over to the presently used dynamical systems variables via the current gauge fixing (see the previous discussion about the correspondence between our variables and those used by [5]). Let us define

$$(V_g,\,V_f) = \exp\left[-r\,\frac{\mathcal{G}_{\gamma\delta}c^\gamma\,\beta^\delta}{\mathcal{G}_{\gamma\delta}c^\gamma\,c^\delta}\right](U_g,\,U_f)\,, \tag{75}$$

then

$$\exp\left[-r\,\frac{\mathcal{G}_{\gamma\delta}c^\gamma\,\beta^\delta}{\mathcal{G}_{\gamma\delta}c^\gamma\,c^\delta}\right]\mathcal{H} = \exp\left[-r\,\frac{\mathcal{G}_{\gamma\delta}c^\gamma\,\beta^\delta}{\mathcal{G}_{\gamma\delta}c^\gamma\,c^\delta}\right]T + V_g + V_f = 0\,, \tag{76}$$

which yields

$$\exp\left[-r\,\frac{\mathcal{G}_{\gamma\delta}c^\gamma\,\beta^\delta}{\mathcal{G}_{\gamma\delta}c^\gamma\,c^\delta}\right] = -\left(\frac{V_g + V_f}{T}\right) \propto \frac{V_g + V_f}{\pi_0^2(1 - \Sigma^2)}\,, \tag{77}$$

and hence

$$M \propto \left(\frac{c^\alpha\pi_\alpha}{\pi_0}\right)^2\frac{V_g + V_f}{(1 - \Sigma^2)} = \left(\frac{c^\alpha\pi_\alpha}{\pi_0}\right)^2 V_f\,\Omega^{-1}\,, \tag{78}$$

since $V_g/V_f = \Omega_k/\Omega$ and $1 - \Sigma^2 = \Omega_k + \Omega$.

It follows from our definitions that $\pi_\alpha = \frac{1}{6}\pi_0(2 - \Sigma_\alpha)$, $\pi_0 = \pi_1 + \pi_2 + \pi_3$, see also [15, 19]. In combination with (72) we find that this yields

$$\frac{c^\alpha\pi_\alpha}{\pi_0} \propto 1 - \frac{1}{8}(1-w)\Sigma_1 - \frac{3}{8}(1-w)\Sigma_2 - \frac{1}{2}w\Sigma_3 = 1 - \frac{\sqrt{3}}{8}(7w-3)\bar{\Sigma} - \frac{1}{8}(1+3w)\Sigma_+\,. \tag{79}$$

Next we need to solve for V_f in terms of the state space variables $(\Sigma_+, \bar{\Sigma}, \check{\Sigma}^2, \Omega_k, v^2)$. This can be done with the help of the constants of motion $\hat{n}_1$, l, $\hat{t}_3$ through equations (75), (68), (67), (60), and (57), using the form (72) of the homothetic vector $\mathbf{c}$. After some algebra one finds that

$$V_f \propto \left[(v^2)^{3-7w}\,\Gamma^{8(1-w)}\,G_+^{2(1+7w)}\left(\frac{\Omega_k}{\Omega}\right)^{-(1-w)}\right]^{1/(3+13w)}\,. \tag{80}$$

Taking $M^{-(3+13w)/2}$ and replacing Ω with Ψ via (8), and v^2 through eq. (2g) yields the monotone function M_H, which we give in the main text. Similar, but much simpler, Hamiltonian methods were used to find the monotone function M_{CS} for the non-tilted Bianchi type II models, which is also given in the main text.

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[7]This is actually the square of the monotone function in [15], but we find this form more convenient in the present context.

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