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Dual conformal constraints and infrared equations from global residue theorems in $\mathcal{N}$=4 SYM theory

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Abstract

Infrared equations and dual conformal constraints arise as consistency conditions on loop amplitudes in $\mathcal{N}$=4 super Yang-Mills theory. These conditions are linear relations between leading singularities, which can be computed in the Grassmannian formulation of $\mathcal{N}$=4 super Yang-Mills theory proposed recently. Examples for infrared equations have been shown to be implied by global residue theorems in the Grassmannian picture.

Both dual conformal constraints and infrared equations are mapped explicitly to global residue theorems for one-loop next-to-maximally-helicity-violating amplitudes. In addition, the identity relating the BCFW and its parity-conjugated form of tree-level amplitudes, is shown to emerge from a particular combination of global residue theorems.

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1 Introduction

Scattering amplitudes in maximally supersymmetric super Yang-Mills (SYM) theory, possess many beautiful properties which are obscured in their local formulations [1]. Remarkably simple formulæ for MHV amplitudes have been known since [2, 3]. More recently, Witten's seminal work on twistor string theory for $\mathcal{N}=4$ SYM [4] has triggered the development and exploration of many new techniques for efficiently computing scattering amplitudes, such as MHV diagrams [5] and the Britto-Cachazo-Feng-Witten (BCFW) recursion relations [6, 7]. In addition, organizing amplitudes in a maximally supersymmetric way has lead to the discovery of dual superconformal symmetry in $\mathcal{N}=4$ SYM theory [8, 9, 10], which can be combined with conventional superconformal symmetry to yield the Yangian symmetry [11]. Recently, in [12, 13] the Grassmannian conjecture was put forward: the Grassmannian integral was argued to encode many of the previously hidden structures and symmetries of $\mathcal{N}=4$ SYM amplitudes.

The Grassmanian integral is conjectured to generate all-loop leading singularities of $\mathcal{N}=4$ SYM theory. Leading singularities [14, 15, 16, 17] are the highest codimension singularities at l-loop level, obtained by cutting $4l$ propagators in the generalized unitarity method [18, 19]. The leading singularities[1] are well-defined and infrared-finite objects. It turns out that the box coefficients from expanding a one-loop amplitude in $\mathcal{N}=4$ SYM into scalar box functions are exactly the one-loop leading singularities. Generalizing this fact, it was conjectured that leading singularities together with their corresponding integral basis can determine complete amplitudes of $\mathcal{N}=4$ SYM theory at any loop order [20].

There is evidence that leading singularities at all loop-orders are related to each other by infrared (IR) equations [21, 22, 23]. For $\mathcal{N}=4$ SYM amplitudes at one-loop level, IR equations can be derived by comparing the IR-divergences of the amplitude with those of its expansion into box functions, as will be reviewed in subsection 2.1. They turn out to be simple linear relations among various box coefficients (one-loop leading singularities) and the tree amplitude, which can be viewed as special version of a leading singularity.

Furthermore, based on anomalous dual conformal symmetry, constraints for one-loop leading singularities have been derived in [24, 25]. While box coefficients as well as tree amplitudes are covariant under dual conformal symmetry, one-loop amplitudes exhibit a dual conformal anomaly due to infrared divergences. Dual conformal constraints can be derived – very similarly to IR equations – by comparing the anomaly of the complete one-loop amplitude to the anomalies of the box functions. Not surprisingly, these constraints, also being linear relations among box coefficients and tree amplitudes, have been found to imply all one-loop IR equations. In addition, there are new relations which originate in the anomalies of the dual superconformal symmetry exclusively.

In [12] evidence has been put forward that one-loop IR equations of $\mathcal{N}=4$ SYM can be traced back to global residue theorems (GRT) in the Grassmannian description of $\mathcal{N}=4$

[1] As common in the literature we will use the terminology "leading singularity" to refer to the discontinuity across a leading singularity.

SYM theory. In this formalism, leading singularities are expressed as residues of the multi-dimensional complex contour integral [12]

$$\mathcal{L}_{n;k}(\mathcal{W}_a) = \int \frac{d^{k\times n} C_{\alpha a}}{(12\cdots k)\,(23\cdots(k+1)\,)\,\cdots(n1\cdots(k-1)\,)} \prod_{\alpha=1}^{k} \delta^{4|4}(C_{\alpha a}\mathcal{W}_a)\,. \tag{1.1}$$

In analogy to Cauchy's theorem in the complex plane, where the residues enclosed in a certain contour sum up to zero, there are GRTs in the multidimensional complex space relating the residues of eq. (1.1). Starting from known IR equations and expressing the leading singularities in terms of residues, it was shown for a couple of examples that those equations indeed can be traced back to GRTs. Despite of these promising results, a general map between IR equations and GRTs has been missing so far. In this article, we will propose that not only IR equations, but in fact all one-loop dual conformal constraints, have their origin in GRTs in the Grassmannian formulation.

Although we conjecture dual conformal constraints (and thus IR equations) to be related to GRTs for any one-loop amplitude, we will limit our considerations to the NMHV sector. In this sector any integration contour for the evaluation of the Grassmannian integral is in one-to-one correspondence with a certain choice of denominator factors in eq. (1.1) to be set to zero. Starting from the N^2MHV level, a choice of vanishing minors does not determine a residue uniquely. Thus this identification can not be made straightforwardly any more. In addition, in the NMHV situation a definite map between one-loop leading singularities and residues is known, while beyond NMHV a complete identification has not yet been achieved.

It should be noted that only a subset of all available GRTs is used to derive all one-loop dual conformal constraints. Since it is now clear [26, 27] that residues in Grassmannian formulation correspond to all-loop leading singularities, some GRTs should have interpretations as relations involving higher-loop leading singularities. In order to map all one-loop constraints onto leading singularities, we need to decide which residue occurs as a contribution to a leading singularity at a certain loop level *first*. Here we will use the *invariant label* introduced below as a criterion, while a similar classification has been performed by considering the twistor support of leading singularities in [27]. In the same reference it was shown that NMHV leading singularities can maximally occur at three-loop level.

By systematically translating all box coefficients appearing in one-loop dual conformal constraints into residues in the Grassmannian formulation, we find a general map between all one-loop dual conformal constraints and combinations of GRTs in the NMHV sector. Additionally, we also propose a general formula to assign a sum of GRTs to any one-loop IR equation. Finally, the mapping is rounded off by identifying a mechanism which provides the highly nontrivial identities relating the BCFW and the P(BCFW) representations of the NMHV tree amplitude. These identities ensure the absence of spurious poles, parity invariance and cyclicity of tree amplitudes [12].

It is difficult to generalize the result to higher-loop level. The absence of a general notion for an integral basis, already seen at two loops, results in the lack of a clear identification of dual conformal constraints and IR equations beyond one-loop. While there are definitely additional relations originating in GRTs which correspond to these higher-loop constraints, the identification and exploration of those structures is left for a future project.

After describing the framework and introducing the conventions in section 2, we will proceed to classify residues by the invariant label in section 3. The mapping of one-loop IR equations and – more generally – dual conformal constraints to GRTs will be presented in section 4.

2 Prerequisites

2.1 Spacetime formulation

We are going to use the $\mathcal{N}$=4 on-shell super space introduced in [3] with Grassmann coordinates η and the usual spinor-helicity variables $\lambda, \tilde{\lambda}$. Kinematical invariants are defined as

$$[\![i]\!]_m = (k_i + k_{i+1} + \cdots + k_{i+m-1})^2 , \tag{2.1}$$

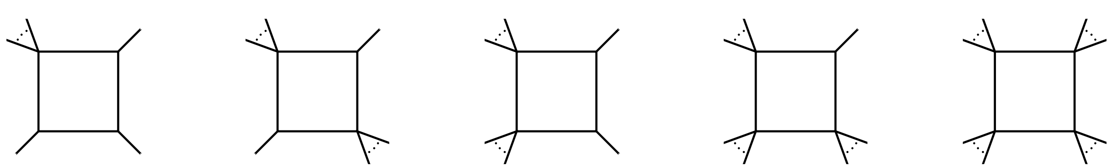
with all momenta being on-shell. This results in $[\![1]\!]_2 = s_{12} = 2k_1 \cdot k_2$ and $[\![2]\!]_3 = t_{234} = 2(k_2 \cdot k_3 + k_2 \cdot k_4 + k_3 \cdot k_4)$ for example. All considerations below are given in a completely super-symmetric invariant form. However, since 4-mass box integrals are IR finite and exhibit no conformal anomaly, they never participate in one-loop IR equations or dual conformal constraints.

As is well known, one-loop superamplitudes in $\mathcal{N}$=4 SYM theory can be expanded into scalar box integrals. However, both in the context of leading singularities and dual conformal symmetry, it is most natural and convenient to use box functions as the basis,

$$M^{\text{1-loop}} = \sum_i C_i F_i , \tag{2.2}$$

where $F_i = -\frac{I_i}{2\sqrt{R_i}}$ are IR-divergent box functions, simply related to scalar box integrals I_i by kinematic factors R_i (see the first paper of ref. [18] for details). Box functions can be identified with certain box configurations. The quantities C_i are called box coefficients, which are exactly one-loop leading singularities.

Considering whether there are one or more legs attached to the corners of a box diagram naturally leads to the following categories:



1-mass (1m) 2-mass easy (2me) 2-mass hard (2mh) 3-mass (3m) 4-mass (4m)

(2.3)

The corresponding box functions and their coefficients are conveniently labeled by the distribution of legs onto the corners of the boxes. Given the known number of legs n, it is sufficient to note the first legs attached to the four corners of the box. If not stated differently, the first entry of the four-element list $r, s, t, u \in \{1, \ldots, n\}$ is assumed to label the massless leg in the upper right corner. In some cases, despite being redundant, the type of box will be noted as a superscript on the coefficient. Employing those conventions, eq. (2.2) reads

$$M_n^{\text{1-loop}}(\lambda, \tilde{\lambda}, \eta, \epsilon) = \sum_{\{rstu\}} C_{rstu}(\lambda, \tilde{\lambda}, \eta) F_{rstu}(\lambda, \tilde{\lambda}, \epsilon)\,, \tag{2.4}$$

where the parameter of dimensional regularization is defined by $4 - 2\epsilon = D$.

In order to obtain the IR equations, one starts from the IR-divergent part of dimensionally regularized one-loop amplitudes of $\mathcal{N}$=4 SYM [21],

$$M_n^{\text{1-loop}}\Big|_{IR} = -\frac{r_\Gamma}{\epsilon^2} \sum_{i=1}^{n} (-[\![i]\!]_2)^{-\epsilon} M^{\text{tree}}, \tag{2.5}$$

where $r_\Gamma := \Gamma(1+\epsilon)\Gamma^2(1-\epsilon)\Gamma(1-2\epsilon)$.

The IR-divergent part of box functions is given by [28]

$$\begin{aligned}
F^{\text{1m}}(p,q,r,P)\big|_{IR} &= -\frac{r_\Gamma}{\epsilon^2}\left((-s)^{-\epsilon} + (-t)^{-\epsilon} - (-P^2)^{-\epsilon}\right)\\
F^{\text{2me}}(p,P,q,Q)\big|_{IR} &= -\frac{r_\Gamma}{\epsilon^2}\left((-s)^{-\epsilon} + (-t)^{-\epsilon} - (-P^2)^{-\epsilon} - (-Q^2)^{-\epsilon}\right)\\
F^{\text{2mh}}(p,q,P,Q)\Big|_{IR} &= -\frac{r_\Gamma}{\epsilon^2}\left(\frac{1}{2}(-s)^{-\epsilon} + (-t)^{-\epsilon} - \frac{1}{2}(-P^2)^{-\epsilon} - \frac{1}{2}(-Q^2)^{-\epsilon}\right)\\
F^{\text{3m}}(p,P,R,Q)\big|_{IR} &= -\frac{r_\Gamma}{\epsilon^2}\left(\frac{1}{2}(-s)^{-\epsilon} + \frac{1}{2}(-t)^{-\epsilon} - \frac{1}{2}(-P^2)^{-\epsilon} - \frac{1}{2}(-Q^2)^{-\epsilon}\right)\,,
\end{aligned} \tag{2.6}$$

where F^{4m} has not been listed because of its IR finiteness. In the notation $F(K_1, K_2, K_3, K_4)$ of eq. (2.6), capital letters define sums of consecutive massless momenta and lower case letters correspond to single (null) momenta. In addition, the two main kinematical invariants are defined as $s = (K_1 + K_2)^2$ and $t = (K_2 + K_3)^2$.

IR equations are obtained by requiring the consistency between eq. (2.5) and eq. (2.6). There is one IR equation for each kinematical invariant, whose IR behavior shall be considered. Thus each IR equation is conveniently labeled by this kinematical invariant. Since $[\![i]\!]_m = [\![i+m]\!]_{n-m}$ by momentum conservation, we can limit the considerations to $2 \le m \le \lfloor n/2 \rfloor$.

After expanding the left hand side of eq. (2.5) using eqs. (2.2) and (2.6), there are two different situations:

- if the considered kinematical invariant is of the form $[\![i]\!]_2$, the sum of box-coefficients from the left hand side has to be proportional to the tree amplitude M^{tree}.
- For kinematical invariants of the form $[\![i]\!]_m$ with $m > 2$ there is no contribution from the right-hand side of eq. (2.5). Thus the total sum of box-coefficients will vanish, which leads to a relation purely between box coefficients themselves.

The resulting IR equations are relations between IR-finite quantities: the tree amplitude and one-loop leading singularities. It is not difficult to count the number of IR equations for a certain number of legs. There are $\frac{n(n-3)}{2}$ IR equations in total, which split up into n equations involving the tree amplitude and $\frac{n(n-5)}{2}$ pure one-loop equations.

The IR equations considered above represent only a subset of all known relations between one-loop leading singularities. The anomalous dual conformal symmetry at one-loop level results in constraints that imply, but are not limited to, the IR equations. Here we only state the result without repeating the derivation, which can be found in [24, 25].

The dual conformal anomaly of one-loop amplitudes can be obtained by applying the shifted dual conformal generator[2] $\hat{K}^\mu$ to both sides of eq. (2.2). The dual conformal anomaly of one-loop amplitudes has been conjectured in [10] to be proportional to the tree amplitude. On the right-hand side, the generator can pass through the box coefficients due to their invariance. Thus $\hat{K}^\mu$ directly acts on the box functions F_{rstu}, whose anomaly structure resembles that of their infrared divergences. Finally one finds $n(n-4)$ independent equations, eqs. (2.7) and (2.10), which fall into two categories.

- n combinations of (2-mass hard and 1-mass) box coefficients equal to tree amplitudes, which are in one-to-one correspondence with the n IR equations of this category,

  $$\mathcal{E}_{i,i-2} = -\mathcal{E}_{i-1,i} = -2M_n^{\text{tree}}\,, \tag{2.7}$$

  where $i = 1, ..., n$ and

  $$\mathcal{E}_{i,i-2} = -\mathcal{E}_{i-1,i} := -\sum_{j=i+1}^{i+n-3} C_{i-2,i-2,i,j} \tag{2.8}$$

  are boundary cases of the more general $\mathcal{E}_{i,k}$ defined in eq. (2.10) below.

- $n(n-3)$ combinations of box coefficients vanish,

  $$\mathcal{E}_{i,k} = 0\,, \tag{2.9}$$

  where $i = 1, ..., n$ and $k = i+2, ..., i+n-3$. The quantities $\mathcal{E}_{i,k}$ are defined as

  $$\mathcal{E}_{i,k} := \sum_{j=k+1}^{i+n-2} C_{i,k,j,i-1} - \sum_{j=i+1}^{k-1} C_{i,j,k,i-1}\,. \tag{2.10}$$

  Since there are $2n$ algebraic identities among them, we have $n(n-5)$ independent constraints. These constraints imply the $\frac{n(n-5)}{2}$ IR equations of the same category but also $\frac{n(n-5)}{2}$ new constraints.

Finally we would like to emphasize that, although in the following we mainly focus on the NMHV sector, the IR equations and dual conformal constraints are valid for any N^{k-2}MHV amplitude.

[2] Box coefficients are invariant [11] under the shifted operator $\hat{K}^\mu$ rather than covariant under the usual K^μ.

2.2 Grassmannian formulation

The object which was suggested by Arkani-Hamed *et al.* to calculate leading singularities of all-loop n-particle amplitudes in the N$^{(k-2)}$MHV sector is a gauged version [12] of eq. (1.1)

$$\mathcal{L}_{n;k} = L_{n;k} \times \delta^4(\sum_a p_a), \tag{2.11}$$

where $L_{n;k}$ is defined as

$$L_{n;k} = J(\lambda, \tilde{\lambda}) \int \frac{d^{(k-2)\times(n-k-2)}\tau}{[(12\cdots k)\,(23\cdots(k+1)\,)\,\cdots(n1\cdots(k-1)\,)]\,(\tau)} \prod_I \delta^4(\tilde{\eta}_I + c_{Ii}(\tau)\tilde{\eta}_i)\,. \tag{2.12}$$

In the above expression, J is a Jacobian prefactor depending on the precise fixing of the gauge. The remaining variables to integrate over are τ_γ, where $\gamma = 1, \ldots, d$ with $d := (k-2) \times (n-k-2)$. The objects in the denominator, called minors,

$$(m_1 \cdots m_k) \equiv \varepsilon^{\alpha_1 \cdots \alpha_k} C_{\alpha_1 m_1} \cdots C_{\alpha_k m_k}\,, \tag{2.13}$$

are obtained from a gauge-fixed $k \times n$ matrix C whose nontrivial elements c_{Ii} are solutions to the kinematical constraints

$$\lambda_i - c_{Ii}(\tau_\gamma)\lambda = 0, \quad \tilde{\lambda}_I + c_{Ii}(\tau_\gamma)\tilde{\lambda}_i = 0. \tag{2.14}$$

A couple of simplifications take place in the NMHV sector: the number of integration variables is just $d = n-5$, and no composite residues occur [12]. More importantly, the degree of the minors eq. (2.13) can be shown to be $\min[k-2, n-k-2]$, which renders the NMHV situation the easiest nontrivial one: for $k = 3$ the minors are linear expressions in τ. Note, that the integrand is a holomorphic function of the (complexified) variables τ_γ. In this light, eq. (2.11) is a contour integral in $\mathbb{C}^d$.

A residue of eq. (2.12) occurs at a multiple zero of degree d. In the NMHV situation this is equivalent to choosing $d = n-5$ minors to vanish simultaneously. Due to the linear degree of the minors there is exactly one solution and thus one residue for each of these choices. Correspondingly, any residue in the NMHV sector can be identified by noting the vanishing minors, where each minor (cf. eq. (2.13)) is referred to by its first entry m_1. The resulting list with $n-5$ elements is enclosed in curly brackets.

In the 8-point NMHV situation a residue can be calculated by choosing $3 = n-5$ minors to vanish. For example the residue obtained by setting the denominator terms (123), (234) and (781) to zero, will be referred to as $\{1, 2, 7\}$.

As shown in [12], the determination of a multi-dimensional residue includes a determinant, which renders the labeling for residues totally antisymmetric, for example

$$\{i, j, k\} = -\{i, k, j\}\,. \tag{2.15}$$

Although the labeling of NMHV residues with $n-5$ coordinates is favorable for lower-point amplitudes, for general considerations we will fall back to the *complementary labeling.*

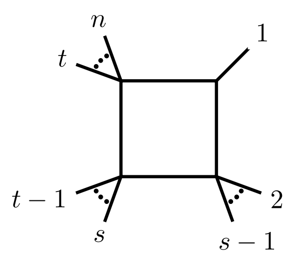
The usual labeling can be obtained from the 5-number complementary labeling by the bar operation

$$\{j_1, \ldots, j_{(n-5)}\} = \overline{\{i_1, \ldots, i_5\}} = \{\Xi\} \cdot \mathrm{sgn}(i_1, \ldots, i_5) \cdot \mathrm{sgn}(i_1, \ldots, i_5, \Xi) \tag{2.16}$$

where Ξ is the ordered complement

$$\{1, \ldots, n\} \backslash \{i_1, \ldots, i_5\}\,. \tag{2.17}$$

In the NMHV sector, there is a clear map between box coefficients and residues. The simplest situation occurs for the 3-mass box, where the following box coefficient is given by:



$$C^{3m}_{12st} \,\widehat{=}\, \overline{\{s-2, s-1, t-2, t-1, n\}}, \tag{2.18}$$

and any other 3-mass boxes can be obtained by cyclic shifts.

The expressions for other box coefficients can be easily obtained as sums of (degenerate) 3-mass boxes by employing the results from [29]:

$$\begin{aligned}
C^{1m}_{r,r+1,r+2,r+3} &= C^{2me}_{r+2,r+3,r,r+1} + C^{3m}_{r+1,r+2,r+3,r} \\
C^{2mh}_{r,r+1,r+2,s} &= C^{3m}_{r+1,r+2,s,r} + C^{3m}_{r,r+1,r+2,s} && (s > r+3,\ r > s+1) \\
C^{2me}_{r,r+1,s,s+1} &= \sum_{\substack{u,v \\ u \geq r+2 \\ u+2 \leq v \leq s}} C^{3m}_{r,r+1,u,v} + \sum_{\substack{u,v \\ u \geq s+2 \\ u+2 \leq v \leq r}} C^{3m}_{s,s+1,u,v} && (s > r+2, r > s+2),
\end{aligned} \tag{2.19}$$

where all indices have to be understood modulo n. By writing '$>$' we mean '$> \bmod n$' and the summations have also to be adapted accordingly. If not stated otherwise, the modulo-n notation will be understood implicitly below.

Finally, the NMHV tree-amplitude can be expressed in terms of residues. With the sums

$$\begin{aligned}
E &= \sum_{k \text{ even}} \{k\} \\
O &= \sum_{k \text{ odd}} \{k\}
\end{aligned} \tag{2.20}$$

and the product[3]

$$\{i_1\} \star \{i_2\} = \left\{ \begin{array}{l} \{i_1, i_2\} \text{ if } \ i_1 < i_2 \\ 0 \quad \text{otherwise} \end{array} \right\} \tag{2.21}$$

[3] Here we use the notation: $\{i\}\{j\} := \{i, j\}$ and $\{i + j\}\{k\} := \{i, j\} + \{i, k\}$.

the BCFW form of the tree-amplitude is given by

$$M^{\text{tree}}_{\text{BCFW}} = E \star O \star E \star \cdots\,, \tag{2.22}$$

and the parity-conjugated (P(BCFW)) form is obtained from

$$M^{\text{tree}}_{\text{P(BCFW)}} = (-1)^{n-5}\, O \star E \star O \star \cdots\,. \tag{2.23}$$

While for tree-level amlitudes and one-loop leading singularities there is a clear map to residues, there is also evidence that certain residues encode higher-loop information [12, 26, 27]. An interesting question is the following: is it possible to distinguish, which residues appear at tree and one-loop level and which participate in higher-loop leading singularities only? Section 3 is devoted to this classification.

As discussed in reference [12], residues of eq. (2.12) are not independent objects, but are subject to global residue theorems (GRTs). In particular, all NMHV GRTs can be generated from basic GRTs, which have the form,

$$\sum_{j=1}^{n} \{j, i_1, ..., i_{n-6}\} = 0\,. \tag{2.24}$$

where $(i_1, ..., i_{n-6})$ is referred to as *source term* and will be used to uniquely label basic GRTs below. It is not difficult to see that any GRT constrains the sum of six residues to vanish. In order to avoid confusion, source terms are enclosed by usual brackets (), while residues will be enclosed by curly brackets {}.

In reference [12] it was shown on a couple of examples that indeed one-loop IR equations and the identity between the BCFW and P(BCFW) form of the tree amplitude can be traced back to GRTs in the Grassmanian formulation. Since at the NMHV level one can unambiguously map all one-loop leading singularities to residues, it is natural to investigate the precise map between all one-loop IR equations, and more generally, dual conformal constraints to sums of GRTs. In addition, we will find out which GRTs imply the equality of the BCFW and the P(BCFW) form of the tree amplitude, which will be referred to as *remarkable identity.* This analysis will be performed in section 4.

3 Classification of residues

As we have discussed above, any residue at NMHV level is labeled by $n-5$ numbers determining the minors to be set to zero in eq. (2.12).

Employing the identifications eqs. (2.18) and (2.19), one can single out all residues which appear in the one-loop leading singularities. This will also include the constituents of the tree amplitude, as those are related to the one-loop leading singularities by eq. (2.5). For $n \leq 7$ this covers all possible residues.

Starting from $n = 8$, there are residues which only contribute to at least two-loop leading singularities, but do not participate in any one-loop leading singularities [12]. However, residues occurring already in one-loop singularities can contribute to two-loop (and higher) leading singularities.

As argued in [27], certain residues appear at three-loop level only for amplitudes with $n \geq 10$. This is also the maximal loop-level for NMHV leading singularities: as shown in the same reference, there are no leading singularities at four-loops and higher.

Assuming the Grassmannian conjecture to be true, it is clear that residues for any NMHV amplitude should be classified by whether they appear at the one-loop, two-loop or three-loop level *first*. Although they can contribute to higher-loop leading singularities, we will refer to those residues in a slightly inaccurate manner as one-loop, two-loop and three-loop residues respectively.

As will be proven below, residues can be classified by using invariant labels. Given any sequence of cyclically ordered numbers $(i_1, \ldots, i_p)$ where $i_l \in \{1, \ldots, n\}$ and $p \leq n$, the invariant label is defined as the set

$$\{(i_2 - i_1) \bmod n,\, (i_2 - i_1) \bmod n,\, \ldots, (i_1 - i_p) \bmod n\}\,. \tag{3.1}$$

As is obvious from the above definition, the name invariant label refers to its invariance under cyclic shifts of $(i_1, \ldots, i_p)$. Since any residue is a sequence of $n-5$ numbers from $1, \ldots, n$, one can determine an invariant label for each of them.

Starting from the fact that any invariant label for a residue is a decomposition of n into $n-5$ numbers, it is easy to prove that, given a sufficiently large n, there are exactly seven distinct invariant labels:

type	invariant label
1	$\{1, ..., 1, 6\}$
2	$\{1, ..., 1, 2, 5\}$
3	$\{1, ..., 1, 3, 4\}$
4	$\{1, ..., 1, 2, 3, 3\}$
5	$\{1, ..., 1, 2, 2, 4\}$
6	$\{1, ..., 1, 2, 2, 2, 3\}$
7	$\{1, ..., 1, 2, 2, 2, 2, 2\}$.

In addition to the classification of the loop level, we will find the invariant label to contain additional information: certain types of box coefficients correspond to particular types of one-loop residues, as will be discussed below.

The mapping of box coefficients to residues is given in terms of the complementary labeling defined in section 2.2. Therefore, in a first step we will have to find classes of complementary labels corresponding to types of residues. As expected from the definition of the invariant label, the criterion for classification is the number of succesive subsequences. Again there are exactly seven types, corresponding to the seven possible invariant labels in the table above:

type	complementary sequence	length of succ. subseq.
1	$\overline{\{i, i+1, i+2, i+3, i+4\}}$	5
2	$\overline{\{i, i+1, i+2, i+3, j_{>i+4}\}}$ $\overline{\{j, i_{>j+1}, i+1, i+2, i+3\}}$	4
3	$\overline{\{i, i+1, i+2, j_{>i+3}, j+1\}}$ $\overline{\{j, j+1, i_{>j+2}, i+1, i+2\}}$	3 and 2
4	$\overline{\{i, i+1, j_{>i+2}, j+1, k_{>j+2}\}}$ $\overline{\{k, i_{>k+1}, i+1, j_{>i+2}, j+1\}}$ $\overline{\{i, i+1, k_{>i+2}, j_{>k+1}, j+1\}}$	2 and 2
5	$\overline{\{i, i+1, i+2, j_{>i+3}, k_{>j+1}\}}$ $\overline{\{j, k_{>j+1}, i_{>k+1}, i+1, i+2\}}$ $\overline{\{j, i_{>j+1}, i+1, i+2, k_{>i+3}\}}$	3
6	$\overline{\{i, i+1, j_{>i+2}, k_{>j+1}, l_{>k+1}\}}$ $\overline{\{j, k_{>j+1}, l_{>k+1}, i_{>l+1}, i+1\}}$	2
7	$\overline{\{i, j_{>i+1}, k_{>j+1}, l_{>k+1}, m_{>l+1}\}}$	0

Comparing this table with the results for one-loop leading singularities, eqs. (2.18) and (2.19), it is straightforward to see that types 1 to 4 correspond to one-loop residues. The invariant labels of types 1 to 4 contain exactly one even number.

Alternatively, we could have considered the BCFW and P(BCFW) form of NMHV tree amplitudes given by eqs. (2.22) and (2.23). Using the fact that their residues are odd/even alternating sequences, it follows immediately that their invariant labels can contain one even number only. Since those representations of the tree amplitudes are built from one-loop residues, we again arrive at the conclusion that they belong to types 1 to 4.

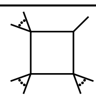
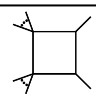
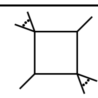
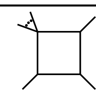
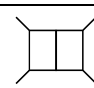
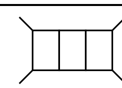
Remembering the results from refs. [12, 27] stated at the beginning of this section, the remaining types 5, 6 and 7 must correspond to higher-loop residues. The fact that all leading singularities for $n \leq 7$ are combinations of one-loop residues nicely agrees with the lack of decompositions of types 5, 6 or 7 at $n \leq 7$.

In addition, since types 5 and 6 can appear for $n \geq 8$ but the last type only appears for $n \geq 10$, we can fit this fact to the results of [27]. We conjecture that types 5 and 6 correspond to two-loop residues while three-loop residues can be assigned to type 7.

n	invariant label
6	$\{6\}$
7	$\{1,6\}$,$\{2,5\}$,$\{3,4\}$
8	$\{1,1,6\}$,$\{1,2,5\}$,$\{1,3,4\}$,$\{2,3,3\}$,$\{2,2,4\}$
9	$\{1,1,1,6\}$,$\{1,1,2,5\}$,$\{1,1,3,4\}$,$\{1,2,2,3\}$,$\{1,2,2,4\}$,$\{2,2,2,3\}$
10	$\{1,1,1,1,6\}$,$\{1,1,1,2,5\}$,$\{1,1,1,3,4\}$,$\{1,1,2,2,3\}$,$\{1,1,2,2,4\}$,$\{1,2,2,2,3\}$,$\{2,2,2,2,2\}$
$\vdots$	$\vdots$
n	$\{1,...,1,6\}$,$\{1,...,1,2,5\}$,$\{1,...,1,3,4\}$,$\{1,...,1,2,3,3\}$, $\{1,...,1,2,2,4\}$,$\{1,...,1,2,2,2,3\}$,$\{1,...,1,2,2,2,2,2\}$.

It is remarkable to see that no further types appear as the number of particles increases, which agrees with the claim that all NMHV leading singularities are combinations of these three types of residues [27].

If we examine the first four cases more carefully, we can see further classifications among these one-loop residues. By comparing them with results in section 2.2, we conclude that 3-mass leading singularities can receive contributions from type 2 and 4 residues and residues for 2-mass hard leading singularities can be type 1 and 3. For 2-mass easy, the corresponding residues are of type 2, 3 and 4, and 1-mass leading singularities can have all four possible cases for one-loop residues. In summary, a complete classification of residues based on invariant labels is presented in table below

	3m	2mh	2me	1m	2-loop	3-loop
						 .
types	2 or 4	(1 + 3) or (3 + 3)	(1 + 2 +3) or (1 + 1)	1,2,3,4	5,6	7

4 Mapping one-loop dual conformal constraints and IR equations to GRTs

After having classified all NMHV residues, all one-loop dual conformal constraints will be translated into residues in this section. First of all, the source terms of the GRTs necessary to show

$$\mathcal{E}_{i,k} = 0 \tag{4.1}$$

for $i = 1, ..., n$ and $k = i + 2, ..., i + n - 3$ shall be investigated.

Starting with $\mathcal{E}(1,4) = 0$ for an amplitude with $n = 9$ legs as an example, the corresponding expression in terms of residues reads

$$\begin{aligned}(&\{1,2,4,9\} + \{1,2,6,9\} + \{1,2,8,9\} + \{1,3,4,9\}\\ &+ \{1,3,6,9\} + \{1,3,8,9\} - \{1,4,5,9\} - \{1,4,7,9\}\\ &+\{1,5,6,9\} + \{1,5,8,9\} - \{1,6,7,9\} + \{1,7,8,9\}) = 0\,.\end{aligned} \tag{4.2}$$

The above equality can be obtained by adding GRTs with the following source terms

$$-(1,4,9) - (1,6,8) - (1,8,9) = 0\,. \tag{4.3}$$

Here and below the global minus sign in the equation will not be noted for these vanishing results.

For finding a general rule describing which GRTs have to be added, it is sufficient to restrict the attention to the case $i = 1$, because all other conformal constraints can be obtained by cyclical shifts. In the table below, a couple of lower-point examples are listed.

part.	$\mathcal{E}(i,k)$	$m{=}k{-}i$	source terms
7	$\mathcal{E}(1,3)$	2	$0 = (7)$
8	$\mathcal{E}(1,3)$	2	$0 = (3,8) + (5,8) + (7,8)$
	$\mathcal{E}(1,4)$	3	$0 = (1,8)$
	$\mathcal{E}(1,5)$	4	trivial vanishing
	$\mathcal{E}(1,6)$	5	trivial vanishing
9	$\mathcal{E}(1,3)$	2	$0 = (3,4,9) + (3,6,9) + (3,8,9) + (5,6,9) + (5,8,9) + (7,8,9)$
	$\mathcal{E}(1,4)$	3	$0 = (1,4,9) + (1,6,9) + (1,8,9)$
	$\mathcal{E}(1,5)$	4	$0 = (1,2,9)$
	$\mathcal{E}(1,6)$	5	trivial vanishing
	$\mathcal{E}(1,7)$	6	trivial vanishing
10	$\mathcal{E}(1,3)$	2	$0 = (3,4,5,10) + (3,4,7,10) + (3,4,9,10) + (3,6,7,10) + (3,6,9,10)$
			$+(3,8,9,10) + (5,6,7,10) + (5,6,9,10) + (5,8,9,10) + (7,8,9,10)$
	$\mathcal{E}(1,4)$	3	$0 = (1,4,5,10) + (1,4,7,10) + (1,4,9,10)$
			$+(1,6,7,10) + (1,6,9,10) + (1,7,9,10)$
	$\mathcal{E}(1,5)$	4	$0 = (1,2,5,10) + (1,2,7,10) + (1,2,9,10)$
	$\mathcal{E}(1,6)$	5	$0 = (1,2,3,10)$
	$\mathcal{E}(1,7)$	6	trivial vanishing
	$\mathcal{E}(1,8)$	7	trivial vanishing
$\vdots$	$\vdots$	$\vdots$	$\vdots$

Based on those examples and further tests, the general rule for obtaining the source terms in the fourth column for a certain m and n legs can be obtained as

$$\sum_{\mathcal{V}} (1,2,..,(m-2), \underbrace{v_1,\ldots,v_{n-m-5}}_{\mathcal{V}}, n) = 0\,, \tag{4.4}$$

where $\mathcal{V}$ is a strictly increasing succession of $n-m-5$ numbers

$$v \in \{m+1,\ldots,n-1\}, \tag{4.5}$$

which has to be chosen such that the whole source term is strictly odd/even alternating, and the summation is over all possible $\mathcal{V}$s.

For example, in order to obtain the GRTs for the vanishing of $\mathcal{E}(1,5)$ in a scenario with $n = 11$, one would start with $(1,2,\mathcal{V},11)$. According to eq. (4.5) the numbers $v \in \mathcal{V}$ have to be in the range $\{5,\ldots,10\}$. Thus all valid choices for $\mathcal{V}$ in this scenario are

$$(5,6),\ (5,8),\ (5,10),\ (7,8),\ (7,10),\ (9,10)\,, \tag{4.6}$$

which finally leads to

$$\mathcal{E}(1,5) = (1,2,5,6,11) + (1,2,5,8,11) + (1,2,5,10,11) \\ + (1,2,7,8,11) + (1,2,7,10,11) + (1,2,9,10,11) = 0\,. \tag{4.7}$$

Considering the other type of dual conformal constraint (cf. eq. (2.7)),

$$\mathcal{E}_{i,i-2} = -\mathcal{E}_{i-1,i} = -2M_n^{\text{tree}}\,, \tag{4.8}$$

one first needs to pick a form of the tree amplitude. For the investigation here it will be useful to choose

$$2M^{\text{tree}} = M^{\text{tree}}_{\text{BCFW}} + M^{\text{tree}}_{\text{P(BCFW)}}\,, \tag{4.9}$$

where the two representations of the tree-level amplitude have been given in eqs. (2.22) and (2.23). The BCFW and the P(BCFW) form of the tree amplitude are cyclically invariant, but in order to show this, it is necessary to employ GRTs. For example, the equality of the seven-point BCFW form of the tree amplitude to its shifted version,

$$\{2,3\} + \{2,5\} + \{2,7\} + \{4,5\} + \{4,7\} + \{6,7\} \\ = \{3,4\} + \{3,6\} + \{3,1\} + \{5,6\} + \{5,1\} + \{7,1\}, \tag{4.10}$$

is not obvious. So we expect that only for one particular choice of i in eq. (4.8) is the expression eq. (4.9) obtained. Translating again box coefficients into residues confirms this expectation. Only for $i = 2$ is the chosen form of the tree amplitude reproduced,

$$\mathcal{E}(2,n) = -\mathcal{E}(1,2) = -(M^{\text{tree}}_{\text{BCFW}} + M^{\text{tree}}_{\text{P(BCFW)}})\,. \tag{4.11}$$

Having mapped all dual conformal constraints to sums of GRTs, it is interesting to make contact to the IR equations. As discussed in [24], any IR equation can be written down as a particular combination of $\mathcal{E}(i,k)$,

$$[\![i]\!]_2 : \mathcal{E}_{i,i+2} + \mathcal{E}_{i+2,i} - \mathcal{E}_{i+3,i} = -2M_n^{\text{tree}}\,, \tag{4.12}$$

and

$$[\![i]\!]_m : \mathcal{E}_{i,i+m} + \mathcal{E}_{i+m,i} - \mathcal{E}_{i+1,i+m} - \mathcal{E}_{i+m+1,i} = 0\,, \tag{4.13}$$

for $m = 3, ..., \lfloor n/2 \rfloor$. It would be straightforward to just add and subtract the appropriate source terms corresponding to the different terms $\mathcal{E}(i,k)$. However, because of cancellations between different GRTs this does not result in the simplest possible expression. Therefore the explicit analysis performed for the dual conformal constraints will be repeated for the IR equations below[4].

[4]Note that the considerations are again limited to the starting point $i = 1$, because one can always obtain results for other i's by cyclic shifts.

As an illustrations for IR equations, let's consider the kinematic invariant $[\![1]\!]_2$ for the 7-point NMHV scenario. Either scanning for the IR divergences or directly using eq. (4.12) delivers

$$C^{\rm 1m}_{1234}+C^{\rm 1m}_{7123}+\tfrac{1}{2}C^{\rm 2mh}_{1235}-\tfrac{1}{2}C^{\rm 2mh}_{6713}+\tfrac{1}{2}C^{\rm 2mh}_{1236}-\tfrac{1}{2}C^{\rm 2mh}_{3451}-C^{\rm 2me}_{7134}-\tfrac{1}{2}C^{\rm 3m}_{3461}-\tfrac{1}{2}C^{\rm 3m}_{7135} \tag{4.14}$$

for the left hand side of eq. (2.5). Translating into residues, one obtains:

$$\begin{aligned}(\{7,1\}+\{5,1\}+\{3,1\}+\{4,5\})+(\{6,7\}+\{4,7\}+\{2,7\}+\{3,4\})\\ +\tfrac{1}{2}(\{2,5\}+\{5,6\})-\tfrac{1}{2}(\{7,3\}+\{3,4\})+\tfrac{1}{2}(\{2,3\}+\{3,6\})-\tfrac{1}{2}(\{4,5\}+\{5,1\})\\ -(\{7,1\})-\tfrac{1}{2}(\{3,1\})-\tfrac{1}{2}(\{7,5\})\end{aligned} \tag{4.15}$$

which by use of GRTs equals

$$\{2,3\}+\{2,5\}+\{2,7\}+\{4,5\}+\{4,7\}+\{6,7\}\,. \tag{4.16}$$

This is precisely the BCFW-form of the 7-point NMHV tree-amplitude.

As a second example, let's consider the invariant $[\![1]\!]_5$ for 9 particles. The corresponding IR equation reads

$$\begin{aligned}-\tfrac{1}{2}C^{\rm 2mh}_{1235}-\tfrac{1}{2}C^{\rm 2mh}_{5671}+C^{\rm 2mh}_{9125}+C^{\rm 2mh}_{4561}-\tfrac{1}{2}C^{\rm 2mh}_{8915}-\tfrac{1}{2}C^{\rm 2mh}_{3451}-C^{\rm 2me}_{1245}-\tfrac{1}{2}C^{\rm 3m}_{5681}+\tfrac{1}{2}C^{\rm 3m}_{9135}\\ +\tfrac{1}{2}C^{\rm 3m}_{4571}+C^{\rm 3m}_{1256}-C^{\rm 3m}_{5691}+C^{\rm 3m}_{9145}+\tfrac{1}{2}C^{\rm 3m}_{1257}+\tfrac{1}{2}C^{\rm 3m}_{4581}+\tfrac{1}{2}C^{\rm 3m}_{1258}+\tfrac{1}{2}C^{\rm 3m}_{5613}-\tfrac{1}{2}C^{\rm 3m}_{9157}=0\,.\end{aligned} \tag{4.17}$$

Translating into residues and sorting out the pre-factors (but not using any GRT) leads to the following result:

$$\begin{aligned}\tfrac{1}{2}\,(&-\{1,2,3,5\}-\{1,2,3,7\}-\{1,2,4,5\}-\{1,2,4,7\}\\ &+\{1,2,5,6\}+\{1,2,5,8\}+\{1,2,5,9\}-\{1,2,6,7\}\\ &+\{1,2,7,8\}+\{1,2,7,9\}-\{1,5,6,7\}-\{2,5,6,7\}\\ &-\{3,5,6,7\}-\{4,5,6,7\}+\{5,6,7,8\}+\{5,6,7,9\})=0\,.\end{aligned} \tag{4.18}$$

It is not difficult to see that this equation arises by adding GRTs with the following source terms:

$$-(1,2,5)-(1,2,7)-(5,6,7)=0\,. \tag{4.19}$$

Now we turn to a general analysis of GRTs for IR equations. For $m=2$, the linear combination of residues should coincide with the expression for the tree amplitude. From eqns. (4.4),(4.11) and (4.12), it is straightforward to see which GRTs lead to the parity-invariant form of the tree amplitude eq. (4.9). With other choices of GRTs, we can also get BCFW, P(BCFW)(eqs. (2.22) and (2.23)) or any other form of the tree amplitude in terms of residues.

In case of $m>2$ there is no infrared divergence on the right hand side of eq. (2.5). Therefore, the sum is expected to vanish by use of certain combinations of GRTs.

The following table shows a couple of examples:

particles	kin. inv	source terms
7	$[\![1]\!]_3$	$0=(1)$
8	$[\![1]\!]_3$	$0=(1,4)+(1,6)$
	$[\![1]\!]_4$	$0=(1,2)+(5,6)$
9	$[\![1]\!]_3$	$0=(1,4,5)+(1,4,7)+(1,6,7)$
	$[\![1]\!]_4$	$0=(1,2,5)+(1,2,7)+(5,6,7)$
10	$[\![1]\!]_3$	$0=(1,4,5,6)+(1,4,5,8)+(1,4,7,8)+(1,6,7,8)$
	$[\![1]\!]_4$	$0=(1,2,5,6)+(1,2,5,8)+(1,2,7,8)+(5,6,7,8)$
	$[\![1]\!]_5$	$0=(1,2,3,6)+(1,2,3,8)+(1,6,7,8)+(3,6,7,8)$
$\vdots$	$\vdots$	$\vdots$
12	$[\![1]\!]_3$	$0=(1,4,5,6,7,8)+(1,4,5,6,7,10)+(1,4,5,6,9,10)$ $+(1,4,5,8,9,10)+(1,4,7,8,9,10)+(1,6,7,8,9,10)$
	$[\![1]\!]_4$	$0=(1,2,5,6,7,8)+(1,2,5,6,7,10)+(1,2,5,6,9,10)$ $+(1,2,5,8,9,10)+(1,2,7,8,9,10)+(5,6,7,8,9,10)$
	$[\![1]\!]_5$	$0=(1,2,3,6,7,8)+(1,2,3,6,7,10)+(1,2,3,6,9,10)$ $+(1,2,3,8,9,10)+(1,6,7,8,9,10)+(3,6,7,8,9,10)$
	$[\![1]\!]_6$	$0=(1,2,3,4,7,8)+(1,2,3,4,7,10)+(1,2,3,4,9,10)$ $+(1,2,7,8,9,10)+(1,4,7,8,9,10)+(3,4,7,8,9,10)$

Inspecting the above table, one initial observation can be made: any one-loop IR equation for an n-point amplitude can be represented as the sum of $n-6$ basic GRTs, each of which is a sum of 6 residues. However, some residues will cancel, thus the final number of terms is smaller than $6\cdot(n-6)$.

Furthermore, based on the examples in the table above and further tests up to $n=20$, we propose the general rule, which is kept odd/even alternating, for $m=3$ IR equations,

$$[\![1]\!]_3\ :\sum_{\mathcal{V}}(1,\underbrace{v_1,\ldots,v_{n-7}}_{\mathcal{V}})=0\,, \tag{4.20}$$

where $\mathcal{V}$ a strictly increasing succession of $n-7$ numbers,

$$v\in\{4,\ldots,n-2\}. \tag{4.21}$$

For $m>3$, another kind of source terms appears. While the type already encountered for $m=3$ starts from the left with $(1,\ldots)$, a second type has the form $(\ldots,n-2)$ starting from the right. Again by testing examples up to $n=20$, for $4\le m\le\lfloor n/2\rfloor$, we find the following general rule,

$$\begin{aligned}[\![1]\!]_m\ :\sum_{\mathcal{V}}&(1,2,\ldots,(m-2)\,,\underbrace{v_1,\ldots,v_{n-m-4}}_{\mathcal{V}})\\ &+\sum_{\mathcal{W}}(\underbrace{w_1,\ldots,w_{n-m-4}}_{\mathcal{W}},(m+1),\ldots,(n-2))=0\,,\end{aligned} \tag{4.22}$$

where $\mathcal{V}$ and $\mathcal{W}$ are again strictly increasing successions of $n-m-4$ numbers satisfying

$$m+1\le v\le n-2\qquad\text{and}\qquad 1\le w\le m-2 \tag{4.23}$$

and chosen to respect the odd/even alternating structure of source terms. The first term in eq. (4.22) is a generalization of eq. (4.20).

The labels of source terms in our general formula eq. (4.22) are strictly odd/even alternating, which nicely connects to the classification of residues: starting from a strictly alternating source term, the residues in the resulting GRT will be either strictly alternating themselves or have an "all-but-one" odd/even alternating structure such as *eoeoo*. Translating this into the language of invariant labels and comparing with the possible invariant labels singles out exactly types 1 to 4, as expected.

One of the most important implications of one-loop IR equations, which is completely obscured in the BCFW formalism is the remarkable identity. This identity relates the BCFW representation of the tree-amplitude eq. (2.22) with the P(BCFW) form eq. (2.23),

$$M_{\mathrm{BCFW}} = M_{\mathrm{P(BCFW)}}\,. \tag{4.24}$$

Being highly nontrivial identities in the BCFW approach, they have an astonishingly simple form in the language of residues.

While it was already shown in [12] that identities

$$E \star O \star E \star \cdots = (-1)^{(n-5)} O \star E \star O \star \cdots \tag{4.25}$$

are implied by GRTs, we have found a way to derive them directly from combinations of GRTs.

The statement is simple: adding all GRTs corresponding to all source terms of the form *oeoe*... for a particular number of legs produces the remarkable identity

$$M_{\mathrm{BCFW}} = M_{\mathrm{P(BCFW)}} : O \star E \star O \cdots = 0, \tag{4.26}$$

where $O = \sum_{\mathrm{k\,odd}}(k)$ and $E = \sum_{\mathrm{k\,even}}(k)$, and we have $n-6$ factors since this is a sum of GRTs labeled by source terms.

For example, adding GRTs with source-terms

$$(1,2),\ (1,4),\ (1,6),\ (1,8),\ (3,4),\ (3,6),\ (3,8),\ (5,6),\ (5,8)\ \text{and}\ (7,8)$$

produces the identity for $n = 8$:

$$\begin{aligned}
&\{2,3,4\} + \{2,3,6\} + \{2,3,8\} + \{2,5,6\} + \{2,5,8\} \\
&+\{2,7,8\} + \{4,5,6\} + \{4,5,8\} + \{4,7,8\} + \{6,7,8\} \\
&= -(\{1,2,3\} + \{1,2,5\} + \{1,2,7\} + \{1,4,5\} + \{1,4,7\} \\
&+\{1,6,7\} + \{3,4,5\} + \{3,4,7\} + \{3,6,7\} + \{5,6,7\})
\end{aligned} \tag{4.27}$$

As expected by parity invariance, one obtains the same result by adding GRTs corresponding to all source terms of the form *eoeo*... .

5 Conclusion

In this article we have identified which global residue theorems in the Grassmannian formulation of $\mathcal{N}$=4 SYM theory imply the recently derived one-loop dual conformal constraints and the well-known one-loop IR equations in the NMHV sector. For both sets of equations the source terms for the corresponding GRTs can be obtained from the general rules eqs. (4.4) and (4.22). In addition, the remarkable identity relating the BCFW and the P(BCFW) form of the tree amplitude emerges from adding all GRTs with an odd/even alternating pattern of source terms eq. (4.26).

According to the classification of NMHV residues performed in the initial part of the article, all one-loop residues are of odd/even alternating or "all-but-one" odd/even alternating structure. This nicely fits to the general rules: all GRTs involved have strictly odd/even alternating source terms, which relate exactly these types of residues.

Even restricting the consideration to the NMHV sector, there are many GRTs beyond the ones employed in the mapping. These are presumably related to higher-loop dual conformal constraints and IR equations. From the classification it is obvious which residues contribute to higher-loop leading singularities. However, without a general formalism to single out an integral basis for two loops and beyond, the identification of higher-loop dual conformal constraints and IR equations is left for a future project.

Furthermore, although dual conformal constraints (and thus IR equations) should be related to GRTs beyond the NMHV sector, a general map has not been found so far. In [30], general contours for N^2MHV tree amplitudes have been derived using ideas from localization in the Grassmannian manifold. It would be extremely interesting to generalize this analysis to leading singularities of loop amplitudes beyond the NMHV sector. Once this is achieved, it should be possible to identify the Grassmannian origin of dual conformal constraints for $k > 3$.

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