

BLACK HOLE NORMAL MODES: A SEMIANALYTIC APPROACH¹

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ABSTRACT

We present a new semianalytic technique for determining the complex normal mode frequencies of black holes. The method is based on the WKB approximation. It yields a simple analytic formula that gives the real and imaginary parts of the frequency in terms of the parameters of the black hole and of the field whose perturbation is under study, and in terms of the quantity $(n + \frac{1}{2})$, where $n = 0, 1, 2, \dots$ and labels the fundamental mode, first overtone mode, and so on. In the case of the fundamental gravitational normal modes of the Schwarzschild black hole, the WKB estimates agree with numerical results to better than 7% in the real part of the frequency and 0.7% in the imaginary part, with the relative agreement improving with increasing angular harmonic. Carried to higher order, the method may provide an accurate and systematic means to study black hole normal modes.

Subject headings: black holes — gravitation

For several years, the normal modes of oscillation of black holes have been of great interest both to gravitation theorists and to gravitational wave experimentalists (Detweiler 1979). These modes are the resonant, nonradial deformations of black holes, analogous to those of the Sun or Earth, that can be induced by external perturbations. They are characterized by a spectrum of discrete, complex frequencies, the real part of the frequencies corresponding to the oscillation frequency, and the imaginary parts corresponding to the rate at which each mode is damped as a result of the emission of radiation. For a given kind of physical perturbation (scalar, electromagnetic, gravitational), the complex frequencies are uniquely determined by the mass and angular momentum of the hole, the angular harmonic index (l, m) of the deformation, and the degree of the harmonic of the mode.

To the gravitational wave astronomer, black hole normal modes may be an interesting source of gravitational waves emitted at discrete frequencies by a deformed black hole left over following a supernova collapse. The identification of the frequencies and damping times of such waves could aid in estimating the parameters of the black hole. Normal modes are also important in analyzing the stability of black holes against external perturbations. Although the nonrotating Schwarzschild black hole is known rigorously to be stable, the situation is not so certain in the case of the rotating Kerr black hole (Detweiler and Ove 1983). Normal modes may also play a role in the quantum mechanical evaporation of black

holes. A recent model for this process developed by York (1983) makes use of the fluctuations of the horizon at the normal mode frequencies as a means to estimate the temperature that characterizes the radiation evaporated from the hole.

Previous approaches to black hole normal modes have relied heavily on numerical techniques (Detweiler 1979). Although the basic equations describing the perturbations of black holes reduce to a single second-order ordinary differential equation that is equivalent to the one-dimensional Schrödinger equation for a particle encountering a potential barrier, the nature of the potential precludes an exact solution in terms of known functions. Numerical integration of the equations requires selecting a value for the complex frequency, integrating the differential equation, and checking whether the boundary conditions for a normal mode are satisfied (outgoing waves at infinity, ingoing waves at the horizon). Since those conditions are not satisfied in general, the complex frequency plane must be surveyed for the discrete values that lead to normal modes. This technique is time consuming and therefore costly, and it makes difficult a systematic survey of normal modes for a wide range of parameter values.

We have developed what we believe is a promising technique for determining the normal mode frequencies semianalytically, using the WKB approximation. Although based on an approximation, we believe this approach will be powerful (a) because the WKB approximation is known in many cases to be more accurate than one has a right to expect *a priori*; (b) because the method can be carried to higher orders, either as a means to improve the accuracy or as a means to estimate the errors explicitly; and (c) because it will allow a more systematic study of normal modes than has been possible

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using outright numerical methods. The purpose of this *Letter* is to lay out the basic elements of this method and to apply it to a simple test case: the fundamental normal mode frequencies of the Schwarzschild black hole. The result is a simple analytic expression for the complex frequency, that, for $l = 2$ agrees with the numerical results of Chandrasekhar and Detweiler (1975) within 7% for the real part and 0.7% for the imaginary part, with the relative agreement improving with increasing l .

The motivation for using the WKB approximation is the similarity between the equations of black hole perturbation theory and the one-dimensional Schrödinger equation for a potential barrier. This similarity has been emphasized and exploited by Chandrasekhar (1983), for example. In both cases, the central equation has the form

$$d^2\psi/dx^2 + Q(x)\psi = 0. \quad (1)$$

In the black hole case, ψ represents the radial part of the perturbation variable, assumed to have time dependence $e^{i\omega t}$, and angular dependence $\Phi(\theta, \phi)$ appropriate to the particular perturbation and black hole under study. The coordinate x is a "tortoise coordinate" r_* which ranges from $-\infty$ at the horizon to $+\infty$ at spatial infinity. The function $-Q(x)$ is constant at $x = \pm\infty$, although not necessarily the same at both ends, and rises to a maximum in the vicinity of $x = 0$ (Fig. 1). It depends on the mass and angular momentum of the black hole, the angular harmonic indices, and the frequency, and it can be complex in general.

In quantum mechanics, $-Q(x) = (2m/\hbar^2)[V(x) - E]$, where E is the energy of the particle of mass m , and $V(x)$ is the potential barrier, assumed to tend to constant values as $x \rightarrow \pm\infty$.

Since $Q(x)$ tends to a constant for large $|x|$, ψ takes the form $e^{\pm i\alpha x}$, $\text{Re } \alpha > 0$. As $x \rightarrow \infty$, outgoing (ingoing) waves correspond to the negative (positive) sign, and as $x \rightarrow -\infty$, outgoing (ingoing) waves correspond to the positive (negative) sign. Here, "outgoing" means moving away from the potential barrier, so in the black hole case, "outgoing as $x \rightarrow -\infty$ " corresponds to waves crossing the horizon into the black hole.

For a wave incident on the barrier from $x = \infty$ with a given amplitude, it is a standard calculation in quantum mechanics to determine the amplitude of the wave reflected back to $x = \infty$ and that transmitted to $x = -\infty$. If the function $-Q(x)$ is positive anywhere, i.e., if the energy is below the peak of the potential, the reflected amplitude is generally comparable to the incident amplitude, while the transmitted amplitude is much smaller. When the WKB approximation is applied to such problems, it leads to an estimate for the transmitted amplitude of e^{-B} , where B is a "barrier penetration factor," given by an integral of $[-Q(x)]^{1/2}$ between the classical turning points of the potential $V(x)$, where Q vanishes. Similar reflection-transmission calculations have been done for black holes (for review and references see Chandrasekhar 1983).

Normal mode problems, on the other hand, involve a rather different set of boundary conditions. A normal mode is a free oscillation of the hole itself, with no incoming radiation driving it. The boundary condition at $x = \infty$ is therefore purely

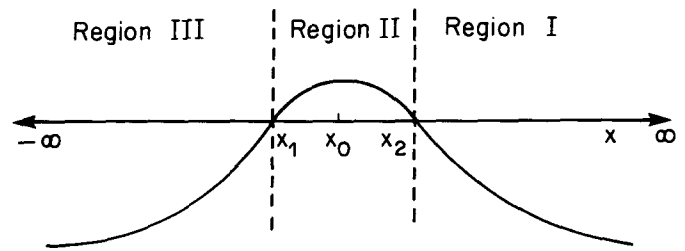


FIG. 1.—The function $-Q(x)$

outgoing waves; moreover, causality demands that at $x = -\infty$ the wave flux be into the horizon, i.e., "outgoing" again. One therefore expects the "reflected" and "transmitted" waves of the standard scattering problem to have comparable amplitudes, with the incident amplitude zero. At first glance, one might expect the WKB approximation to be useless for normal modes, because it always seems to lead to an exponentially small factor e^{-B} relating the transmitted to the reflected amplitude, rather than a factor of order unity. However, there is at least one case in which this is not true, namely that in which the maximum value of $-Q(x)$ is precisely zero. In quantum mechanics, this occurs when the energy coincides with the peak of the potential $V(x)$. In this "second-order turning point" problem, the WKB approximation leads to equal magnitudes for the two outgoing waves, each a factor $2^{-1/2}$ times the incident amplitude. This suggests that, if normal modes exist for a given potential, they must exist "nearby"; in other words, for complex frequencies such that $[-Q(x)]_{\text{max}} \approx 0$. However, if $[-Q(x)]_{\text{max}} \geq 0$, the classical turning points will in general be too close together to allow application of the standard WKB approach, which involves matching of two WKB approximations to the solution across each turning point. Nevertheless, a simple modification of the matching procedure allows a complete solution of the normal mode problem.

The modification involves matching two WKB solutions across both of the turning points simultaneously. Outside the turning points (regions I and III of Fig. 1), the WKB functions are given by (see Bender and Orszag 1978 for discussion)

$$\psi_{\text{I}}(x) \approx Q^{-1/4} \exp\left\{\pm i \int_{x_2}^x [Q(t)]^{1/2} dt\right\},$$

$$\psi_{\text{III}}(x) \approx Q^{-1/4} \exp\left\{\pm i \int_x^{x_1} [Q(t)]^{1/2} dt\right\}. \quad (2)$$

In region II, we approximate $Q(x)$ by a parabola. This is justified provided the turning points are closely spaced, i.e., provided $[-Q(x)]_{\text{max}} \ll |Q(\pm\infty)|$. Then Q has the form $Q(x) = Q_0 + \frac{1}{2}Q_0''(x - x_0)^2 + O(x - x_0)^3$, where $Q_0 \equiv Q(x_0) < 0$, and $Q_0'' \equiv d^2Q/dx^2|_{x_0} > 0$. The definitions

$$k \equiv \frac{1}{2}Q_0'', \quad t \equiv (4k)^{1/4} e^{i\pi/4} (x - x_0),$$

$$\nu + \frac{1}{2} \equiv -iQ_0/(2Q_0'')^{1/2}, \quad (3)$$

bring equation (1) into the form

$$d^2\psi/dt^2 + (\nu + \frac{1}{2} - \frac{1}{4}t^2)\psi = 0, \tag{4}$$

whose solutions are parabolic cylinder functions $D_\nu(t)$, with the general solution given by $\psi = AD_\nu(t) + BD_{-\nu-1}(it)$. For large $|t|$ the asymptotic forms of these solutions (Bender and Orszag 1978) yield

$$\psi \approx Be^{-3i\pi(\nu+1)/4}(4k)^{-(\nu+1)/4}(x-x_0)^{-(\nu+1)}e^{ik^{1/2}(x-x_0)^2/2}$$

$$+ [A + B(2\pi)^{1/2}e^{-i\pi\nu/2}/\Gamma(\nu+1)]e^{i\pi\nu/4}(4k)^{\nu/4}$$

$$\times (x-x_0)^\nu e^{-ik^{1/2}(x-x_0)^2/2},$$

$x \gg x_2$

$$\psi \approx Ae^{-3i\pi\nu/4}(4k)^{\nu/4}(x_0-x)^\nu e^{-ik^{1/2}(x-x_0)^2/2}$$

$$+ [B - iA(2\pi)^{1/2}e^{-i\pi\nu/2}/\Gamma(-\nu)]e^{i\pi(\nu+1)/4}$$

$$\times (4k)^{-(\nu+1)/4}(x_0-x)^{-(\nu+1)}e^{ik^{1/2}(x-x_0)^2/2},$$

$x \ll x_1$, (5)

where $\Gamma(\nu)$ is a gamma function. It is straightforward to show that the $\exp[-ik^{1/2}(x-x_0)^2/2]$ pieces of both solutions in equation (5) match to the outgoing wave of the WKB solutions of equation (2). For a normal mode, the coefficients of the $\exp[ik^{1/2}(x-x_0)^2/2]$ pieces must therefore vanish: this can only be achieved if $B=0$ and if $\Gamma(-\nu)=\infty$. The latter condition implies that ν must be an integer. This leads to the simple condition for a normal mode,

$$Q_0/(2Q_0'')^{1/2} = i(n + \frac{1}{2}), \quad n = 0, 1, 2, \dots \tag{6}$$

Since Q is frequency dependent, this condition will lead to discrete, complex values for the normal mode frequencies.

This result is completely general. It applies to any one-dimensional potential problem of the form of equation (1). As a simple application, we apply it to the Schwarzschild black hole.

The radial equation for scalar, electromagnetic, or gravitational perturbations of a Schwarzschild black hole is given by

$$d^2\psi/dr_*^2 + \{ \sigma^2 - [1 - (2/r)] [\lambda r^2 + (2\beta/r^3)] \} \psi = 0, \tag{7}$$

where $\lambda = l(l+1)$, where l is the angular harmonic index; $\beta = 1, 0, -3$ for the three types of perturbation, respectively; and $\sigma = M\omega$, where M is the mass of the black hole. The radial coordinates have been expressed in units of M , and r_* is related to r by $dr/dr_* = 1 - (2/r)$. With Q identified as

TABLE 1
NORMAL MODE FREQUENCIES FOR SCHWARZSCHILD:
WKB VERSUS NUMERICAL^a

l	Re σ			Im σ		
	WKB	Numerical	%	WKB	Numerical	%
2	0.3988	0.3737	7.0	0.0883	0.0889	0.7
3	0.6166	0.5994	2.9	0.0923	0.0927	0.4
4	0.8223	0.8092	1.6	0.0939	0.0941	0.2

^a Numerical results from Chandrasekhar and Detweiler 1975.

the quantity in braces in equation (7), we obtain

$$Q'(r_*) = [1 - (2/r)] \{ (\lambda/r^3)[2 - (6/r)] + (2\beta/r^4) \times [3 - (8/r)] \},$$

$$Q''(r_*) = [1 - (2/r)] \{ -(\lambda/r^4)[6 - (40/r) + (60/r^2)] - (2\beta/r^5)[12 - (70/r) + (96/r^2)] \}. \tag{8}$$

The peak of $-Q$ occurs at $r = r_0 = \frac{3}{2}\lambda^{-1} \{ \lambda - \beta + [\lambda^2 + (14/9)\lambda\beta + \beta^2]^{1/2} \}$. Then from equation (6) we have

$$\sigma^2 = [1 - (2/r_0)] \{ (\lambda/r_0^2) + (2\beta/r_0^3) \} + i(2Q_0'')^{1/2} (n + \frac{1}{2}). \tag{9}$$

The results for σ for gravitational perturbations ($\beta = -3$), for $l = 2, 3, 4$, and for $n = 0$ (the fundamental mode) are listed in Table 1. For comparison we have shown the values obtained from the numerical calculations of Chandrasekhar and Detweiler (1975), together with the percentage agreement. It turns out that, for the first overtone mode ($n = 1$), the agreement breaks down, a consequence, we suspect, of the inadequacy of the parabolic approximation. This question is currently under investigation. For large l , equations (8) and (9) imply that $\text{Im } \sigma \approx \frac{1}{2}(27)^{-1/2} \approx 0.096225$ for any β , a limit which has been observed numerically (Detweiler 1979).

The method presented here is to be contrasted with that adopted by Mashhoon and co-workers (Blome and Mashhoon 1984; Ferrari and Mashhoon 1984) in which the function $-Q(x)$ is replaced by an alternative function that approximates $-Q(x)$, but for which exact analytic solutions of equation (1) are known. The normal mode frequencies obtained agree with the numerical results where available, but there is no systematic way in this method to estimate the errors or to improve the accuracy. This may be a drawback when one works in uncharted areas, such as regimes where unstable modes might exist.

In future papers in this series, we will discuss (a) the second-order WKB approximation for normal modes, (b) application of the method to overtone ($n \geq 1$) modes, (c) application to Kerr and a search for unstable modes, and (d)

the possible use of this technique in other normal mode problems in physics and astrophysics, such as the normal modes of stars.

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