Minimal data at a given point of space for solutions to certain geometric systems

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Abstract
We consider a geometrical system of equations for a three-dimensional
Riemannian manifold. This system of equations has been constructed as to
include as particular cases several physically interesting systems of equations,
such as the stationary Einstein vacuum field equations or harmonic maps
coupled to gravity in three dimensions. We give a characterization of its
solutions in a neighbourhood of a given point through sequences of symmetric
trace-free tensors (referred to as 'null data'). We show that the null data
determine a formal expansion of the solution and we obtain necessary and
sufficient growth estimates on the null data for the formal expansion to be
absolutely convergent in a neighbourhood of the given point. This provides a
complete characterization of all the solutions to the given system of equations
around that point.

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1. Introduction

Einstein's equations form the core of general relativity. Solutions to this system of equations
represent physically feasible spacetimes. Therefore, analysing the properties of such solutions
is one of the most important tasks in general relativity. It is, at the same time, one of the
utmost difficulty. One of the ways to be able to analyse properties of the solutions is to
make simplifying assumptions, which restrict the possible solutions. Typical assumptions are
those of staticity or stationarity, where the existence of a timelike Killing field is assumed,
allowing for a dimensional reduction of the problem. The literature is full of examples of such
procedures, the simplest being the static Einstein vacuum field equations. More complicated
examples are given by the stationary Einstein–Maxwell fields in arbitrary dimensions [5] (here
stationary means that the dimensionally reduced problem is three dimensional) and static
Einstein–Maxwell-dilaton fields [6]. Also if dimensions higher than four are considered, as

in Kaluza–Klein-type theories, dimensional reduced versions of the theories are also usually analysed. As said, this practice is due to the complicated structure of Einstein's equations and because the static or stationary solutions provide important information about some properties of the solutions.

Even though simpler, the arising systems of equations are still complicated and need detailed analysis. The main objective of the present work is to answer the question of existence and characterization of solutions to the arising systems of equations. That is, we want to show that given suitable data, these data are enough to characterize uniquely the solution and that there is a solution with such a data. As we do not want to enter in a case-by-case discussion, the first step is to look at the shared properties of those systems. In general they have the following features.

- The base manifold is three dimensional.
- The metric is Riemannian.
- On the base manifold there are some scalar fields.
- The metric satisfies an equation whose principal part is given by the Ricci tensor of the metric.
- The fields satisfy Laplace-type equations.

There are two features that are central for our analysis. The first is that in a suitable gauge the systems are elliptic. This implies that the solutions are analytic in this gauge. Second, in this gauge the systems of equations are diagonal, i.e. for each system the principal symbol has only diagonal elements (the off-diagonal elements vanish) and all the diagonal elements are the same. This implies that the complex null cone (characteristic cone) is the same for all the unknowns in each system. This is very important for the geometrical analysis of the problem.

We consider now a more general system of equations that includes all the named features. The setting is the following.

Let N be a three-dimensional manifold with negative definite metric h and n scalar fields on it, ϕ^α, $\alpha = 1, \ldots, n$, that satisfy the following system of equations:

$$\Delta_h \phi^\alpha = f^\alpha(\phi^\gamma, D_c\phi^\gamma), \qquad \alpha = 1, \ldots, n, \tag{1}$$

$$R_{ab}[h] = F^\alpha(\phi^\gamma) D_a D_b \phi^\alpha + F^{\alpha\beta}(\phi^\gamma) D_a\phi^\alpha D_b\phi^\beta + f(\phi^\gamma, D_c\phi^\gamma) h_{ab}, \tag{2}$$

where D is the covariant derivative associated with h, Δ_h is the Laplace operator with respect to h and $R_{ab}[h]$ is the Ricci tensor of h. $F^\alpha(\phi^\gamma)$, $F^{\alpha\beta}(\phi^\gamma)$, $\alpha, \beta = 1, \ldots, n$, are analytic functions of the scalar fields and $f(\phi^\gamma, D_c\phi^\gamma)$, $f^\alpha(\phi^\gamma, D_c\phi^\gamma)$, $\alpha = 1, \ldots, n$, are analytic functions of the scalar fields and its first derivatives. In order not to burden the text, the span of greek indices is not always stated. We assume that the summation rule for repeated indices holds for both Latin and Greek indices, the latter being written always in the upper position, as no metric is associated with them. We assume $F^{\alpha\beta} = F^{\beta\alpha}$, so we do not to have to symmetrize on the a, b indices. The functions f, f^α, F^α, $F^{\alpha\beta}$ are scalars if considered as functions on N through their dependence on the fields and their first derivatives. This is necessary for (1), (2) to be a geometrically meaningful system of equations. Later on, we will see that the functions f, f^α, F^α and $F^{\alpha\beta}$ are not independent but have to satisfy some relations (discussed in section 7). We refer to (1), (2) as the field equations.

The setting above includes all the named static or stationary cases of Einstein's equations. A particular class of systems that has received increasing attention over the years and that is also included corresponds to harmonic maps coupled to gravity (also called σ-models) whose base space is a three-dimensional manifold or where the spacetime is stationary [3, 10]. Of course the present listing of included systems of equations is far from complete, but it clearly shows that many physically interesting systems of equations are included.

Now we can state clearly the objective of this work: to give a complete characterization of the solutions to the field equations above in a neighbourhood of a given point.

So, let us take a point q in N. As said before, in a suitable gauge the system of equations is elliptic with analytic coefficients; therefore the solution, if it exists, is analytic [7] (cf [2, 8] for application of the result in [7] to the stationary vacuum Einstein equations). This suggest prescribing as data the coefficients of the Taylor expansion of the fields at q. This is not possible, as the Taylor expansion coefficients are not independent of each other. Instead we consider the following n sequences of symmetric trace-free tensors at q:

$$\mathcal{D}^\alpha = \left\{ \phi^\alpha(q), D_{a_1}\phi^\alpha(q), \mathcal{C}(D_{a_2}D_{a_1}\phi^\alpha)(q), \mathcal{C}(D_{a_3}D_{a_2}D_{a_1}\phi^\alpha)(q), \dots \right\}, \quad (3)$$

where $\mathcal{C}$ means taking the symmetric trace-free part of the tensor to which it is applied. These sequences are called the *null data* for our system of equations. It turns out that the null data are indeed a minimal set of data for the field equations, that is, it determines a formal expansion of the solution and its components are independent of each other. Our purpose can then be stated as to derive necessary and sufficient conditions for the null data to determine apart from gauge conditions (unique) real analytic solutions of (1) and (2). This will show that the null data do indeed provide the searched characterization.

Prescribing a minimal set of data at a point as characterization for the solutions to a gauge-elliptic problem is certainly different from posing a standard boundary value problem. A particular advantage of our approach is that the data have a geometrical meaning and do not depend on the choice of an arbitrary hypersurface or coordinate system. This makes the characterization intrinsic to the geometry of the solution.

Another advantage arises if there is a geometrically distinguished point. Our approach then allows for a complete control and analysis of the solution at the given point. The existence of a geometrically distinguished point happens in the case of asymptotically flat static or stationary spacetimes (cf Friedrich [4] and Aceña [1]), where there is a point in the manifold that represents infinity. Unfortunately those cases are not included in our present treatment because besides the metric there are further tensor fields among the unknowns, while we only consider scalar fields. An extension of the present result to include tensors in the unknowns is possible although not entirely straightforward, as one possibly has to take into account integrability conditions that may arise for the system of equations to be consistent. We do not want to get involved in such a discussion here.

It is convenient to express the tensors in $\mathcal{D}^\alpha$ in terms of an h-orthonormal frame $c_\mathbf{a}$, $\mathbf{a} = 1, 2, 3$, centred at q. Denoting by $D_\mathbf{a}$ the covariant derivative in the direction of $c_\mathbf{a}$,

$$\mathcal{D}^{\alpha*} = \left\{ \phi^\alpha(q), D_{\mathbf{a_1}}\phi^\alpha(q), \mathcal{C}(D_{\mathbf{a_2}}D_{\mathbf{a_1}}\phi^\alpha)(q), \mathcal{C}(D_{\mathbf{a_3}}D_{\mathbf{a_2}}D_{\mathbf{a_1}}\phi^\alpha)(q), \dots \right\}. \quad (4)$$

These tensors will be called the *null data in the frame $c_\mathbf{a}$* and are defined uniquely up to rigid rotations in $\mathbb{R}^3$.

If the metric h and the potentials ϕ^α exist, then they are real analytic near q and one has Cauchy estimates on the derivatives of the potentials. The Cauchy estimates imply that there exist positive constants M, r, such that the components of the null data satisfy

$$\left| \mathcal{C}(D_{\mathbf{a}_p} \dots D_{\mathbf{a_1}}\phi^\alpha)(q) \right| \leqslant \frac{Mp!}{r^p}, \qquad p \geqslant 0, \qquad \mathbf{a}_p, \dots, \mathbf{a_1} = 1, 2, 3. \quad (5)$$

Our main result corresponds to the statement that these estimates are not only necessary but also sufficient to have an analytic solution and is presented in the following theorem.

Theorem 1.1.

Suppose

$$\hat{\mathcal{D}}^\alpha = \left\{ \psi^\alpha_{\mathbf{a_1}}, \psi^\alpha_{\mathbf{a_2}\mathbf{a_1}}, \psi^\alpha_{\mathbf{a_3}\mathbf{a_2}\mathbf{a_1}}, \dots \right\}, \qquad \alpha = 1, \dots, n, \quad (6)$$

are n infinite sequences of symmetric, trace-free tensors given in an orthonormal frame at the origin of a three-dimensional Euclidean space. If there exist positive constants M, r such that the components of these tensors satisfy the estimates

$$\left| \psi^{\alpha}_{\mathbf{a}_p \ldots \mathbf{a}_1} \right| \leqslant \frac{Mp!}{r^p}, \qquad p \geqslant 0, \qquad \mathbf{a}_p, \ldots, \mathbf{a}_1 = 1, 2, 3, \qquad \alpha = 1, \ldots, n,$$

then there exists an analytic solution h, ϕ^{α}, $\alpha = 1, \ldots, n$, of the field equations near q, unique up to isometries, so that the null data implied by it in a suitable frame $c_{\mathbf{a}}$ as described above satisfy

$$\mathcal{C}\left(D_{\mathbf{a}_p} \ldots D_{\mathbf{a}_1} \phi^{\alpha} \right)(q) = \psi^{\alpha}_{\mathbf{a}_p \ldots \mathbf{a}_1}, \qquad p \geqslant 0, \qquad \mathbf{a}_p, \ldots, \mathbf{a}_1 = 1, 2, 3, \qquad \alpha = 1, \ldots, n.$$

The sequences (6), not necessarily satisfying any estimates, will be referred to as *abstract null data*. As the type of estimates imposed here on the abstract null data do not depend on the orthonormal frame in which they are given, and since these estimates are necessary as well as sufficient, then all possible solutions of (1), (2) are characterized by the null data.

A few words on the requirement on the manifold N to be three-dimensional are in order here. The proof that we present of the theorem above relies heavily on the use of space spinors, which can be used only on a three-dimensional manifold. This may seem as the main reason for such a requirement, in fact it is not. There is a one to one correspondence between space-spinor analysis and the more usual tensor analysis on a three-dimensional manifold. Therefore, all the proofs could be translated and expressed in terms of tensor analysis, and then one could try to extend them to higher dimensions. The main reason for the requirement is that in three dimensions the Riemann tensor is completely determined by the Ricci tensor. If one wants to extend the result to higher dimensions one is forced to include in the system of equations an equation for the Weyl tensor, which makes the system of equations more complicated and also introduces complications regarding the integrability conditions.

The use of null data in general relativity was first presented by Penrose and Rindler [9], where it is used as data for fields coupled to gravity in spacetime (four-dimensional Lorentzian manifold). It is important to note that although there are similarities in the use of the null data in the Lorenzian case and in the Riemannian case, treated in the present work, there is a profound difference. Namely, in the Lorenzian case, one has to impose analyticity on the solution to be able to characterize it using the null data, while in the Riemannian case analyticity is a natural consequence of the equations.

In the context of three-dimensional Riemannian spaces, the null data were first introduced by Friedrich [4] as a way to characterize static asymptotically flat solutions to the vacuum Einstein field equations. They were also used by Aceña [1] for the stationary asymptotically flat vacuum case. The techniques that we use to prove the result of the present work are similar to those introduced by Friedrich and used by Aceña. Therefore, we will not present the procedure in full detail, but we will state the important steps and the features that are distinctive for the case that we are treating here.

2. The exact sets of equation argument

We have defined the null data $\mathcal{D}^{\alpha}$ (3) and we want to use them to characterize solutions to the field equations. Therefore, an important first step is to show that the null data can actually be used to construct formal solutions to the field equations. For this we construct expansions of the fields in normal coordinates.

We assume from now on N to be small enough to coincide with a convex h-normal neighbourhood of q. Let $c_{\mathbf{a}}$, $\mathbf{a} = 1, 2, 3$, be an h-orthonormal frame field on N which is

parallelly transported along the h-geodesics through q and let x^a denote normal coordinates centred at q so that $c^b{}_{\mathbf{a}} \equiv \langle \mathrm{d}x^b, c_{\mathbf{a}} \rangle = \delta^b{}_{\mathbf{a}}$ at q. We refer to such a frame as a *normal frame centred at q*. Its dual frame will be denoted by $\chi^{\mathbf{c}} = \chi^{\mathbf{c}}{}_b \mathrm{d}x^b$. In the following, all tensor fields, except the frame field $c_{\mathbf{a}}$ and the coframe field $\chi^{\mathbf{c}}$, will be expressed in terms of this frame field, so that the metric is given by $h_{\mathbf{ab}} \equiv h(c_{\mathbf{a}}, c_{\mathbf{b}}) = -\delta_{\mathbf{ab}}$. With $D_{\mathbf{a}} \equiv D_{c_{\mathbf{a}}}$ denoting the covariant derivative in the $c_{\mathbf{a}}$ direction, the connection coefficients with respect to $c_{\mathbf{a}}$ are defined by $D_{\mathbf{a}} c_{\mathbf{c}} = \Gamma_{\mathbf{a}}{}^{\mathbf{b}}{}_{\mathbf{c}} c_{\mathbf{b}}$.

An analytic tensor field $T_{\mathbf{a}_1 \dots \mathbf{a}_k}$ on N has in the normal coordinates x^a a *normal expansion* at q, which can be written as

$$T_{\mathbf{a}_1 \dots \mathbf{a}_k}(x) = \sum_{p \geqslant 0} \frac{1}{p!} x^{b_p} \dots x^{b_1} D_{\mathbf{b}_p} \dots D_{\mathbf{b}_1} T_{\mathbf{a}_1 \dots \mathbf{a}_k}(q), \tag{7}$$

where we assume from now on that the summation convention does not distinguish between bold face and other indices.

Since $h_{\mathbf{ab}} = -\delta_{\mathbf{ab}}$, it remains to be seen how to obtain normal expansions for the ϕ^{α}'s using the field equations and the null data. That is, we need to see how to obtain

$$D_{\mathbf{a}_p} \dots D_{\mathbf{a}_1} \phi^{\alpha}(q), \qquad p \geqslant 0, \qquad \mathbf{a}_p, \dots, \mathbf{a}_1 = 1, 2, 3.$$

The algebra necessary for doing this simplifies considerably in the space-spinor formalism. How to do the transition is explained in [4]. Here we recall a few important properties.

- A space-spinor field $T_{A_1 B_1 \dots A_p B_p} = T_{(A_1 B_1) \dots (A_p B_p)}$ arises from a real tensor field $T_{\mathbf{a}_1 \dots \mathbf{a}_p}$ if and only if

  $$T_{A_1 B_1 \dots A_p B_p} = (-1)^p \tau_{A_1}{}^{A'_1} .. \tau_{B_p}{}^{B'_p} \bar{T}_{A'_1 B'_1 \dots A'_p B'_p}, \tag{8}$$

  where $\tau^{AA'} = \epsilon_0{}^A \epsilon_0{}^{A'} + \epsilon_1{}^A \epsilon_1{}^{A'}$. ϵ_{AB} is the constant ϵ-spinor, which satisfies $\epsilon_{AB} = -\epsilon_{BA}$, $\epsilon_{01} = 1$, and it is used to move indices according to the rules $\iota_B = \iota^A \epsilon_{AB}$, $\iota^A = \epsilon^{AB} \iota_B$.

- Any spinor field $T_{A \dots H}$ admits a decomposition into products of totally symmetric spinor fields and ϵ-spinors which can be written schematically in the form

  $$T_{A \dots H} = T_{(A \dots H)} + \sum \epsilon's \times symmetrized\ contractions\ of\ T. \tag{9}$$

- The operation of taking the symmetric trace-free part of a tensor translates into taking the totally symmetric part of the corresponding spinor. So the null data translate into n sequences of totally symmetric spinors.

- We also have a complex frame field c_{AB} (related to $c_{\mathbf{a}}$), such that $h(c_{AB}, c_{CD}) = h_{ABCD} \equiv -\epsilon_{A(C} \epsilon_{D)B}$, and its dual 1-form field χ^{AB} (related to $\chi^{\mathbf{a}}$). The covariant derivative of a spinor field ι^C in the direction of c_{AB} is given by

  $$D_{AB} \iota^C = c_{AB}(\iota^C) + \Gamma_{AB}{}^C{}_D \iota^D,$$

  where $\Gamma_{ABCD} = \Gamma_{(AB)(CD)}$ are the spinor connection coefficients.

- The commutator of derivatives are given in terms of the curvature spinor

  $$(D_{CD} D_{EF} - D_{EF} D_{CD}) \iota^A = r^A{}_{BCDEF} \iota^B, \tag{10}$$

  where

  $$r_{ABCDEF} = \tfrac{1}{2} \big[\big(s_{ABCE} - \tfrac{1}{6} r h_{ABCE} \big) \epsilon_{DF} + \big(s_{ABDF} - \tfrac{1}{6} r h_{ABDF} \big) \epsilon_{CE} \big], \tag{11}$$

  r being the Ricci scalar of h and $s_{ABCD} = s_{(ABCD)}$ the trace-free part of the Ricci tensor of h. In tensor notation, the decomposition of the Ricci tensor reads $R_{ab}[h] = s_{ab}[h] + \tfrac{1}{3} r[h] h_{ab}$.

Equations (1), (2) take in the space-spinor formalism the form

$$D^P{}_A D_{BP}\phi^\alpha = -\tfrac{1}{2}\epsilon_{AB}f^\alpha, \tag{12}$$

$$r_{ABCDEF} = \tfrac{1}{2}\big[\big(S_{ABCE} - \tfrac{1}{6}Rh_{ABCE}\big)\epsilon_{DF} + \big(S_{ABDF} - \tfrac{1}{6}Rh_{ABDF}\big)\epsilon_{CE}\big], \tag{13}$$

where

$$S_{ABCD} = F^\alpha D_{AB}D_{CD}\phi^\alpha + F^{\alpha\beta}D_{AB}\phi^\alpha D_{CD}\phi^\beta + \tilde{f}h_{ABCD}, \tag{14}$$

$$R = \hat{f}, \tag{15}$$

with

$$\tilde{f} = -\tfrac{1}{3}(F^{\alpha\beta}D_a\phi^\alpha D^a\phi^\beta + F^\alpha f^\alpha),$$
$$\hat{f} = F^{\alpha\beta}D_a\phi^\alpha D^a\phi^\beta + 3f + F^\alpha f^\alpha.$$

Using these equations and the theory of 'exact sets of fields' is possible to prove the following result.

Lemma 2.1. *Let there be n given sequences*

$$\hat{\mathcal{D}}^\alpha = \big\{\psi^\alpha, \psi^\alpha_{A_1 B_1}, \psi^\alpha_{A_2 B_2 A_1 B_1}, \psi^\alpha_{A_3 B_3 A_2 B_2 A_1 B_1}, \cdots\big\}$$

of totally symmetric spinors satisfying the reality condition (8). Assume that there exists a solution h, ϕ^α, to the field equations so that the spinors given by $\hat{\mathcal{D}}^\alpha$ coincide with the null data $\mathcal{D}^{\alpha*}$ given by (4) in terms of an h-orthonormal normal frame centred at q, i.e.

$$\psi^\alpha_{A_p B_p \ldots A_1 B_1} = D_{(A_p B_p} \ldots D_{A_1 B_1)}\phi^\alpha(q), \qquad p \geqslant 0.$$

Then the coefficients of the normal expansions (7) of the fields ϕ^α, i.e.

$$D_{A_p B_p} \ldots D_{A_1 B_1}\phi^\alpha(q), \qquad p \geqslant 0,$$

are uniquely determined by the data $\hat{\mathcal{D}}^\alpha$ and satisfy the reality condition.

Proof. The proof is by induction. It holds $\phi^\alpha(q) = \psi^\alpha$, $D_{AB}\phi^\alpha(q) = \psi^\alpha_{AB}$.

To discuss the induction step we assume that the expansion coefficients of ϕ^α up to order p are known and start with $D_{A_{p+1}B_{p+1}} \ldots D_{A_1 B_1}\phi^\alpha(q)$ and its decomposition in the form (9). By assumption, the totally symmetric part of it is given by $\psi^\alpha_{A_{p+1}B_{p+1}\ldots A_1 B_1}$. The other terms in the decomposition contain contractions. Let us consider a general contraction, say A_i contracted with A_j. We can commute the operators $D_{A_i B_i}$ and $D_{A_j B_j}$ with other covariant derivatives, generating by (10) and (13) only terms of lower order, until we have

$$D_{A_{p+1}B_{p+1}} \ldots D_{A_{i+1}B_{i+1}} D_{A_{i-1}B_{i-1}} \ldots D_{A_{j+1}B_{j+1}} D_{A_{j-1}B_{j-1}} \ldots D_{A_1 B_1} D^P_{B_i} D_{PB_j}\phi^\alpha(q).$$

Equation (12) then shows how to express the resulting term by quantities of lower order that are already known.

That the expansion coefficients satisfy the reality condition is a consequence of the formalism and the fact that they are satisfied by the data. $\qquad\square$

In order to show the convergence of the formal series determined in the previous lemma we need to impose estimates on the free coefficients given by $\hat{\mathcal{D}}^\alpha$. For the necessary part we have the following result.

Lemma 2.2. *A necessary condition for the formal series determined in lemma 2.1 to be absolutely convergent near the origin is that the data given by $\hat{\mathcal{D}}^\alpha$ satisfy estimates of the type*

$$\big|\psi^\alpha_{A_p B_p \ldots A_1 B_1}\big| \leqslant \frac{p!M}{r^p}, \qquad p \geqslant 0, \tag{16}$$

with some positive constants M, r.

The proof of the last lemma is very similar to the respective lemma in [4] and is not repeated here.

A more clarifying way of stating the last two lemmas is as follows.

- If we have two solutions of the field equations and the null data (4) of one of the solutions are related to the null data of the other solution by a rigid rotation in $\mathbb{R}^3$, then the two solutions are related by an isomorphism and are therefore geometrically equivalent.
- If we have a solution of the field equations, then its null data (4) satisfy estimates of the form (5).

3. The characteristic initial value problem

After showing that the null data determine the solution, one would have to show that the estimates (16) imply Cauchy estimates for the expansion coefficients

$$\left| D_{A_p B_p} \dots D_{A_1 B_1} \phi^\alpha(q) \right| \leqslant \frac{p! M}{r^p}, \qquad p \geqslant 0.$$

This would ensure the convergence of the normal expansion in a neighbourhood of q and the existence of the solution. But deriving estimates on the expansion coefficients from estimates on the null data using the procedure described in the proof of lemma 2.1 has not been possible. Instead, one can use the intrinsic geometric nature of the problem and the data to formulate the problem as a boundary value problem to which Cauchy–Kowalevskaya-type arguments apply. The formalism necessary for this has been developed in [4] and used in [1]. Here we present the important facts for the present work, following the notation in [1]. The reader is referred to [1, 4] for details.

The fields h, ϕ^α can be extended near q by analyticity into the complex domain and considered as holomorphic fields on a complex analytic manifold N_c. Under the analytic extension, and choosing N_c to be a sufficiently small neighbourhood of q, the main differential geometric concepts and formulae remain valid. The extended coordinates and the extended frame, again denoted by x^a and c_{AB}, satisfy the same defining equations and the extended fields, denoted again by h, ϕ^α, satisfy the field equations as before.

The *null cone at q* is defined as the set

$$\mathcal{N}_q = \{ p \in N_c | \Gamma(p) = 0 \},$$

where $\Gamma = \delta_{ab} x^a x^b$. It is the cone swept out by the complex null geodesics through q. For the type of systems we consider one very specific and important feature of this cone is that it is the characteristic cone not only for the Ricci operator in a harmonic gauge but also for the Laplace operators that act on the scalar fields. The coincidence of the characteristic cones for all the unknowns is due to the special form of the principal symbol of the system, it is diagonal and all the diagonal elements are the same. If this were not the case then the construction used in the present work would not be possible. It turns out for our problem that knowing the null data is equivalent to knowing the restriction of the holomorphic functions ϕ^α to the null cone. The identification is made explicit later on.

Our problem can thus be formulated as the boundary value problem for the field equations with data given on the null cone by the functions $\phi^\alpha|_{\mathcal{N}_q}$. The difficulty with this formulation is that $\mathcal{N}_q$ is not a smooth hypersurface but an analytic set with a vertex at the point q; therefore, an adapted coordinate and frame field is needed. The construction is done rigorously in [4] in terms of the principal bundle of spin frames over N_c, $SL(N_c)$. There a three-dimensional submanifold $\hat{N}$ of $SL(N_c)$ is constructed in such a way that together with the projection map $\pi : SL(N_c) \to N_c$ it induces coordinates on N_c. Here we describe how the construction is seen on N_c.

We start by taking the null vector c_{11} at q and constructing the null geodesic γ with affine parameter w whose tangent vector at q is c_{11}. That is, $\gamma(w)$ is the affinely parametrized null geodesic that has $\gamma(0) = q$ and $\gamma'(0) = c_{11}$.

We define now a family of frames e_{AB} at q in terms of the frame c_{AB} and a parameter $v \in \mathbb{C}$ in the following way:

$$e_{00}(v) = c_{00} + 2vc_{01} + v^2 c_{11}, \qquad e_{01}(v) = c_{01} + vc_{11}, \qquad e_{11}(v) = c_{11}.$$

These frames are orthonormal in the sense that $h(e_{AB}, e_{CD}) = h_{ABCD}$. As v varies e_{00} covers all null directions at q except c_{11}.

We parallelly propagate the frames $e_{AB}(v)$ over $\gamma(w)$. So now at each point of $\gamma(w)$ we have a family of frames $e_{AB}(v, w)$.

Consider a fixed value for $v = v_0$ and $w = w_0$. At $\gamma(w_0)$ we construct the null geodesic that goes through this point and has a tangent vector $e_{00}(v_0, w_0)$ at $\gamma(w_0)$. We call u the affine parameter on this null geodesic that vanishes at $\gamma(w_0)$ and we assign to a point on this geodesic the corresponding value of u and the values $v = v_0$ and $w = w_0$. We also parallelly propagate the frame $e_{AB}(v_0, w_0)$ over the null geodesic. Doing this for all possible values of v and w covers all of N_c and defines the frame field e_{AB} there.

The functions $z^1 = u$, $z^2 = v$, $z^3 = w$ define holomorphic coordinates on $\hat{N}$. We denote again π the restriction of the projection to $\hat{N}$. The map π induces a biholomorphic diffeomorphism of $\hat{N}' \equiv \hat{N} \backslash U_0$, where $U_0 \equiv \{u = 0\}$, onto $\pi(\hat{N}')$, but the gauge is singular in the sense that the set U_0, which is a two-surface on $\hat{N}$, projects into the curve γ on N_c. We also need to distinguish the set $I \equiv \{u = 0, w = 0\}$, where $\pi(I) = q$. Finally, the set $W_0 \equiv \{w = 0\}$ projects onto $\mathcal{N}_q \backslash \gamma$ and will therefore define the initial data set for our problem.

It is important to recall that although the coordinates z^a are well adapted to the geometry of the problem we are dealing with, and are holomorphic on $\hat{N}$, they are not a good coordinate system in all of N_c. The set $\{u = 0\}$ corresponds to the curve γ instead of being a hypersurface on N_c, and thus there is no way of assigning a v coordinate to points on γ. For this the null curve γ will be referred to as *the singular generator of $\mathcal{N}_q$ in the gauge determined by the frame c_{AB} at q*. The singularity of the gauge is reflected in the following properties, deduced in [4], and which are important for the present work.

Regarding the frame field, as seen on $\hat{N}$, if one writes $e_{AB} = e^a{}_{AB} \partial_{z^a}$, then on $\hat{N}'$,

$$\left(e^a{}_{AB}\right) = \begin{pmatrix} 1 & e^1{}_{01} & e^1{}_{11} \\ 0 & e^2{}_{01} & e^2{}_{11} \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & \mathcal{O}(u^2) & \mathcal{O}(u^2) \\ 0 & \frac{1}{2u} + \mathcal{O}(u) & \mathcal{O}(u) \\ 0 & 0 & 1 \end{pmatrix} \quad \text{as} \quad u \to 0.$$

We shall write

$$e^a{}_{AB} = e^{*a}{}_{AB} + \hat{e}^a{}_{AB},$$

with the singular part

$$e^{*a}{}_{AB} = \delta^a_1 \epsilon_A{}^0 \epsilon_B{}^0 + \delta^a_2 \frac{1}{u} \epsilon_{(A}{}^0 \epsilon_{B)}{}^1 + \delta^a_3 \epsilon_A{}^1 \epsilon_B{}^1,$$

and holomorphic functions $\hat{e}^a{}_{AB}$ on $\hat{N}$ which satisfy

$$\hat{e}^a{}_{AB} = \mathcal{O}(u) \quad \text{as} \quad u \to 0. \tag{17}$$

The connection coefficients $\Gamma_{ABCD} = \Gamma_{(AB)(CD)}$ satisfy

$$\Gamma_{00AB} = 0 \quad \text{on } \hat{N}, \quad \Gamma_{11AB} = 0 \quad \text{on } U_0,$$

and also

$$\Gamma_{ABCD} = \Gamma^*_{ABCD} + \hat{\Gamma}_{ABCD},$$

with the singular part

$$\Gamma^*_{ABCD} = -\frac{1}{u}\epsilon_{(A}{}^{0}\epsilon_{B)}{}^{1}\epsilon_{C}{}^{0}\epsilon_{D}{}^{0},$$

and holomorphic functions $\hat{\Gamma}_{ABCD}$ on $\hat{N}$ which satisfy

$$\hat{\Gamma}_{ABCD} = \mathcal{O}(u) \quad \text{as} \quad u \to 0. \tag{18}$$

We need a couple of more definitions and properties to be able to deal with spinor functions on $\hat{N}$.

In general, a holomorphic spinor field ψ on N_c is represented on $SL(N_c)$ by a holomorphic spinor-valued function $\psi_{A_1\ldots A_j}$, given by the components of ψ in the corresponding spin frame. We shall use the notation $\psi_k = \psi_{(A_1\ldots A_j)_k}, k = 0, \ldots, j$, where $(\cdots)_k$ denotes the operation *'symmetrize and set k indices equal to* 1, *the rest equal to* 0'. These functions completely specify ψ if ψ is symmetric. They are then referred to as the *essential components of ψ*.

As the induced map π of $\hat{N}$ into N_c is singular on U_0, not every holomorphic function of the z^a can arise as a pull-back to $\hat{N}$ of a holomorphic function on N_c. The former must have a special type of expansion in terms of the z^a which reflects the particular relation between the 'angular' coordinate v and the 'radial' coordinate u. The following definition and lemma, taken from [4], are needed for manipulating spinor functions on $\hat{N}$.

Definition 3.1. *A holomorphic function g on $\hat{N}$ is said to be of v-finite expansion type k_g, with k_g being an integer, if it has in terms of the coordinates u, v and w a Taylor expansion at the origin of the form*

$$g = \sum_{p=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{2m+k_g} g_{m,n,p}u^m v^n w^p,$$

where it is assumed that $g_{m,n,p} = 0$ if $2m + k_g < 0$.

Lemma 3.1. *Let $\phi_{A_1\ldots A_j}$ be a holomorphic, symmetric, spinor-valued function on $SL(N_c)$. Then the restrictions of its essential components $\phi_k = \phi_{(A_1\ldots A_j)_k}, 0 \leqslant k \leqslant j$, to $\hat{N}$ satisfy*

$$\partial_v\phi_k = (j - k)\phi_{k+1}, \qquad k = 0, \ldots, j, \qquad \text{on } U_0,$$

(where we set $\phi_{j+1} = 0$) and ϕ_k is of the expansion type $j-k$.

As stated at the beginning of this section, prescribing the null data is equivalent to knowing $\phi^\alpha|_{\mathcal{N}_q}$. Following [4] it is possible to see how this fit into our particular gauge. Consider the normal frame c_{AB} on N_c near q and denote the null data of h in this frame by

$$\mathcal{D}^{\alpha*} = \left\{ D_{(A_pB_p}\ldots D_{A_1B_1)}\phi^\alpha(q), p = 1, 2, 3, \ldots \right\},$$

then on W_0

$$\phi^\alpha(u, v) = \sum_{m=0}^{\infty}\sum_{n=0}^{2m} \psi^\alpha_{m,n}u^m v^n, \tag{19}$$

where

$$\psi^\alpha_{m,n} = \frac{1}{m!}\binom{2m}{n} D_{(A_mB_m}\ldots D_{A_1B_1)_n}\phi^\alpha(q), \qquad 0 \leqslant n \leqslant 2m.$$

This shows how to determine $\phi^\alpha(u, v)$ from the null data $\mathcal{D}^{\alpha*}$ and vice versa.

4. The field equations on $\hat{N}$

Now, having the coordinates and frame field, we can use the frame calculus in its standard form. Given the fields ϕ^α and using the frame e_{AB} and the connection coefficients Γ_{ABCD} on $\hat{N}$, we set

$$r_{ABCDEF} \equiv e_{CD}(\Gamma_{EFAB}) - e_{EF}(\Gamma_{CDAB}) + \Gamma_{EF}{}^K{}_C\Gamma_{DKAB}$$
$$+ \Gamma_{EF}{}^K{}_D\Gamma_{CKAB} - \Gamma_{CD}{}^K{}_E\Gamma_{KFAB} - \Gamma_{CD}{}^K{}_F\Gamma_{EKAB}$$
$$+ \Gamma_{EF}{}^K{}_B\Gamma_{CDAK} - \Gamma_{CD}{}^K{}_B\Gamma_{EFAK} - t_{CD}{}^{GH}{}_{EF}\Gamma_{GHAB}$$

and define there the quantities $t_{AB}{}^{EF}{}_{CD}$, R_{ABCDEF}, σ^α_{AB}, Σ^α_{AB} by

$$t_{AB}{}^{EF}{}_{CD}e^a{}_{EF} \equiv 2\Gamma_{AB}{}^E{}_{(C}e^a{}_{D)E} - 2\Gamma_{CD}{}^E{}_{(A}e^a{}_{B)E} - e^a{}_{CD,b}e^b{}_{AB} + e^a{}_{AB,b}e^b{}_{CD},$$
$$R_{ABCDEF} \equiv r_{ABCDEF} - \tfrac{1}{2}\left[\left(S_{ABCE} - \tfrac{1}{6}Rh_{ABCE}\right)\epsilon_{DF} + \left(S_{ABDF} - \tfrac{1}{6}Rh_{ABDF}\right)\epsilon_{CE}\right],$$
$$\sigma^\alpha_{AB} \equiv D_{AB}\phi^\alpha - \phi^\alpha_{AB},$$
$$\Sigma^\alpha_{AB} \equiv D^P{}_A\phi^\alpha_{BP} + \tfrac{1}{2}\epsilon_{AB}f^\alpha.$$

It is important to note that R and S_{ABCD}, defined in (15), (14), represent functions of the fields that need not bear relation with the geometric Ricci scalar and trace-free part of the Ricci tensor. The equality among the quantities defined in terms of the fields ϕ^α and its derivatives and the geometric quantities comes when the field equations are solved.

The tensor fields on the left-hand side have been introduced as labels for the equations and for discussing in an ordered manner their interdependences. In terms of these tensor fields, the field equations read

$$t_{AB}{}^{EF}{}_{CD}e^a{}_{EF} = 0, \qquad R_{ABCDEF} = 0, \qquad \sigma^\alpha_{AB} = 0, \qquad \Sigma^\alpha_{AB} = 0.$$

The first equation is Cartan's first structural equation with the requirement that the metric connection be torsion free. The second equation is equation (13). The third equation define the symmetric spinors ϕ^α_{AB}. The last equation is the field equation (12) in terms of ϕ^α_{AB}.

We want to see how to calculate in our particular gauge a formal expansion of the fields using the initial data in the form $\phi^\alpha(u, v)$. As the system of field equations is overdetermined we have to choose a subsystem of it. In the rest of this section, we choose a particular subsystem, writing the chosen equations in our gauge, and at the end we see how a formal expansion is determined by these equations and the initial data.

4.1. The $\sigma^\alpha_{00} = 0$ equations

The first set of equations that needs particular attention are the equations $\sigma^\alpha_{00} = 0$. In our gauge they read

$$\partial_u\phi^\alpha = \phi^\alpha_{00}.$$

These equations are used in the following to calculate ϕ^α_{00} each time we know ϕ^α as a function of u. In particular, as ϕ^α will be prescribed on W_0 as part of the initial data, this equation allows us to calculate ϕ^α_{00} there immediately.

4.2. The '∂_u-equations'

We now present what we will refer to as the '∂_u-equations'. These equations are chosen because they have the following features. They are a system of PDEs for the set of functions $\hat{e}^a{}_{A1}$, $\hat{\Gamma}_{A1CD}$ and ϕ^α_{A1}, which comprise all the unknowns with the exceptions of the free data ϕ^α and the derived functions ϕ^α_{00}. They are all interior equations on the hypersurfaces $\{w = w_0\}$

in the sense that only derivatives in the directions of u and v are involved; in particular, if we consider the hypersurface W_0, they are all inner equations in $\mathcal{N}_q$. The possibility of choosing such a subsystem of inner equations in $\mathcal{N}_q$ is due to the coincidence of the characteristic cones for all the unknowns. Also they split into a hierarchy that will be presented in the following section. The '∂_u-equations' are as follows.

Equations $t_{AB}{}^{EF}{}_{00}e^a{}_{EF} = 0$:

$$\partial_u \hat{e}^1{}_{01} + \frac{1}{u}\hat{e}^1{}_{01} = -2\hat{\Gamma}_{0101} + 2\hat{\Gamma}_{0100}\hat{e}^1{}_{01},$$

$$\partial_u \hat{e}^2{}_{01} + \frac{1}{u}\hat{e}^2{}_{01} = \frac{1}{u}\hat{\Gamma}_{0100} + 2\hat{\Gamma}_{0100}\hat{e}^2{}_{01},$$

$$\partial_u \hat{e}^1{}_{11} = -2\hat{\Gamma}_{1101} + 2\hat{\Gamma}_{1100}\hat{e}^1{}_{01},$$

$$\partial_u \hat{e}^2{}_{11} = \frac{1}{u}\hat{\Gamma}_{1100} + 2\hat{\Gamma}_{1100}\hat{e}^2{}_{01}.$$

Equations $R_{AB00EF} = 0$:

$$\partial_u \hat{\Gamma}_{0100} + \frac{2}{u}\hat{\Gamma}_{0100} - 2\hat{\Gamma}^2_{0100} = \frac{1}{2}\left(F^\alpha \partial_u \phi^\alpha_{00} + F^{\alpha\beta}\phi^\alpha_{00}\phi^\beta_{00}\right),$$

$$\partial_u \hat{\Gamma}_{0101} + \frac{1}{u}\hat{\Gamma}_{0101} - 2\hat{\Gamma}_{0100}\hat{\Gamma}_{0101} = \frac{1}{2}\left(F^\alpha \partial_u \phi^\alpha_{01} + F^{\alpha\beta}\phi^\alpha_{00}\phi^\beta_{01}\right),$$

$$\partial_u \hat{\Gamma}_{0111} + \frac{1}{u}\hat{\Gamma}_{0111} - 2\hat{\Gamma}_{0100}\hat{\Gamma}_{0111} = \frac{1}{2}\left(F^\alpha \partial_u \phi^\alpha_{11} + F^{\alpha\beta}\phi^\alpha_{00}\phi^\beta_{11} + \tilde{f}\right) - \frac{1}{12}\hat{f},$$

$$\partial_u \hat{\Gamma}_{1100} + \frac{1}{u}\hat{\Gamma}_{1100} - 2\hat{\Gamma}_{0100}\hat{\Gamma}_{1100} = F^\alpha \partial_u \phi^\alpha_{01} + F^{\alpha\beta}\phi^\alpha_{00}\phi^\beta_{01},$$

$$\partial_u \hat{\Gamma}_{1101} - 2\hat{\Gamma}_{1100}\hat{\Gamma}_{0101} = F^\alpha \partial_u \phi^\alpha_{11} + F^{\alpha\beta}\phi^\alpha_{00}\phi^\beta_{11} + \tilde{f} + \frac{1}{12}\hat{f},$$

$$\partial_u \hat{\Gamma}_{1111} - 2\hat{\Gamma}_{1100}\hat{\Gamma}_{0111} = F^\alpha \left(\frac{1}{2u}\partial_v \phi^\alpha{}_{11} + \hat{e}^1{}_{01}\partial_u \phi^\alpha_{11} + \hat{e}^2{}_{01}\partial_v \phi^\alpha_{11} - 2\hat{\Gamma}_{0111}\phi^\alpha_{01} + 2\hat{\Gamma}_{0101}\phi^\alpha_{11}\right)$$
$$+ F^{\alpha\beta}\phi^\alpha_{01}\phi^\beta_{11}.$$

Equations $\Sigma^\alpha_{A0} = 0$:

$$\partial_u \phi^\alpha_{01} = \frac{1}{2u}\left(\partial_v \phi^\alpha_{00} - 2\phi^\alpha_{01}\right) + \hat{e}^1{}_{01}\partial_u \phi^\alpha_{00} + \hat{e}^2{}_{01}\partial_v \phi^\alpha_{00} - 2\hat{\Gamma}_{0101}\phi^\alpha_{00} + 2\hat{\Gamma}_{0100}\phi^\alpha_{01},$$

$$\partial_u \phi^\alpha_{11} = \frac{1}{2u}\left(\partial_v \phi^\alpha_{01} - \phi^\alpha_{11}\right) + \hat{e}^1{}_{01}\partial_u \phi^\alpha_{01} + \hat{e}^2{}_{01}\partial_v \phi^\alpha_{01} - \hat{\Gamma}_{0111}\phi^\alpha_{00} + \hat{\Gamma}_{0100}\phi^\alpha_{11} + \frac{1}{2}f^\alpha.$$

4.3. The ∂_u-equations hierarchy

The system of ∂_u-equations splits into a hierarchy of subsystems as follows:

(H1) $R_{000001} = 0$,
(H2) $t_{01}{}^{EF}{}_{00}e^2{}_{EF} = 0$,
(H3) $t_{01}{}^{EF}{}_{00}e^1{}_{EF} = 0$, $R_{010001} = 0$, $\Sigma^\alpha_{00} = 0$,
(H4) $R_{000011} = 0$,
(H5) $t_{11}{}^{EF}{}_{00}e^2{}_{EF} = 0$,
(H6) $R_{110001} = 0$, $\Sigma^\alpha_{10} = 0$,
(H7) $R_{010011} = 0$,
(H8) $R_{110011} = 0$,
(H9) $t_{11}{}^{EF}{}_{00}e^1{}_{EF} = 0$.

The defining property of the hierarchy is the following. If ϕ^α are prescribed on $\{w = w_0\}$ and using $\sigma^\alpha_{00} = 0$, then H1 reduces to an ODE. Once we have its solution, H2 reduces to an ODE. Given its solution, H3 reduces to a system of ODEs, with coefficients that are calculated by operations interior to $\{w = w_0\}$ from the previously known or calculated functions. This procedure continues till the end of the hierarchy. So, given ϕ^α on $\{w = w_0\}$ and the appropriate initial data on $U_0 \cap \{w = w_0\}$, all the unknowns can be determined on $\{w = w_0\}$ by solving a sequence of ODEs in the independent variable u.

4.4. The '∂_w-equations'

The initial data ϕ^α are prescribed on W_0, and to determine their evolution off W_0 we need the equations $\sigma^\alpha_{11} = 0$, referred to as the ∂_w-equations, which read

$$\partial_w \phi^\alpha = \phi^\alpha_{11} - \hat{e}^1{}_{11} \partial_u \phi^\alpha - \hat{e}^2{}_{11} \partial_v \phi^\alpha.$$

4.5. Calculating the formal expansion

Let us call X any of the unknowns that we are solving for, i.e. $\hat{e}^a{}_{AB}$, $\hat{\Gamma}_{ABCD}$, ϕ^α, ϕ^α_{AB}. We want to see that it is possible to obtain a formal expansion of X in a neighbourhood of q using the σ^α_{00}-equations, the ∂_u-equations and the ∂_w-equations, given ϕ^α on W_0 as our datum. We give here an inductive argument showing that with our setting $\partial^k_w X|_{W_0}$ can be determined for all k.

We need the following conditions, obtained from the gauge requirements (17), (18)

$$\partial^k_w \hat{e}^a{}_{A1}\big|_I = 0, \qquad a = 1, 2, \qquad A = 0, 1, \qquad k \geqslant 0,$$
$$\partial^k_w \hat{\Gamma}_{A1CD}\big|_I = 0, \qquad A, C, D = 0, 1, \qquad k \geqslant 0,$$

and from the $\sigma^\alpha_{00} = 0$ equations and the spinorial behaviour as discussed in lemma 3.1,

$$\partial^k_w \phi^\alpha{}_{A1}\big|_I = \tfrac{1}{2} \partial_u \partial^{1+A}_v \partial^k_w \phi\big|_I, \qquad A = 0, 1, \qquad k \geqslant 0,$$

which are our initial conditions for the ∂^k_w-derivatives of the ∂_u-equations.

Using the initial conditions for $k = 0$ and following what has been said in subsection 4.3 we successively integrate the subsystems H1 to H9 to determine all of X on W_0.

As inductive hypothesis we assume as known $\partial^p_w X|_{W_0}$, $0 \leqslant p \leqslant k - 1$, $k \geqslant 1$. Applying formally ∂^{k-1}_w to the ∂_w-equations, and restricting them to W_0, we find $\partial^k_w \phi^\alpha|_{W_0}$ in terms of known functions. We apply formally ∂^k_w to the ∂_u-equations. This is a system of PDEs where the unknowns are $\partial^k_w X$. Keeping the discussed hierarchy and considering the functions that we already know on W_0, it again becomes a sequence of ODEs, which together with the initial conditions on I can be integrated on W_0. Thus, we know $\partial^k_w X|_{W_0}$ and the induction step is completed.

The procedure just stated shows that we know $\partial^k_w X|_{W_0}$ for all k. Expanding these functions around $q = \{u = 0, v = 0, w = 0\}$ gives

$$\partial^m_u \partial^n_v \partial^p_w X\big|_q \qquad \forall\, m, n, p,$$

and the procedure gives a unique sequence of expansion coefficients for all the functions in X.

Lemma 4.1. *The procedure described above determines at the point $O = (u = 0, v = 0, w = 0)$ from the data ϕ^α given on W_0 according to (19) a unique sequence of expansion coefficients*

$$\partial^m_u \partial^n_v \partial^p_w X(O), \qquad m, n, p = 0, 1, 2, \ldots,$$

where X stands for any of the functions $\hat{e}^a{}_{AB}$, $\hat{\Gamma}_{ABCD}$, ϕ^α, ϕ^α_{AB}.

If the corresponding Taylor series are absolutely convergent in some neighbourhood Q of O, they define a solution to the equation $\sigma_{00}^\alpha = 0$, to the ∂_u-equations and to the ∂_w-equations on Q.

By lemma 3.1 we know that all spinor-valued functions should have a specific v-finite expansion type. The following lemma, whose proof is quite similar to the proof in [4], is important for handling the estimates in the following section.

Lemma 4.2. *If the data ϕ^α are given on W_0 as in (19) the formal expansions of the fields obtained in lemma 4.1 correspond to the ones of functions of v-finite expansion types given by*

$$
\begin{aligned}
k_{\hat{e}^1{}_{AB}} &= -A - B, & k_{\hat{e}^2{}_{AB}} &= 3 - A - B, & AB &= 01, 11, \\
k_{\hat{\Gamma}_{01AB}} &= 2 - A - B, & k_{\hat{\Gamma}_{11AB}} &= 1 - A - B, & A, B &= 0, 1, \\
k_{\phi^\alpha} &= 0, & k_{\phi^\alpha_{AB}} &= 2 - A - B, & A, B &= 0, 1.
\end{aligned}
$$

5. Convergence of the formal expansion

In the previous section we have seen how to calculate a formal expansion for $\hat{e}^a{}_{AB}$, $\hat{\Gamma}_{ABCD}$, ϕ^α, ϕ^α_{AB} given $\phi^\alpha|_{W_0}$, or, what is the same, given the null data. From lemma 2.2 we know which are the necessary conditions on the null data in order to have analytic solutions of the conformal field equations. In this section we show that those conditions are also sufficient for the formal expansion determined in the previous section to be absolutely convergent.

We consider the abstract null data as given by n sequences

$$
\hat{\mathcal{D}}^\alpha = \left\{ \psi^\alpha_{A_1 B_1}, \psi^\alpha_{A_2 B_2 A_1 B_1}, \psi^\alpha_{A_3 B_3 A_2 B_2 A_1 B_1}, \cdots \right\}
$$

of totally symmetric spinors satisfying the reality condition (8) and we construct $\phi^\alpha|_{W_0}$ by setting in the expansions (19)

$$
D_{(A_m B_m} \ldots D_{A_1 B_1)} \phi^\alpha(q) = \psi^\alpha_{A_m B_m \ldots A_1 B_1}, \qquad m \geqslant 0.
$$

Observing lemma 2.2 one finds as a necessary condition for the functions ϕ^α on W_0 to determine an analytic solution to the conformal static vacuum field equations that its non-vanishing Taylor coefficients at the point O satisfy estimates of the form

$$
\left| \partial_u^m \partial_v^n \phi^\alpha(O) \right| \leqslant \binom{2m}{n} m! n! \frac{M}{r^m}, \qquad m \geqslant 0, \qquad 0 \leqslant n \leqslant 2m. \tag{20}
$$

These conditions are also sufficient for $\phi^\alpha(u, v)$ to be holomorphic functions on W_0. So the null data give rise to n analytic functions ϕ^α on W_0.

From $\sigma_{00}^\alpha = 0$ we have $\phi_{00}^\alpha = \partial_u \phi^\alpha$, so having $\phi^\alpha|_{W_0}$ we have $\phi_{00}^\alpha|_{W_0}$, which is also an analytic function on W_0.

Following lemma 6.1 in [4], we can derive from (20) slightly different type of estimates for $\phi^\alpha(u, v)$ which are more convenient in our case.

Lemma 5.1. *Let e be the Euler number. For given ρ_{ϕ^α} in $\mathbb{R}$, such that $0 < \rho_{\phi^\alpha} < e^2$, there exist positive constants c_{ϕ^α}, r_{ϕ^α} so that (20) imply estimates of the form*

$$
\left| \partial_u^m \partial_v^n \phi^\alpha(O) \right| \leqslant c_{\phi^\alpha} \frac{r_{\phi^\alpha}^m m! \rho_{\phi^\alpha}^n n!}{(m+1)^2 (n+1)^2}, \qquad m \geqslant 0, \qquad 0 \leqslant n \leqslant 2m. \tag{21}
$$

We present our estimates.

Lemma 5.2. *Assume $\phi^\alpha = \phi^\alpha(u, v)$ are holomorphic functions defined on some open neighbourhood U of $O = \{u = 0, v = 0, w = 0\}$ in $W_0 = \{w = 0\}$ which have expansions of the form*

$$\phi^\alpha(u, v) = \sum_{m=0}^{\infty} \sum_{n=0}^{2m} \psi_{m,n}^\alpha u^m v^n$$

so that its Taylor coefficients at the point O satisfy estimates of the type (21) with some positive constants c_{ϕ^α}, r_{ϕ^α} and $\rho_{\phi^\alpha} < \frac{1}{8}$. Then there exist positive constants r, ρ, $c_{\hat{e}^a{}_{AB}}$, $c_{\hat{\Gamma}_{ABCD}}$, $c_{\phi_{AB}^\alpha}$ so that the expansion coefficients determined from ϕ^α in lemma 4.1 satisfy for $m, n, p = 0, 1, 2, \ldots$

$$|\partial_u^m \partial_v^n \partial_w^p X(O)| \leqslant c_X \frac{r^{m+p+q_X}(m+p)!\rho^n n!}{(m+1)^2(n+1)^2(p+1)^2}, \tag{22}$$

where X stands for any of the functions $\hat{e}^a{}_{AB}$, $\hat{\Gamma}_{ABCD}$, ϕ^α, ϕ_{AB}^α and

$$q_{\hat{e}^a{}_{AB}} = q_{\hat{\Gamma}_{ABCD}} = -1, \qquad q_{\phi^\alpha} = q_{\phi_{AB}^\alpha} = 0.$$

Remark. Taking into account the v-finite expansion types of the functions X obtained in lemma 4.2, we can replace the right-hand sides in the estimates above by zero if n is large enough relative to m. This will not be pointed out at each step and for convenience the estimates will be written as above.

We take the following four lemmas from [4]. The first states the necessary part of the estimates, and the other three are needed in order to manipulate the estimates in the proof of lemma 5.2.

Lemma 5.3. *If g is a holomorphic function near O, then there exist positive constants c, r_0, ρ_0 such that*

$$\left|\partial_u^m \partial_v^n \partial_w^p g(O)\right| \leqslant c \frac{r^{m+p}(m+p)!\rho^n n!}{(m+1)^2(n+1)^2(p+1)^2}, \qquad m, n, p = 0, 1, 2, \ldots$$

for any $r \geqslant r_0$, $\rho \geqslant \rho_0$. If in addition $g(0, v, 0) = 0$, the constants can be chosen such that

$$\left|\partial_u^m \partial_v^n \partial_w^p g(O)\right| \leqslant c \frac{r^{m+p-1}(m+p)!\rho^n n!}{(m+1)^2(n+1)^2(p+1)^2}, \qquad m, n, p = 0, 1, 2, \ldots$$

for any $r \geqslant r_0$, $\rho \geqslant \rho_0$.

Lemma 5.4. *For any non-negative integer n there is a positive constant C, $C > 1$, independent of n so that*

$$\sum_{k=0}^{n} \frac{1}{(k+1)^2(n-k+1)^2} \leqslant C \frac{1}{(n+1)^2}.$$

In the following, C will always denote the constant above.

Lemma 5.5. *For any integers m, n, k, j, with $0 \leqslant k \leqslant m$, and $0 \leqslant j \leqslant n$ resp. $0 \leqslant j \leqslant n-1$ holds*

$$\binom{m}{k}\binom{n}{j} \leqslant \binom{m+n}{k+j} \text{ resp. } \binom{m}{k}\binom{n-1}{j} \leqslant \binom{m+n}{k+j}.$$

14

Lemma 5.6. *Let m, n, p be non-negative integers and g_i, $i = 1, \ldots, N$, be smooth complex valued functions of u, v, w on some neighbourhood U of O whose derivatives satisfy on U (resp. at a given point $q \in U$) estimates of the form*

$$\left| \partial_u^j \partial_v^k \partial_w^l g_i(O) \right| \leqslant c_i \frac{r^{j+l+q_i}(j+l)!\rho^k k!}{(j+1)^2(k+1)^2(l+1)^2}$$

for $0 \leqslant j \leqslant m$, $0 \leqslant k \leqslant n$, $0 \leqslant l \leqslant p$, with some positive constants c_i, r, ρ and some fixed integers q_i (independent of j, k, l). Then one has on U (resp. at q) the estimates

$$\left| \partial_u^m \partial_v^n \partial_w^p (g_1 \cdot \ldots \cdot g_N)(O) \right| \leqslant C^{3(N-1)} c_1 \cdot \ldots \cdot c_N \frac{r^{m+p+q_1+\cdots+q_N}(m+p)!\rho^n n!}{(m+1)^2(n+1)^2(p+1)^2}. \tag{23}$$

Remark. This lemma remains true if *m, n, p* are replaced in (23) by integers m', n', p' with $0 \leqslant m' \leqslant m$, $0 \leqslant n' \leqslant n$, $0 \leqslant p' \leqslant p$.

The factor $C^{3(N-1)}$ in (23) can be replaced by $C^{(3-s)(N-1)}$ if *s* of the integers *m, n, p* vanish.

From now on we consider that a function in a modulus sign is evaluated at the origin *O*.

Proof of lemma 5.2. The proof is by induction, following the inductive procedure which led to lemma 4.1. A general outline is as follows. We start leaving the choice of the constants *r*, ρ, c_f open (except for c_{ϕ^α} and $c_{\phi^{\alpha_{00}}}$, which are determined by the data). We use the induction hypothesis and the equations that lead to lemma 4.1 to derive estimates for the derivatives of the next order. These estimates are of the form

$$\left| \partial_u^m \partial_v^n \partial_w^p X \right| \leqslant c_X \frac{r^{m+p+q_X}(m+p)!\rho^n n!}{(m+1)^2(n+1)^2(p+1)^2} A_X \tag{24}$$

with certain constants A_X which depend on *m, n, p* and the constants c_X, *r* and ρ. Sometimes superscripts will indicate to which order of differentiability particular constants A_X refer. In the way we will have to make assumptions on *r* to proceed with the induction step. We shall collect these conditions and the constants A_X, or estimates for them, and at the end it will be shown that the constants c_X, *r* and ρ can be adjusted so that all conditions are satisfied and $A_X \leqslant 1$. This will complete the induction proof.

In order not to write long formulae that do not add to the understanding of the procedure, we state here some properties that are used to simplify the estimates.

- During the procedure we need estimates on the derivatives of the functions f, f^α, F^α and $F^{\alpha\beta}$, which are analytic functions of the fields ϕ^α and ϕ_{AB}^α. The functions are also scalars if we consider them as functions on the manifold through their dependence on the fields. This and the Cauchy estimates for derivatives of analytic functions allow us to get the following result. Let us denote by g any of the functions f, f^α, F^α and $F^{\alpha\beta}$. If, for $r \geqslant r_0$, $0 \leqslant j \leqslant m$, $0 \leqslant k \leqslant n$, $0 \leqslant l \leqslant p$,

$$\left| \partial_u^j \partial_v^k \partial_w^l \phi^\alpha \right| \leqslant c_{\phi^\alpha} \frac{r^{j+l}(j+l)!\rho^k k!}{(j+1)^2(k+1)^2(l+1)^2},$$

$$\left| \partial_u^j \partial_v^k \partial_w^l \phi_{AB}^\alpha \right| \leqslant c_{\phi_{AB}^\alpha} \frac{r^{j+l}(j+l)!\rho^k k!}{(j+1)^2(k+1)^2(l+1)^2}$$

then there exists positive constants c_g, $R_0 \geqslant r_0$ such that for $R \geqslant R_0$

$$\left| \partial_u^m \partial_v^n \partial_w^p g \right| \leqslant c_g \frac{R^{m+p}(m+p)!\rho^n n!}{(m+1)^2(n+1)^2(p+1)^2}. \tag{25}$$

Using this for calculating the estimates leads to terms with the factor $\frac{R}{r}$. We can then fix R by making $R = R_0$ and add to the requirements for r that $r \geqslant R_0$. We have then

$$\frac{R}{r} \leqslant 1. \tag{26}$$

- After calculating the estimates and using (25) and (26) we find that all the A's satisfy inequalities of the form

$$A \leqslant \alpha + \frac{\hat{\alpha}}{r},$$

where α, $\hat{\alpha}$ are constants that do not depend on r. If $\hat{\alpha} = 0$ then we have to show that we can make $\alpha \leqslant 1$. If the $\hat{\alpha}$'s are not zero we can take a constant a, $0 < a < 1$, and require that $\alpha \leqslant a$ and then choose r large enough such that $\frac{\hat{\alpha}}{r} \leqslant 1 - a$. In the estimates that follow, we shall not write the explicit expressions for the $\hat{\alpha}$'s, as they do not play any role if we are able to make r big enough at the end of the procedure.

As the ϕ^{α}'s are analytic functions of u and v on W_0, also the ϕ^{α}_{00}'s are. Then there exist positive constants $0 < \rho_{\phi^{\alpha}}, \rho_{\phi^{\alpha}_{00}} \leqslant \frac{1}{8}$ and $c_{\phi^{\alpha}}, c_{\phi^{\alpha}_{00}}, r_{\phi^{\alpha}}, r_{\phi^{\alpha}_{00}}$ such that

$$\left| \partial_u^m \partial_v^n \phi^{\alpha} \right| \leqslant c_{\phi^{\alpha}} \frac{r_{\phi^{\alpha}}^m m! \rho_{\phi^{\alpha}}^n n!}{(m+1)^2 (n+1)^2},$$

$$\left| \partial_u^m \partial_v^n \phi^{\alpha}_{00} \right| \leqslant c_{\phi^{\alpha}_{00}} \frac{r_{\phi^{\alpha}_{00}}^m m! \rho_{\phi^{\alpha}_{00}}^n n!}{(m+1)^2 (n+1)^2}.$$

As the inequalities do not change if the constants are changed for bigger constants, we choose (but leaving the precise value open)

$$r \geqslant \max \left\{ r_{\phi^{\alpha}}, r_{\phi^{\alpha}_{00}} \right\}, \qquad \rho \geqslant \max \left\{ \rho_{\phi^{\alpha}}, \rho_{\phi^{\alpha}_{00}} \right\},$$

then

$$\left| \partial_u^m \partial_v^n \partial_w^0 \phi^{\alpha} \right| \leqslant c_{\phi^{\alpha}} \frac{r^m m! \rho^n n!}{(m+1)^2 (n+1)^2},$$

$$\left| \partial_u^m \partial_v^n \partial_w^0 \phi^{\alpha}_{00} \right| \leqslant c_{\phi^{\alpha}_{00}} \frac{r^m m! \rho^n n!}{(m+1)^2 (n+1)^2}.$$

Using how the frame fields and the coordinates were constructed,

$$\hat{e}^a{}_{AB} \big|_{U_0} = 0 \Rightarrow \left| \partial_u^0 \partial_v^n \partial_w^p \hat{e}^a{}_{AB} \right| = 0,$$

$$\hat{\Gamma}_{ABCD} \big|_{U_0} = 0 \Rightarrow \left| \partial_u^0 \partial_v^n \partial_w^p \hat{\Gamma}_{ABCD} \right| = 0.$$

From the required spinorial behaviour we have

$$\phi^{\alpha}_{01} = \tfrac{1}{2} \partial_v \phi^{\alpha}_{00}, \qquad \phi^{\alpha}_{11} = \partial_v \phi^{\alpha}_{01}, \qquad \text{on} \quad U_0,$$

then

$$\left| \partial_u^0 \partial_v^n \partial_w^0 \phi^{\alpha}_{01} \right| = \frac{1}{2} \left| \partial_v^{n+1} \phi^{\alpha}_{00} \right| \leqslant \frac{1}{2} c_{\phi^{\alpha}_{00}} \frac{\rho^{n+1}(n+1)!}{(n+2)^2} = c_{\phi^{\alpha}_{01}} \frac{\rho^n n!}{(n+1)^2} A^{m=0,\,p=0}_{\phi^{\alpha}_{01}},$$

where

$$A^{m=0,\,p=0}_{\phi^{\alpha}_{01}} = \frac{1}{2} \frac{c_{\phi^{\alpha}_{00}}}{c_{\phi^{\alpha}_{01}}} \rho \frac{(n+1)^3}{(n+2)^2} \leqslant \frac{1}{2} \frac{c_{\phi^{\alpha}_{00}}}{c_{\phi^{\alpha}_{01}}} \rho,$$

as $k_{\phi^{\alpha}_{01}} = 1$ and then $n = 0, 1$. Now that we have this inequality, and in the same way, we get

$$A^{m=0,\,p=0}_{\phi^{\alpha}_{11}} \leqslant \frac{1}{4} \frac{c_{\phi^{\alpha}_{01}}}{c_{\phi^{\alpha}_{11}}} \rho.$$

16

Now we have estimates for $\partial_u^0 \partial_v^n \partial_w^0 X$, with $n \in \mathbb{N}_0$. We assume as inductive hypothesis that we have estimates for $\partial_u^l \partial_v^n \partial_w^0 X$, with $0 \leqslant l \leqslant m-1$, $n \in \mathbb{N}_0$. We already have those estimates for ϕ^α and ϕ_{00}^α. We use the ∂_u-equations to get estimates for $\partial_u^m \partial_v^n \partial_w^0 X$ for the rest of the unknowns. The estimates obtained in this step are less restrictive than the estimates for general p, so we skip their enumeration and go on to the next step.

We assume that we have estimates for $\partial_u^m \partial_v^n \partial_w^l X$, with $0 \leqslant l \leqslant p-1$, $m \in \mathbb{N}_0$, $n \in \mathbb{N}_0$, and use the system of equations and the properties stated on the lemmas of this section to get estimates for $\partial_u^m \partial_v^n \partial_w^p X$.

Using the ∂_w-equations we get

$$A_{\phi^\alpha}^{p \geqslant 1} \leqslant 4C^3 c_{\hat{e}^1{}_{11}} + \frac{\hat{\alpha}_{\phi^\alpha}^{p \geqslant 1}}{r}.$$

Using the equation $D_{11}\phi_{00}^\alpha = D_{00}\phi_{11}^\alpha$, which follows from $\sigma_{00}^\alpha = 0$ and $\sigma_{11}^\alpha = 0$,

$$A_{\phi_{00}^\alpha}^{p \geqslant 1} \leqslant \frac{4}{c_{\phi_{00}^\alpha}} \left(c_{\phi_{11}^\alpha} + C^3 c_{\hat{e}^1{}_{11}} c_{\phi_{00}^\alpha} \right) + \frac{\hat{\alpha}_{\phi_{00}^\alpha}^{p \geqslant 1}}{r}.$$

For $m = 0$ and $p \geqslant 1$ it is not possible to use the ∂_u-equations to get estimates, so it is necessary to use the spinorial behaviour to get, as before,

$$A_{\phi_{01}^\alpha}^{m=0} \leqslant \frac{1}{2} \frac{c_{\phi_{00}^\alpha}}{c_{\phi_{01}^\alpha}} \rho, \qquad A_{\phi_{11}^\alpha}^{m=0} \leqslant \frac{1}{4} \frac{c_{\phi_{01}^\alpha}}{c_{\phi_{11}^\alpha}} \rho.$$

Now we use the ∂_u-equations, obtaining

$$A_{\hat{e}^1{}_{01}}^{m \geqslant 1} \leqslant \frac{\hat{\alpha}_{\hat{e}^1{}_{01}}^{m \geqslant 1}}{r}, \qquad\qquad A_{\hat{e}^2{}_{01}}^{m \geqslant 1} \leqslant \frac{c_{\hat{\Gamma}_{0100}}}{2 c_{\hat{e}^2{}_{01}}} + \frac{\hat{\alpha}_{\hat{e}^2{}_{01}}^{m \geqslant 1}}{r},$$

$$A_{\hat{e}^1{}_{11}}^{m \geqslant 1} \leqslant \frac{\hat{\alpha}_{\hat{e}^1{}_{11}}^{m \geqslant 1}}{r}, \qquad\qquad A_{\hat{e}^2{}_{11}}^{m \geqslant 1} \leqslant \frac{c_{\hat{\Gamma}_{1100}}}{c_{\hat{e}^2{}_{11}}} + \frac{\hat{\alpha}_{\hat{e}^2{}_{11}}^{m \geqslant 1}}{r},$$

$$A_{\hat{\Gamma}_{0100}}^{m \geqslant 1} \leqslant \frac{C^3 c_{F^\alpha} c_{\phi_{00}^\alpha}}{2 c_{\hat{\Gamma}_{0100}}} + \frac{\hat{\alpha}_{\hat{\Gamma}_{0100}}^{m \geqslant 1}}{r}, \qquad\qquad A_{\hat{\Gamma}_{0101}}^{m \geqslant 1} \leqslant \frac{C^3 c_{F^\alpha} c_{\phi_{01}^\alpha}}{2 c_{\hat{\Gamma}_{0101}}} + \frac{\hat{\alpha}_{\hat{\Gamma}_{0101}}^{m \geqslant 1}}{r},$$

$$A_{\hat{\Gamma}_{0111}}^{m \geqslant 1} \leqslant \frac{C^3 c_{F^\alpha} c_{\phi_{11}^\alpha}}{2 c_{\hat{\Gamma}_{0111}}} + \frac{\hat{\alpha}_{\hat{\Gamma}_{0111}}^{m \geqslant 1}}{r}, \qquad\qquad A_{\hat{\Gamma}_{1100}}^{m \geqslant 1} \leqslant \frac{C^3 c_{F^\alpha} c_{\phi_{01}^\alpha}}{c_{\hat{\Gamma}_{1100}}} + \frac{\hat{\alpha}_{\hat{\Gamma}_{1100}}^{m \geqslant 1}}{r},$$

$$A_{\hat{\Gamma}_{1101}}^{m \geqslant 1} \leqslant \frac{C^3 c_{F^\alpha} c_{\phi_{11}^\alpha}}{c_{\hat{\Gamma}_{1101}}} + \frac{\hat{\alpha}_{\hat{\Gamma}_{1101}}^{m \geqslant 1}}{r}, \qquad\qquad A_{\hat{\Gamma}_{1111}}^{m \geqslant 1} \leqslant \frac{C^3 \left(\rho + C^3 c_{\hat{e}^1{}_{01}} \right) c_{F^\alpha} c_{\phi_{11}^\alpha}}{c_{\hat{\Gamma}_{1111}}} + \frac{\hat{\alpha}_{\hat{\Gamma}_{1111}}^{m \geqslant 1}}{r},$$

$$A_{\phi_{01}^\alpha}^{m \geqslant 1} \leqslant \frac{\left(\rho + C^3 c_{\hat{e}^1{}_{01}} \right) c_{\phi_{00}^\alpha}}{c_{\phi_{01}^\alpha}} + \frac{\hat{\alpha}_{\phi_{01}^\alpha}^{m \geqslant 1}}{r}, \qquad\qquad A_{\phi_{11}^\alpha}^{m \geqslant 1} \leqslant \frac{\left(\rho + C^3 c_{\hat{e}^1{}_{01}} \right) c_{\phi_{01}^\alpha}}{c_{\phi_{11}^\alpha}} + \frac{\hat{\alpha}_{\phi_{11}^\alpha}^{m \geqslant 1}}{r}.$$

We take a constant a, $0 < a < 1$, whose precise value will be fixed later. We see that if we can make r arbitrarily large, then all the A's are less or equal to 1 if the following inequalities are satisfied:

17

$$\frac{c_{\hat{\Gamma}_{0100}}}{2c_{\hat{e}^2{}_{01}}} \leqslant a, \qquad\qquad \frac{c_{\hat{\Gamma}_{1100}}}{c_{\hat{e}^2{}_{11}}} \leqslant a, \qquad\qquad \frac{C^3 c_{F\alpha} c_{\phi^\alpha_{00}}}{2c_{\hat{\Gamma}_{0100}}} \leqslant a,$$

$$\frac{C^3 c_{F\alpha} c_{\phi^\alpha_{01}}}{2c_{\hat{\Gamma}_{0101}}} \leqslant a, \qquad\qquad \frac{C^3 c_{F\alpha} c_{\phi^\alpha_{11}}}{2c_{\hat{\Gamma}_{0111}}} \leqslant a, \qquad\qquad \frac{C^3 c_{F\alpha} c_{\phi^\alpha_{01}}}{c_{\hat{\Gamma}_{1100}}} \leqslant a,$$

$$\frac{C^3 c_{F\alpha} c_{\phi^\alpha_{11}}}{c_{\hat{\Gamma}_{1101}}} \leqslant a, \qquad\qquad \frac{C^3 \big(\rho + C^3 c_{\hat{e}^1{}_{01}}\big) c_{F\alpha} c_{\phi^\alpha_{11}}}{c_{\hat{\Gamma}_{1111}}} \leqslant a, \qquad\qquad \frac{4\big(c_{\phi^\alpha_{11}} + C^3 c_{\hat{e}^1{}_{11}} c_{\phi^\alpha_{00}}\big)}{c_{\phi^\alpha_{00}}} \leqslant a,$$

$$\frac{\big(\rho + C^3 c_{\hat{e}^1{}_{01}}\big) c_{\phi^\alpha_{00}}}{c_{\phi^\alpha_{01}}} \leqslant a, \qquad\qquad \frac{\big(\rho + C^3 c_{\hat{e}^1{}_{01}}\big) c_{\phi^\alpha_{01}}}{c_{\phi^\alpha_{11}}} \leqslant a.$$

We define

$$\rho \equiv \max\big\{\rho_{\phi^\alpha}, \rho_{\phi^\alpha_{00}}\big\} \leqslant \tfrac{1}{8}$$

and

$$a \equiv (32\rho^2)^{\frac{1}{3}} < 1.$$

Considering the last two inequalities, we define

$$c_{\hat{e}^1{}_{01}} \equiv \frac{\rho}{C^3}, \qquad c_{\phi^\alpha_{01}} \equiv \frac{2\rho c_{\phi^\alpha_{00}}}{a}, \qquad c_{\phi^\alpha_{11}} \equiv \frac{2\rho c_{\phi^\alpha_{01}}}{a}.$$

With the previous definitions and

$$c_{\hat{e}^1{}_{11}} \equiv \frac{4\rho^2}{C^3 a^2}$$

the third to last inequality is satisfied. The rest of the inequalities are satisfied by defining

$$c_{\hat{\Gamma}_{0100}} \equiv \frac{C^3 c_{F\alpha} c_{\phi^\alpha_{00}}}{2a}, \qquad c_{\hat{\Gamma}_{0101}} \equiv \frac{C^3 c_{F\alpha} c_{\phi^\alpha_{01}}}{2a}, \qquad c_{\hat{\Gamma}_{0111}} \equiv \frac{C^3 c_{F\alpha} c_{\phi^\alpha_{11}}}{2a},$$

$$c_{\hat{\Gamma}_{1100}} \equiv \frac{C^3 c_{F\alpha} c_{\phi^\alpha_{01}}}{a}, \qquad c_{\hat{\Gamma}_{1101}} \equiv \frac{C^3 c_{F\alpha} c_{\phi^\alpha_{11}}}{a}, \qquad c_{\hat{\Gamma}_{1111}} \equiv \frac{C^3 \big(\rho + C^3 c_{\hat{e}^1{}_{01}}\big) c_{F\alpha} c_{\phi^\alpha_{11}}}{a},$$

$$c_{\hat{e}^2{}_{01}} \equiv \frac{c_{\hat{\Gamma}_{0100}}}{2a}, \qquad c_{\hat{e}^2{}_{11}} \equiv \frac{c_{\hat{\Gamma}_{1100}}}{a}.$$

Now we can take *r* big enough so that all the *A*'s are less or equal than 1.   □

The following lemma states the convergence result. The proof follows as the one given in [4].

Lemma 5.7.   *The estimates (22) for the derivatives of the functions X and the expansion types given in lemma 4.2 imply that the associated Taylor series are absolutely convergent in the domain $|v| < \frac{1}{\alpha\rho}$, $|u| + |w| < \frac{\alpha^2}{r}$, for any real number α, $0 < \alpha \leqslant 1$. It follows that the formal expansions determined in lemma 4.1 define indeed a (unique) holomorphic solution to the ∂_u-equations, the ∂_w-equations and the $\sigma^\alpha_{00} = 0$ equations, which induces the data ϕ^α on W_0.*

6. The complete set of equations on $\hat{N}$

We have seen in section 4 how to calculate a formal expansion for our fields using a subset of the field equations. In the previous section, we have shown that these formal expansions are convergent in a neighbourhood of *q*. In this section we shall show that these fields satisfy the complete system of field equations. First, we prove that the field equations are satisfied in the limit as $u \to 0$. Second, we derive a subsidiary system of equations, for which the first

18

result provides the initial conditions, and which allows us to prove that the complete system is satisfied.

Lemma 6.1.   *The functions $\hat{e}^a{}_{AB}$, $\hat{\Gamma}_{ABCD}$, ϕ^α, ϕ^α_{AB}, whose expansion coefficients are determined by lemma 4.1, with expansions that converge on an open neighbourhood of the point O, neighbourhood that we assume to coincide with $\hat{N}$, satisfy the complete set of field equations on the set U_0 in the sense that the fields $t_{AB}{}^{CD}{}_{EF}$, R_{ABCDEF}, σ^α_{AB}, Σ^α_{AB} calculated from these functions on $\hat{N}\backslash U_0$ have vanishing limit as $u \to 0$.*

Proof.   Taking into account that the equations used to calculate the expansion coefficients are already satisfied, it is left to show that $t_{01}{}^{AB}{}_{11} = 0$, $R_{AB0111} = 0$, $\sigma^\alpha_{01} = 0$ and $\Sigma^\alpha_{AB} = 0$ in the limit $u \to 0$. From the definition (see [4] for details) and using the way in which the coordinates and the frame field were constructed,

$$\lim_{u \to 0} t_{01}{}^{AB}{}_{11} = 0.$$

Using the definitions of R_{ABCDEF}, r_{ABCDEF}, $t_{AB}{}^{CD}{}_{EF}$ and the way in which the coordinates and the frame field were constructed, we have near $u = 0$

$$R_{AB0111} = \frac{1}{2u}\left[\partial_v \hat{\Gamma}_{11AB} - 2\hat{\Gamma}_{111(A}\epsilon^0_{B)} + \epsilon^0_A \epsilon^0_B \left(-\frac{2}{u}\hat{e}^1_{11} - \partial_v \hat{e}^2_{11} + 4\hat{\Gamma}_{0111}\right)\right]$$
$$- \frac{1}{2}\left[F^\alpha D_{AB}\phi^\alpha_{11} + F^{\alpha\beta}\phi^\alpha_{AB}\phi^\beta_{11} + \epsilon^0_A \epsilon^0_B \left(\tilde{f} - \frac{1}{6}\hat{f}\right)\right] + O(u).$$

Taking the limit $u \to 0$ and using on U_0 the ∂_u-equations, the ∂_w-equation and the spinorial behaviour of the quantities involved, we get

$$\lim_{u \to 0} R_{AB0111} = 0.$$

Using that $\phi^\alpha_{01} = \frac{1}{2}\partial_v \phi^\alpha_{00}$ in U_0 and $\phi^\alpha_{00} = \partial_u \phi^\alpha$ as $\sigma^\alpha_{00} = 0$,

$$\lim_{u \to 0} \sigma^\alpha_{01} = \left(\tfrac{1}{2}\partial_u \partial_v \phi^\alpha - \phi^\alpha_{01}\right)\big|_{u=0} = 0.$$

Now, as $\sigma^\alpha_{AB}\big|_{u=0} = 0$, $\Sigma^\alpha_{AB}\big|_{u=0} = \left(-\Sigma^\alpha_{BA} + t_{AG}{}^{EFG}{}_B D_{EF}\phi^\alpha\right)\big|_{u=0}$, which imply

$$\lim_{u \to 0} \Sigma^\alpha_{AB} = 0.$$

This concludes the proof showing that the complete system of field equations are satisfied in the limit as $u \to 0$.   □

Lemma 6.2.   *The functions $\hat{e}^a{}_{AB}$, $\hat{\Gamma}_{ABCD}$, ϕ^α, ϕ^α_{AB}, corresponding to the expansions determined in lemma 4.1, satisfy the complete set of field equations on the set $\hat{N}$.*

Proof.   We have seen that the field equations are satisfied in the limit $u \to 0$; we proceed to deduce a system of equations that those quantities satisfy.

Following the proof of lemma 5.5 in [4] we find that

$$\left(\partial_u + \frac{1}{u}\right)t_{01}{}^{AB}{}_{11} = 2\hat{\Gamma}_{0100}t_{01}{}^{AB}{}_{11} + 2R^{(A}{}_{00111}\epsilon_0{}^{B)}. \tag{27}$$

Also following the proof of lemma 5.5 in [4] we get in our case that

$$\left(\partial_u + \frac{1}{u}\right)R_{AB0111} = 2\hat{\Gamma}_{0100}R_{AB0111} - \left(S_{AB00} - \frac{1}{6}R\epsilon_A{}^1\epsilon_B{}^1\right)t_{01}{}^{01}{}_{11}$$
$$+ \frac{1}{2}\left(D^{EF}S_{ABEF} - \frac{1}{6}D_{AB}R\right). \tag{28}$$

19

Let us consider the coupled system of PDEs (27) and (28), with the initial conditions already derived. Taking first (27) with $AB = 11$ we see that $t_{01}{}^{11}{}_{11} = 0$. Now (27) with $AB = 01$ and (28) with $AB = 00$ show that $t_{01}{}^{01}{}_{11} = 0$ and $R_{000111} = 0$ if and only if $D^{EF}S_{00EF} - \frac{1}{6}D_{00}R = 0$. If $D^{EF}S_{01EF} - \frac{1}{6}D_{01}R = 0$, equation (28) with $AB = 01$ implies $R_{010111} = 0$, and then (27) with $AB = 00$ implies that $t_{01}{}^{00}{}_{11} = 0$. Finally, if $D^{EF}S_{11EF} - \frac{1}{6}D_{11}R = 0$ then (28) with $AB = 11$ implies $R_{110111} = 0$.

In the equations $R_{AB00EF} = 0$, which were used to calculate the unknowns as part of the ∂_u-equations, the quantities $S_{0000}, S_{0001}, S_{0011}, S_{0111}$ and R are equal to the corresponding components of the trace-free part of the Ricci spinor and the Ricci scalar. Then the equations $D^{EF}S_{00EF} - \frac{1}{6}D_{00}R = 0$ and $D^{EF}S_{01EF} - \frac{1}{6}D_{01}R = 0$ are automatically satisfied, as they only include $S_{0000}, S_{0001}, S_{0011}, S_{0111}$ and R, and if these quantities are replaced for their expression in terms of the field frame coefficients and the connection coefficients, then the two equations are the components of the contracted Bianchi identity. The equation that is not automatically satisfied is $D^{EF}S_{11EF} - \frac{1}{6}D_{11}R = 0$, as it includes S_{1111}, which has not been used as part of the procedure to calculate the unknowns. So we need to include

$$D^{EF}S_{11EF} - \tfrac{1}{6}D_{11}R = 0 \tag{29}$$

as requirement for the complete system of field equations to be satisfied. So we have that if and only if (29) is satisfied then

$$t_{AB}{}^{EF}{}_{CD} = 0, \qquad R_{ABCDEF} = 0.$$

From now on we assume (29) to be satisfied and analyse the implications in the following section.

From the definition of σ^{α}_{AB},

$$D_{AB}\sigma^{\alpha}_{CD} - D_{CD}\sigma^{\alpha}_{AB} = -t_{AB}{}^{EF}{}_{CD}D_{EF}\phi^{\alpha} + \epsilon_{BC}\Sigma^{\alpha}_{AD} + \epsilon_{AD}\Sigma^{\alpha}_{CB},$$

and then

$$\left(\partial_u + \frac{1}{u}\right)\sigma^{\alpha}_{01} = 2\hat{\Gamma}_{0100}\sigma^{\alpha}_{01},$$

which together with the initial condition for $u = 0$ gives $\sigma^{\alpha}_{01} = 0$, and together with the quantities that are already known to be zero from the ∂_u-equations gives

$$\sigma^{\alpha}_{AB} = 0.$$

The last equation implies that

$$\Sigma^{\alpha}_{AB} = -\Sigma^{\alpha}_{BA} + t_{AG}{}^{EFG}{}_{B}D_{EF}\phi^{\alpha},$$

using the ∂_u-equations and what has already been deduced,

$$\Sigma^{\alpha}_{AB} = 0.$$

So the full system of field equations is satisfied. $\qquad\qquad\square$

7. Conditions on f, f^{α}, F^{α}, $F^{\alpha\beta}$

We need to consider now the requirement (29). If we write it in full using the field equations, it takes the form

$$J^{\alpha AB}D_{11}D_{AB}\phi^{\alpha} + K^{\alpha}D_{11}\phi^{\alpha} = 0, \tag{30}$$

where $J^{\alpha AB}$ is the spinor version of

$$J^{\alpha a} = -\frac{1}{2}\frac{\partial f}{\partial(D_a\phi^{\alpha})} + \frac{1}{2}F^{\beta}\frac{\partial f^{\beta}}{\partial(D_a\phi^{\alpha})} + \left(\frac{\partial F^{\alpha}}{\partial\phi^{\beta}} + F^{\alpha}F^{\beta}\right)D^a\phi^{\beta} \tag{31}$$

and

$$K^\alpha = -\frac{1}{2}\frac{\partial f}{\partial \phi^\alpha} + fF^\alpha + \frac{1}{2}F^\beta\frac{\partial f^\beta}{\partial \phi^\alpha} - \frac{1}{2}f^\beta\frac{\partial F^\beta}{\partial \phi^\alpha} + f^\beta F^{\beta\alpha}$$
$$+ \left(\frac{\partial F^{\alpha\beta}}{\partial \phi^\gamma} - \frac{1}{2}\frac{\partial F^{\beta\gamma}}{\partial \phi^\alpha} + F^{\alpha\gamma}F^\beta\right)D_a\phi^\beta D^a\phi^\gamma. \tag{32}$$

As

$$D_{AB}D_{CD}\phi^\alpha = D_{(AB}D_{CD)}\phi^\alpha + \tfrac{1}{3}f^\alpha h_{ABCD},$$

we can write (30) in the following form:

$$J^\alpha_{00}D_{(11}D_{11)}\phi^\alpha - 2J^\alpha_{01}D_{(01}D_{11)}\phi^\alpha + J^\alpha_{11}\left(\tfrac{1}{3}f^\alpha + D_{(00}D_{11)}\phi^\alpha\right) + K^\alpha D_{11}\phi^\alpha = 0.$$

If we now evaluate this expression at the origin, all the quantities involved depend only on the free initial data. That is, J^α_{AB}, K^α and f^α are functions of $\phi^\alpha|_0$ and $D_{AB}\phi^\alpha|_0$. As $D_{(AB}D_{CD)}\phi^\alpha|_0$ are also part of the free initial data, in order for the equality to be satisfied for all initial data we need that

$$J^\alpha_{AB} = 0. \tag{33}$$

So from (30) we are left with

$$K^\alpha D_{11}\phi^\alpha = 0.$$

Again evaluating this expression at the origin, as K^α is a scalar function, and considering that the equality should be satisfied for all orthonormal frames at the origin, we also need that

$$K^\alpha = 0. \tag{34}$$

Conditions (33) and (34) imply that the contracted Bianchi identity needs to be satisfied by both sides of (2) from the beginning.

    Let us consider (33). F^α does not depend on $D_a\phi^\alpha$, and then the condition using the expression (31) can be written as

$$\frac{\partial}{\partial(D_a\phi^\alpha)}(f - F^\beta f^\beta) = 2\left(\frac{\partial F^\alpha}{\partial \phi^\beta} + F^\alpha F^\beta\right)D^a\phi^\beta.$$

This equation can only be integrated if

$$\frac{\partial F^\alpha}{\partial \phi^\beta} = \frac{\partial F^\beta}{\partial \phi^\alpha}, \tag{35}$$

and then

$$f - F^\alpha f^\alpha = \left(\frac{\partial F^\alpha}{\partial \phi^\beta} + F^\alpha F^\beta\right)D_a\phi^\alpha D^a\phi^\beta + G(\phi^\gamma),$$

where $G(\phi^\gamma)$ is a free function of ϕ^γ. We can consider this equation as an expression for f in terms of the other functions, having

$$f = F^\alpha f^\alpha + \left(\frac{\partial F^\alpha}{\partial \phi^\beta} + F^\alpha F^\beta\right)D_a\phi^\alpha D^a\phi^\beta + G(\phi^\gamma).$$

If we put this into (34) using expression (32) we get

$$\frac{1}{2}\frac{\partial G}{\partial \phi^\alpha} - GF^\alpha + f^\beta\left(\frac{\partial F^\beta}{\partial \phi^\alpha} - F^\alpha F^\beta - F^{\alpha\beta}\right) + \left(\frac{1}{2}\frac{\partial^2 F^\gamma}{\partial \phi^\alpha \partial \phi^\beta} - F^\alpha\frac{\partial F^\beta}{\partial \phi^\gamma}\right.$$
$$\left. + F^\beta\frac{\partial F^\gamma}{\partial \phi^\alpha} - \frac{\partial F^{\alpha\beta}}{\partial \phi^\gamma} + \frac{1}{2}\frac{\partial F^{\beta\gamma}}{\partial \phi^\alpha} - F^\alpha F^\beta F^\gamma - F^{\alpha\beta}F^\gamma\right)D^a\phi^\beta D_a\phi^\gamma = 0.$$

This can be written in the following form:

$$G^{\alpha\beta} f^\beta = G^\alpha + G^{\alpha\beta\gamma} D^a\phi^\beta D_a\phi^\gamma, \tag{36}$$

where

$$G^\alpha = -\frac{1}{2}\frac{\partial G}{\partial\phi^\alpha} + F^\alpha G,$$

$$G^{\alpha\beta} = \frac{\partial F^\beta}{\partial\phi^\alpha} - F^\alpha F^\beta - F^{\alpha\beta},$$

$$G^{\alpha\beta\gamma} = -\frac{1}{2}\frac{\partial^2 F^\gamma}{\partial\phi^\alpha\partial\phi^\beta} + F^\alpha\frac{\partial F^\beta}{\partial\phi^\gamma} - F^\beta\frac{\partial F^\gamma}{\partial\phi^\alpha} + \frac{\partial F^{\alpha\beta}}{\partial\phi^\gamma} - \frac{1}{2}\frac{\partial F^{\beta\gamma}}{\partial\phi^\alpha} + F^\alpha F^\beta F^\gamma + F^{\alpha\beta} F^\gamma.$$

If $\det(G^{\alpha\beta}) \neq 0$ then

$$f^\rho = (G^{\rho\alpha})^{-1}(G^\alpha + G^{\alpha\beta\gamma} D^a\phi^\beta D_a\phi^\gamma).$$

Equation (35) means that the form corresponding to F^α is closed, and therefore there exists a function $F(\phi^\gamma)$ such that $F^\alpha = \frac{\partial F}{\partial\phi^\alpha}$.

So we are left with the possibility of choosing $\frac{1}{2}(n^2 + n + 4)$ free functions of the fields ϕ^γ, namely G, F, $F^{\alpha\beta}$. Once we have these functions, we calculate $F^\alpha = \frac{\partial F}{\partial\phi^\alpha}$, and if $\det(G^{\alpha\beta}) \neq 0$ we calculate f^α. If $\det(G^{\alpha\beta}) = 0$ then some of the f^α's are free functions, and we have conditions on the rhs of (36).

8. Analyticity at q

The last needed step is to show that the holomorphic solution of lemma 5.7 can be extended as to cover a full neighbourhood of the point q. This is not obvious from our construction as our gauge is singular. The proof that we can indeed get a holomorphic solution in a whole neighbourhood of q is in all similar to the respective proofs in [1, 4]; therefore, here we only explain the steps of the proof.

The first step is to show that the obtained solution can be expressed in terms of the normal coordinates x^a and the frame field c_{AB} based on the frame c_{AB} at q. This can be done by showing that for small $|x^c|$ the coordinate transformation $x^a \to z^a(x^c)$, where defined, is nondegenerate. This means that all the tensor fields entering the field equations can be expressed in terms of the normal coordinates x^a and the normal frame field c_{AB}. The coordinates x^a which cover a domain U in $\mathbb{C}^3$ on which the frame vector fields $c_{AB} = c^a{}_{AB}\partial_{x^a}$ exist, are linearly independent and holomorphic. Also in U the other tensor fields expressed in terms of the x^a and c_{AB} are holomorphic. However, by the singularity of our gauge, U does not contain the hypersurface $x^1 + ix^2 = 0$ but the boundary of U becomes tangent to this hypersurface at $x^a = 0$.

The second step is to note that the construction of the submanifold $\hat{N}$ was done based on the frame c_{AB} at q. Starting with a different frame $\tilde{c}_{AB}$ at q all the previous constructions and derivations can be repeated if the estimates for the null data in the c_{AB}-gauge can be translated into the same type of estimates for the null data in the $\tilde{c}_{AB}$-gauge. Indeed, the estimates do translate into each other, and hence all the statements made about the solution in the c_{AB}-gauge apply to the solution in the $\tilde{c}_{AB}$-gauge, in particular statements about domains of convergence.

Now, the solution in the $\tilde{c}_{AB}$-gauge can be expressed in terms of normal coordinates $\tilde{x}^a$ based on the frame $\tilde{c}_{AB}$ at q, which cover a domain $\tilde{U}$ in $\mathbb{C}^3$. Again $\tilde{U}$ does not contain the hypersurface $\tilde{x}^1 + i\tilde{x}^2 = 0$ but the boundary of $\tilde{U}$ becomes tangent to this hypersurface at $\tilde{x}^a = 0$.

By the uniqueness statements made so far, the solution in the $\tilde{c}_{AB}$-gauge and in normal coordinates $\tilde{x}^a$ is related to the solution in the c_{AB}-gauge and in normal coordinates x^a, on the domain $U \cap \tilde{U}$, by the rotation that takes $\tilde{c}_{AB}$ to c_{AB}, corresponding to the rotation of normal coordinates. We can extend this as a coordinate and frame transformation to the solution obtained in the $\tilde{c}_{AB}$-gauge to express all fields in terms of x^a and c_{AB}. Then the solution obtained in the c_{AB}-gauge and the solution in the $\tilde{c}_{AB}$-gauge are genuine holomorphic extensions of each other, as one covers the singular generator of the other one away from the origin in a regular way.

Therefore, the set U can be extended in such a way as to contain a punctured neighbourhood of the origin in which the solution is holomorphic in the normal coordinates x^a and the normal frame c_{AB}. Then the solution is in fact holomorphic on a full neighbourhood of the origin $x^a = 0$, which represents the point q, as holomorphic functions in more than one dimension cannot have isolated singularities.

By lemma 2.1 we have from null data satisfying the reality conditions a formal expansion of the solution with expansion coefficients satisfying the reality conditions. By the various uniqueness statements obtained in the lemmas, this expansion must coincide with the expansion in normal coordinates of the solution obtained above. This implies the existence of a three-dimensional real slice on which the tensor fields satisfy the reality conditions. It is obtained by requiring the coordinates x^a to assume values in $\mathbb{R}^3$. This completes the proof of theorem 1.1.

9. Conclusions

We have seen how to determine a formal expansion of the solution to certain types of elliptic systems of equations at a given point using a minimal set of freely specifiable data, the null data. We have also obtained necessary and sufficient conditions on the null data for the formal expansion to be absolutely convergent, thus showing that the null data characterize all possible solutions in a neighbourhood of the given point.

The system of equations is general enough as to include as particular cases stationary Einstein–Maxwell fields, static Einstein–Maxwell-dilaton fields and harmonic maps coupled to gravity whose base space is three dimensional or where the spacetime is stationary.

One interesting outcome of the analysis is the conditions that the functions of the fields that enter the field equations need to satisfy for the existence of solutions. These conditions can be read as that the contracted Bianchi identity needs to be fulfilled by the system of equations. This seems to indicate that only geometrically well-behaved systems of equations allow for general data and solutions. It could be interesting to analyse if this forces the system of equations to be the Euler–Lagrange equations for some Lagrangian. Along the same line, it would also be interesting to analyse whether the considered system of equations can generally arise from a dimensional reduction. This would be the case for example if one starts with a four-dimensional stationary spacetime or in Kaluza–Klein-type theories.

Acknowledgments

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