TWO-POINT PROBLEM FOR SYSTEMS SATISFYING THE CONTROLLABILITY CONDITION WITH LIE BRACKETS OF THE SECOND ORDER

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We study a two-point control problem for systems linear in control. The class of problems under consideration satisfies a controllability condition with Lie brackets up to the second order, inclusively. To solve the problem, we use trigonometric polynomials whose coefficients are computed by expanding the solutions into the Volterra series. The proposed method allows one to reduce the two-point control problem to a system of algebraic equations. It is shown that this algebraic system has (locally) at least one real solution. The proposed method for the construction of control functions is illustrated by several examples.

1. Introduction

The problem of planning of motions of nonholonomic mechanical systems occupies an important place in the contemporary theory of control in connection with the nontrivial geometric properties of the trajectories and applications to robotics. Despite a large number of works on the control of motion of nonholonomic systems, the problem of constructive synthesis of control functions under sufficiently general assumptions about the vector fields in the system remains open. In the present, work, we consider the most significant, from our point of view, results obtained in this field.

Brockett solved the problem of optimal control with quadratic quality functional for a system in the canonical form satisfying the rank condition with Lie brackets of the first order [2]. Murray and Sastry [14] extended this result and represented a family of software controls in the form of combinations of sine curves for the solution of two-point problems with one and several chains of integrators. A more general method was proposed by Sussmann and Liu for a class of systems linear in control. For these systems, the control functions with large amplitudes can be used for the solution of the problem of approximate tracking of the trajectories [16]. The sinunosoids with large amplitudes were also used in the work [9] to find the time-periodic solutions of the problem of planning of motion with bypassing the obstacles. A method aimed at the solution of the two-point control problem with piecewise constant controls was proposed in [12]. Functions of this kind are used both in nilpotent systems and for the approximate control over the general classes of systems. In [6], the two-point control problem was solved for several examples of systems linear in control with piecewise constant inputs. The sinusoidal and polynomial control functions for systems with two chains of integrators and three inputs were constructed in [3]. The globally convergent algorithms of motion planning were described in [5, 10]. Another method based on the use of the Lie algebra and the generalized Campbell–Baker–Hausdorff–Dynkin formula was proposed in [7].

In the present work, we consider the systems linear in control whose vector fields together with their Lie brackets of the first and second orders satisfy the rank condition. To solve the two-point control problem, we use the expansion of solutions of the system with trigonometric time-dependent controls in the Volterra series. This representation enables us to compute the coefficients of control functions via the solution of the system of algebraic equations. Note that a similar approach was considered in [8] for systems satisfying the condition of controllability with Lie brackets of the first order.

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We now briefly describe the structure of the present work. In Section 2, we give the statement of the problem and present some facts related to the representation of solutions in the form of Volterra series. The main result concerning the construction of controls is formulated in Subsection 3.1 and proved in Subsection 3.2. In Section 4, we consider some examples illustrating the obtained results. Some technical details can be found in the Appendix.

2. Auxiliary Constructions

2.1. Statement of the Problem. Consider a class of systems linear in control of the form

$$\dot{x} = \sum_{i=1}^{m} u_i f_i(x), \tag{1}$$

where $x = (x_1, \ldots, x_n)^T \in \mathbb{R}^n$ is a state, $u = (u_1, \ldots, u_m)^T \in \mathbb{R}^m$ is a control, and $f_i(x)$ are smooth mappings from $\mathbb{R}^n$ into $\mathbb{R}^n$, $m < n$.

We pose the two-point control problem for system (1) in the following way:

Given initial $x^0 \in \mathbb{R}^n$ and final $x^1 \in \mathbb{R}^n$ states, it is necessary to find an admissible control $u(t) \in \mathbb{R}^m$, $t \in [0, \varepsilon]$ that transfers system (1) from x^0 to x^1 for time ε.

Assume that system (1) satisfies the condition of controllability with Lie brackets up to the second order, inclusively, at a point x^0, i.e.,

$$\operatorname{span}\left\{f_i(x^0), [f_{j_1}, f_{j_2}](x^0), \left[[f_{l_1}, f_{l_2}], f_{l_3}\right](x^0)\right\} = \mathbb{R}^n, \tag{2}$$

where

$$i \in \{1, \ldots, m\} = S_1,\ (j_1, j_2) \in S_2 \subseteq \{1, \ldots, m\}^2,\ (l_1, l_2, l_3) \in S_3 \subseteq \{1, \ldots, m\}^3,\ |S_2| + |S_3| = n - m,$$

and

$$[f_i, f_j] = \frac{\partial f_j}{\partial x} f_i - \frac{\partial f_i}{\partial x} f_j$$

are the Lie brackets of the vector fields f_i and f_j. Here and in what follows, $\dfrac{\partial f_i(x)}{\partial x}$ stands for the Jacobi matrix. Elements of the sets S_2 and S_3 are ordered so that $j_1 < j_2$ for all $(j_1, j_2) \in S_2$ and $l_1 < l_2$ for $(l_1, l_2, l_3) \in S_3$.

2.2. Representation of Solutions in the Form of Volterra Series. We now expand the solution $x(t)$ of system (1) with initial condition $x(0) = x^0 \in \mathbb{R}^n$ in a Volterra series by assuming that the function $u(t)$ is continuous for $t \in [0, \varepsilon]$ (see [15]):

$$x(t) = x^0 + \sum_{i=1}^{m} f_i(x^0) \int_0^t u_i(s)ds + \sum_{i,j=1}^{m} \frac{\partial f_i(x)}{\partial x} f_j(x)\bigg|_{x=x^0} \int_0^t \int_0^\tau u_i(\tau)u_j(p)dpd\tau$$

$$+ \sum_{i,j,l=1}^{m} \frac{\partial}{\partial x}\left(\frac{\partial f_i(x)}{\partial x} f_j(x)\right) f_l(x)\bigg|_{x=x^0} \int_0^t \int_0^\tau \int_0^s u_i(\tau)u_j(s)u_l(p)dpdsd\tau + R(t),\ t \in [0, \varepsilon]. \tag{3}$$

In representation (3), the function $R(t)$ denotes the residual term, which is equal to zero for a class of nilpotent systems. For systems of the general form, we get the following assertion on the estimation of the residual term:

Lemma 1. *Let $D \subset \mathbb{R}^n$ be a convex region and let the function $x(t) \in D$, $0 \le t \le \varepsilon$, be a solution of system (1) with the initial condition $x(0)=x^0 \in D$ and control $u \in C[0,\varepsilon]$. If the vector fields $f_1(x), \ldots, f_m(x)$ satisfy the conditions*

$$\left\| \frac{\partial f_i}{\partial x}(x) \right\| \le M_1, \quad \left\| \frac{\partial^2 f_{ij}}{\partial x^2}(x) \right\| \le M_2,$$

$$\left(\sum_{k,p,z=1}^{n} \left(\frac{\partial^3 f_{ij}(x)}{\partial x_k \partial x_p \partial x_z} \right)^2 \right)^{1/2} \le M_3, \quad x \in D, \quad i,j \in S_1,$$

with some positive constants M_1, M_2, and M_3, then the residual term in expansion (3) satisfies the following inequality:

$$\|R(t)\| \le \frac{m^4 U^4 M_0}{24} \left(M_1^3 + M_0(M_0 M_3 + 4M_1 M_2)\sqrt{n} \right) t^4 + O(t^5), \tag{4}$$

where

$$M_0 = \max_{1 \le i \le m} \|f_i(x^0)\| \quad \text{and} \quad U = \max_{1 \le i \le m} \|u_i\|_{L^\infty(0,\varepsilon)}.$$

This result is proved by analogy with Lemma 2.2 in [8].

For the subsequent investigations, we represent relation (3) as follows:

$$x(t) = x^0 + \sum_{i=1}^{m} f_i\left(x^0\right) \int_0^t u_i(s)ds + \frac{1}{2} \sum_{i<j} [f_i, f_j]\left(x^0\right) \int_0^t \int_0^\tau (u_j(\tau)u_i(s) - u_i(\tau)u_j(s))ds\,d\tau$$

$$+ \frac{1}{3} \sum_{i \le j} \sum_{l=1}^{m} [[f_i, f_j], f_l]\left(x^0\right) \int_0^t \int_0^\tau \int_0^s \left(u_l(\tau)\left(u_j(s)u_i(p) - u_i(s)u_j(p)\right)\right) dp\,ds\,d\tau$$

$$+ G(t) + R(t). \tag{5}$$

The formula for $G(t)$ is presented in the Appendix.

3. Solution of the Two-Point Control Problem

3.1. Construction of Control Functions. In order to solve the two-point control problem, we use the following family of trigonometric polynomials:

$$u_i(t) = a_i + \sum_{(q,r) \in S_2} a_{qr} \left(\delta_{iq} \cos \frac{2\pi K_{qr}}{\varepsilon} t + \delta_{ir} \sin \frac{2\pi K_{qr}}{\varepsilon} t \right)$$

$$+ \sum_{(q,r,s)\in S_3} a_{qrs} \left(\delta_{iq} \cos \frac{2\pi K_{1qrs}}{\varepsilon} t + \delta_{ir} \sin \frac{2\pi K_{2qrs}}{\varepsilon} t + \delta_{is} \sin \frac{2\pi K_{3qrs}}{\varepsilon} t \right), \tag{6}$$

$$i \in S_1, \quad t \in [0, \varepsilon],$$

where a_i, a_{qr}, and a_{qrs} are real parameters, K_{qr}, K_{1qrs}, K_{2qrs}, and K_{3qrs} are integers, $\varepsilon > 0$, and δ_{ij} stands for the Kronecker delta.

Assumption 1. *For any* $(q, r) \in S_2$, $(q_k, r_k, s_k) \in S_3$, $k = 1, 2, 3$, *the conditions*

$$c_1 K_{qr} + \sum_{k=1}^{3} c_{k+1} K_{q_k r_k s_k} = 0, \quad c_i \in \mathbb{Z}, \quad 0 < \sum_{k=1}^{4} |c_i| \le 3,$$

imply that $(q_1, r_1, s_2) = (q_2, r_2, s_3) = (q_3, r_3, s_3)$ *and* $c_1 = 0$, $c_2 = c_3 = -c_4 = 1$.

By using formula (5) with controls (6), we obtain

$$x(\varepsilon) = x^0 + \varepsilon \sum_{i=1}^{m} f_i\left(x^0\right) a_i + \frac{\varepsilon^2}{4\pi} \sum_{(i,j)\in S_2} [f_i, f_j]\left(x^0\right) \frac{a_{ij}^2}{K_{ij}} + \frac{\varepsilon^2}{2} \Omega_1(a, x^0)$$

$$+ \frac{\varepsilon^3}{16\pi^2} \sum_{(i,j,l)\in S_3} \frac{a_{ijl}^3}{K_{2ijl}} \left(\frac{\left[[f_i, f_j], f_l\right](x^0)}{K_{3ijl}} - \frac{\left[f_i, [f_j, f_l]\right](x^0)}{K_{1ijl}} \right) + \frac{\varepsilon^3}{6} \Omega_2(a, x^0) + R(\varepsilon), \tag{7}$$

where Ω_1 and Ω_2 depend on the coefficients of functions (6) and the initial value x^0 (the explicit formulas are presented in the Appendix).

Representation (7) and Lemma 1 now yield the main result of the present work:

Theorem 1. *For given states* $x^0 \in \mathbb{R}^n$ *and* $x^1 \in \mathbb{R}^n$ *and time* $\varepsilon > 0$, *a control*

$$u(t) = (u_1(t), \ldots, u_m(t))^T$$

of the form (6) with coefficients satisfying the system of equations

$$\varepsilon \sum_{i=1}^{m} f_i(x^0) a_i + \frac{\varepsilon^2}{4\pi} \sum_{(i,j)\in S_2} [f_i, f_j](x^0) \frac{a_{ij}^2}{K_{ij}} + \frac{\varepsilon^2}{2} \Omega_1(a, x^0)$$

$$+ \frac{\varepsilon^3}{16\pi^2} \sum_{(i,j,l)\in S_3} \frac{a_{ijl}^3}{K_{2ijl}} \left(\frac{\left[[f_i, f_j], f_l\right](x^0)}{K_{3ijl}} - \frac{\left[f_i, [f_j, f_l]\right](x^0)}{K_{1ijl}} \right)$$

$$+ \frac{\varepsilon^3}{6} \Omega_2(a, x^0) = x^1 - x^0, \tag{8}$$

transfers system (1) from the initial position $x(0) = x^0$ *into an* $R(\varepsilon)$*-neighborhood of the point* x^1, *where* $R(\varepsilon)$ *is estimated by formula (4).*

3.2. Solvability of the Algebraic System. Although system (8) is quite cumbersome, it can be represented in a more compact form by using the properties of the Kronecker delta in the formulas for Ω_1 and Ω_2. In this section, we prove that there exists at least one real solution of system (8):

$$a = (a_1, \dots, a_m, a_{j_1 l_1}, \dots, a_{j_\alpha l_\alpha}, a_{q_1 r_1 S_2}, \dots, a_{q_\beta r_\beta s_\beta})^T \in \mathbb{R}^n,$$

where $\alpha = |S_2|$ and $\beta = |S_3|$. Following the approach proposed in [8], we define new variables by the formulas

$$\tilde{a} = (\tilde{a}_1, \dots, \tilde{a}_m, \tilde{a}_{j_1 l_1}, \dots, \tilde{a}_{j_\alpha l_\alpha}, \tilde{a}_{q_1 r_1 s_2}, \dots, \tilde{a}_{q_\beta r_\beta s_\beta})^T \in \mathbb{R}^n,$$

where $\tilde{a}_i = \varepsilon a_i$ for $i \in S_1$,

$$\tilde{a}_{jl} = \frac{\varepsilon^2 a_{jl}^2}{4\pi K_{jl}}$$

for $(j, l) \in S_2$, and

$$\tilde{a}_{qrs} = \frac{\varepsilon^3 a_{qrs}^3}{16\pi^2 K_{2qrs}}$$

for $(q, r, s) \in S_3$.

We now assume that

$$\operatorname{sign}(K_{jl}) = \operatorname{sign}(\tilde{a}_{jl})$$

for nonzero $\tilde{a}_{jl}$ and $\operatorname{sign}(K_{jl}) = 1$ for $\tilde{a}_{jl} = 0$, which guarantees that the definition of $\tilde{a}_{jl}$ is correct.

We now rewrite system (8) in the form

$$\sum_{i=1}^{m} \tilde{a}_i f_i(x^0) + \sum_{(i,j)\in S_2} \tilde{a}_{ij}[f_i, f_j](x^0) + \sum_{(i,j,l)\in S_3} \tilde{a}_{ijl} \left(\frac{[[f_i, f_j], f_l](x^0)}{K_{3ijl}} - \frac{[f_i, [f_j, f_l](x^0)]}{K_{1ijl}} \right)$$
$$+ \widetilde{\Omega}(\tilde{a}, x^0) = x^1 - x^0, \tag{9}$$

where

$$\widetilde{\Omega}(\tilde{a}, x^0) = \Omega_1(\tilde{a}, x^0) + \frac{1}{3} \Omega_2(\tilde{a}, x^0).$$

We can show that the quantity $\widetilde{\Omega}(\tilde{a}, x^0)$ is independent of the signs of K_{qr} and contains no terms of the order of smallness lower than $4/3$ with respect to $\tilde{a}$ as $\tilde{a} \to 0$. For each multiindex $(q, r, s) \in S_3$, we select an integer K_{1qrs} such that the following matrix is nonsingular:

$$\mathcal{F}(x^0) = \Big(f_1(x), \dots, f_m(x^0), [f_{j_1}, f_{l_1}](x^0), \dots, [f_{j_\alpha}, f_{l_\alpha}](x^0),$$

$$\frac{[[f_{q_1}, f_{r_1}], f_{S_2}](x^0)}{K_{3q_1r_1S_2}} - \frac{[f_{q_1}, [f_{r_1}, f_{S_2}]](x^0)}{K_{1q_1r_1S_2}}$$

$$\ldots, \frac{[[f_{q_\beta}, f_{r_\beta}], f_{s_\beta}](x^0)}{K_{3q_\beta r_\beta s_\beta}} - \frac{[f_{q_\beta}, [f_{r_\beta}, f_{s_\beta}]](x^0)}{K_{1q_\beta r_\beta s_\beta}}\Bigg). \tag{10}$$

where $\alpha = |S_2|$ and $\beta = |S_3|$. Note that this choice of the parameters K_{1qrs} is always possible due to the rank condition (2). Thus, we can choose the integer values of $|K_{qr}|$, K_{1qrs}, and K_{3qrs} in agreement with Assumption 1. Hence, for the solution $\tilde{a}$ of system (9) with given $x^0 \in \mathbb{R}^n$, the components of the solution of system (8) are determined as follows:

$$a_i = \varepsilon^{-1}\tilde{a}_i \quad \text{for} \quad i \in S_1, \quad a_{jl} = 2\varepsilon^{-1}\operatorname{sign}(\tilde{a}_{jl})\sqrt{\pi|K_{jl}||\tilde{a}_{jl}|} \quad \text{for} \quad (j,l) \in S_2,$$

$$a_{qrs} = 2\sqrt[3]{2}\varepsilon^{-1}\sqrt[3]{\pi^2 K_{2qrs}\tilde{a}_{qrs}} \quad \text{for} \quad (q,r,s) \in S_3,$$

where K_{jl} are positive for $a_{jl} \geq 0$ and negative otherwise.

With the help of these transformations, the problem of solvability of system (8) is reduced to the investigation of system (9).

Theorem 2. *There exists $r > 0$ such that system (9) is solvable with respect to $\tilde{a} \in \mathbb{R}^n$ for all $x^1 \in \mathbb{R}^n$ satisfying the inequality $\|x^1 - x^0\| \leq r$.*

Proof. Let K_{1qrs} and $(q,r,s) \in S_3$ be chosen to guarantee that matrix (10) is nonsingular for all $x \in \mathbb{R}^n$. Multiplying both sides of system (9) by the matrix $\mathcal{F}^{-1}(x^0)$, we get $\Phi(\tilde{a}) = 0$, where

$$\Phi(\tilde{a}) = \tilde{a} + \mathcal{F}^{-1}(x^0)\left(\widetilde{\Omega}(\tilde{a}, x^0) + x^0 - x^1\right).$$

We now estimate the norm of the function $\Phi(\tilde{a}) - \tilde{a}$:

$$\|\Phi(\tilde{a}) - \tilde{a}\| = \left\|\mathcal{F}^{-1}(x^0)\left(\widetilde{\Omega}(\tilde{a}, x^0) + x^0 - x^1\right)\right\| \leq \left\|\mathcal{F}^{-1}(x^0)\right\|\left(\left\|\widetilde{\Omega}(\tilde{a}, x^0)\right\| + \left\|x^0 - x^1\right\|\right).$$

By virtue of the properties of $\widetilde{\Omega}(\tilde{a}, x^0)$, there exists $C(x^0) > 0$ such that

$$\|\widetilde{\Omega}(\tilde{a}, x^0)\| \leq C(x^0)\|\tilde{a}\|^{4/3}$$

for all $\tilde{a} \in \mathbb{R}^n : \|\tilde{a}\| \leq 1$. Hence,

$$\|\Phi(\tilde{a}) - \tilde{a}\| \leq \|\mathcal{F}^{-1}(x^0)\|\left(C(x^0)\|\tilde{a}\|^{4/3} + \|x^0 - x^1\|\right).$$

Assume that a number $\gamma \in (0, 1)$ satisfies the inequality

$$\gamma^{1/3} < \frac{1}{C(x^0)\|\mathcal{F}^{-1}(x^0)\|}.$$

Then there exists a number $r > 0$ such that

$$\|\mathcal{F}^{-1}(x^0)\| \left(C(x^0)\gamma^{4/3} + \|x^0 - x^1\|\right) < \gamma.$$

This yields the estimate

$$\|\Phi(\tilde{a}) - \tilde{a}\| \le \|\tilde{a}\| \quad \text{for} \quad \tilde{a} \in S_\gamma = \{\tilde{a} \in \mathbb{R}^n : \|\tilde{a}\| = \gamma\}.$$

In other words, the mappings $\Phi(\tilde{a})$ and $\Psi(\tilde{a}) = \tilde{a}$ are homotopic on the sphere S_γ, and the rotation of the vector field $\Phi(\tilde{a})$ on S_γ is equal to 1. The principle of nonzero rotation [11] yields the existence of a vector $\tilde{a} \in B_\gamma(0)$ such that $\Phi(\tilde{a}) = 0$.

Theorem 2 implies the existence of a real solution $a \in \mathbb{R}^n$ of system (8) for sufficiently close points x^0 and x^1. In what follows, we demonstrate the applicability of Theorem 1 on several examples for which the system of algebraic equations (8) can be numerically solved.

4. Examples

4.1. Controlled System with Quadratic Vector Fields. Consider the two-point control problem for the following system:

$$\begin{aligned} \dot{x}_1 &= u_1, \\ \dot{x}_2 &= u_2, \\ \dot{x}_3 &= x_2^2 u_1 - x_1^2 u_2, \end{aligned} \tag{11}$$

where $x \in \mathbb{R}^3$ and $u \in \mathbb{R}^2$. This system was considered in works [4, 17] as a model of motion of a plane solid body with two oscillators. Here, we explicitly construct the controls $u_1(t)$ and $u_2(t)$ transferring system (11) from zero to the point $x^1 \in \mathbb{R}^3$. The vector fields for system (11) are as follows:

$$f_1(x) = (1, 0, x_2^2)^T, \quad f_2(x) = (0, 1, -x_1^2)^T, \quad [f_1, f_2](x) = (0, 0, -2(x_1 + x_2))^T.$$

Let $x^0 = 0 \in \mathbb{R}^3$. Then $f_1(x^0) = f_2(x^0) = [f_1, f_2](x^0) = 0$. In other words, the Lie brackets of the first order are insufficient in order that the rank condition of controllability be satisfied. Therefore, we also find the Lie bracket of the second order:

$$[[f_1, f_2], f_1](x) = (0, 0, 2)^T.$$

Then the rank condition (2) holds with $S_2 = \varnothing$ and $S_3 = \{(1, 2, 1)\}$.

In this case, controls (6) take the form

$$u_1(t) = a_1 + a_{121}\left(\cos\frac{2\pi K_1}{\varepsilon}t + \sin\frac{2\pi K_3}{\varepsilon}t\right) \quad \text{and} \quad u_2(t) = a_2 + a_{121}\sin\frac{2\pi K_2}{\varepsilon}t, \tag{12}$$

where the parameters $K_1 = K_{1121}$, $K_2 = K_{2121}$, and $K_3 = K_{3121}$ satisfy Assumption 1.

Substituting (12) in relation (7), we get

$$x_1(\varepsilon) = \varepsilon a_1, \quad x_2(\varepsilon) = \varepsilon a_2,$$

$$x_3(\varepsilon) = \frac{\varepsilon^3}{2\pi^2 K_1}\left(-\frac{a_{121}^3}{4K_3} + \frac{a_{121}^2}{4(K_2+K_3)}\left(\frac{a_1}{K_2^2}(4K_2K_3 + 3K_1(K_2+K_3)) - \frac{a_2}{K_1K_3^2}(2K_2^2K_3 + K_1(K_1+K_3)^2)\right) + \frac{a_{121}(a_1+a_2)}{K_1K_2K_3}\left(\pi K_1^2(a_1K_3 - a_2K_2) + a_2K_2K_3\right) + \frac{2\pi^2K_1}{3}a_1a_2(a_2-a_1)\right). \tag{13}$$

Note that all Lie brackets whose orders are higher than two are equal to zero, and the proposed representation of solutions with the help of the Volterra series is exact. It is easy to see that, for given $\varepsilon > 0$ and $x(\varepsilon) \in \mathbb{R}^3$ and nonzero integers K_1, K_2, and K_3, the system of algebraic equations (13) has real solutions a_1, a_2, and a_{121}.

Let $K_1 = 2$, $K_2 = 3$, and $K_3 = 5$. Then the following functions solve the two-point control problem for $x^0 = (0, 0, 0)^T$ and $x^1 = x(\varepsilon)$ for any $\varepsilon > 0$:

$$u_1(t) = \frac{x_1^1}{\varepsilon} + \frac{\hat{a}_{121}}{\varepsilon}\left(\cos\frac{4\pi}{\varepsilon}t + \sin\frac{10\pi}{\varepsilon}t\right), \quad u_2(t) = \frac{x_2^1}{\varepsilon} + \frac{\hat{a}_{121}}{\varepsilon}\sin\frac{6\pi}{\varepsilon}t,$$

where $\hat{a}_{121}$ is a real root of the cubic equation

$$a_{121}^3 - \frac{75x_1^1 - 22x_2^1}{10}a_{121}^2 - \frac{2(x_1^2 + x_2^1)}{3}\left(20\pi x_1^1 + (15 - 12\pi)x_2^1\right)a_{121} - \frac{80\pi^2}{3}\left(x_1^1x_2^1\left(x_2^1 - x_1^1\right) - 3x_3^1\right) = 0.$$

In Fig. 1, we present the plot of a solution $x(t)$ of system (11) with $x^1 = (1, 1, 1)^T$ for $\varepsilon = 1$.

4.2. Rolling Disk. Consider the kinematic equations of a disk rolling along a plane without sliding (see, e.g., [13]):

$$\dot{x}_1 = u_1\cos x_3, \quad \dot{x}_2 = u_1\sin x_3, \quad \dot{x}_3 = u_2, \quad \dot{x}_4 = u_1, \tag{14}$$

where $x = (x_1, x_2, x_3, x_4)^T \in \mathbb{R}^4$, $u = (u_1, u_2)^T \in \mathbb{R}^2$.

For system (14), the following rank condition is satisfied:

$$\operatorname{span}\{f_1(x), f_2(x), [f_1, f_2](x), [[f_1, f_2], f_2](x)\} = \mathbb{R}^4 \quad \forall x \in \mathbb{R}^4.$$

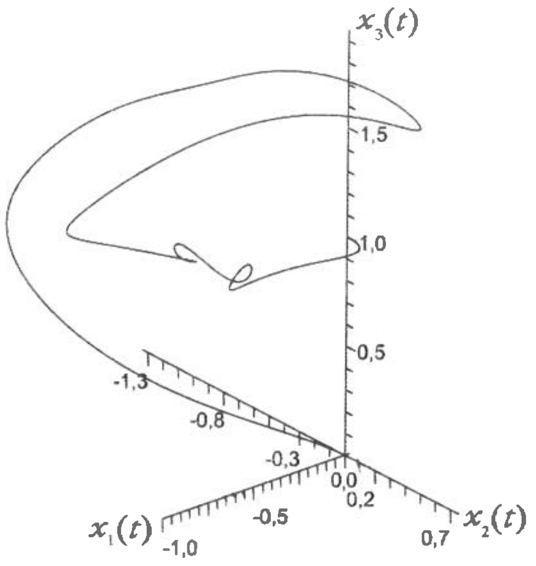

Fig. 1. Trajectory of system (11), (12) with the boundary conditions $x(0) = (0, 0, 0)^T$ and $x(1) = (0, 0, 1)^T$.

Here,

$$f_1(x) = (\cos x_3, \sin x_3, 0, 1)^T \quad \text{and} \quad f_2(x) = (0, 0, 1, 0)^T.$$

Hence, $S_1 = \{1, 2\}$, $S_2 = \{(1, 2)\}$, and $S_3 = \{(1, 2, 2)\}$, and the control functions (6) take the form

$$u_1(t) = a_1 + a_{12} \cos \frac{2\pi K_{12}}{\varepsilon} t + a_{122} \cos \frac{2\pi K_1}{\varepsilon} t,$$
$$u_2(t) = a_2 + a_{12} \sin \frac{2\pi K_{12}}{\varepsilon} t + a_{122} \left(\sin \frac{2\pi K_2}{\varepsilon} t + \sin \frac{2\pi K_3}{\varepsilon} t \right). \qquad (15)$$

Now let $x^0 \in \mathbb{R}^4$, $x^1 \in \mathbb{R}^4$, $\varepsilon > 0$, $K_1 = 2$, $K_2 = 3$, $K_3 = 5$, and $|K_{12}| = 9$, and let a_1, a_2, a_{12}, and a_{122} be real solutions of the equations

$$x_1^1 - x_1^0 = \varepsilon a_1 \cos x_3^0 - \varepsilon^2 I_1 \sin x_3^0 - \varepsilon^3 I_2 \cos x_3^0,$$
$$x_2^1 - x_2^0 = \varepsilon a_1 \sin x_3^0 + \varepsilon^2 I_1 \cos x_3^0 - \varepsilon^3 I_2 \sin x_3^0,$$
$$x_3^1 - x_3^0 = \varepsilon a_2, \quad x_4^1 - x_4^0 = \varepsilon a_1,$$

where

$$I_1 = -\frac{a_{12}^2}{\pi K_{12}} + \frac{a_1 a_{12}}{2\pi K_{12}} + \frac{4 a_1 a_{122}}{15\pi} + \frac{a_1 a_2}{2},$$

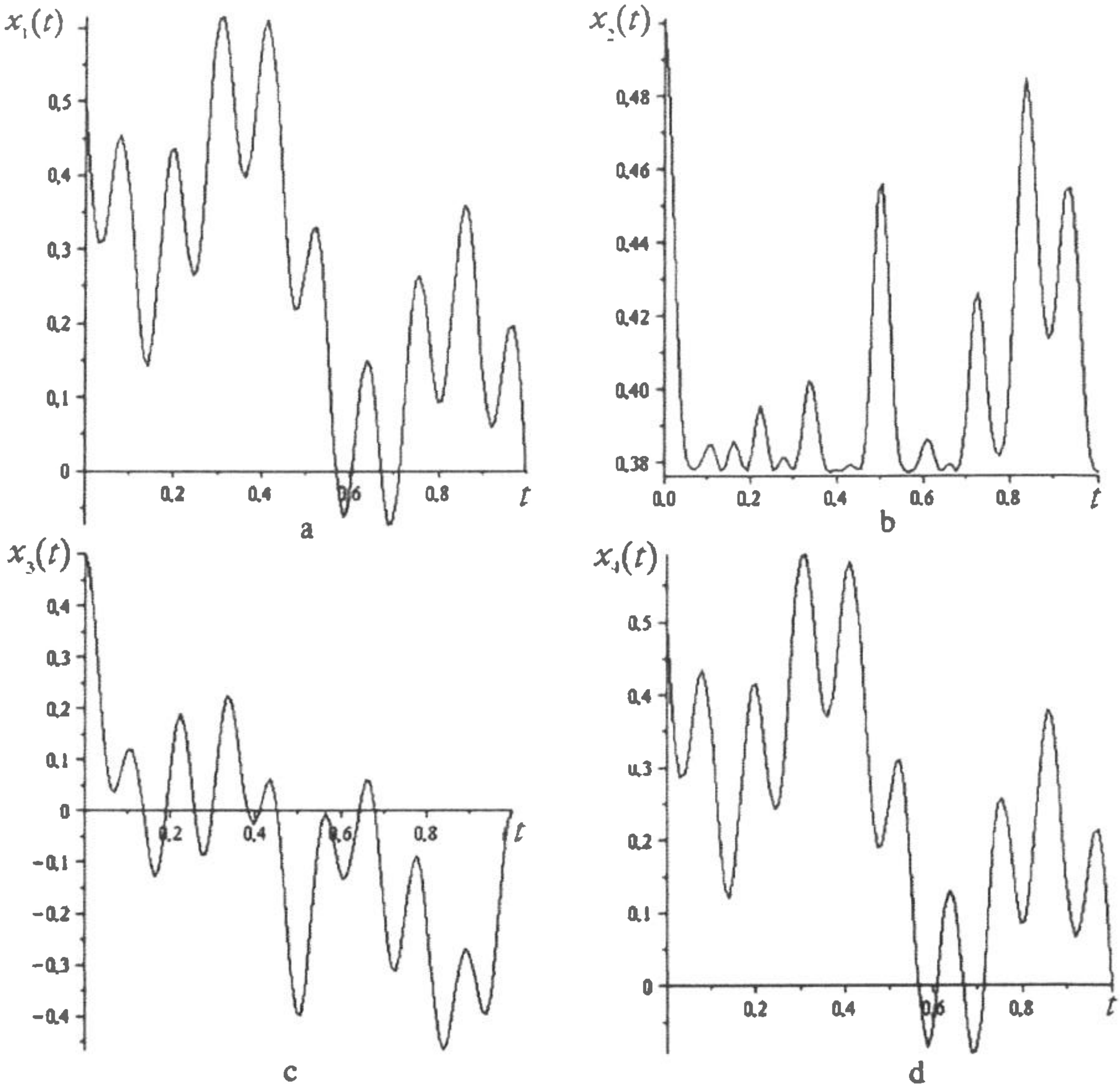

Fig. 2. Components of the solution of system (14): (a) $(t, x_1(t))$; (b) $(t, x_2(t))$; (c) $(t, x_3(t))$; (d) $(t, x_4(t))$.

$$I_2 = \frac{a_{122}^3}{240\pi^2} + \frac{9a_{122}^2 a_1}{200\pi^2} - \frac{a_{122}a_{12}^2}{15\pi^2 K_{12}} + \frac{2a_{122}a_{12}a_1}{15K_{12}\pi^2} + \frac{a_{122}a_2^2}{16\pi^2} + \frac{2a_{122}a_1 a_2}{15\pi}$$

$$- \frac{a_{12}^3}{8\pi^2 K_{12}^2} + \frac{3a_{12}^2 a_1}{16\pi^2 K_{12}^2} - \frac{a_{12}^2 a_2}{8\pi K_{12}} + \frac{a_{12}a_1 a_2}{4\pi K_{12}} + \frac{a_{12}a_2^2}{4\pi^2 K_{12}^2} + \frac{a_1 a_2^2}{6}.$$

Then controls (15) transfer system (14) from x^0 to x^1 for time ε. In particular, for $x^0 = (0.5, 0.5, 0.5, 0.5)^T$, $x^1 = (0, 0, 0, 0)$, and $\varepsilon = 1$, the behavior of the solution of system (14) with

$$u_1(t) \approx -0.5 - 7.42\cos 18\pi t - 2.3\sin 10\pi t$$

and

$$u_2(t) = -0.5 - 7.42 \sin 18\pi t - 2.3(\sin 6\pi t + \sin 10\pi t)$$

is shown in Fig. 2.

4.3. Rotational Motion of a Solid Body. Theorem 1 and Lemma 1 substantiate the possibility of application of controls of the form (6) to the systems linear in control. We now show that the proposed approach can also be extended to some systems affine in control. To illustrate the efficiency of application of trigonometric controls to these systems, we consider the following well-known example of rotational motion of a solid body under the action of two independent controlling moments (see, e.g., [1, 18]):

$$\begin{aligned} \dot{x}_1 &= A_1 x_2 x_3 + b_1 u_1 + c_1 u_2, \\ \dot{x}_2 &= A_2 x_1 x_3 + b_2 u_1 + c_2 u_2, \\ \dot{x}_3 &= A_3 x_1 x_2 + b_3 u_1 + c_3 u_2, \end{aligned} \tag{16}$$

where

$$x = (x_1, x_2, x_3)^T \in \mathbb{R}^3$$

is the vector of angular velocity of the solid body, u_1 and u_2 are controls, and A_i, b_i, c_i, $i = 1, 2, 3$, are real parameters.

We now construct controls u_1 and u_2 of the form (6) that transfer system (16) from $x^0 = 0$ into a small neighborhood of the point $x^1 \in \mathbb{R}^3$ for time ε. We denote

$$f_0(x) = (A_1 x_2 x_3, A_2 x_1 x_3, A_3 x_1 x_2)^T, \quad f_1(x) = (b_1, b_2, b_3)^T, \quad \text{and} \quad f_2(x) = (c_1, c_2, c_3)^T$$

and assume that

$$\det \begin{pmatrix} b_1 & c_1 & A_1(c_2 b_3 + c_3 b_2) \\ b_2 & c_2 & A_2(c_1 b_3 + c_3 b_1) \\ b_3 & c_3 & A_3(c_1 b_2 + c_2 b_1) \end{pmatrix} \neq 0.$$

Then

$$\operatorname{rank}\{f_1(x^0), f_2(x^0), [[f_0, f_1], f_1](x^0)\} = 3.$$

Representation (5) can be rewritten in the form

$$x(t) = f_1 \int_0^t u_1(\tau) d\tau + f_2 \int_0^t u_2(\tau) d\tau + \sum_{i,j=1}^{2} [[f_0, f_i], f_j] \int_0^t \int_0^\tau \int_0^s u_i(s) u_j(p) dp ds d\tau + R(t). \tag{17}$$

Let $S_1 = \{1, 2\}$, $S_2 = \varnothing$, and $S_3 = \{(0, 1, 2)\}$. By using the control functions

$$u_i = a_i + a \sin \frac{2\pi K_i}{\varepsilon} t, \quad i = 1, 2, \tag{18}$$

and relation (17) for $x^1 = x(\varepsilon)$, we find

$$\begin{aligned} x_1^1 &= \varepsilon(a_1 b_1 + a_2 c_1) + 2A_1 b_2 b_3 I_{11} + A_1(b_2 c_3 + b_3 c_2) I_{12} + 2a_1 c_2 c_3 I_{22}, \\ x_2^2 &= \varepsilon(a_1 b_2 + a_2 c_2) + 2A_2 b_1 b_3 I_{11} + A_2(b_1 c_3 + b_3 c_1) I_{12} + 2a_2 c_1 c_3 I_{22}, \\ x_1^3 &= \varepsilon(a_1 b_3 + a_2 c_3) + 2A_3 b_1 b_2 I_{11} + A_3(b_1 c_2 + b_2 c_1) I_{12} + 2a_3 c_1 c_2 I_{22}, \end{aligned} \tag{19}$$

where

$$I_{ii} = \varepsilon^3 \left(\frac{a_i^2}{3} + \frac{a_i a}{4\pi K_i} + \frac{3a^2}{16\pi^2 K_i^2} \right), \quad i = 1, 2,$$

$$I_{12} = \varepsilon^3 \left(\frac{a_1 a_2}{3} + \frac{a}{4\pi} \left(\frac{a_1}{K_1} + \frac{a_2}{K_2} \right) + \frac{3a^2}{8\pi^2 K_1 K_2} \right).$$

Computing the parameters in (18) as a real solution of system (19), we get the controls approximately transferring system (16) from zero to x^1 for time ε.

As an example, we set $A_1 = -1/3$, $A_2 = 1/2$, $A_3 = -1/5$, $b_1 = c_2 = 1$, and $b_2 = b_3 = c_1 = c_3 = 0$. In this case, the parameters of controls (18) have the form

$$K_1 = 1, \quad K_2 = \begin{cases} -\operatorname{sign}\left(\varepsilon x_1^1 x_2^1 + 24 x_3^1\right), & \varepsilon x_1^1 x_2^1 \neq -24 x_3^1, \\ 1, & \varepsilon x_1^1 x_2^1 = -24 x_3^1, \end{cases}$$

$$a_1 = \varepsilon^{-1} x_1^1, \quad a_2 = \varepsilon^{-1} x_2^1,$$

$$a = -(2\varepsilon)^{-1}\pi(x_1^1 + K_2 x_2^1) + (6\varepsilon^2)^{-1}\left(9\pi^2\varepsilon^2\left({x_1^1}^2 + {x_2^1}^2\right) + 30\pi^2\varepsilon\left|\varepsilon x_1^1 x_2^1 + 24 x_3^1\right|\right)^{1/2}.$$

In Fig. 3, we present the plot of the solution of system (16) for $x^1 = x(1) = (1, 1, 0)^T$.

4.4. Condition of Controllability with Lie Brackets of the High Order. The following example illustrates one more possible direction of development of the proposed approach. Consider a system

$$\begin{aligned} \dot{x}_1 &= x_2^3, \\ \dot{x}_2 &= -x_1 x_2^2 + u, \end{aligned} \tag{20}$$

$$x = (x_1, x_2) \in \mathbb{R}^2, \quad u \in \mathbb{R}.$$

Let $x^0 = (0, 0)$. Then

$$\operatorname{span}\{f_1, [[[f_0, f_1], f_1] f_1]\} = \mathbb{R}^2$$

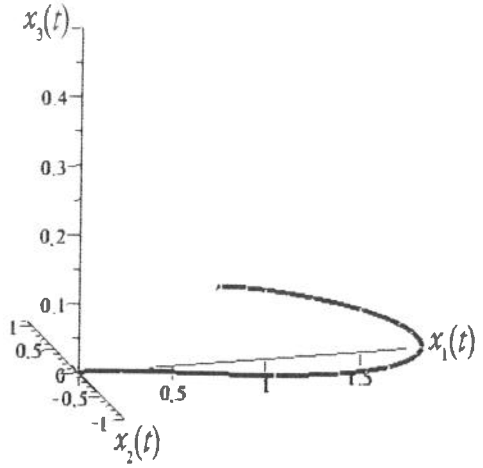

Fig. 3. Trajectory of system (16) with control (18) for the boundary conditions $x(0) = (0, 0, 0)$ and $x(1) = (1, 1, 0)$.

with $f_0 = (x_2^3, -x_1 x_2^2)^T$ and $f_1 = (0, 1)^T$.

As in the previous examples, we now write the representation of the solution of system (20) in terms of the Volterra series but with Lie brackets of the third order:

$$x(t) = f_1(x^0)\int_0^t u(\tau)d\tau + [[[f_0, f_1], f_1] f_1](x^0)\int_0^t\int_0^\tau\int_0^s\int_0^p u(s)u(p)u(r)drdpdsd\tau + R(t). \tag{21}$$

Let

$$u(t) = a + b\sin\frac{2\pi K}{\varepsilon}t, \quad K \in \mathbb{Z}\setminus\{0\}. \tag{22}$$

Then, for $x^1 = x(\varepsilon)$, relation (21) takes the form

$$x_1^1 = \frac{\varepsilon^4}{16\pi^3 K^3}\left(4\pi^3 K^3 a^3 + 4\left(2\pi^2 K^2 - 3\right)a^2 b + 9\pi K a b^2 + 5b^3\right), \quad x_2^1 = \varepsilon a.$$

We set $K = 1$ and $a = \varepsilon^{-1}x_2^1$ and assume that b satisfies the cubic equation

$$5b^3 + \frac{9\pi x_2^1}{\varepsilon}b^2 + \frac{4(2\pi^2-3){x_2^1}^2}{\varepsilon^2}b + \frac{4\pi^3(\varepsilon {x_2^1}^3 - 4x_1^1)}{\varepsilon^4} = 0.$$

Then control (22) approximately solves the two-point control problem for system (20) with the initial position at zero. Thus, for $x^0 = (0, 0)^T$, $x^1 = (1, 1)^T$, and $\varepsilon = 0.1$, we get the following control: $u(t) \approx 10 + 79.54\sin 20\pi t$ (Fig. 4).

This example demonstrates that the family of trigonometric polynomials (22) can also be used for the solution of the two-point problem for the nonlinear system (20) (with good accuracy) in the case where it is necessary to use Lie brackets of higher orders in order to guarantee the validity of the condition of controllability.

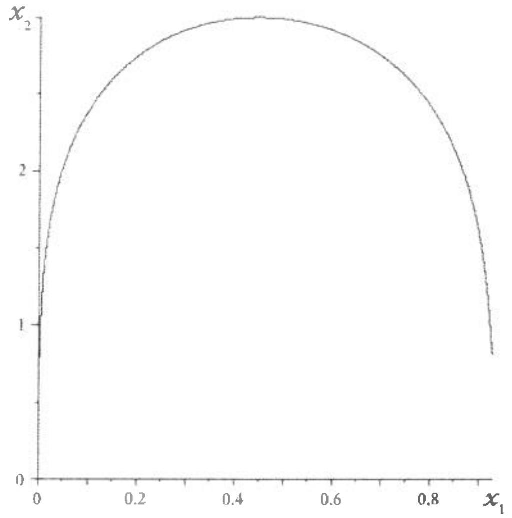

Fig. 4. Trajectory of system (20) with controls (22).

5. Conclusions

In the present work, we study the two-point control problem for a class of systems linear in control and satisfying the rank condition (2). As the key feature of our work, we can mention the reduction of the boundary-value problem with control to a system of algebraic equations of the third order, which is formulated in Theorem 1. To the best of our knowledge, no general results on the solvability of systems of this kind have been published up to now. Thus, the theorem on local solvability (Theorem 2) is a novel result concerning the substantiation of the applicability of trigonometric controls to the local problem of motion planning. Note that the proof of Theorem 2 is based on the principle of nonzero rotation because the theorem of implicit function is not applicable in the analyzed case. In addition, the proposed method requires no changes of variables for the transition to canonical forms, which is convenient from the practical viewpoint. By an example of nilpotent system, we show that the proposed scheme of control transfers the system exactly to the target point. In the general case, in order to estimate the errors, it is necessary to consider the residual term of the Volterra series. The problem of errors of the method remains open for subsequent investigations. Another possible extension of the method is connected with the use of high-order Lie brackets and with the problem of motion planning with bypassing the obstacles.

Appendix.

$$G(t) = \frac{1}{2}\sum_{i,j=1}^{m} \frac{\partial f_i(x)}{\partial x} f_j(x)\Big|_{x=x^0} \int_0^t u_i(s)ds \int_0^t u_j(s)ds$$

$$+\frac{1}{6}\sum_{i,j,l=1}^{m} \frac{\partial}{\partial x}\left(\frac{\partial f_i(x)}{\partial x} f_j(x)\right) f_l(x)\Big|_{x=x^0}$$

$$\times \int_0^t u_i(s)ds \int_0^t u_j(s)ds \int_0^t u_l(s)ds$$

$$+\frac{1}{6}\sum_{i<j}\sum_{l=1}^{m}\left(\frac{\partial f_l(x)}{\partial x}[f_i,f_j](x)+2\frac{\partial}{\partial x}([f_i,f_j](x))f_l(x)\right)\Bigg|_{x=x^0}$$

$$\times\int_0^t u_l(s)ds\int_0^t\int_0^\tau(u_j(s)u_i(p)-u_i(s)u_j(p))dpds.$$

$$\Omega_1=\sum_{i,j=1}^{m}\frac{\partial f_i(x)}{\partial x}f_j(x)\Bigg|_{x=x^0}a_ia_j+\frac{1}{\pi}\sum_{i<j}[f_i,f_j](x^0)\left(\sum_{(q,r)\in S_2}\frac{a_{qr}}{K_{qr}}A_{ij}^r\right.$$

$$\left.+\sum_{(q,r,s)\in S_3}a_{qrs}\left(\frac{A_{ij}^r}{K_{2qrs}}+\frac{A_{ij}^s}{K_{3qrs}}\right)\right),\quad A_{ij}^q=\delta_{iq}\tilde{a}_j-\delta_{jq}\tilde{a}_i,$$

$$\Omega_2=\sum_{i,j,l=1}^{m}\frac{\partial}{\partial x}\left(\frac{\partial f_i(x)}{\partial x}f_j(x)\right)f_l(x)\Bigg|_{x=x^0}a_ia_ja_l$$

$$+\frac{1}{\pi}\sum_{i<j}\sum_{l=1}^{m}\left(\frac{\partial f_l(x)}{\partial x}[f_i,f_j](x)+2\frac{\partial}{\partial x}([f_i,f_j](x))f_l(x)\right)\Bigg|_{x=x^0}$$

$$\times a_l\left\{\sum_{(q,r)\in S_2}\frac{a_{qr}}{K_{qr}}\left(\delta_{iq}\delta_{jr}\frac{a_{qr}}{2}+A_{ij}^r\right)+\sum_{(q,r,s)\in S_3}a_{qrs}\left(\frac{A_{ij}^r}{K_{2qrs}}+\frac{A_{ij}^s}{K_{3qrs}}\right)\right\}$$

$$+\frac{1}{2\pi}\sum_{i<j}\sum_{l=1}^{m}[[f_i,f_j],f_l]\left\{\sum_{(q,r)\in S_2}\frac{a_{qr}}{K_{qr}}\left\{\delta_{iq}\delta_{jr}a_la_{qr}\right.\right.$$

$$+A_{ij}^q\left(-\frac{\delta_{lr}a_{qr}}{2}+\frac{3}{\pi K_{qr}}\left(a_l-\frac{\delta_{lq}a_{qr}}{4}\right)\right)+A_{ij}^r\left(\frac{a_{qr}\delta_{lq}}{2}+a_l-\frac{5\delta_{lr}a_{qr}}{4\pi K_{qr}}\right)$$

$$-\frac{1}{\pi}\sum_{(k,p)\in S_2}\frac{a_{kp}}{K_{kp}}\left(\delta_{lp}A_{ij}^r\right.$$

$$\left.+\frac{a_{qr}}{2}(2\delta_{lp}\delta_{iq}\delta_{jr}+\delta_{lq}(\delta_{ip}\delta_{jr}-\delta_{jp}\delta_{ir})+\delta_{lr}(\delta_{iq}\delta_{jp}-\delta_{jq}\delta_{ip}))\right)$$

$$+\frac{1}{\pi}\sum_{\substack{(k,p)\in S_2\\(k,p)\neq(q,r)}}\frac{a_{kp}}{K_{kp}^2-K_{qr}^2}\left(K_{qr}\left(a_l(\delta_{ik}\delta_{jq}-\delta_{jk}\delta_{iq})-\delta_{lk}A_{ij}^q\right)\right.$$

$$+ K_{kp}\left(a_l\left(\delta_{ip}\delta_{jr}-\delta_{jp}\delta_{ir}\right)-\delta_{lp}A_{ij}^r\right)\Big)\Bigg\}$$

$$+\sum_{(q,r,s)\in S_3} a_{qrs}\left\{\frac{3A_{ij}^q}{\pi K_{1qrs}^2}\left(a_l-\frac{a_{qrs}\delta_{lq}}{4}\right)\right.$$

$$+\frac{A_{ij}^r}{K_{2qrs}}\left(a_l-\frac{5a_{qrs}\delta_{lr}}{4\pi K_{2qrs}}\right)-\frac{A_{ij}^s}{K_{3qrs}}\left(a_l-\frac{5a_{qrs}\delta_{ls}}{4\pi K_{3qrs}}\right)$$

$$-\frac{1}{\pi}\sum_{(k,p,z)\in S_3} a_{kpz}\left(\frac{A_{ij}^r}{K_{2qrs}}\left(\frac{\delta_{lp}}{K_{2kpz}}+\frac{\delta_{lz}}{K_{3kpz}}\frac{2K_{3kpz}^2-K_{2qrs}^2}{K_{3kpz}^2-K_{2qrs}^2}\right)\right.$$

$$+\frac{A_{ij}^s}{K_{3qrs}}\left(\frac{\delta_{lz}}{K_{3kpz}}+\frac{\delta_{lp}(2K_{2kpz}^2-K_{3qrs}^2)}{K_{2kpz}(K_{2kpz}^2-K_{3qrs}^2)}\right)$$

$$\left.+a_l\left(\frac{K_{3kpz}(\delta_{iz}\delta_{jr}-\delta_{jz}\delta_{ir})}{K_{2qrs}(K_{3kpz}^2-K_{2qrs}^2)}+\frac{K_{2kpz}(\delta_{ip}\delta_{js}-\delta_{jp}\delta_{is})}{K_{3qrs}(K_{2kpz}^2-K_{3qrs}^2)}\right)\right)$$

$$+\frac{1}{\pi}\sum_{\substack{(k,p,z)\in S_3\\(k,p,z)\neq(q,r,s)}} a_{kpz}\left(\frac{a_l(\delta_{ik}\delta_{jq}-\delta_{jk}\delta_{iq})-\delta_{lk}A_{ij}^q}{K_{1kpz}^2-K_{1qrs}^2}\right.$$

$$+\frac{K_{2kpz}}{K_{2qrs}}\frac{a_l(\delta_{ip}\delta_{jr}-\delta_{jp}\delta_{ir})-\delta_{lp}A_{ij}^r}{K_{2kpz}^2-K_{2qrs}^2}$$

$$\left.\left.+\frac{K_{3kpz}}{K_{3qrs}(K_{3kpz}^2-K_{3qrs}^2)}(a_l(\delta_{iz}\delta_{js}-\delta_{jz}\delta_{is})-\delta_{lz}A_{ij}^s)\right)\right\}$$

$$+\sum_{(q,r)\in S_2}\sum_{(k,p,z)\in S_3} a_{qr}a_{kpz}\left\{\frac{1}{K_{1kpz}^2-K_{qr}^2}\left(2a_l(\delta_{ik}\delta_{jq}-\delta_{jk}\delta_{iq})-\delta_{lk}A_{ij}^q+\delta_{lq}A_{ij}^k\right)\right.$$

$$+\frac{1}{K_{2kpz}K_{qr}\left(K_{2kpz}^2-K_{qr}^2\right)}\left(\left(a_l(K_{2kpz}^2+K_{qr}^2)-\frac{a_{qr}\delta_{lq}}{2}(K_{2kpz}^2-K_{qr}^2)\right)\right.$$

$$\times(\delta_{ip}\delta_{jr}-\delta_{jp}\delta_{ir})+a_{qr}\left(K_{2kpz}^2-K_{qr}^2\right)\left(\delta_{lp}(\delta_{ir}\delta_{jq}-\delta_{jr}\delta_{iq})+\frac{\delta_{lr}}{2}(\delta_{ip}\delta_{jq}-\delta_{iq}\delta_{jp})\right)$$

$$-\delta_{lp}A_{ij}^r(2K_{2kpz}^2-K_{qr}^2)-\delta_{lr}A_{ij}^p\left(K_{2kpz}^2-2K_{qr}^2\right)\Big)$$

$$+\frac{1}{K_{3kpz}K_{qr}(K_{3kpz}^2-K_{qr}^2)}\Bigg(\bigg(a_l(K_{3kpz}^2+K_{qr}^2)-\frac{a_{qr}\delta_{lq}}{2}\left(K_{3kpz}^2-K_{qr}^2\right)\bigg)$$

$$\times(\delta_{iz}\delta_{jr}-\delta_{jz}\delta_{ir})+a_{qr}\left(K_{3kpz}^2-K_{qr}^2\right)\left(\delta_{ls}(\delta_{ir}\delta_{jq}-\delta_{jr}\delta_{iq})+\frac{\delta_{lr}}{2}(\delta_{iz}\delta_{jq}-\delta_{iq}\delta_{jz})\right)$$

$$-\delta_{lz}A_{ij}^r(2K_{3kpz}^2-K_{qr}^2)-\delta_{lr}A_{ij}^z(K_{3kpz}^2-2K_{qr}^2)\Bigg)\Bigg\}.$$

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