Minkowski Flux Vacua of Type II Supergravities

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We study flux compactifications of 10D type II supergravities to 4D Minkowski space-time, supported by parallel orientifold $O_p$ planes with $3 \leq p \leq 8$. With some geometric restrictions, the 4D Ricci scalar can be written as a negative sum of squares involving Bogomol’nyi-Prasad-Sommerfield-like conditions. Setting all squares to zero provides automatically a solution to 10D equations of motion. This way we characterize a broad class, if not the complete set, of Minkowski flux vacua with parallel orientifolds. We conjecture an extension with nongeometric fluxes. None of our results rely on supersymmetry.

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Context and results.—We consider vacua of ten-dimensional (10D) type II supergravities on a 4D maximally symmetric space-time times a 6D compact manifold, with fluxes. Such vacua are a major framework for string phenomenology, given the role of fluxes in moduli stabilization. Having a complete classification of these vacua would thus be a significant achievement. We make here an essential step towards this goal: with few requirements, we reveal and characterize a broad class of Minkowski flux vacua.

The best understood flux compactifications of type II supergravities are those with just fluxes and no sources. Well-known no-go theorems [1] imply that such reductions only lead to anti-de Sitter (AdS) vacua. Even more, those AdS vacua are no genuine compactifications since they lack a tunable hierarchy between the AdS length scale and the Kaluza-Klein scale [2]. Space-time filling orientifold planes are a natural way out of this problem and allow one or AdS ones with scale separation (see, e.g., Ref.[6]). The Kaluza-Klein scale [2]. Space-time filling orientifold planes (and $O_p$ planes) are a natural way out of this problem and allow one to find Minkowski vacua [3,4] (see Ref. [5] for a review) or AdS ones with scale separation (see, e.g., Ref. [6]). However such reductions come with an extra level of complication, the backreaction of the orientifolds themselves. In the case of nonintersecting $O_p$ planes (and possibly $D_p$ branes) the backreaction is rather well understood: it introduces a warp factor $e^\Delta$ in the 10D metric

$$ds_{10}^2 = e^{2A}(d\tilde{s}_4^2 + d\tilde{s}_6^2) + e^{-2A}d\tilde{s}_{6,\perp}^2, \tag{1}$$

with inverse powers along parallel and transverse directions to the sources; $d\tilde{s}_6^2$ and $d\tilde{s}_{6,\perp}^2$ still depend on all internal coordinates [7]. So we work here in this context: type II supergravities supplemented by space-time filling, parallel (i.e., nonintersecting), backreacted $O_p/D_p$ sources, with a fixed size $3 \leq p \leq 8$, and the standard dilaton value $e^\Delta = g_\ast e^{A(p-3)}$ with a constant $g_\ast$. In this framework, we study Minkowski flux vacua; note we do not capture nonperturbative solutions. We follow conventions of a companion paper [7].

$O_p$ planes.—Let us recall known results on $O_3$ compactifications in IIB, following Refs. [4,7,8]. The sourced flux is the 5-form $F_5$, the other Ramond-Ramond (RR) fluxes being the 1- and 3-forms $F_1$, $F_3$, and the NSNS 3-form flux is denoted $H$. For our purposes, we need two equations: the first one is a combination of the dilaton equation of motion (EOM), the 10D and the 4D traces of the Einstein equation, giving an expression for the Ricci scalar $\mathcal{R}_4$ of $d\tilde{s}_4^2$; the second equation is the $F_5$ Bianchi identity (BI)

$$e^{-2A}\mathcal{R}_4 = -\frac{1}{2}|H|^2 - \frac{e^{2\Delta}}{2}|F_3|^2 - e^{2\Delta}|F_5|^2 + \frac{e^{\Delta}}{4} T_{10} + \cdots, \tag{2}$$

$$dF_5 = H \wedge F_3 + \frac{e^{3}}{4} T_{10} \text{vol}_6, \tag{2}$$

where the dots stand for terms explicitly dependent on $\partial \phi$ and $\partial A$, and we leave the definitions of the square of forms, the sign $e_p$, and the sources contributions $T_{10}$ to Ref. [9]. We see from above that, at least in the smeared limit where $\partial \phi = \partial A = 0$, the flux contributions cannot be canceled to find a Minkowski vacuum without the $O_3$ contribution in $T_{10}$. We now combine both equations of Eq. (2) and rewrite the result as in Ref. [7]

$$e^{-2A}\mathcal{R}_4 = -|e^{4A} \ast_6 de^{-4A} - e_5 e^{\Phi} F_5|^2$$

$$- \frac{1}{2} |\ast_6 H + e_3 e^{\Phi} F_3|^2 + e^{-2A}\partial(\cdots), \tag{3}$$

where $\partial(\cdots)$ denotes a total derivative over the (unwarped) compact manifold. Upon integration, we deduce that a Minkowski vacuum requires both squares to vanish. One leads to the well-known ISD condition [4]:

$$H = e_3 e^{\Phi} \ast_6 F_3.$$
This also fixes $F_5$, and relates eventually $dF_5$ to $\Delta_6 e^{-4A}$ as expected from the BI. With further combinations of EOM, one can show that $F_1 = 0$ and $\phi$ is constant (see below). Using the $H$ and $F_1$ BI (i.e., that these fluxes are closed), one then shows that all EOM are solved, provided the (unwarped) compact manifold is Ricci flat. In other words, given this geometric requirement, and assuming satisfied BI, one finds all Minkowski flux vacua, and those are characterized through the expression (3) by setting the squares of BPS-like conditions to zero.

$O_p$ planes with $p > 3$.—The essence of the present Letter is to derive analogous results for $O_p$ planes with $p > 3$. First, up to little additional structure, $\hat{R}_d$ can again be written in terms of squares of BPS-like conditions; with few geometric restrictions, asking for a Minkowski vacuum then requires us to set these conditions to zero, fixing the sourced flux $F_{8-p}$, $3 \leq p \leq 8$, and relating $H$ to $F_{6-p}$. We further show that these three fluxes are the only nontrivial ones. Finally, assuming their BI, we show that all EOM are satisfied, giving us a broad class of Minkowski flux vacua.

Let us point out the main differences when $p > 3$. For $p = 3$, all internal directions are transverse to the sources, while for $p > 3$, some are parallel, as in Eq. (1). $A$ is then only dependent on transverse directions, and $\epsilon_6$ gets replaced by $\epsilon_{\perp} \Delta_6$ by $\Delta_\parallel$, etc. Fluxes may now have different components: $F_I = F_I^{(0)} + F_I^{(1)} + \cdots$, where $n$ in $F_I^{(n)}$ denotes the number of parallel indices of the component. We will as well introduce the projection on the transverse subspace $F_I^{(\perp)} = F_I^{(0)}$. A second difference is the possibility of RR fluxes $F_I$ with $I > 8 - p$. Taking these features into account, a lengthy computation [7] leads to an expression for $\Delta_{\perp}$, analogous to Eq. (3), given by

$$e^{-2A} \Delta_{\perp} = -|e^{4A} \pm_4 de^{-4A} - e_p e^\phi F_8 |_{8-p}^2 |^2$$

$$- \frac{1}{2} \left| \pm_4 H \right|^2 + (e_p e^\phi F_8 |_{8-p}^2) |^2$$

$$- \frac{1}{2} \sum_{a} \left| \pm_4 \left( de^{a} \right) \right| - e_p e^\phi \left( t_{0a} F_8^{(1)} \right)^2$$

$$- (\text{flux})^2 - (\Delta_\parallel + \Delta_\perp) + e^{-2A} \partial(\ldots),$$

for $0 \leq 8 - p \leq 5$, where we use the internal one-form basis $e^a = e^{a} \delta y^m$ constructed with vielbeins, and the contraction $t_{ij}$ by a vector $V$ such that $t_{ij} e^b = \delta^b_{ij}$. We use from now on the shorthand $t_{ij} \equiv t_{ij}$, and the 6D metric in flat indices $\delta_{\alpha \beta}$. One has $de^a = -\frac{1}{2} \epsilon^{ab} e^c \wedge e^c$, where the anholonomicity symbol (from now on “geometric flux”) $\epsilon$ does not need to be constant. Thus, by definition, $(de^{a})_{\perp} = -\frac{1}{2} \epsilon^{ab} \epsilon^{c_1} \epsilon^{c_2} \wedge e_{c_1} \wedge e_{c_2}$. This expression (4) is derived in [7] where on top of the analogue of Eq. (2), one uses the trace of the Einstein equation along internal parallel directions. Differences between Eqs. (4) and (3) are related to those mentioned above: there is a third BPS-like condition involving $F_{8-p}^{(1)}$, the (flux)$^2$ are squares of $F_{8-p}^{(p-1)}$ and of $F_{1 \geq 8-p}$, and $\Delta_\parallel + \Delta_\perp$ are curvature terms given in terms of $\tilde{F}_{ab} |_{b_{c_1}}$, $\tilde{F}_{a_{b_{c_1}}} |_{b_{c_1}}$, $\tilde{F}_{a_{b_{c_1}}} |_{b_{c_1}}$, $\tilde{F}_{a_{b_{c_1}}} |_{b_{c_1}}$, i.e., involving parallel directions; we refer to Ref. [7] for precise definitions.

$\Delta_\parallel$ and $\Delta_\perp$ are the only quantities with indefinite sign. We thus set them here to zero, and will do the same for purely transverse directions. For computational convenience, we actually set the corresponding (unwarped) geometric fluxes to zero

$$\tilde{F}_{a_{b_{c_1}}} |_{b_{c_1}} = 0, \quad (5)$$

$$\tilde{F}_{a_{b_{c_1}}} |_{b_{c_1}} = 0, \quad (6)$$

where Eq. (5) implies $\Delta_\parallel = \Delta_\perp = 0$; note the latter holds for all known Minkowski flux vacua on twisted tori [10]. The geometric requirements (5), (6), are the only ones made on the vacua, and can be viewed as the analogue of the Ricci flatness for $p = 3$. Asking for the transverse subspace to have no boundary amounts to $\tilde{F}_{a_{b_{c_1}}} |_{b_{c_1}} = 0$, implied by Eq. (6); we then integrate Eq. (4) over this subspace, giving $\hat{R}_d$ only in terms of BPS-like conditions and squares of fluxes. A Minkowski vacuum requires us to set them to zero, making the total derivative vanish, as for $p = 3$. These conditions are enough to characterize a broad class of Minkowski flux vacua, as we show in the next sections.

Fluxes.—Without the curvature terms, asking for a Minkowski vacuum sets the BPS-like conditions of Eq. (4) and the squares of fluxes to zero. This, together with the trace of the Einstein equation along internal parallel directions [7], implies that fluxes (and components) within the BPS-like conditions are the only ones allowed to be nonzero, i.e., $H = H |_{\perp}, F_{6-p} = F_{6-p} |_{\perp}, F_{8-p} = F_{8-p} + F_{8-p}^{(1)}, \quad (7)$

Indeed, we get that $F_{1 \geq 8-p} = 0$, and that the lowest RR flux $F_{4-p}$, meaning $F_0$ for $p = 4$ and $F_1$ for $p = 3$, vanishes [11]. The only possible flux content is then Eq. (7).

The BI for these fluxes [12] are given by

$$dH = df_{6-p} = 0, \quad (dF_{8-p} - H \wedge F_{6-p}) = \frac{e^{T_1} |_{1} \text{vol}_{1}}{p+1} \quad (8)$$

and projecting the $F_{8-p}$ BI on the transverse directions, together with Eq. (7), gives

$$e^{\phi} \frac{T_{10}}{p+1} = e^{6A} \Delta_\perp e^{-4A} + e^{2\phi} |F_{6-p}^2|^2 + e^{2\phi} |F_{8-p}^{(1)}|^2 \quad (9)$$

$$= e^{6A} \Delta_\perp e^{-4A} + |H|^2 + |F_{8-p}^a_{b_{c_1}}|^2, \quad (10)$$
the last term being defined in Ref. [13]. Upon integration of the last equation one recovers the RR tadpole condition that expresses the net charge in terms of the fluxes. Both the $H$ flux and the geometric flux contribute to cancel the $O_p$ charge, and the two contributions are schematically viewed as $T$ dual [8,14]. While $F^{(1)}_{8-p}$ is related to the geometric flux, $e^{(0)}_{8-p}$ leads to $\delta_\perp e^{-4A}$ generating the $\delta$ functions, and can thus be interpreted as related to the sources’ backreaction. Explicit vacua with both pieces are given, e.g., in Refs. [10,15], while a vacuum with all fluxes (7) turned on is given in Eq. (6.42) of Ref. [15].

We now show that the fluxes (7) solve their EOM [7], given here by

$$e^{-4A}d(e^{4A}*_6 F_{8-p}) = 0 \quad (4 \leq p \leq 7),$$

$$e^{-4A}d(e^{4A}*_6 F_{6-p}) + H \wedge *_6 F_{8-p} = 0 \quad (3 \leq p \leq 5),$$

$$e^{-4A}d(e^{4A-2\phi}_6 H) - F_{6-p} \wedge *_6 F_{8-p} = 0,$$  \hspace{1cm} (11)

provided relevant BI are satisfied. To that end, it is convenient to rewrite $F_{8-p}$ as follows (details in Ref. [16]):

$$F_{8-p} = (-1)^p e_p e^{-4A} *_6 d(e^{4A-\phi} vol)_{\perp},$$  \hspace{1cm} (12)

with the internal parallel volume form, and use for an $l$-form $A_l$ that $*_6 A_l|_{\perp} = (-1)^{(p+1)}*vol_{\perp} \wedge *_6 A_l|_{\perp}$. It is then straightforward to see that the fluxes (7) solve the three above EOM, given the $H$ and $F_{6-p}$ BI. With the $F_{8-p}$ BI, we now prove the remaining EOM are also satisfied. Note it is the case for the branes own EOM: Eq. (12) is the calibration condition that minimizes their energy [7].

Satisfying all EOM.—The remaining EOM are the dilaton and the Einstein equation; we split the latter into the 4D one, the transverse internal one, and, for $p \neq 3$, the internal parallel and off-diagonal ones. Verifying that they are satisfied is purely technical; we give here and in the Appendix only the main insights. For all these EOM, we use the 10D Einstein trace [7], leading us to consider in the following the trace reversed Einstein equation. We also use Eq. (9) and the dilaton and warp factor formulas of Appendix C of Ref. [7]. The dilaton EOM and the 4D Einstein equation are then straightforward to verify. The latter boils down to

$$\mathcal{R}|_{\mu \nu} + 2\nabla|_{\mu} \partial|_{\nu} \Phi - g_{\mu \nu} (2|\partial \Phi|^2 - \Delta \Phi)$$

$$= g_{\mu \nu}/8 \left( e^{\phi} T_7 \right) - |H|^2 - e^{2\phi} \left( \sum_{q=0}^6 (1-q)|F_q|^2 \right),$$  \hspace{1cm} (13)

with even or odd RR fluxes for IIA/IIB, and each side can be shown to be equal to Eq. (A1). We have introduced the shorthand $|_{\mu = \perp} = M = p$, and similarly, $|_{\alpha = A = p}$.

Internal Einstein equations are treated using flat indices. The Ricci tensor is computed using formulas of Appendix C of Ref. [7] and $f^{a\perp bc} = \delta^a_b \delta^c_f f^{a\perp bc}$. Starting from the general tensor expression, one should pay attention going from 10D to 6D indices, parallel and transverse, and further to warped versus unwarped quantities. We also use the source contribution $T_{ab} = \delta^a_b \delta^f_p \delta^c_f T_{10}/(p + 1)$. The internal parallel Einstein equation requires the dilaton derivatives (A2). One should then prove that the Ricci tensor is given by Eq. (A3), which is achieved using assumption (5). We also use Eq. (5) for the internal off-diagonal Einstein equation and compute for the dilaton Eq. (A4); this equation eventually becomes

$$\mathcal{R}|_{ab|\perp} = (p - 2) \delta^a_b d^\perp \delta^a_b c^f \delta^e_{\perp} e^{-4A} \partial e^A.$$  \hspace{1cm} (14)

The computed Ricci tensor (A5) is, however, different. Interestingly, the match is achieved using the $F_{8-p}$ BI along nontransverse directions, i.e., $dF_{8-p} - (dF_{8-p})_{\perp} = 0$, which has two components due to $F_{8-p}$. Setting the one along $e^a_{\perp} = e^a_{\perp} \wedge e^a_{\perp}$, respectively, along $e^a_{\perp} \wedge e^a_{\perp}$, to zero gives identities (A6), respectively, (A7), using Eq. (5) and $\delta^a_{\perp} b_{\perp} = 0$. Identity (A7) allows us to solve the off-diagonal Einstein equation. Finally, using the expression (A8) for the dilaton, the internal transverse Einstein equation is given by

$$\mathcal{R}|_{a b | \perp} = - \frac{1}{2} \delta^a_{b} f_{\alpha | \perp} |\alpha_{\perp} = 1/2 | h_{b} | F_{6-p} - \delta^a_{b} \frac{\varepsilon^2 \phi}{2} | F_{6-p} |^2.$$  \hspace{1cm} (15)

This holds as well for $p = 3$, where $F_5 = f_3(0)$, using Eq. (7). The last three rows of Eq. (15) vanish using Eqs. (6) and (7). Computing the Ricci tensor with Eqs. (5) and (6), one then obtains a match, up to $\frac{1}{2} \partial | \alpha_{\perp} f_{\alpha | \perp}$, which is zero due to the identity (A6). All EOM are thus satisfied.

Nongeometric fluxes extension.—A natural extension of Eq. (4) (without curvature terms) including nongeometric NSNS fluxes would be

$$2 e^{-2A} \tilde{\mathcal{R}}_g = -2 e^{4A} *_{\perp} de^{4A} - e_p e^{\phi} F^{(0)}_{8-p} - \left( H_{\alpha_{\perp} c_{\perp} | b_{\perp} a_{\perp}} e^a_{\perp} c_{\perp} e^{a_{\perp}} / 3 + e_p e^{\phi} F^{(0)}_{8-p} \right)$$

$$- \left( \sum_{a} \left( f_{\alpha_{\perp} c_{\perp} | b_{\perp} a_{\perp}} e^{a_{\perp}} c_{\perp} e^{a_{\perp}} / 2 + e_p e^{\phi} (t_{\alpha_{\perp} c_{\perp} | b_{\perp} a_{\perp}} F^{(1)}_{8-p}) \right) \right)$$

$$- \left( \sum_{a} \left( Q_{\alpha_{\perp} c_{\perp} | b_{\perp} a_{\perp}} e^{a_{\perp}} c_{\perp} e^{a_{\perp}} + e_p e^{\phi} (t_{\alpha_{\perp} c_{\perp} | b_{\perp} a_{\perp}} F^{(2)}_{10-p})) \right) \right)$$

$$- (\text{flux})^2 + e^{-2A} \partial (...)$$  \hspace{1cm} (16)
This combination of BPS-like conditions is T-duality invariant from the 4D perspective: there one lifts or lowers T-dualized indices of NSNS fluxes [17], while parallel and transverse directions get exchanged when T dualized. Setting \( \mathcal{R}_|| + \mathcal{R}_\perp = 0 \), or even conditions (5), (6), may also have analogues with nongeometric fluxes. Finally, one could extend the \( F_{8-p} \) BI on transverse directions (9) by adding \( + |Q_c a_i h_j|^2 + |R^a_i h_{i c}|^2 \). In these extended expressions, all terms are, however, not present for any \( p \), due to the number of parallel and transverse directions. All fluxes are allowed for \( p = 6 \), but not for the others: on top of the first BPS-like condition, only the one in \( H \) is present for \( p = 3 \), \( H, f \) for \( p = 4 \), \( H, f, Q \) for \( p = 5 \), \( f, Q, R \) for \( p = 7 \), and \( Q, R \) for \( p = 8 \). This restriction can also be understood schematically by T dualizing an \( O_3 \) vacuum on \( T^6 \) with \( H \) flux.

A first setting to derive these conjectured expressions would be 4D gauged supergravities, replacing \( \mathcal{R}_4 \) by the vacuum value of the scalar potential. A more natural option would be the 10D formalism of \( \beta \) supergravity [18], or extensions thereof, including Ref. [19]. But this goes beyond the scope of this Letter.

**Discussion.**—In this Letter, we have characterized a broad class of Minkowski flux vacua with parallel localized \( O_p/D_p \) sources; this is achieved thanks to the general rewriting Eq. (4) of the 4d Ricci scalar as a negative sum of squares, extending a well-known result for \( O_3 \) planes [4] to general \( O_p \) reductions. The fluxes are Eq. (7) (their BI are assumed to be satisfied), the metric is Eq. (1) with restrictions (5) and (6), and the dilaton is given after Eq. (1). Interestingly, requiring only the geometric restrictions (5) and (6), we have shown that these are the only Minkowski flux vacua.

Our analysis does not rely on supersymmetry (SUSY) so our vacua can capture both SUSY and non-SUSY solutions. To get SUSY \( O_3 \) vacua, the Ricci flat condition should be supplemented by Calabi-Yau (see Ref. [10] for a Ricci flat non-Calabi-Yau solvmanifold), and the ISD 3-form flux has to be (1,2) and primitive [4]. Here, it would be interesting to compare our geometric restrictions to the generalized Calabi-Yau condition and our fluxes to the SUSY ones [10,15], given the integrability result of Ref. [20]. Already, our sourced flux takes the value (12) of a calibrated brane [7], which would be the SUSY value in the presence of an \( SU(3) \times SU(3) \) structure.

A natural question is whether there exists other Minkowski flux vacua with parallel sources than the class found here, thus meaning some with the geometric restrictions not satisfied (see a related discussion in Ref. [7]). We may capture all no-scale vacua, but still wonder whether other vacua exist, which do not arise from the BPS-like conditions. We can, for instance, look at the set of SUSY Minkowski flux vacua on twisted tori: all known solutions are reviewed in Ref. [10]. All listed vacua with parallel sources verify Eqs. (5) and (6) (in particular those of Ref. [15]), except for the new ones found in Ref. [10]. For the latter, however, the Ricci tensor still vanishes and \( \mathcal{R}_|| = \mathcal{R}_\perp = 0 \), indicating a suspected possible refinement of our geometric conditions. Then, up to this detail, we do not find vacua beyond the class presented here.

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**APPENDIX: RICCI TENSOR AND DILATON DERIVATIVES**

Each side of the 4D Einstein equation (13) is equal to

\[
\frac{g_{\mu \nu}}{16} (7 - p)(e^{6A} \Delta \epsilon e^{-4A} - e^{8A} |de^{-4A}|^2). \tag{A1}
\]

For the internal parallel Einstein equation, we use

\[
\nabla|a_i| \partial_{b_j} \phi = -\frac{\delta_{a_i b_j}}{4} e^{2\phi - 2A} \delta^e_{-d_1} \partial_{e_i} e^{2A} \partial_{d_2} e^{-2\phi} \tag{A2}
\]

\[
\mathcal{R}|a_i| b_j = \frac{\delta_{a_i b_j}}{8} (2 e^{6A} \Delta \epsilon e^{-4A} - (p - 1)e^{8A} |de^{-4A}|^2) + \frac{\delta_{a_i c_k} \delta_{b_i d_l}}{4} \delta^{e_i d_2} \delta^{f_j}_{c_k d_1} f^{c_k f_j g_l}_{e_i h_k l_s}. \tag{A3}
\]

where the warp factor terms of (A3) can be rewritten as

\[-\delta_{a_i b_j} \delta^e_{-d_1} (e^A \partial_{e_i} \partial_{d_2} e^A + (2p - 7) \partial_{e_i} e^{4A} \partial_{d_2} e^A). \]

For the internal off-diagonal Einstein equation, we compute with Eq. (5) and \( 2 \partial_{a_i} \partial_{b_j} = f^c ab \partial_c \),

\[
\nabla|a_i| b_j \partial_{b_j} \phi = \nabla|b_j| a_i \partial_{a_i} \phi = -\frac{\delta_{a_i c_k} \delta_{b_j d_l}}{2} f^{c_k f_j g_l}_{e_i h_k l_s} \tag{A4}
\]

\[
\mathcal{R}|e_i| d_1 = (p - 3) \delta^{b_i c_i} \delta^{a_i b_i} f^{h_i}_{e_i d_1} \partial_{d_2} e^A + \frac{1}{2} \delta^{b_i c_i} \delta^{a_i b_i} \partial_{d_2} f^{h_i}_{e_i d_1} e^A \tag{A5}
\]

\[
\mathcal{R}|e_i| d_1 = (p - 3) \delta^{b_i c_i} \delta^{a_i b_i} f^{h_i}_{e_i d_1} \partial_{d_2} e^A + \frac{1}{2} \delta^{b_i c_i} \delta^{a_i b_i} \partial_{d_2} f^{h_i}_{e_i d_1} e^A \tag{A5}
\]

and the \( F_{8-p} \) BI along nontransverse directions gives

\[
\partial_{a_i} f^{c_k}_{e_i h_k l_s} = 0. \tag{A6}
\]
For the transverse Einstein equation, we compute

\[ -2\delta^{\perp d} f^b_{\perp 1} \partial_{d_1} e^A + \delta^{d_1} \partial_{d_1} f^b_{\perp 1} = 0. \]  

(A7)

For the transverse Einstein equation, we compute

\[
\nabla|_{a_\perp} \partial b_\perp f_{\perp 1} \partial_{a_\perp} \phi = \partial_a \partial_b \phi - \omega_{a_\perp} b_\perp \mid_{(\partial A = 0)} \partial_{\perp 1} \phi + \partial_{a_\perp} \phi \partial_{b_\perp} e^A - \delta^{a_\perp b_\perp} \delta^{\perp d_1} \partial_{e_\perp 1} \phi \partial_{d_1} e^A. \quad (A8)
\]


[12] We define the square of the q-form A_q as |A_q|^2 = A_{m_1...m_q}A^{m_1...m_q}/q!, the sign \epsilon_p = (-1)^{(9-p)/2}p+1, and sources are localized in transverse directions through T_{10} = (2\kappa_0^2/\sqrt{|g_{11}|})(p+1)T_p(\sum_{a_\perp} 2^{p-5} \delta(y_{a_\perp}) - \sum_{b_\perp} \delta(y_{b_\perp})). We consider BPS sources where tension and charge are related as T_p = \mu_p, and without world-volume flux.


[14] Proving that F_{10-p} = 0 actually requires another combination of EOM worked out in Ref. [7], namely, the one that allows us to conclude on the no go for p = 7, 8.

[15] The nonzero fluxes may also enter the H\wedge F_{8-p}, but this quantity vanishes as it should, given its number of transverse components.

[16] [f_{a_{\perp b_{\perp 1}}}]^2 = (\delta_{a_{\perp b_{\perp 1}}})^2 \delta^{\perp d_1} f^b_{\perp 1} f_{a_{\perp d_1}} = \sum_{a_{\perp 1}} |(d\omega_{a_{\perp 1}})|_{\perp 1}^2.


[19] The internal manifold and transverse subspace having no boundary, one has \tilde{f}_{a_{\perp d_1}} = 0. One then shows that d\tilde{\omega}_{a_{\perp 1} d_1} = e^{-A_{a_{\perp} \perp 1}} \sum_{a_{\perp 1}} \epsilon_{\perp 1} (d\omega_{a_{\perp 1}})_{\perp 1}, allowing us to prove that F_{8-p} in Eq. (7) can be rewritten as Eq. (12).


