GLOBAL REGULARITY FOR THE 2+1 DIMENSIONAL EQUIVARIANT EINSTEIN-WAVE MAP SYSTEM

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Abstract. In this paper we consider the equivariant 2+1 dimensional Einstein-wave map system and show that if the target satisfies the so called Grillakis condition, then global existence holds. In view of the fact that the 3+1 vacuum Einstein equations with a spacelike translational Killing field reduce to a 2+1 dimensional Einstein-wave map system with target the hyperbolic plane, which in particular satisfies the Grillakis condition, this work proves global existence for the equivariant class of such spacetimes.

Contents

1. Introduction

In this paper we shall prove that global existence holds for the maximal Cauchy development of asymptotically flat initial data for the equivariant 2+1 dimensional Einstein-wave map system assuming that the target (N, h) is a rotationally symmetric 2-manifold with metric satisfying the Grillakis condition, see (1.5) below. The Grillakis condition holds in particular if h has negative sectional curvature. Therefore, our result applies in the important special case obtained by considering the 3+1 vacuum Einstein equations with a spacelike translational Killing field which reduces to a 2+1 dimensional Einstein-wave map system with target the hyperbolic plane $\mathbb{H}^2$, see [22] and also [1, 23] and references therein. It follows that global existence holds for an equivariant solution of the 3+1 vacuum Einstein equations with a spacelike translational Killing field.

Before discussing the equivariant 2+1 dimensional Einstein-wave map system, we first provide some background on the equivariant wave map problem.

1.1. **Equivariant critical wave maps.** Let $(M, \mathbf{g}_{\mu\nu})$ be a Lorentzian spacetime and (N, h_{AB}) a Riemannian manifold. The action defined for a map $\Phi : M \to N$

by

$$S_{\mathrm{WM}} := -\frac{1}{2}\int_M \mathbf{g}^{\mu\nu}\partial_\mu\Phi^A\partial_\nu\Phi^B h_{AB}\circ\Phi \tag{1.1}$$

has Euler-Lagrange equation

$$\Box_{\mathbf{g}}\Phi^A + {}^{(h)}\Gamma^A_{BC}\circ\Phi\,\partial_\mu\Phi^B\partial_\nu\Phi^C\mathbf{g}^{\mu\nu} = 0 \tag{1.2}$$

where, denoting by ∇ the Levi-Civita covariant derivative of $\mathbf{g}$, $\Box_{\mathbf{g}} = \nabla^\alpha\nabla_\alpha$ is the d'Alembertian, and where ${}^{(h)}\Gamma^A_{BC}$ denote the Christoffel symbols of h. The action (1.1) is the Lorentzian analogue of the Dirichlet integral, or harmonic map energy, and if M is static, time independent solutions of (1.2) are simply harmonic maps. In the particular case where the target is a compact Lie group, this system is known in the physics literature as a σ-model, and in the mathematics literature (with general target), it is known as the wave map equation.

Next, we restrict ourselves to the equivariant class. We assume M is a globally hyperbolic $2+1$-dimensional spacetime with Cauchy surface diffeomorphic to $\mathbb{R}^2$ and that N is a complete Riemannian manifold of dimension 2 with metric h of the form

$$h = d\rho^2 + g^2(\rho)d\theta^2$$

for an odd function $g:\mathbb{R}\to\mathbb{R}$ with $g'(0)=1$. Let $e^{i\theta}$, $\theta\in\mathbb{S}^1=\mathbb{R}/2\pi\mathbb{Z}$ denote a semifree circle action on M and N. We assume that the S^1 action on M is generated by a hypersurface orthogonal Killing field ∂_θ, that it has a non-empty fixed point set[1], and that the non-trivial orbits of this action in M are spatial.Then we may write $\mathbf{g}$ in the form[2]

$$\mathbf{g} = \check{\mathbf{g}} + r^2 d\theta^2 \tag{1.3}$$

where $\check{\mathbf{g}}$ is a metric on the orbit space $\mathcal{Q} = M/\mathbb{S}^1$ and r is the radius function, defined such that $2\pi r(p)$ is the length of the $\mathbb{S}^1$ orbit through p. We assume that M has Cauchy surface Σ diffeomorphic to $\mathbb{R}^2$, which we may, without loss of generality, assume to be symmetric[3].

A map $\Phi: M\to N$ is equivariant, with rotation number $k\in\mathbb{Z}$ if

$$\Phi\circ e^{i\theta} = e^{ik\theta}\circ\Phi.$$

Let the function ϕ be defined by

$$\phi = \rho\circ\Phi$$

where $\rho: N\to\mathbb{R}_+$ is the radial coordinate function on N. With the above definitions, the wave maps equation takes the form

$$\Box_{\mathbf{g}}\phi - \frac{k^2 g(\phi)g'(\phi)}{r^2} = 0. \tag{1.4}$$

The Cauchy problem for equivariant wave maps with base $M=\mathbb{R}^{2+1}$ was studied by Shatah and Tahvildar-Zadeh [30] who proved that for targets satisfying[4]

$$g'(s)\geq 0, \quad \text{for } s\geq 0$$

[1] It follows that the fixed point set is a timelike line, see [6] and references therein.

[2] As an example, consider $\mathbb{R}^{2+1}$ with the metric

$$\mathbf{g} = -dt^2 + dr^2 + r^2 d\theta^2$$

In this case, the orbit space is $\mathcal{Q} = \{(t,r),\ r\geq 0\}$ with metric $\check{\mathbf{g}} = -dt^2 + dr^2$.

[3] To see this, note that $\mathcal{Q}$ is globally hyperbolic, and hence by [5], there is a Cauchy time function $\check{t}$ on $\mathcal{Q}$ which may be lifted to a symmetric Cauchy time function t on M.

[4] This condition is equivalent to the assumption that the target (N,h) is geodesically convex.

global well-posedness holds for the equivariant wave map problem. It was then shown by Grillakis in [13] that it suffices for the target to satisfy the Grillakis condition[5]

$$sg'(s) + g(s) > 0, \quad \text{for } s > 0. \tag{1.5}$$

Let us also mention subsequent developments by Shatah and Struwe in [29], Shatah and Tahvildar-Zadeh in [31], and by Struwe in [33]. Finally, let us mention the work of Christodoulou and Tahvildar-Zadeh for the case of spherically symmetric solutions [9]. Note that in these works, the proof consists of two main steps, a proof of energy non-concentration and a proof of global existence for small energy initial data.

Remark 1.1. *(1) In [4], it was established that vacuum Einstein's equations for G_2-symmetric 3+1 dimensional spacetimes reduce to spherically symmetric wave maps from $U : \mathbb{R}^{2+1} \to \mathbb{H}^2$. Consequently, the aforementioned work of Christodoulou and Tahvildar-Zadeh [9] was applied in [4] to prove global regularity for large data for these spacetimes. In the context of our problem, we would like to emphasize that the nonzero homotopy degree prevents a similar reduction to flat-space wave maps. Thus we are forced to consider the coupling with Einstein's equations.*

(2) More recent work shows that large data global existence holds for the wave map problem (1.2) with $M = \mathbb{R}^{2+1}$ and target[6] $N = \mathbb{H}^2$ even in the absence of equivariant symmetry, see [34], [32] and [20].

(3) It is known that for targets which are not geodesically convex, e.g. $N = \mathbb{S}^2$, singularities may form, see [27, 26].

1.2. **The equivariant Einstein-wave map problem.** Let $R_{\mathbf{g}}$ denote the scalar curvature of the Lorentzian metric $\mathbf{g}$ on M, and let $\kappa > 0$ a constant. Let

$$S_{\text{grav}} := \frac{1}{2\kappa} \int_M R_{\mathbf{g}}$$

denote the Einstein-Hilbert action, then the Euler-Lagrange equation for an Einstein-wave map with action

$$S_{\text{grav}} + S_{\text{WM}}$$

consists of (1.2) coupled to the Einstein equation

$$G_{\mu\nu} = \kappa S_{\mu\nu} \tag{1.6}$$

where $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R \mathbf{g}_{\mu\nu}$ is the Einstein tensor for the metric $\mathbf{g}$ and

$$S_{\mu\nu} = \partial_\mu \Phi^A \partial_\nu \Phi^B h_{AB} - \frac{1}{2} \partial_\alpha \Phi^C \partial_\beta \Phi^D \mathbf{g}^{\alpha\beta} h_{CD} \mathbf{g}_{\mu\nu} \tag{1.7}$$

is the stress-energy tensor for the wave map.

Remark 1.2. *As emphasized above, the main motivation for considering the Einstein-wave map problem is that the 3+1 vacuum Einstein equations with a spacelike translational Killing field reduces to a 2+1 dimensional Einstein-wave map system with target the hyperbolic plane $\mathbb{H}^2$, see [22] and also [1, 23] and references therein.*

In this paper, we restrict ourselves to the equivariant class and recall some of the notations already introduced in section 1.1.

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[5]For example, the Grillakis condition is satisfied in the important particular case $N = \mathbb{H}^2$, with $g(\rho) = \sinh(\rho)$.

[6]Note that more general targets are considered in [32].

Definition 1.3 (Equivariant critical Einstein-wave map)**.** *Let $(M, \mathbf{g})$ be a globally hyperbolic spacetime with an $\mathbb{S}^1$ action by isometries $e^{i\theta}$, with hypersurface orthogonal generator ∂_θ which is spacelike away from fixed points. Let the metric h on N be of the form $h = d\rho^2 + g^2(\rho)d\theta^2$. Assume that M has Cauchy surface diffeomorphic to $\mathbb{R}^2$. Let $\Phi : M \to N$ be an equivariant map, with rotation number $k \in \mathbb{Z}$, i.e. $\Phi \circ e^{i\theta} = e^{ik\theta} \circ \Phi$, and let $\phi = \rho \circ \Phi$.*

An equivariant critical Einstein-wave map spacetime with target N is a triple $(M, \mathbf{g}, \Phi)$ solving

$$G_{\mu\nu} = \kappa S_{\mu\nu} \tag{1.8a}$$

$$\Box_{\mathbf{g}}\phi - \frac{k^2 f(\phi)}{r^2} = 0 \tag{1.8b}$$

where $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}R\mathbf{g}_{\mu\nu}$ is the Einstein tensor for the metric $\mathbf{g}_{\mu\nu}$, r is the radius function, and $f(\phi) = g(\phi)g'(\phi)$.

Remark 1.4.
1. *We shall throughout the paper restrict to the case when the generator ∂_θ of the S^1 action on M is hypersurface orthogonal.*
2. *See section 1.1 for the technical conditions on $g(\rho)$ which will be assumed to hold throughout the paper.*
3. *In this work we shall assume $k = 1$, however the arguments easily extend to the general case $k \in \mathbb{Z}$.*

For a Cauchy surface Σ, let T^μ be the future directed unit normal. Denote also by R the induced scalar curvature and K_{ab} the second fundamental form of the embedding, defined by $K(X, Y) = \mathbf{g}(\nabla_X T, Y)$ for vector fields X, Y tangent to Σ. It is well known that the Cauchy data for the Einstein equations (1.8a) must satisfy some compatibility conditions known as the constraint equations[7]

$$R + (K^c{}_c)^2 - K_{ab}K^{ab} = 2\kappa S_{\mu\nu}T^\mu T^\nu, \tag{1.9a}$$

$$D^c K_{ac} - D_a K^c{}_c = \kappa S_{a\nu}T^\nu, \tag{1.9b}$$

where D_a is the intrinsic covariant derivative on Σ.

We shall consider metrics $\mathbf{g}$ of the form

$$\mathbf{g} = -e^{2\alpha(t,r)}dt^2 + e^{2\beta(t,r)}dr^2 + r^2 d\theta^2. \tag{1.10}$$

It will follow from our results that this is not a restriction, see section 4.1. A calculation shows that with $\mathbf{g}$ of the form (1.10), the second fundamental form K of a Cauchy t-level set Σ is of the form

$$K = K_{rr}dr^2$$

with $K_{rr} = e^{-\alpha+2\beta}\partial_t\beta$.

Definition 1.5 (Cauchy data set for the $2+1$ equivariant Einstein-wave map system)**.** *A Cauchy data set for the $2+1$ equivariant Einstein-wave map system with target (N, h) is a 5-tuple $(\Sigma, q, K, \phi_0, \phi_1)$ consisting of*

1. *a Riemannian 2-manifold (Σ, q) with an isometric action by $e^{i\theta}$ as above and a 2-tensor K of the form $K_{rr}dr^2$ symmetric under the same action,*
2. *rotationally symmetric functions $\phi_0 : \Sigma \to \mathbb{R}_+$, $\phi_1 : \Sigma \to \mathbb{R}$,*

such that the constraint equations (1.9) hold.

[7] They correspond respectively to $G_{TT} = \kappa S_{TT}$ and $G_{Ta} = \kappa S_{Ta}$.

The proof by Choquet-Bruhat and Geroch [7] of existence and uniqueness of maximal solutions to the Cauchy problem for the vacuum Einstein equations, together with the equivariance of the Cauchy data, is readily generalized to give the following result.

Theorem 1.6 (Maximal Cauchy development for the 2+1 equivariant Einstein-wave map problem)**.** *Let* $(\Sigma, q, K, \phi_0, \phi_1)$ *be an equivariant Cauchy data set for the* $2+1$ *Einstein-wave map system. Then there is a unique, maximal Cauchy development* $(M, \mathbf{g}, \Phi)$ *satisfying the equivariant Einstein-wave-map system* (1.8)*.*

1.3. **Asymptotic flatness.** Let H^s_δ be the weighted L^2 Sobolev spaces[8] on $\mathbb{R}^2$. A 2-dimensional rotationally symmetric Cauchy data set (Σ, q, K) is asymptotically flat if

$$q = e^{2\beta} dr^2 + r^2 d\theta^2$$

with $\beta = \beta_\infty + \tilde{\beta}$ and $(\tilde{\beta}, K) \in H^{s+1}_\delta \times H^s_{\delta+1}$ for some $\delta \in (-1, 0)$. This is compatible with the setup in [15], specialized to the rotationally symmetric case. Note that the existence of such asymptotically flat solutions to the constraint equations without rotational symmetry is proved in [15] [16] (and used in [17] to prove stability in exponential time of the Minkowski space-time in this setting).

1.4. **Global existence conjecture.** A major open problem in the field of general relativity is given by the Cosmic Censorship conjectures formulated for large data solutions of the Einstein equations by Penrose in 1969 [24] (republished as [25], see also the discussion in [21]), see for example [2] for a precise statement. These fundamental conjectures are still widely open in general, but have been proved in some cases when assuming certain symmetries, see in particular the seminal proof of Christodoulou of the Cosmic Censorship conjectures for the Einstein equations coupled to a scalar field in spherical symmetry (see [8] and references therein). An intermediate goal toward the general case would be to assume the presence of only one Killing field, and prove global regularity for the 3+1 vacuum Einstein equations with a spacelike translational Killing field.

Conjecture 1.7 (Global existence for the 3+1 vacuum Einstein equations with a spacelike translational Killing field)**.** *Maximal Cauchy developments of asymptotically flat solutions to the 3+1 vacuum Einstein equations with a spacelike translational Killing field are regular and geodesically complete.*

Recall from point 2 of remark 1.1that large data global existence holds for the corresponding semilinear analog, namely the wave map problem with $M = \mathbb{R}^{2+1}$ and target $N = \mathbb{H}^2$. A proof of Conjecture 1.7 would likely require a local existence result at the critical level which seems currently out of reach[9] for quasilinear wave equations in dimensions higher than 1+1. As a first step towards Conjecture 1.7, we prove in this paper the special case of equivariant symmetry (see Remark 1.9 below).

1.5. **Large data global regularity for the equivariant Einstein-wave map problem.** We are now ready to state our main result.

[8] We here use the same conventions as Huneau [15]. In particular on $\mathbb{R}^n$, $u = o(r^{-\delta-1})$ if $u \in H^s_\delta$ for $s > n/2$.

[9] Note for instance that in the absence of symmetry, the resolution of the bounded L^2-curvature conjecture in [19] for the 3+1 Einstein vacuum equations provides a local existence result which is 1/2 derivative above the scaling.

Theorem 1.8 (Global regularity of equivariant Einstein-wave maps)**.** *Let* $(M, \mathbf{g}, \phi)$ *be the maximal Cauchy development of an asymptotically flat and regular Cauchy data set for the* $2+1$ *equivariant Einstein-wave map problem* (1.8) *with target* (N, h)*. Assume that the metric* h *has the form*

$$h = d\rho^2 + g^2(\rho)d\theta^2$$

for an odd function $g : \mathbb{R} \to \mathbb{R}$ *with* $g'(0) = 1$*. Assume that* g *satisfies*

$$\int_0^s g(s')ds' \to \infty \quad \textit{when } s \to \infty \tag{1.11}$$

and the Grillakis condition (1.5)*. Then,* $(M, \mathbf{g})$ *is regular and geodesically complete, and global regularity holds for* ϕ*.*

Remark 1.9. *As mentioned in Remark 1.2, an important motivation for studying the Einstein-wave map system arises from the fact that this system with target* $N = \mathbb{H}^2$ *arises naturally as the reduction of the 3+1 vacuum Einstein equations with a spacelike translational Killing field. In particular, Theorem 1.8 proves Conjecture 1.7 in the special case of equivariant symmetry and should be seen as the analog of the proof of the Cosmic Censorships in this setting.*

The proof of Theorem 1.8 follows, as in the semilinear analog of free wave maps on Minkowski space, from non-concentration of energy and small energy global existence. The proof of non-concentration of energy and the initial framework of the global existence problem is contained in the PhD Thesis of the second author [14]. Let us emphasize in particular the following

- The non-concentration of energy relies on the vectorfield method. Unlike the semilinear case where one relies on vectorfields of Minkowski, the vectorfields we use here have to be carefully constructed and controlled. In particular, we exhibit a vectorfield[10] T, which is not Killing but leads nevertheless to a conserved current.
- The small energy global existence relies in a fundamental way on the null structure of the equations written in null coordinates. Indeed, derivatives along outgoing null cones of ϕ as well as the metric coefficients behave better, while the null structure allows to integrate by parts derivatives along ingoing null cones such that the new terms generated behave better.

The structure of the paper is as follows. In section 2, we introduce null coordinates $(u, \underline{u})$ and a notion of mass. In section 3, we prove the absence of trapped surfaces and that the first singularity, if it exists, must lie on the axis of symmetry. In section 4, we introduce (t, r) coordinates. In section 5, we prove the non-concentration of the energy. ln section 6, we state a result on small energy global existence and use it in conjunction with non-concentration of energy to prove Theorem 1.8. The rest of the paper is then devoted to the proof of small energy global existence. In section 7 we derive a uniform weighted upper bound for ϕ. In section 8, we rely on the upper bound of section 7 to derive a uniform upper bound for $\partial\phi$. Finally, we rely on the upper bound of section 8 to prove small energy global existence in section 9.

[10] the analog of $\frac{\partial}{\partial t}$ in Minkowski.

2. Null coordinates

We assume that all objects are smooth, unless otherwise stated. In this section we introduce a null coordinate system and a notion of mass in 2+1 dimension. This setup will be used in the next section to prove that the first singularity for the critical, equivariant Einstein-wave map system occurs on the axis of rotation.

2.1. **Existence of null coordinates.**

Recall from the discussion in section 1.1 that the orbit space $(\mathcal{Q}, \breve{\mathbf{g}})$ is a 2-dimensional globally hyperbolic Lorentzian space and in particular conformally flat. Hence we may introduce a global null coordinate system with respect to which $\breve{\mathbf{g}}$ takes the form

$$\breve{\mathbf{g}} = -\Omega^2(u, \underline{u}) du d\underline{u}$$

and hence we have shown that $(M, \mathbf{g})$ admits a coordinate system $(u, \underline{u}, \theta)$ such that $\mathbf{g}$ takes the form

$$\mathbf{g} = -\Omega^2 du d\underline{u} + r^2(u, \underline{u}) d\theta^2$$

where now $d\theta^2$ is the line element on the $\mathbb{S}^1$ symmetry orbit. By redefining the coordinates $u, \underline{u}$ we may without loss of generality assume that the conditions

$$r = 0, \ \partial_{\underline{u}} r = \frac{1}{2}, \ \partial_u r = -\frac{1}{2} \text{ and } \Omega = 1 \text{ on } \Gamma. \tag{2.1}$$

are valid on the axis Γ. Also, the volume element is $\mu_{\mathbf{g}} = \Omega^2 r/2$ and the wave operator on symmetric functions (i.e. $\partial_\theta \phi = 0$) is

$$\Box_{\mathbf{g}} \phi = -\frac{2}{\Omega^2 r} \Big(\partial_u (r \partial_{\underline{u}} \phi) + \partial_{\underline{u}} (r \partial_u \phi) \Big). \tag{2.2}$$

2.2. **The stress energy tensor in null coordinates.**

The components of $S_{\alpha\beta}$ in the coordinate system $(u, \underline{u}, \theta)$ are

$$S_{uu} = \partial_u \phi \partial_u \phi, \tag{2.3a}$$
$$S_{\underline{u}\underline{u}} = \partial_{\underline{u}} \phi \partial_{\underline{u}} \phi, \tag{2.3b}$$
$$S_{u\underline{u}} = \frac{\Omega^2}{4} \frac{g^2(\phi)}{r^2}, \tag{2.3c}$$
$$S_{\theta\theta} = \frac{r^2}{2} \left(\frac{4}{\Omega^2} \partial_u \phi \partial_{\underline{u}} \phi + \frac{g^2(\phi)}{r^2} \right). \tag{2.3d}$$

The stress energy tensor satisfies the dominant energy condition since

$$S_{uu} \geq 0, \quad S_{u\underline{u}} \geq 0, \quad S_{\underline{u}\underline{u}} \geq 0. \tag{2.4}$$

2.3. **The Einstein equation.**

The components in the coordinate system $(u, \underline{u}, \theta)$ of the Einstein tensor $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R \mathbf{g}_{\mu\nu}$ are

$$G_{uu} = -\Omega^2 r^{-1} \partial_u (\Omega^{-2} \partial_u r), \tag{2.5a}$$
$$G_{\underline{u}\underline{u}} = -\Omega^2 r^{-1} \partial_{\underline{u}} (\Omega^{-2} \partial_{\underline{u}} r), \tag{2.5b}$$
$$G_{u\underline{u}} = r^{-1} \partial_u \partial_{\underline{u}} r, \tag{2.5c}$$
$$G_{\theta\theta} = 4 r^2 \Omega^{-4} (\partial_u \Omega \partial_{\underline{u}} \Omega - \Omega \partial_u \partial_{\underline{u}} \Omega). \tag{2.5d}$$

We can now write the $u, \underline{u}$ components of the Einstein equations $G_{\alpha\beta} = \kappa S_{\alpha\beta}$ in the form

$$\partial_u(\Omega^{-2}\partial_u r) = -\Omega^{-2} r\kappa S_{uu}, \tag{2.6a}$$

$$\partial_{\underline{u}}(\Omega^{-2}\partial_{\underline{u}} r) = -\Omega^{-2} r\kappa S_{\underline{uu}}, \tag{2.6b}$$

$$\partial_u \partial_{\underline{u}} r = r\kappa S_{u\underline{u}}, \tag{2.6c}$$

$$\Omega^{-2}(\partial_u \Omega \partial_{\underline{u}} \Omega - \Omega \partial_u \partial_{\underline{u}} \Omega) = \frac{1}{4} r^{-2} \Omega^2 \kappa S_{\theta\theta}. \tag{2.6d}$$

Here, the equation (2.6c) is special to $2 + 1$ dimensions.

2.4. **The mass.** Define the quantity

$$m = 1 + 4\Omega^{-2}\partial_u r \partial_{\underline{u}} r. \tag{2.7}$$

Remark 2.1. *The quantity m defined by* (2.7) *has a form related to the Hawking mass in 3+1 dimensional spherically symmetric gravity. In 3+1 dimensions and spherical symmetry, the Hawking mass m_H is given by*

$$m_H = \frac{r}{2}(1 + 4\Omega^{-2}\partial_u r \partial_{\underline{u}} r).$$

In $2 + 1$ dimensions, the quantity m defined in (2.7), when evaluated at infinity, is a function of the mass defined by Ashetkar and Varadarajan [3]*.*

Lemma 2.2. *The quantity m admits a limit along any a space like asymptotically flat curve, which does not depend on the particular curve. We denote this limit by m_∞. We have furthermore*

$$m_\infty \in [0, 1).$$

Proof. See [15] for the proof of this lemma, where m_∞ is called the deficit angle. □

The mass m satisfies the following equations which are analogous to the ones satisfied by the Hawking mass in the $3 + 1$ dimensional case,

$$\partial_u m = 4\kappa\Omega^{-2} r(S_{u\underline{u}}\partial_u r - S_{uu}\partial_{\underline{u}} r) \tag{2.8a}$$

$$\partial_{\underline{u}} m = 4\kappa\Omega^{-2} r(S_{u\underline{u}}\partial_{\underline{u}} r - S_{\underline{uu}}\partial_u r) \tag{2.8b}$$

3. The first singularity occurs on the axis

3.1. **Absence of trapped surfaces.** Following Dafermos [11], we define the regions

$$\mathcal{R} = \{p \in \mathcal{Q} \text{ such that } \partial_{\underline{u}} r > 0, \quad \partial_u r < 0\},$$
$$\mathcal{T} = \{p \in \mathcal{Q} \text{ such that } \partial_{\underline{u}} r < 0, \quad \partial_u r < 0\},$$
$$\mathcal{A} = \{p \in \mathcal{Q} \text{ such that } \partial_{\underline{u}} r = 0, \quad \partial_u r < 0\}.$$

Then $\mathcal{R}, \mathcal{T}, \mathcal{A}$ are the non-trapped (or regular), trapped and marginally trapped regions, respectively. Due to work of Ida [18], one expects that in a 2+1 dimensional spacetime satisfying the dominant energy condition, trapped or marginally trapped surfaces occur only in exceptional cases. In fact, as shown by Galloway et al. [12] a 2+1 dimensional spacetime satisfying the dominant energy condition and a mild asymptotic condition, weaker than asymptotic flatness, cannot contain any marginally trapped surfaces. We give below a direct proof that in the case under consideration, there are no trapped or marginally trapped surfaces.

Theorem 3.1 (Absence of trapped surfaces)**.** *We have*

(1) $\mathcal{Q} = \mathcal{R}$

(2) For $q \in \mathcal{Q}$*,*

$$0 \leq m(q) < m_\infty < 1. \tag{3.1}$$

In particular, the spacetimes under consideration contain no trapped or marginally trapped surfaces, i.e. $\mathcal{T} = \emptyset$*,* $\mathcal{A} = \emptyset$*.*

Proof. Let $\check{\Sigma}$ be a Cauchy curve in $\mathcal{Q}$. Note that each $p \in \mathcal{Q}$ is on such a Cauchy curve. Let s be a coordinate on $\check{\Sigma}$ and let $x(s)$ be the point in $\check{\Sigma}$ with coordinate value s. We may without loss of generality assume $\check{\Sigma}$ has one endpoint $x(0)$ on Γ corresponding to $s = 0$ and an "asymptotically flat" end corresponding to $s \to \infty$ so that the coordinate s takes values in $[0, \infty)$. By our normalizations, see (2.1), we have $m(x(0)) = 0$.

Now $V = \partial_s$ is a vectorfield tangent to $\check{\Sigma}$ and in particular is spatial. Therefore, since V points towards increasing values of s, $V = a\partial_{\underline{u}} - b\partial_u$ for positive functions a, b. It follows from the assumption of asymptotic flatness that $x(s)$ is contained in $\mathcal{R}$ for s large enough. Due to the dominant energy condition, see (2.4), and equations (2.8), we have

$$Vm \geq 0 \tag{3.2}$$

in the regular region $\mathcal{R}$. Now consider a point $q \in \check{\Sigma} \cap \partial\mathcal{R}$, where $\partial\mathcal{R}$ denotes the boundary of $\mathcal{R}$. At such a point, one of the equations $\partial_u r = 0$ or $\partial_{\underline{u}} r = 0$ holds, and hence $m(q) = 1$. Due to asymptotic flatness, $\lim_{s\to\infty} m(x(s)) = m_\infty \in [0, 1)$. Hence due to the monotonicity of m, see (3.2), we get a contradiction from $m(q) = 1$. Therefore $\check{\Sigma} \cap \partial\mathcal{R} = \emptyset$. This argument also shows that $\check{\Sigma} \subset \mathcal{R}$. Also, since $m(x(0)) = 0$ with $x(0) = \check{\Sigma} \cap \Gamma$, then we have $0 \leq m < 1$ on $\check{\Sigma}$.

The properties of the mass discussed above, together with the fact that each point of $\mathcal{Q}$ is on a Cauchy curve, and the maximality of $\mathcal{Q}$, allow us to conclude the proof of the theorem. □

3.2. **First singularities.** We now restrict our consideration to the future $\mathcal{Q}^+$ of $\check{\Sigma}$. Due to Theorem 3.1, $\mathcal{Q}^+ = \mathcal{Q} \cap J^+(\check{\Sigma})$. We now introduce some notions following Dafermos [11, 10].

Definition 3.2. *Let* $p \in \overline{\mathcal{Q}^+}$*. The indecomposable past subset* $J^-(p) \cap \mathcal{Q}^+$ *is said to be* **eventually compactly generated** *if there is a compact subset* $\mathcal{X} \subset \mathcal{Q}^+$ *such that*

$$J^-(p) \cap \mathcal{Q}^+ \subset D^+(\mathcal{X}) \cup J^-(\mathcal{X}). \tag{3.3}$$

We will say that in this situation $\mathcal{X}$ *generates* $J^-(p) \cap \mathcal{Q}^+$*.*

Definition 3.3. *A point* $p \in \overline{\mathcal{Q}^+} \setminus \mathcal{Q}^+$ *is said to be a* **first singularity** *if* $J^-(p) \cap \mathcal{Q}^+$ *is eventually compactly generated and if any eventually compactly generated indecomposable subset of* $J^-(p) \cap \mathcal{Q}^+$ *is of the form* $J^-(q) \cap \mathcal{Q}^+$ *for some* $q \in \mathcal{Q}^+$*.*

We will now state an extension criterion, which is a direct consequence of the well posedness of the characteristic initial value problem, see [10, Prop. 1.1]. To state this we need to introduce for a subset $Y \subset \mathcal{Q}^+ \setminus \Gamma$, the quantity $N(Y)$,

$$N(Y) = \sup\{|\Omega|_1, |\Omega^{-1}|_0, |r|_2, |r^{-1}|_0, |\phi|_1\} \tag{3.4}$$

where $|f|_k = \max(|f|_{C^k(u)}, |f|_{C^k(v)})$.

We can now state the extension criterion

Proposition 3.4 ([10, Property 1.1])**.** *Let* $p \in \overline{\mathcal{Q}^+} \setminus \overline{\Gamma}$ *be a first singularity. Then, for any compact* $\mathcal{X} \subset \mathcal{Q}^+ \setminus \Gamma$*, generating* $J^-(p)$*, i.e. which satisfies* (3.3)*, we have*

$$N(D^+(\mathcal{X}) \setminus \{p\}) = \infty.$$

The following theorem states that the first singularity occurs on the axis.

Theorem 3.5 (The first singularity occurs on the axis)**.** *Let* $p \in \overline{\mathcal{Q}^+} \setminus \mathcal{Q}^+$ *be a first singularity. Then* $p \in \overline{\Gamma} \setminus \Gamma$*.*

Remark 3.6. *Theorem 3.5 should be compared to* [10, Theorem 3.1]*, which states that a first singularity occurs either on the axis or has a trapped surface in its past.*

Proof. Let us introduce the notations

$$\nu := \partial_u r, \tag{3.5}$$

$$\lambda := \partial_{\underline{u}} r, \tag{3.6}$$

$$\zeta := r\partial_u \phi, \tag{3.7}$$

$$\vartheta := r\partial_{\underline{u}} \phi, \tag{3.8}$$

$$\varkappa := -\frac{1}{4}\Omega^2 \nu^{-1}. \tag{3.9}$$

In the present, 2+1 dimensional case, m is given by (2.7), which using the above notation takes the form

$$m = 1 + 4\Omega^{-2}\nu\lambda.$$

Note that we have by Theorem 3.1 $m < 1$ and also, since $\mathcal{Q}^+ \subset \mathcal{R}$, it holds that $\nu < 0$, $\lambda > 0$, $\varkappa > 0$ everywhere in $\mathcal{Q}^+$. Further, note that from the definitions $r > 0$ in $\mathcal{Q}^+ \setminus \Gamma$. We may assume without loss of generality that $\mathcal{X} \subset \mathcal{Q}^+ \setminus \Gamma$. If $p = (u_s, \underline{u}_s)$ denotes first singularity, we may further assume that $\mathcal{X}$ is given by

$$\mathcal{X} = \Big(\{u_1\} \times [\underline{u}_1, \underline{u}_s]\Big) \cup \Big([u_1, u_s] \times \{\underline{u}_1\}\Big)$$

where $u_0 < u_s$, $\underline{u}_0 < \underline{u}_s$ and $u_s < \underline{u}_0$ to ensure $\mathcal{X} \subset \mathcal{Q}^+ \setminus \Gamma$. Note that we have

$$[u_0, u_s] \times [\underline{u}_0, \underline{u}_s] = D^+(\mathcal{X}) = J^-(p) \cap D^+(\mathcal{X}).$$

In view of the compactness of $\mathcal{X}$ the following bounds hold on $\mathcal{X}$,

$$0 < r_0 \le r \le R \tag{3.10a}$$

$$0 < \lambda \le \Lambda, \qquad 0 > \nu \ge -N \tag{3.10b}$$

$$|\phi| \le P, \qquad |\vartheta| \le \Theta, \qquad |\zeta| \le Z$$

$$0 < \varkappa \le K \tag{3.10c}$$

$$|\partial_u \Omega| \le H \qquad |\partial_{\underline{u}} \Omega| \le H$$

$$|\partial_u \nu| \le H \qquad |\partial_{\underline{u}} \lambda| \le H$$

Equation (2.6a) yields

$$\partial_u \varkappa = \kappa \frac{1}{r}\left(\frac{\zeta}{\nu}\right)^2 \nu\varkappa \tag{3.11}$$

Due to (3.11) and $\nu < 0$, it follows that inequality (3.10c) holds in all of $D^+(\mathcal{X})\setminus\{p\}$. Since $\nu < 0$, $\lambda > 0$, it follows that inequality (3.10a) holds throughout $D^+(\mathcal{X})\setminus\{p\}$.

Now consider $p^* = (u^*, \underline{u}^*) \in D^+(\mathcal{X}) \setminus \{p\}$. The past null curves starting at p^* intersect $\mathcal{X}$ at $(u^*, \underline{u}_0)$ and $(u_0, \underline{u}^*)$, respectively, see figure 1. Integrating (3.8) yields

$$\begin{aligned} |(\phi(u^*, \underline{u}^*)| &\le |\phi(u^*, \underline{u}_0)| + \left| \int_{\underline{u}_0}^{\underline{u}^*} \frac{\vartheta}{r}(u^*, \underline{u}) d\underline{u} \right| \\ &\le P + \sqrt{\int_{\underline{u}_0}^{\underline{u}^*} \frac{\vartheta^2}{r\varkappa} d\underline{u}} \sqrt{\int_{\underline{u}_0}^{\underline{u}^*} \frac{\varkappa}{r} d\underline{u}}. \end{aligned} \tag{3.12}$$

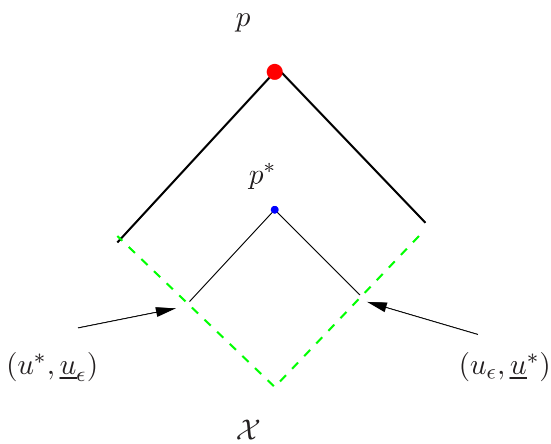

Figure 1. $D^+(\mathcal{X})$

Equation (2.8b) gives, using the present notation

$$\partial_{\underline{u}} m = \kappa \left(\frac{g^2(\phi)}{r} \lambda + \frac{\vartheta^2}{r\varkappa} \right).$$

In view of $\lambda > 0$, this gives, integrating along the same null curve as above,

$$\begin{aligned} \int_{\underline{u}_0}^{\underline{u}^*} \frac{\vartheta^2}{r\varkappa} d\underline{u} &\leq \frac{1}{\kappa} \left(m(u^*, \underline{u}^*) - m(u^*, \underline{u}_\epsilon) \right) \\ &\leq \frac{1}{\kappa} \end{aligned} \tag{3.13}$$

where we used (3.1). We can now use the inequality (3.13) together with the previous estimates of $\varkappa$ and r and (3.12) to show that ϕ is uniformly bounded in $D^+(\mathcal{X}) \setminus \{p\}$.

We next estimate λ and ν. First, use the relation

$$\varkappa(1 - m) = \lambda$$

and the previous estimates for $m, \varkappa$, to get the inequality $0 < \lambda < K$ on $D^+(\mathcal{X}) \setminus \{p\}$. In order to estimate ν, recall that $\nu < 0$ on $\mathcal{Q}$ by theorem 3.1. Next, note that in view of (2.6c) and (2.4) we have $\partial_{\underline{u}} \nu > 0$ and hence integrating as above gives

$$\nu(u^*, \underline{u}_\epsilon) < \nu(u^*, \underline{u}^*) < 0.$$

This means that the inequalities (3.10b) hold on $D^+(\mathcal{X}) \setminus \{p\}$.

From the definition of $\varkappa$, cf. (3.9), we have

$$\Omega^2 = -4\nu\varkappa$$

which in view of the above estimates gives

$$\Omega^2 \leq 4NK \quad \text{on } D^+(\mathcal{X}) \setminus \{p\}. \tag{3.14}$$

To estimate the first derivative of ϕ, we write (1.8) in the form

$$\partial_u \vartheta = \frac{1}{2} r^{-1} \nu \vartheta - \frac{1}{2} r^{-1} \lambda \zeta + \varkappa \nu \frac{f(\phi)}{r} \tag{3.15}$$

$$\partial_{\underline{u}} \zeta = \frac{1}{2} r^{-1} \lambda \zeta - \frac{1}{2} r^{-1} \nu \vartheta + \varkappa \nu \frac{f(\phi)}{r} \tag{3.16}$$

Integrating these relations as above yields uniform bounds for ϑ, ζ in $D^+(\mathcal{X}) \setminus \{p\}$.

Next, observe that (2.6d) takes the form

$$-\partial_u \partial_{\underline{u}} \log(\Omega) = \frac{1}{8} r^{-2} \kappa \left(4\vartheta\zeta + \Omega^2 g^2(\phi)\right) \tag{3.17}$$

in the current notation. The right hand side of (3.17) is uniformly bounded on $D^+(\mathcal{X}) \setminus \{p\}$ by the above estimates. Integrating as above along curves the null curves $\{(u, \underline{u}^*), u_\epsilon \leq u \leq u^*\}$ and $\{(u^*, \underline{u}), \underline{u}_\epsilon < \underline{u} < \underline{u}^*\}$ yields uniform bounds on $\partial_u \log(\Omega)$ and $\partial_{\underline{u}}\Omega$ on $D^+(\mathcal{X}) \setminus \{p\}$, and hence in view of (3.14) also on $\partial_u \Omega$ and $\partial_{\underline{u}}\Omega$. A second integration of $\partial_u \log(\Omega)$ or $\partial_{\underline{u}} \log(\Omega)$ allows us to give a uniform bound on $|\log(\Omega)|$, and hence also on $|\Omega^{-1}|$, in $D^+(\mathcal{X}) \setminus \{p\}$.

Now we have uniform bounds in $D^+(\mathcal{X}) \setminus \{p\}$ for the quantities $|r^{-1}|, |\Omega^{-1}|$, $|\partial_u r|$, $|\partial_{\underline{u}} r|$, $|\phi|$, $|\partial_u \phi|$, $|\partial_{\underline{u}}\phi|$, $|\partial_u \Omega|$, $|\partial_{\underline{u}}\Omega|$. A bound on $|\partial_u \partial_{\underline{u}} r|$ follows in view of these estimates directly from (2.6c). It remains only to estimate $\partial_u \partial_u r = \partial_u \nu$ and $\partial_{\underline{u}} \partial_{\underline{u}} r = \partial_{\underline{u}} \lambda$. In order to do this, we can use equations (2.6a) and (2.6b) since all occuring terms are bounded by our previous estimates.

This completes the proof that if p is a first singularity in $\overline{\mathcal{Q}^+} \setminus \overline{\Gamma}$, we have $N(D^+(\mathcal{X}) \setminus \{p\}) < \infty$ which by proposition 3.4 gives a contradiction. This shows that every first singularity occurs in $\overline{\Gamma} \setminus \Gamma$, i.e. on the axis, and hence concludes the proof of Theorem 3.5. □

4. (t, r) COORDINATES

4.1. **Construction of (t, r) coordinates.** Let $(M, \mathbf{g}, \phi)$ be the maximal Cauchy development of an asymptotically flat Cauchy data set for the Einstein-wave map problem. Let $\Gamma = \{r = 0\}$ be the axis of rotation in M. If Γ is incomplete to the future, we let p_Γ be the first singularity.

Lemma 4.1. *Let t be the parameter on Γ such that $\dot{\Gamma} = d\Gamma/dt$ satisfies*

$$\mathbf{g}_{\alpha\beta} \dot{\Gamma}^\alpha \dot{\Gamma}^\beta = -1 \text{ for } t < 0 \text{ and } \lim_{t \nearrow 0} \Gamma(t) = p_\Gamma.$$

Extend t to be constant on the maximal orbit $\check{\Sigma}_t$ of ∇r starting at $\Gamma(t) \in \Gamma \cap \mathcal{R}$. Then, (t, r) is a regular coordinates system on $\cup_{t<0} \check{\Sigma}_t$ and

$$\check{\mathbf{g}} = -e^{2\alpha} dt^2 + e^{2\beta} dr^2$$

for some functions $\alpha = \alpha(t, r)$, $\beta = \beta(t, r)$. Furthermore, we have

$$\alpha = \beta = 0 \text{ on } \Gamma.$$

Proof. Recall that the radius function r is well-defined and smooth on the regular part of M and hence also on $\mathcal{Q}$. Let ∇r be the gradient field of r on $(\mathcal{Q}, \check{\mathbf{g}})$. We have

$$\nabla r = -2\Omega^{-2}(\partial_u r \partial_{\underline{u}} + \partial_{\underline{u}} r \partial_u)$$

and

$$\check{\mathbf{g}}(\nabla r, \nabla r) = -4\Omega^{-2} \partial_u r \partial_{\underline{u}} r.$$

This means that

$$\check{\mathbf{g}}(\nabla r, \nabla r) = 1 - m$$

where m is the mass as defined in section 2.4. In view of (3.1), we have that $m \in [0, 1)$ in $\mathcal{Q}$. Thus we have $\check{\mathbf{g}}(\nabla r, \nabla r) > 0$ on $\mathcal{Q}$.

Consider a maximal orbit $\check{\Sigma}_t$ of ∇r starting at some point $\Gamma(t) \in \Gamma \cap \mathcal{R}$. Since $\check{\mathbf{g}}(\nabla r, \nabla r) > 0$ on $\mathcal{Q}$, the radius function r is a parametrization of $\check{\Sigma}$. By Cauchy stability for the ODE

$$\frac{dx}{dr} = \nabla r, \tag{4.1}$$

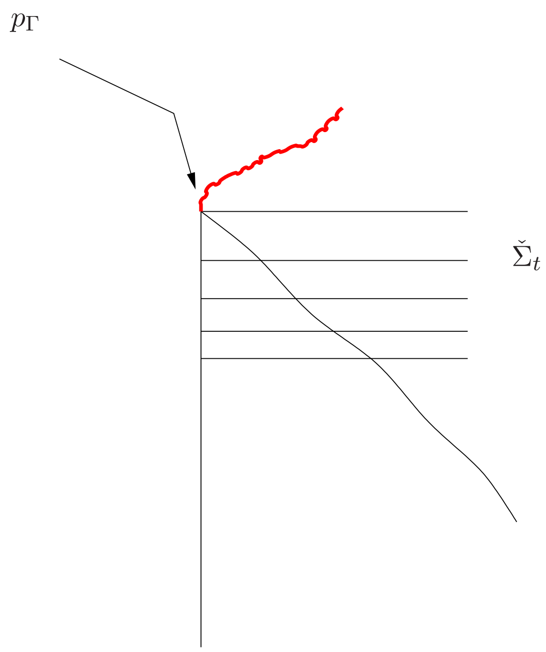

FIGURE 2. The $\check{\Sigma}_t$ foliation

we have that $\check{\Sigma}_t$ defines a foliation in $\mathcal{Q}$. This foliation does not cover all of $\mathcal{Q}$, but the domain of the foliation includes the past domain of influence of the first singularity, cf. figure 2. Let $\tilde{\mathcal{Q}}$ denote the domain of the foliation.

We can now extend the coordinate t from the axis Γ to the domain of the $\check{\Sigma}_t$ foliation. This defines a function t on $\tilde{\mathcal{Q}}$. Recall that the $\check{\Sigma}_t$ are orbits of a vector field ∇r on $\mathcal{Q}$. By uniqueness for (4.1) we have that the function t has non-vanishing gradient. Furthermore, we have by construction that $\check{\mathbf{g}}(\nabla t, \nabla r) = 0$ on the domain of the time foliation. Together with the fact that t has non-vanishing gradient and $\check{\mathbf{g}}(\nabla r, \nabla r) > 0$ on $\mathcal{Q}$, we infer

$$\check{\mathbf{g}}(\nabla t, \nabla r) = 0, \ \check{\mathbf{g}}(\nabla t, \nabla t) < 0, \ \check{\mathbf{g}}(\nabla r, \nabla r) > 0 \text{ on } \cup_{t<0} \check{\Sigma}_t.$$

It follows that (t, r) as coordinate functions on the domain of the foliation, and in this coordinate system, we have

$$\check{\mathbf{g}} = -e^{2\alpha} dt^2 + e^{2\beta} dr^2$$

where $\alpha = -\log(-\check{\mathbf{g}}(\nabla t, \nabla t))/2$ and $\beta(t, r) = -\log(\check{\mathbf{g}}(\nabla r, \nabla r))/2$. Furthermore, note that in view of our choice for t in Γ and the fact that $\check{\mathbf{g}}(\nabla t, \nabla r) = 0$, we have $\check{\mathbf{g}}(\nabla t, \nabla t) = -1$ on Γ. Also, we have $\check{\mathbf{g}}(\nabla r, \nabla r) = 1$ on Γ. We infer that

$$\alpha = \beta = 0 \text{ on } \Gamma.$$

This concludes the proof of the lemma. □

The above construction lifts to $(M, \mathbf{g})$ to give a foliation Σ_t. We denote the domain of this foliation $\tilde{M}$. On $\tilde{M}$ we have coordinates $(x^\alpha) = (t, r, \theta)$, and the metric $\mathbf{g}$ takes the form

$$\mathbf{g} = -e^{2\alpha} dt^2 + e^{2\beta} dr^2 + r^2 d\theta^2. \tag{4.2}$$

4.2. **Einstein Tensor.** The components in the polar coordinates (t, r, θ) of the Einstein tensor $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}R\mathbf{g}_{\mu\nu}$ are

$$\begin{aligned}
G_{tt} &= e^{2(\alpha-\beta)}\beta_r r^{-1},\\
G_{tr} &= \beta_t r^{-1},\\
G_{rr} &= \alpha_r r^{-1},\\
G_{\theta\theta} &= r^2\big(e^{-2\beta}(-\beta_r\alpha_r + \alpha_r^2 + \alpha_{rr}) - e^{-2\alpha}(\beta_t^2 - \beta_t\alpha_t + \beta_{tt})\big),\\
G_{t\theta} &= 0,\\
G_{r\theta} &= 0.
\end{aligned}$$

4.3. **Stress-energy Tensor.** Recall that the energy-momentum tensor $S(\Phi)$ for a wave map $\Phi : (M, g) \to (N, h)$ is as follows

$$S_{\mu\nu}(\Phi) := \langle \partial_\mu \Phi, \partial_\nu \Phi\rangle_{h(\Phi)} - \frac{1}{2}g_{\mu\nu}\langle \partial^\sigma \Phi, \partial_\sigma \Phi\rangle_{h(\Phi)}, \tag{4.3}$$

where μ,ν,$\sigma = 0, 1, 2$. In the following we will calculate each of the components of the energy momentum tensor in (t, r, θ) coordinates. Note,

$$\langle \partial^\sigma U, \partial_\sigma U\rangle_{h(\Phi)} = -e^{-2\alpha}\phi_t^2 + e^{-2\beta}\phi_r^2 + \frac{g^2(\phi)}{r^2}. \tag{4.4}$$

Now we proceed to calculate $S_{\mu\nu}$

$$\begin{aligned}
S_{tt} &= \frac{1}{2}e^{2\alpha}\left(e^{-2\alpha}\phi_t^2 + e^{-2\beta}\phi_r^2 + \frac{g^2(\phi)}{r^2}\right),\\
S_{tr} &= \phi_t\phi_r,\\
S_{rr} &= \frac{1}{2}e^{2\beta}\left(e^{-2\alpha}\phi_t^2 + e^{-2\beta}\phi_r^2 - \frac{g^2(\phi)}{r^2}\right),\\
S_{\theta\theta} &= \frac{1}{2}r^2\left(e^{-2\alpha}\phi_t^2 - e^{-2\beta}\phi_r^2 + \frac{g^2(\phi)}{r^2}\right),\\
S_{t\theta} &= 0,\\
S_{r\theta} &= 0.
\end{aligned}$$

Let T and R be the normalization of ∂_t and ∂_r

$$T := e^{-\alpha}\partial_t \text{ and } R := e^{-\beta}\partial_r.$$

We define the energy density $\mathbf{e} := S(T, T)$ and momentum density $\mathbf{m} := S(T, R)$

$$\begin{aligned}
\mathbf{e} &= \frac{1}{2}\left(e^{-2\alpha}\,\phi_t^2 + e^{-2\beta}\,\phi_r^2 + \frac{g^2(\phi)}{r^2}\right)\\
&= \frac{1}{2}\left((T(\phi))^2 + (R(\phi))^2 + \frac{g^2(\phi)}{r^2}\right)\\
\mathbf{m} &= e^{-(\alpha+\beta)}\phi_t\,\phi_r\\
&= T(\phi)\,R(\phi).
\end{aligned}$$

We further define

$$\mathbf{e_0} := (T(\phi))^2 + (R(\phi))^2\,, \quad \mathbf{f} := \frac{g^2(\phi)}{r^2}.$$

4.4. **Einstein equivariant wave map system of equations.** Using the above expressions for $G_{\mu\nu}$ and $S_{\mu\nu}$ we have the system of equations

$$\beta_r = \frac{1}{2} r\,\kappa\, e^{2\beta} \left(e^{-2\alpha}\,\phi_t^2 + e^{-2\beta}\,\phi_r^2 + \frac{g^2(\phi)}{r^2} \right), \tag{4.5a}$$

$$\beta_t = r\,\kappa\,\phi_t\,\phi_r, \tag{4.5b}$$

$$\alpha_r = \frac{1}{2} r\,\kappa\, e^{2\beta} \left(e^{-2\alpha}\,\phi_t^2 + e^{-2\beta}\,\phi_r^2 - \frac{g^2(\phi)}{r^2} \right), \tag{4.5c}$$

$$\Box_{\mathbf{g}}\phi = \frac{g'(\phi)g(\phi)}{r^2}, \tag{4.5d}$$

where

$$\Box_{\mathbf{g}}\phi = -e^{-2\alpha}(\phi_{tt} + (\beta_t - \alpha_t)\phi_t) + e^{-2\beta}\left(\phi_{rr} + \frac{\phi_r}{r} + (\alpha_r - \beta_r)\phi_r\right).$$

We remark that the full system (1.8) yields some redundant equations. The system (4.5) is a subset containing the equations which are relevant for our purposes.

5. Non-concentration of energy

Let us define the energy on a Cauchy surface Σ_t

$$E(\Phi)(t) := \int_{\Sigma_t} \mathbf{e}\,\bar{\mu}_q = 2\pi \int_0^\infty \mathbf{e}(t,r) r e^{\beta(t,r)}\, dr\,,$$

the energy in a coordinate ball B_r

$$E(\Phi)(t,r) := \int_{B_r} \mathbf{e}\,\bar{\mu}_q\,, = 2\pi \int_0^r \mathbf{e}(t,r') r' e^{\beta(t,r')}\, dr'$$

the energy inside the causal past $J^-(O)$ of O

$$E^O(t) := \int_{\Sigma_t \cap J^-(O)} \mathbf{e}\,\bar{\mu}_q\,.$$

The goal of this section is to prove the following result.

Theorem 5.1 (Non-concentration of energy)**.** *Let $(M, \mathbf{g}, \Phi)$ be a smooth, globally hyperbolic, equivariant maximal development of smooth, compactly supported equivariant initial data set $(\Sigma, q, K, \Phi_0, \Phi_1)$ with finite initial energy and satisfying the constraint equations, and let (N, h) be a rotationally symmetric, complete, connected Riemannian manifold satisfying the Grillakis condition* (1.5) *as well as* (1.11)*. Then the energy of the Einstein-wave map system* (1.8) *cannot concentrate, i.e., $E^O(t) \to 0$, where O is the first (hypothetical) singularity of M.*

5.1. **Energy conservation.** We start by proving the energy is conserved.

Lemma 5.2. *The energy $E(\phi)(t)$ is conserved.*

Proof. Consider two Cauchy surfaces Σ_s and Σ_τ at $t = s$ and $t = \tau$ respectively, with $-1 \le \tau \le s < 0$. The asymptotically flat initial data ensures that each Σ_t is asymptotically flat and each component of $S_{\mu\nu} \to 0$ as $r \to \infty$. We shall now construct a divergence free vector field P_T as follows. Consider the Einstein's equations (4.5a) and (4.5b). They can be rewritten as follows

$$-\partial_r \left(e^{-\beta}\right) = r\,\kappa e^{\beta}\mathbf{e},$$
$$-\partial_t \left(e^{-\beta}\right) = r\,\kappa e^{\alpha}\mathbf{m}.$$

From the smoothness of β we have $-\partial^2_{rt}\left(e^{-\beta}\right) = -\partial^2_{tr}\left(e^{-\beta}\right)$, which implies

$$-\partial_t \left(re^{\beta}\mathbf{e}\right) + \partial_r(re^{\alpha}\mathbf{m}) = 0. \tag{5.1}$$

Now define a vectorfield

$$P_T := -e^{-\alpha}\,\mathbf{e}\,\partial_t + e^{-\beta}\,\mathbf{m}\,\partial_r,$$

then the divergence of P_T is given by

$$\begin{aligned} \nabla_\nu P^\nu_T &= \frac{1}{\sqrt{|\mathbf{g}|}}\,\partial_\nu\left(\sqrt{|\mathbf{g}|}\,P^\nu_T\right) \\ &= \frac{1}{re^{\beta+\alpha}}\left(-\partial_t\left(re^{\beta}\,\mathbf{e}\right) + \partial_r\left(re^{\alpha}\,\mathbf{m}\,\right)\right) \\ &= 0 \end{aligned} \tag{5.2}$$

from (5.1). Now let us apply the Stokes' theorem in the space-time region whose boundary is $\Sigma_s \cup \Sigma_\tau$, then we have

$$0 = \int_{\Sigma_s} e^{\alpha} P^t_T \bar{\mu}_q - \int_{\Sigma_\tau} e^{\alpha} P^t_T \bar{\mu}_q. \tag{5.3}$$

Therefore, it follows that

$$E(\phi)(\tau) = E(\phi)(s) \tag{5.4}$$

for any τ, s such that $-1 \le \tau \le s < 0$. □

In the following lemma we shall prove that the metric functions $\beta(t, r)$ and $\alpha(t, r)$ are uniformly bounded during the evolution of the Einstein-wave map system.

Lemma 5.3. *There exist constants $c^-_\beta, c^+_\beta, c^-_\alpha, c^+_\alpha$ depending only on the initial data and the universal constants such that the following uniform bounds on the metric functions $\beta(t, r)$ and $\alpha(t, r)$ hold*

$$c^-_\beta \le \beta(t, r) \le c^+_\beta,$$
$$c^-_\alpha \le \alpha(t, r) \le c^+_\alpha.$$

Proof. For simplicity of notation, we use a generic constant c for the estimates on $\beta(t, r)$ and $\alpha(t, r)$. The Einstein equation (4.5a) for β_r can be rewritten as

$$-(e^{-\beta})_r = \kappa\, r\, e^{\beta}\mathbf{e}.$$

Integrating with respect to r and recalling that $\beta_{|\Gamma} = 0$, we get

$$1 - e^{-\beta} = \kappa \int_0^r \mathbf{e}\, r'\, e^{\beta} d\,r' = \frac{\kappa}{2\pi} E(\phi)(t, r)$$

so,

$$e^{\beta} = \left(1 - \frac{\kappa}{2\pi}E(\phi)(t, r)\right)^{-1}.$$

Let us introduce the notation $\beta_\infty(t) = \lim_{r\to\infty}\beta(r, t)$. Then we have

$$e^{\beta_\infty(t)} = \left(1 - \frac{\kappa}{2\pi}E(\phi)(t)\right)^{-1}.$$

Since $E(\phi)(t,r)$ is a nondecreasing function of r, then so is $\beta(t,r)$

$$1 = e^{\beta(t,0)} \leq e^{\beta(t,r)} \leq e^{\beta_\infty(t)}.$$

Furthermore, since the energy is conserved $E(\phi)(t) = E(\phi)(-1)$, $\beta_\infty(t) = \beta_\infty(-1)$ is also conserved during the evolution of the Einstein wave map system and hence

$$0 \leq \beta(t,r) \leq \beta_\infty(-1).$$

Similarly let us consider the Einstein's equation (4.5c) for α_r

$$\alpha_r = r\,\kappa\, e^{2\beta}(\mathbf{e} - \mathbf{f}).$$

Integrating with respect to r and recalling that $\alpha_{|\Gamma} = 0$, we get

$$\begin{aligned}\alpha(t,r) &\leq c\int_0^r (\mathbf{e} - \mathbf{f}) r e^{\beta} d\,r\\ &\leq c\int_0^r \mathbf{e}\, r\, e^{\beta} d\,r\\ &\leq c\end{aligned}$$

and

$$\begin{aligned}\alpha(t,r) &\geq -c\int_0^r \frac{\mathbf{f}}{2} r e^{\beta}\, d\,r\\ &\geq -c\int_0^r \mathbf{e}\, r\, e^{\beta}\, d\,r\\ &\geq -c.\end{aligned}$$

This concludes the proof of the lemma. □

Lemma 5.4. *Assume that the target manifold (N,h) satisfies*

$$\wp(\phi) := \int_0^{\phi} g(s)\,ds \to \infty \quad as \quad \phi \to \infty, \tag{5.5}$$

then there exists a constant c dependent only on the initial data and the universal constants such that

$$\phi \in L^\infty \quad with \quad \|\phi\|_\infty \leq c$$

for every solution ϕ of the equivariant wave map equation.

Proof. Extending the technique used in Lemma 8.1 in [29], we consider

$$\begin{aligned}\wp(\phi(t,r)) &= \int_0^r \partial_r(\wp(\phi(t,r)))\,dr\\ &= \int_0^r g(\phi)\partial_r\phi\,dr\\ &= \int_0^r \left(g(\phi)(re^{-\beta})^{-1/2}\right)\left(\partial_r\phi(re^{-\beta})^{1/2}\right)\,dr.\end{aligned}$$

Consequently,

$$\begin{aligned}|\wp(\phi(t,r))| &\leq \left(\int_0^r (g(\phi))^2(re^{-\beta})^{-1}\,dr\right)^{1/2}\left(\int_0^r (\partial_r\phi)^2 re^{-\beta}\,dr\right)^{1/2}\\ &\leq \left(\int_0^\infty \frac{g(\phi)^2}{r^2} re^{\beta}\,dr\right)^{1/2}\left(\int_0^\infty e^{-2\beta}(\partial_r\phi)^2 re^{\beta}\,dr\right)^{1/2}\\ &\leq c(E_0).\end{aligned}$$

Arguing via contradiction, the result follows. □

5.2. **The vectorfield method.** Let X be a space-time vectorfield. The corresponding momentum P_X is given by the contraction of S with X i.e.,

$$P^{\mu}_{X} = S^{\mu}{}_{\nu} X^{\nu}. \tag{5.6}$$

We have,

$$\nabla_{\nu} P^{\nu}_{X} = X^{\mu}\, \nabla_{\nu} S^{\nu}{}_{\mu} + S^{\nu}{}_{\mu}\, \nabla_{\nu} X^{\mu}. \tag{5.7}$$

Since the stress energy tensor S is divergence free, the first term in the right hand side of (5.7) drops out, therefore

$$\begin{aligned}\nabla_{\nu} P^{\nu}_{X} =& S^{\mu\nu}\, \nabla_{\mu} X_{\nu} \\ =& \frac{1}{2}\, {}^{(X)}\boldsymbol{\pi}_{\mu\nu} S^{\mu\nu},\end{aligned}$$

where the deformation tensor ${}^{(X)}\boldsymbol{\pi}_{\mu\nu}$ is given by

$$\begin{aligned}{}^{(X)}\boldsymbol{\pi}_{\mu\nu} :=& \nabla_{\mu} X_{\nu} + \nabla_{\nu} X_{\mu} \\ =& \mathbf{g}_{\sigma\nu} \partial_{\mu} X^{\sigma} + \mathbf{g}_{\sigma\mu} \partial_{\nu} X^{\sigma} + X^{\sigma} \partial_{\sigma} \mathbf{g}_{\mu\nu}.\end{aligned}$$

Construction of useful identities using suitably chosen multipliers X and Stokes' theorem is central to our method to prove non-concentration of energy of equivariant Einstein-wave maps. In the following let us calculate the divergence of P_X for various choices of X.
Consider $T = e^{-\alpha}\partial_t$. The corresponding momentum P_T is

$$P_T = -\, e^{-\alpha} \mathbf{e}\, \partial_t + e^{-\beta} \mathbf{m}\, \partial_r. \tag{5.8}$$

Then, we have,

$$\begin{aligned}\nabla_{\nu} P^{\nu}_{T} &= \frac{1}{2} e^{-2\alpha} \left(e^{\alpha}\beta_t\right) \phi_t^2 + \frac{1}{2} e^{-2\beta} \left(e^{\alpha}\beta_t\right) \phi_r^2 \\ &\quad - \frac{1}{2} \left(e^{\alpha}\beta_t\right) \frac{g^2(\phi)}{r^2} - \alpha_r e^{-\alpha-2\beta} \phi_t \phi_r \\ &= e^{-\alpha} \Big(\beta_t(\mathbf{e} - \mathbf{f}) - \alpha_r e^{-\beta} \mathbf{m}\Big) \\ &= 0 \end{aligned} \tag{5.9}$$

after the usage of Einstein's equations (4.5b) and (4.5c). Also, recall from (5.2) that

$$0 = \nabla_{\nu} P^{\nu}_{T} = \frac{1}{r e^{\beta+\alpha}} \Big(-\partial_t (r e^{\beta} \mathbf{e}) + \partial_r (r e^{\alpha} \mathbf{m})\Big). \tag{5.10}$$

For $R = e^{-\beta}\partial_r$ and

$$P_R = -\, e^{-\alpha} \mathbf{m}\, \partial_t + e^{-\beta} (\mathbf{e} - \mathbf{f})\, \partial_r, \tag{5.11}$$

the divergence $\nabla_{\nu} P^{\nu}_{R}$ is

$$\begin{aligned}\nabla_{\nu} P^{\nu}_{R} &= \frac{1}{2}\, {}^{(R)}\boldsymbol{\pi}_{\mu\nu} S^{\mu\nu} \\ &= -e^{-\beta} \alpha_r\, \mathbf{e} + \frac{1}{2r} e^{-\beta} (e^{-2\alpha} \phi_t^2 - e^{-2\beta} \phi_r^2 + \mathbf{f}) + e^{-\alpha} \beta_t \mathbf{m}. \end{aligned} \tag{5.12}$$

Equivalently,

$$\begin{aligned}\nabla_{\nu} P^{\nu}_{R} &= \frac{1}{\sqrt{-\mathbf{g}}} \partial_{\nu} (\sqrt{-\mathbf{g}}\, P^{\nu}_{R}) \\ &= \frac{1}{r e^{\beta+\alpha}} \Big(-\partial_t (r e^{\beta} \mathbf{m}) + \partial_r ((\mathbf{e} - \mathbf{f}) r e^{\alpha})\Big). \end{aligned} \tag{5.13}$$

Similarly for the choice $\mathcal{R}_a := r^a\,\partial_r$, we have

$$
\begin{aligned}
P_{\mathcal{R}_a} =& - e^{\beta-\alpha} r^a\,\mathbf{m}\partial_t + r^a\,(\mathbf{e}-\mathbf{f})\partial_r,\\
\nabla_\nu P^\nu_{\mathcal{R}_a} =& \frac{1}{2}\left(r^a(-\alpha_r+\beta_r)+(1+a)r^{a-1}\right)e^{-2\alpha}\phi_t^2\\
&+\frac{1}{2}\left(r^a(-\alpha_r+\beta_r)+(a-1)r^{a-1}\right)e^{-2\beta}\phi_r^2\\
&+\frac{1}{2}\left(-r^a(\alpha_r+\beta_r)+(1-a)r^{a-1}\right)\frac{g^2(\phi)}{r^2}\\
=&\frac{1}{2}\left((1+a)r^{a-1}\right)e^{-2\alpha}\phi_t^2+\frac{1}{2}\left((a-1)r^{a-1}\right)e^{-2\beta}\phi_r^2\\
&+\frac{1}{2}\left((1-a)r^{a-1}\right)\frac{g^2(\phi)}{r^2}
\end{aligned} \tag{5.14}
$$

where we used Einstein equations (4.5c) and (4.5a) for α_r and β_r respectively. In particular, we have $\mathcal{R}_1 := r\,\partial_r$ and

$$
\begin{aligned}
P_{\mathcal{R}_1} =& - re^{\beta-\alpha}\,\mathbf{m}\,\partial_t + r(\mathbf{e}-\mathbf{f})\partial_r,\\
\nabla_\nu P^\nu_{\mathcal{R}_1} =& e^{-2\alpha}\phi_t^2.
\end{aligned} \tag{5.15}
$$

Let $J^-(O)$ be the causal past of the the point O and $I^-(O)$ the chronological past of O. We will need the following definitions

$$
\begin{aligned}
\Sigma_t^O &:= \Sigma_t \cap J^-(O)\\
K(t) &:= \cup_{t\le t'<0}\,\Sigma_{t'}\cap J^-(O)\\
C(t) &:= \cup_{t\le t'<0}\,\Sigma_{t'}\cap (J^-(O)\setminus I^-(O))\\
K(t,s) &:= \cup_{t\le t'<s}\,\Sigma_{t'}\cap J^-(O)\\
C(t,s) &:= \cup_{t\le t'<s}\,\Sigma_{t'}\cap (J^-(O)\setminus I^-(O))
\end{aligned}
$$

for $-1\le t<s<0$. In the following we will try to understand the behaviour of various quantities of the wave map as one approaches O in a limiting sense. For this we will use the Stokes' theorem in the region $K(\tau,s), -1\le\tau\le s<0$ (as shown in the figure 3) for divergence of vector fields P_X with suitable choices of the vector field X. The volume 3-form of (M,g) is given by

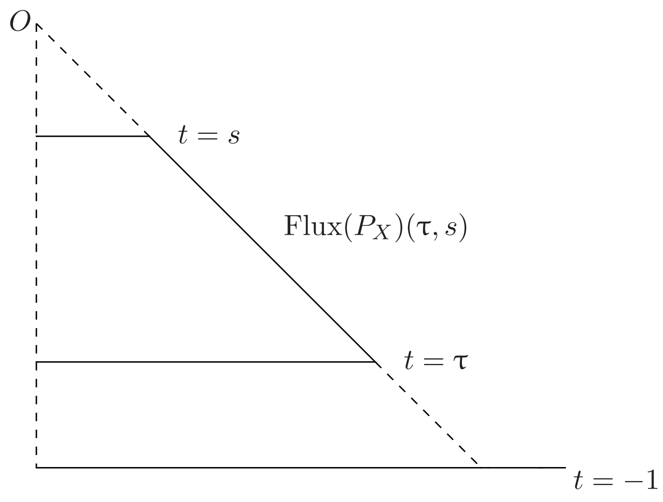

FIGURE 3. Application of the Stokes' theorem for the divergence of P_X

$$\bar{\mu}_g := re^{\beta+\alpha} dt \wedge dr \wedge d\theta$$

and the area 2-form of (Σ, q) by

$$\bar{\mu}_q = re^{\beta} dr \wedge d\theta.$$

Let us define 1-forms $\widetilde{\ell}$, $\widetilde{n}$ and $\widetilde{m}$ as follows

$$\begin{aligned} \widetilde{\ell} &:= -e^{\alpha} dt + e^{\beta} dr, \\ \widetilde{n} &:= -e^{\alpha} dt - e^{\beta} dr, \\ \widetilde{m} &:= r d\theta, \end{aligned}$$

so we have,

$$\bar{\mu}_g = \frac{1}{2}\left(\widetilde{\ell} \wedge \widetilde{n} \wedge \widetilde{m}\right).$$

Let us also define the 2-forms $\bar{\mu}_{\widetilde{\ell}}$ and $\bar{\mu}_{\widetilde{n}}$ such that

$$\begin{aligned} \bar{\mu}_{\widetilde{\ell}} &:= -\frac{1}{2} \widetilde{n} \wedge \widetilde{m}, \\ \bar{\mu}_{\widetilde{n}} &:= \frac{1}{2} \widetilde{\ell} \wedge \widetilde{m}, \end{aligned}$$

so that

$$\begin{aligned} \bar{\mu}_g &= -\widetilde{\ell} \wedge \bar{\mu}_{\widetilde{\ell}}, \\ \bar{\mu}_g &= -\widetilde{n} \wedge \bar{\mu}_{\widetilde{n}}. \end{aligned}$$

We now apply the Stokes' theorem for the $\bar{\mu}_g$-divergence of P_X in the region $K(\tau, s)$ to get

$$\int_{K(\tau,s)} \nabla_\nu P_X^\nu \, \bar{\mu}_g = \int_{\Sigma_s^O} e^{\alpha} \, P_X^t \, \bar{\mu}_q - \int_{\Sigma_\tau^O} e^{\alpha} \, P_X^t \, \bar{\mu}_q + \mathrm{Flux}(P_X)(\tau, s) \tag{5.16}$$

where

$$\mathrm{Flux}(P_X)(\tau, s) = -\int_{C(\tau,s)} \widetilde{n}(P_X) \, \bar{\mu}_{\widetilde{n}}.$$

5.3. **Monotonicity of energy.**

Lemma 5.5. *We have* $E^O(\tau) \geq E^O(s)$ *for* $-1 \leq \tau < s < 0$.

Proof. Let us apply the Stokes' theorem (5.16) to the vector field P_T. We have

$$0 = -\int_{\Sigma_s^O} \mathbf{e} \, \bar{\mu}_q + \int_{\Sigma_\tau^O} \mathbf{e} \, \bar{\mu}_q + \mathrm{Flux}(P_T)(\tau, s) \tag{5.17}$$

and

$$\begin{aligned} \mathrm{Flux}(P_T)(\tau, s) &= -\int_{C(\tau,s)} \widetilde{n}\,(P_T) \, \bar{\mu}_{\widetilde{n}} \\ &= -\int_{C(\tau,s)} (\mathbf{e} - \mathbf{m}) \, \bar{\mu}_{\widetilde{n}}. \end{aligned}$$

Note that we have $\mathbf{e} \geq |\mathbf{m}|$. Hence, we have $\mathrm{Flux}(P_T)(\tau, s) \leq 0$ which implies

$$E^O(\tau) - E^O(s) \geq 0 \;\forall\; -1 \leq \tau \leq s < 0.$$

This concludes the proof of the lemma. □

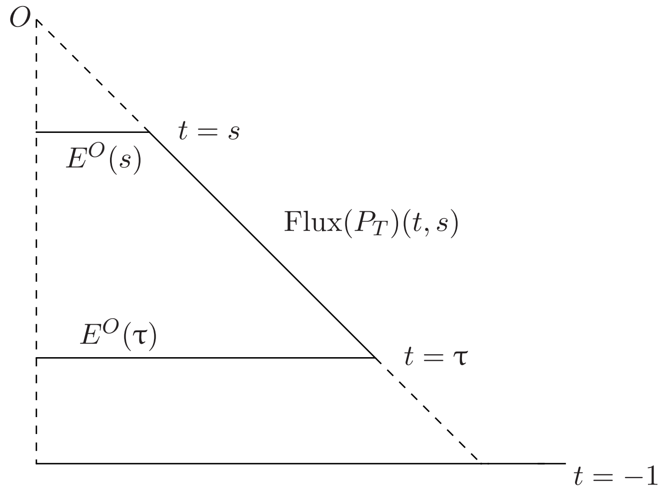

FIGURE 4. Monotonicity of Energy inside the past null cone of O

Let us define

$$E^O_{\text{conc}} := \inf_{\tau \in [-1,0)} E^O(\tau). \tag{5.18}$$

As a consequence of Lemma 5.5, (5.18) is equivalent to

$$E^O_{\text{conc}} = \lim_{\tau \to 0} E^O(\tau). \tag{5.19}$$

We say that the energy of equivariant Cauchy problem concentrates if $E^O_{\text{conc}} \neq 0$ and does not concentrate if $E^O_{\text{conc}} = 0$.

Corollary 5.6. *For a vectorfield X, let*

$$\text{Flux}(P_X)(\tau) := \lim_{s \to 0} \text{Flux}(P_X)(\tau, s).$$

Then, we have

$$\text{Flux}(P_T)(\tau) \to 0 \text{ as } \tau \to 0.$$

Proof. Recall the equation (5.17). For $s \to 0$, we have

$$0 = -E^O_{\text{conc}} + \int_{\Sigma^O_\tau} \mathbf{e}\ \bar{\mu}_q + \text{Flux}(P_T)(\tau). \tag{5.20}$$

Now by the definition (5.18), as $\tau \to 0$ we get

$$\lim_{\tau \to 0} \int_{\Sigma^O_\tau} \mathbf{e}\ \bar{\mu}_q \to E^O_{\text{conc}}. \tag{5.21}$$

Therefore, it follows from (5.20) that $\text{Flux}(P_T)(\tau) \to 0$ as $\tau \to 0$. □

5.4. L^∞ **estimate for the Jacobian.** The goal of this section is to derive uniform bounds for the Jacobian transformation between (t, r, θ) and $(u, \underline{u}, \theta)$ coordinates. Recall that we defined the 1-forms $\widetilde{\ell}$ and $\widetilde{n}$. Their corresponding vectors are null, given by

$$\widetilde{\ell} = e^{-\alpha}\partial_t + e^{-\beta}\partial_r$$
$$\widetilde{n} = e^{-\alpha}\partial_t - e^{-\beta}\partial_r\,.$$

Lemma 5.7. *There exists two scalar functions $\mathcal{F}$ and $\mathcal{G}$ such that*

$$\partial_{\underline{u}} = \frac{1}{2}e^{\mathcal{F}}\widetilde{\ell}, \quad \partial_u = \frac{1}{2}e^{\mathcal{G}}\widetilde{n}, \tag{5.22}$$

with the normalization on Γ

$$\mathcal{F} = \mathcal{G} = 0 \text{ on } \Gamma.$$

Furthermore, $\mathcal{F}$ *and* $\mathcal{G}$ *satisfy*

$$\partial_{\underline{u}}(\mathcal{G}) = e^{\mathcal{F}} r\kappa e^{\beta}(\mathbf{e}+\mathbf{m}-\mathbf{f}), \tag{5.23a}$$

$$\partial_u(\mathcal{F}) = -e^{\mathcal{G}} r\kappa e^{\beta}(\mathbf{e}-\mathbf{m}-\mathbf{f}). \tag{5.23b}$$

Proof. In view of various normalizations on Γ, note that we have

$$\partial_{\underline{u}} r = \frac{1}{2}, \ \partial_u r = -\frac{1}{2}, \ \widetilde{\ell}(r) = 1, \ \widetilde{n}(r) = -1 \text{ on } \Gamma.$$

Furthermore, ∂_u and $\partial_{\underline{u}}$ are also null, and ∂_u, $\partial_{\underline{u}}$, $\widetilde{\ell}$ and $\widetilde{n}$ are all future directed. We infer that there exists two scalar functions $\mathcal{F}$ and $\mathcal{G}$ such that

$$\partial_{\underline{u}} = \frac{1}{2}e^{\mathcal{F}}\widetilde{\ell}, \quad \partial_u = \frac{1}{2}e^{\mathcal{G}}\widetilde{n},$$

with the normalization on Γ

$$\mathcal{F} = \mathcal{G} = 0 \text{ on } \Gamma.$$

Next, we derive equations for $\mathcal{F}$ and $\mathcal{G}$. We have

$$[\widetilde{\ell}, \widetilde{n}] = 2e^{-(\beta+\alpha)}(-\alpha_r\partial_t + \beta_t\partial_r).$$

We infer

$$\begin{aligned}
[\partial_{\underline{u}}, \partial_u] &= \frac{e^{(\mathcal{F}+\mathcal{G})}}{4}\Big([\widetilde{\ell}, \widetilde{n}] + \widetilde{\ell}(\mathcal{G})\widetilde{n} - \widetilde{n}(\mathcal{F})\widetilde{\ell}\Big)\\
&= \frac{e^{(\mathcal{F}+\mathcal{G})}}{2}e^{-(\beta+\alpha)}(-\alpha_r\partial_t + \beta_t\partial_r) + \frac{e^{\mathcal{G}}}{2}\partial_{\underline{u}}(\mathcal{G})(e^{-\alpha}\partial_t - e^{-\beta}\partial_r)\\
&\quad -\frac{e^{\mathcal{F}}}{2}\partial_u(\mathcal{F})(e^{-\alpha}\partial_t + e^{-\beta}\partial_r).
\end{aligned}$$

Since $[\partial_{\underline{u}}, \partial_u] = 0$, $\mathcal{F}$ and $\mathcal{G}$ are such that

$$\begin{aligned}
e^{-\mathcal{F}}\partial_{\underline{u}}(\mathcal{G}) - e^{-\mathcal{G}}\partial_u(\mathcal{F}) &= r\kappa e^{\beta}(\mathbf{e}-\mathbf{f}),\\
e^{-\mathcal{F}}\partial_{\underline{u}}(\mathcal{G}) + e^{-\mathcal{G}}\partial_u(\mathcal{F}) &= r\kappa e^{\beta}\mathbf{m},
\end{aligned}$$

and hence

$$\begin{aligned}
\partial_{\underline{u}}(\mathcal{G}) &= e^{\mathcal{F}} r\kappa e^{\beta}(\mathbf{e}+\mathbf{m}-\mathbf{f}),\\
\partial_u(\mathcal{F}) &= -e^{\mathcal{G}} r\kappa e^{\beta}(\mathbf{e}-\mathbf{m}-\mathbf{f}).
\end{aligned}$$

This concludes the proof of the lemma. □

Let us revisit the Stokes' theorem for $\bar{\mu}_g$-divergence of P_X in $K(\tau, s)$. We have

$$d\,\underline{u} = -e^{-\mathcal{F}}\widetilde{n}, \ d\,u = -e^{-\mathcal{G}}\widetilde{\ell}.$$

The volume 3-form of (M, g) is

$$\bar{\mu}_g = r\,\Omega^2 d\,u \wedge d\,\underline{u} \wedge d\,\theta.$$

Let us introduce the 2-forms $\bar{\mu}_{\underline{u}}$ and $\bar{\mu}_u$ as follows

$$\bar{\mu}_g = d\,\underline{u} \wedge \bar{\mu}_{\underline{u}}, \quad \bar{\mu}_g = d\,u \wedge \bar{\mu}_u$$

so that

$$\bar{\mu}_{\underline{u}} = -r\,\Omega^2\,(d\,u \wedge d\,\theta)\,, \quad \bar{\mu}_u = \,r\,\Omega^2\,(d\,\underline{u} \wedge d\,\theta)\,.$$

Now,

$$\mathrm{Flux}(P_X)(\tau, s) = \int_{C(\tau,s)} d\,\underline{u}(P_X)\,\bar{\mu}_{\underline{u}},$$

for instance,

$$\begin{aligned}\mathrm{Flux}(P_T)(\tau, s) &= \int_{C(\tau,s)} d\,\underline{u}(P_T)\,\bar{\mu}_{\underline{u}},\\ &= -\int_{C(\tau,s)} e^{-\mathcal{F}}(\mathbf{e}-\mathbf{m})\bar{\mu}_{\underline{u}}.\end{aligned}$$

Lemma 5.8. *There exist constants $c_{\mathcal{G}}^-$, $c_{\mathcal{G}}^+$, $c_{\mathcal{F}}^-$ and $c_{\mathcal{F}}^+$ depending only on the initial data and the universal constants such that the following uniform bounds hold*

$$\begin{aligned} c_{\mathcal{G}}^- \,&\leq \mathcal{G} \,\leq\, c_{\mathcal{G}}^+\\ c_{\mathcal{F}}^- \,&\leq \mathcal{F} \,\leq\, c_{\mathcal{F}}^+.\end{aligned}$$

Proof. We integrate (5.23) using the fact that $\mathcal{F} = \mathcal{G} = 0$ on Γ. We infer

$$\begin{aligned}\mathcal{G}(u,\underline{u}) &= \int_u^{\underline{u}} e^{\mathcal{F}} r\kappa e^{\beta}(\mathbf{e}+\mathbf{m}-\mathbf{f})(u,\underline{u}')d\underline{u}',\\ \mathcal{F}(u,\underline{u}) &= \int_u^{\underline{u}} e^{\mathcal{G}} r\kappa e^{\beta}(\mathbf{e}-\mathbf{m}-\mathbf{f})(u',\underline{u})du'.\end{aligned}$$

Next, note that

$$-\frac{\Omega^2}{2} = \mathbf{g}(\partial_u, \partial_{\underline{u}}) = \frac{e^{\mathcal{F}+\mathcal{G}}}{4}\mathbf{g}(\widetilde{n}, \widetilde{\ell}) = -\frac{e^{\mathcal{F}+\mathcal{G}}}{2}$$

and hence

$$\Omega^2 = e^{\mathcal{F}+\mathcal{G}}. \tag{5.26}$$

We infer

$$\begin{aligned}\mathcal{G}(u,\underline{u}) &= \kappa\int_u^{\underline{u}} e^{\beta} e^{-\mathcal{G}}(\mathbf{e}+\mathbf{m}-\mathbf{f})r\Omega^2 d\underline{u}',\\ \mathcal{F}(u,\underline{u}) &= \kappa\int_u^{\underline{u}} e^{\beta} e^{-\mathcal{F}}(\mathbf{e}-\mathbf{m}-\mathbf{f})r\Omega^2 du'.\end{aligned}$$

Using the fact that $|\mathbf{e} \pm \mathbf{m} - \mathbf{f}| \leq \mathbf{e} \pm \mathbf{m}$ and $u \leq \underline{u}$, we infer

$$\begin{aligned}|\mathcal{G}(u,\underline{u})| &\leq c\int_u^{\underline{u}} e^{-\mathcal{G}}(\mathbf{e}+\mathbf{m})r\Omega^2 d\underline{u}',\\ |\mathcal{F}(u,\underline{u})| &\leq c\int_u^{\underline{u}} e^{-\mathcal{F}}(\mathbf{e}-\mathbf{m})r\Omega^2 du'.\end{aligned}$$

Since $d\,\underline{u} = -e^{-\mathcal{F}}\widetilde{n}$ and $d\,u = -e^{-\mathcal{G}}\widetilde{\ell}$, we infer

$$\begin{aligned}|\mathcal{G}(u,\underline{u})| &\leq c\int_u^{\underline{u}} d\,u(P_T)r\Omega^2 d\underline{u}',\\ |\mathcal{F}(u,\underline{u})| &\leq c\int_u^{\underline{u}} d\,\underline{u}(P_T)r\Omega^2 du'.\end{aligned}$$

After integration in θ, the right-hand sides are bounded by fluxes which in turn are bounded by the energy, and hence

$$|\mathcal{G}| \,\leq c, \;|\mathcal{F}| \,\leq\, c.$$

This concludes the proof of the lemma. □

Let us consider the Jacobian $\mathbf{J}$ of the transition functions between (t, r, θ) and $(\underline{u}, u, \theta)$

$$\mathbf{J} := \begin{pmatrix} \partial_{\underline{u}} t & \partial_u t & \partial_\theta t \\ \partial_{\underline{u}} r & \partial_u r & \partial_\theta r \\ \partial_{\underline{u}} \theta & \partial_u \theta & \partial_\theta \theta \end{pmatrix}$$

$$= \frac{1}{2} \begin{pmatrix} e^{\mathcal{F}-\alpha} & e^{\mathcal{G}-\alpha} & 0 \\ e^{\mathcal{F}-\beta} & -e^{\mathcal{G}-\beta} & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

then the inverse Jacobian $\mathbf{J}^{-1}$ is given by

$$\mathbf{J}^{-1} = \begin{pmatrix} e^{-\mathcal{F}+\alpha} & e^{-\mathcal{F}+\beta} & 0 \\ e^{-\mathcal{G}+\alpha} & -e^{-\mathcal{G}+\beta} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Therefore,

$$\partial_t \underline{u} = \frac{1}{2} e^{-\mathcal{F}+\alpha}, \ \partial_r \underline{u} = \frac{1}{2} e^{-\mathcal{F}+\beta}, \ \partial_t u = \frac{1}{2} e^{-\mathcal{G}+\alpha}, \ \partial_r u = -\frac{1}{2} e^{-\mathcal{G}+\beta}, \tag{5.30}$$

and

$$d\underline{u} = e^{(-\mathcal{F}+\alpha)} dt + e^{(-\mathcal{F}+\beta)} dr, \quad du = e^{(-\mathcal{G}+\alpha)} dt - e^{(-\mathcal{G}+\beta)} dr.$$

Corollary 5.9. *There exist constants $c^-_{\mu\nu}, c^+_{\mu\nu}$ and $C^-_{\mu\nu}, C^+_{\mu\nu}$ depending only on the inital data and the universal constants such that all the entries of the Jacobian $\mathbf{J}$ and its inverse $\mathbf{J}^{-1}$ are uniformly bounded*

$$c^-_{\mu\nu} \leq \mathbf{J}_{\mu\nu} \leq c^+_{\mu\nu}$$
$$C^-_{\mu\nu} \leq \mathbf{J}^{-1}_{\mu\nu} \leq C^+_{\mu\nu}$$

for $\mu, \nu = 0, 1, 2$.

Proof. The proof follows from Lemmas 5.3 and 5.8. □

Corollary 5.10. *There exist constants c^-_Ω and c^+_Ω depending only on the initial energy and the universal constants such that the following uniform bounds hold on the metric function Ω in null coordinates.*

$$c^-_\Omega \leq \Omega \leq c^+_\Omega. \tag{5.31}$$

Proof. This follows immediately from (5.26) and Lemma 5.8. □

Corollary 5.11. *Let us introduce the notation*

$$\tau = \frac{u + \underline{u}}{2}, \ \varrho = \frac{\underline{u} - u}{2}.$$

Then, there exist constants c_1, c_2, c_3, c_4 such that the pointwise bounds

$$r \geq c_1 \varrho, \qquad t \geq c_3 \tau,$$
$$r \leq c_2 \varrho \quad \text{and} \quad t \leq c_4 \tau$$

hold for the scalar functions r, t, ϱ and τ.

Proof. We have

$$|\partial_\varrho r| = \frac{1}{2}|\partial_{\underline{u}} r - \partial_u r| = \frac{1}{4}|e^{\mathcal{F}-\beta} + e^{\mathcal{G}-\beta}| \le c_1, \tag{5.32a}$$

$$|\partial_r \varrho| = \frac{1}{2}|\partial_r \underline{u} - \partial_r u| = \frac{1}{2}|e^{-\mathcal{G}-\beta} - e^{-\mathcal{G}+\beta}| \le c_2, \tag{5.32b}$$

$$|\partial_\tau t| = \frac{1}{2}|\partial_{\underline{u}} t + \partial_u t| = \frac{1}{4}|e^{\mathcal{F}-\alpha} + e^{\mathcal{G}-\alpha}| \le c_3, \tag{5.32c}$$

$$|\partial_t \tau| = \frac{1}{2}|\partial_t \underline{u} + \partial_t u| = \frac{1}{2}|e^{-\mathcal{F}+\alpha} + e^{-\mathcal{G}+\alpha}| \le c_4. \tag{5.32d}$$

The proof follows by applying the fundamental theorem of calculus to each of (5.32) in the region $J^-(O)$ and noting that at O, $t = \tau = 0$ and $r = \varrho = 0$ on the axis. ☐

5.5. **Non-concentration away from the axis.** In this section we shall use the vector fields method introduced previously to prove that energy does not concentrate. We start with proving that the energy does not concentrate away from the axis using the divergence free vector P_T.

Lemma 5.12 (Non-concentration away from the axis)**.** *We have*

$$E^O_{ext}(\mathrm{\tau}) := \int_{B_{r_2(\mathrm{\tau})} \setminus B_{r_1(\mathrm{\tau})}} \mathbf{e}\, \bar{\mu}_q \to 0 \ \text{as}\ \mathrm{\tau} \to 0,$$

where $r = r_2(\mathrm{\tau})$ is the radius where the $t = \mathrm{\tau}$ slice intersects the $\varrho = |\tau|$ curve i.e the mantel of the null cone $J^-(O)$ and $r = r_1(\mathrm{\tau})$ is the radius where the $\varrho = \lambda|\tau|$ curve intersects the $t = \mathrm{\tau}$ slice, for any real $\lambda \in (0,1)$. Observe that both $r_1(\mathrm{\tau})$ and $r_2(\mathrm{\tau}) \to 0$ as $\mathrm{\tau} \to 0$.

Proof. Consider a tubular region $\mathcal{S}$ with triangular cross section (as shown in the figure 5) in $\varrho > \lambda\tau$, $\lambda \in (0,1)$ of the spacetime i.e., the "exterior" part of the interior of the past null cone of O. As shown in the figure 5, let us use the divergence-free

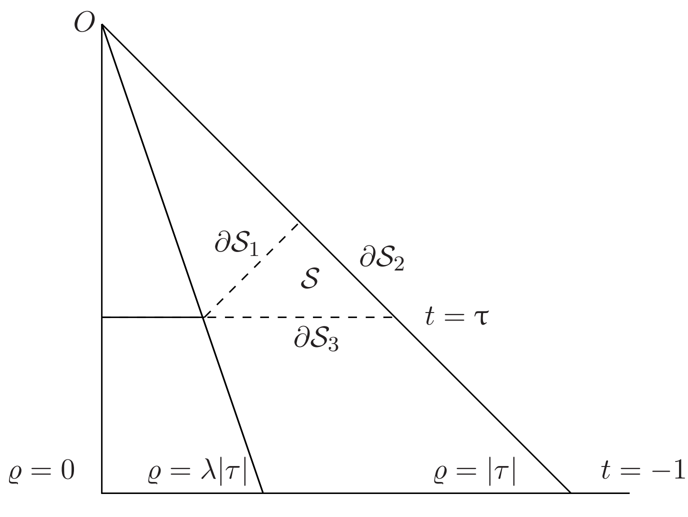

FIGURE 5. Application of Stokes' theorem on the $\bar{\mu}_g$-divergence free P_T to relate the fluxes through surfaces $\partial\mathcal{S}_1, \partial\mathcal{S}_2$ and $\partial\mathcal{S}_3$

vector field P_T and the Stokes' theorem in the region $\mathcal{S}$ to relate the fluxes through the three boundary segments $\partial\mathcal{S}_1, \partial\mathcal{S}_2$ and $\partial\mathcal{S}_3$. We have

$$\begin{aligned} 0 =& \int_{\partial\mathcal{S}_1} d\,u(P_T)\bar{\mu}_u + \int_{\partial\mathcal{S}_2} d\,\underline{u}(P_T)\bar{\mu}_{\underline{u}} - \int_{\partial\mathcal{S}_3} e^{\alpha} P_T^t\,\bar{\mu}_q \\ =& -\frac{1}{2}\int_{\partial\mathcal{S}_1} e^{-\mathcal{G}}(\mathbf{e}+\mathbf{m})\,\bar{\mu}_u - \frac{1}{2}\int_{\partial\mathcal{S}_2} e^{-\mathcal{F}}(\mathbf{e}-\mathbf{m})\,\bar{\mu}_{\underline{u}} + \int_{\partial\mathcal{S}_3} \mathbf{e}\,\bar{\mu}_q. \end{aligned} \tag{5.33}$$

To analyze the behaviour of the flux terms $\int_{\partial\mathcal{S}_1}$ and $\int_{\partial\mathcal{S}_2}$ in (5.33) close to O, let us define

$$\begin{aligned} \widehat{l} \,&:= e^{\beta+\alpha}\widetilde{\ell} = e^{\beta}\partial_t + e^{\alpha}\partial_r, \\ \widehat{n} \,&:= e^{\beta+\alpha}\widetilde{n} = e^{\beta}\partial_t - e^{\alpha}\partial_r, \\ \mathcal{A}^2 \,&:= r(\mathbf{e}-\mathbf{m}), \\ \mathcal{B}^2 \,&:= r(\mathbf{e}+\mathbf{m}). \end{aligned}$$

From (5.12) and (5.13), we have

$$\begin{aligned} &\frac{1}{re^{\beta+\alpha}}\Big(-\partial_t(re^{\beta}\mathbf{m}) + \partial_r((\mathbf{e}-\mathbf{f})re^{\alpha})\Big) \\ &= -\,e^{-\beta}\alpha_r\,\mathbf{e} + \frac{1}{2r}e^{-\beta}(e^{-2\alpha}\phi_t^2 - e^{-2\beta}\phi_r^2 + \mathbf{f}) + e^{-\alpha}\beta_t\mathbf{m}. \end{aligned} \tag{5.34}$$

We have the following identities from (5.10) and (5.34)

$$\partial_t(re^{\beta}\mathbf{e}) - \partial_r(re^{\alpha}\mathbf{m}) = 0, \tag{5.35a}$$

$$\partial_t(re^{\beta}\mathbf{m}) - \partial_r(re^{\alpha}\mathbf{e}) = L, \tag{5.35b}$$

where

$$L \,:= \frac{re^{\alpha}\alpha_r}{2}\left((T\phi)^2 + (R\phi)^2 - \mathbf{f}\right) + e^{\alpha}L_0 - r\beta_t e^{\beta}\mathbf{m}$$

for

$$L_0 \,:= \frac{1}{2}\left(-(Tu)^2 + (Ru)^2 + \mathbf{f}\right) - \frac{2g(\phi)g'(\phi)\phi_r}{r}.$$

Furthermore, we can construct the following using the identities in (5.35)

$$\partial_\nu\left(re^{\beta+\alpha}(\mathbf{e}-\mathbf{m})\widetilde{\ell}^{\nu}\right) = \partial_\nu(\mathcal{A}^2\widehat{\ell}^{\nu}) = -L, \tag{5.36a}$$

$$\partial_\nu\left(re^{\beta+\alpha}(\mathbf{e}+\mathbf{m})\widetilde{n}^{\nu}\right) = \partial_\nu(\mathcal{B}^2\widehat{n}^{\nu}) = L. \tag{5.36b}$$

Let us try to express L in terms of $\mathcal{A}^2$ $\mathcal{B}^2$ after using the Einstein equations

$$\begin{aligned} L &= e^{\alpha}L_0 + \kappa r^2 e^{2\beta+\alpha}\,(\mathbf{e}-\mathbf{f})^2 - \kappa r^2 e^{2\beta+\alpha}\mathbf{m}^2 \\ &= e^{\alpha}L_0 + \kappa r^2 e^{2\beta+\alpha}\left(\mathbf{e}^2 - 2\,\mathbf{e}\,\mathbf{f} + \mathbf{f}^2 - \mathbf{m}^2\right) \\ &= e^{\alpha}L_0 + \kappa e^{2\beta+\alpha}\left(\mathcal{A}^2\,\mathcal{B}^2 - 2\,r^2\,\mathbf{e}\,\mathbf{f} + r^2\mathbf{f}^2\right). \end{aligned} \tag{5.37}$$

We would like to set up a Grönwall estimate for $\mathcal{A}$ and $\mathcal{B}$ using the identities in (5.36). However, the quantity L as shown in (5.37) has nonlinear terms involving $\mathbf{e}$ and $\mathbf{f}$. Therefore, in what follows we use Einstein equations to estimate these terms.

Firstly note that

$$\begin{aligned} \widehat{\ell}^{\mu}\partial_\mu e^{2\beta} =& 2e^{2\beta}(e^{\beta}\beta_t + e^{\alpha}\beta_r) \qquad & \widehat{n}^{\mu}\partial_\mu e^{2\beta} =& 2e^{2\beta}(e^{\beta}\beta_t - e^{\alpha}\beta_r) \\ =& 2e^{2\beta}\kappa r e^{2\beta+\alpha}(\mathbf{m}+\mathbf{e}) & =& 2e^{2\beta}\kappa r e^{2\beta+\alpha}(\mathbf{m}-\mathbf{e}) \\ =& 2\kappa e^{2\beta}e^{2\beta+\alpha}\mathcal{B}^2, & =& -\,2\kappa e^{2\beta}e^{2\beta+\alpha}\mathcal{A}^2, \end{aligned}$$

and

$$\partial_\mu \widehat{\ell}^\mu = e^\beta \beta_t + e^\alpha \alpha_r = r\kappa e^{2\beta+\alpha}(\mathbf{e}+\mathbf{m}-\mathbf{f}) = \kappa e^{2\beta+\alpha}\left(\mathcal{B}^2 - r\,\mathbf{f}\right), \qquad \partial_\mu \widehat{n}^\mu = e^\beta \beta_t - e^\alpha \alpha_r = r\kappa e^{2\beta+\alpha}(-\mathbf{e}+\mathbf{m}+\mathbf{f}) = \kappa e^{2\beta+\alpha}\left(-\mathcal{A}^2 + r\,\mathbf{f}\right).$$

Now consider the quantities $\partial_\mu(e^{2\beta}\mathcal{A}^2\widehat{\ell}^\mu)$ and $\partial_\mu(e^{2\beta}\mathcal{B}^2\widehat{n}^\mu)$,

$$\begin{aligned}
\widehat{\ell}^\mu \partial_\mu(e^{2\beta}\mathcal{A}^2) &= \partial_\mu(e^{2\beta}\mathcal{A}^2\widehat{\ell}^\mu) - e^{2\beta}\mathcal{A}^2\partial_\mu\widehat{\ell}^\mu \\
&= e^{2\beta}\partial_\mu(\mathcal{A}^2\widehat{\ell}^\mu) + \mathcal{A}^2\widehat{\ell}^\mu\partial_\mu e^{2\beta} - \kappa e^{2\beta}e^{2\beta+\alpha}\mathcal{A}^2\mathcal{B}^2 + r\kappa e^{2\beta}e^{2\beta+\alpha}\mathcal{A}^2\,\mathbf{f} \\
&= -e^{2\beta}L + \kappa e^{2\beta}e^{2\beta+\alpha}\mathcal{A}^2\mathcal{B}^2 + r\kappa e^{2\beta}e^{2\beta+\alpha}\mathcal{A}^2\,\mathbf{f} \\
&= e^{2\beta}e^\alpha\left(-L_0 + 2\,r^2\,\mathbf{e}\,\mathbf{f} - r^2\mathbf{f}^2 + r\mathcal{A}^2\mathbf{f}\right) \\
&= e^{2\beta}e^\alpha\Big(-L_0 + \kappa r^2 e^{2\beta}\left(3\mathbf{e}\,\mathbf{f} - \mathbf{f}^2 - \mathbf{m}\,\mathbf{f}\right)\Big),
\end{aligned}$$

$$\begin{aligned}
\widehat{n}^\mu \partial_\mu(e^{2\beta}\mathcal{B}^2) &= \partial_\mu(e^{2\beta}\mathcal{B}^2\widehat{n}^\mu) - e^{2\beta}\mathcal{B}^2\partial_\mu\widehat{n}^\mu \\
&= e^{2\beta}\partial_\mu(\mathcal{B}^2\widehat{n}^\mu) + \mathcal{B}^2\widehat{n}^\mu\partial_\mu e^{2\beta} + \kappa e^{2\beta}e^{2\beta+\alpha}\mathcal{A}^2\mathcal{B}^2 - r\kappa e^{2\beta}e^{2\beta+\alpha}\mathcal{B}^2\mathbf{f} \\
&= e^{2\beta}L - \kappa e^{2\beta}e^{2\beta+\alpha}\mathcal{A}^2\mathcal{B}^2 - r\kappa e^{2\beta}e^{2\beta+\alpha}\mathcal{B}^2\,\mathbf{f} \\
&= e^{2\beta}e^\alpha\Big(L_0 + \kappa e^{2\beta}\left(-2r^2\mathbf{f}\mathbf{e} + r^2\mathbf{f}^2 - r\mathcal{B}^2\mathbf{f}\right)\Big) \\
&= e^{2\beta}e^\alpha\Big(L_0 + \kappa r^2 e^{2\beta}\left(-3\mathbf{e}\,\mathbf{f} + \mathbf{f}^2 - \mathbf{m}\,\mathbf{f}\right)\Big).
\end{aligned}$$

Let us define

$$\begin{aligned}
S_1 :&= 3\mathbf{e}\,\mathbf{f} - \mathbf{f}^2 - \mathbf{m}\,\mathbf{f} \\
&= (\mathbf{e} - \mathbf{m})\,\mathbf{f} + \mathbf{e_0}\,\mathbf{f} \\
&\geq 0.
\end{aligned}$$

Note that we have $\mathbf{e} \geq |\mathbf{m}|$. Similarly define

$$\begin{aligned}
S_2 :&= -3\mathbf{e}\,\mathbf{f} + \mathbf{f}^2 - \mathbf{m}\,\mathbf{f} \\
&= -(\mathbf{e} + \mathbf{m})\,\mathbf{f} - \mathbf{e_0}\,\mathbf{f} \\
&\leq 0.
\end{aligned}$$

Let us now introduce the quantities $\widehat{\mathcal{A}}$ and $\widehat{\mathcal{B}}$ such that

$$\widehat{\mathcal{A}} := e^\beta\mathcal{A}, \quad \widehat{\mathcal{B}} := e^\beta\mathcal{B}.$$

In the following we will try to estimate L_0^2 by $\mathbf{e}^2 - \mathbf{m}^2$. We will use the following identities which are valid for any real numbers a, b, c

$$(a+b+c)^2 \leq 3(a^2+b^2+c^2), \quad \frac{1}{4}(-a^2+b^2)^2 = \frac{1}{4}(a^2+b^2)^2 - a^2b^2.$$

Consider,

$$\begin{aligned} L_0^2 &\le 3\left(\frac{1}{4}\left(-(T\phi)^2+(R\phi)^2\right)^2+4g'(\phi)^2\phi_r^2\,\mathbf{f}+\frac{1}{4}\,\mathbf{f}^2\right)\\ &=3\left(\frac{1}{4}\mathbf{e_0}^2+4g'(\phi)^2\phi_r^2\,\mathbf{f}+\frac{1}{4}\,\mathbf{f}^2-\mathbf{m}^2\right)\\ &\le 3\left(\frac{1}{4}\mathbf{e_0}^2+\frac{c}{2}(R\phi)^2\,\mathbf{f}+\frac{1}{4}\,\mathbf{f}^2-\mathbf{m}^2\right)\\ &\le c\left(\frac{1}{4}\mathbf{e_0}^2+\frac{1}{2}\,\mathbf{e_0}\,\mathbf{f}+\frac{1}{4}\,\mathbf{f}^2-\mathbf{m}^2\right)\\ &=c(\mathbf{e}^2-\mathbf{m}^2)\\ &\le c\frac{\widehat{\mathcal{A}}^2\,\widehat{\mathcal{B}}^2}{r^2} \end{aligned}$$

where we have used the fact that both $\|\phi\|_{L^\infty}$ and $\|\beta\|_{L^\infty}\le c$. Consequently,

$$\begin{aligned} \partial_{\underline{u}}\widehat{\mathcal{A}}^2 =&\frac{1}{2}e^{\beta+\mathcal{F}}\left(-L_0+\kappa\, r^2\,e^{2\beta}S_1\right)\\ \ge&-\frac{1}{2}e^{\beta+\mathcal{F}}L_0. \end{aligned}$$

So,

$$\widehat{\mathcal{A}}\,\partial_{\underline{u}}\widehat{\mathcal{A}}\ge -c|L_0|\ge -c\frac{\widehat{\mathcal{A}}\,\widehat{\mathcal{B}}}{r}$$

that gives us

$$\partial_{\underline{u}}\widehat{\mathcal{A}}\ge -c\frac{\widehat{\mathcal{B}}}{r}$$

and similarly,

$$\partial_u\widehat{\mathcal{B}}\le c\frac{\widehat{\mathcal{A}}}{r}.$$

The rest of the proof is comparable to the case of wave maps on the Minkowski background as in [30] and [9]. Consider the region of spacetime $[\underline{u},0]\times[u_0,u]$ where $\underline{u},u\le 0$ and $u_0<0$ will be chosen later. Using the fundamental theorem of

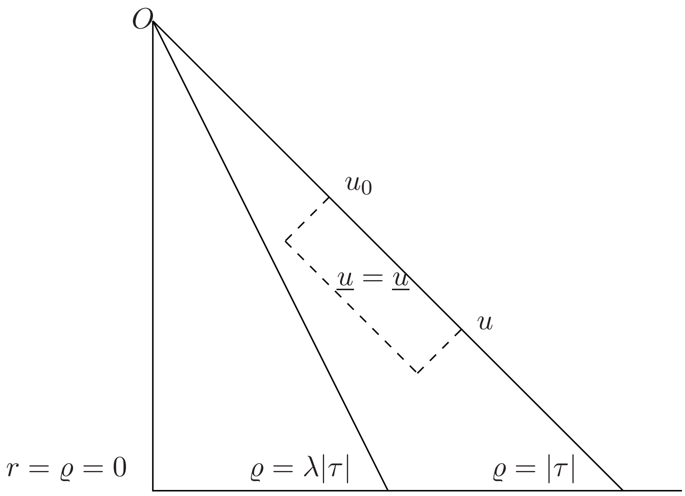

Figure 6. Application of the fundamental theorem of calculus for $\widehat{A}$ and $\widehat{B}$ in the region $[\underline{u},0]\times[u_0,u]$

calculus,

$$\begin{aligned}\widehat{\mathcal{B}}(\underline{u},u) &= \widehat{\mathcal{B}}(\underline{u},u_0) + \int_{u_0}^{u} \partial_{u'}\widehat{\mathcal{B}}(\underline{u},u')\,du' \\ &\le \widehat{\mathcal{B}}(\underline{u},u_0) + c\int_{u_0}^{u} \frac{\widehat{\mathcal{A}}(\underline{u},u')}{r(\underline{u},u')}\,du', \qquad (5.38)\\ \widehat{\mathcal{A}}(\underline{u},u') &= \widehat{\mathcal{A}}(0,u') - \int_{\underline{u}}^{0} \partial_{\underline{u}'}\widehat{\mathcal{A}}(\underline{u}',u')\,d\underline{u}' \\ &\le \widehat{\mathcal{A}}(0,u') + c\int_{\underline{u}}^{0} \frac{\widehat{\mathcal{B}}(\underline{u}',u')}{r(\underline{u}',u')}\,d\underline{u}'. \qquad (5.39)\end{aligned}$$

After plugging (5.39) in (5.38) we get

$$\begin{aligned}\widehat{\mathcal{B}}(\underline{u},u) &\le \widehat{\mathcal{B}}(\underline{u},u_0) + c\left(\int_{u_0}^{u} \frac{\widehat{\mathcal{A}}(0,u')}{r(\underline{u},u')}\,du' + \int_{u_0}^{u} \frac{1}{r(\underline{u},u')}\left(\int_{\underline{u}}^{0} \frac{\widehat{\mathcal{B}}(\underline{u}',u')}{r(\underline{u}',u')}\,d\underline{u}'\right)du'\right) \\ &= \widehat{\mathcal{B}}(\underline{u},u_0) + c\left(\int_{u_0}^{u} \frac{\widehat{\mathcal{A}}(0,u')}{r(\underline{u},u')}\,du'\right) + c\left(\int_{u_0}^{u}\int_{\underline{u}}^{0} \frac{\widehat{\mathcal{B}}(\underline{u}',u')}{r(\underline{u},u')r(\underline{u}',u')}\,d\underline{u}'\,du'\right). \qquad (5.40)\end{aligned}$$

Now consider the second term in the right hand side of (5.40), firstly recall

$$r(\underline{u},u') \ge c\varrho(\underline{u},u') = c\frac{1}{2}(\underline{u}-u'),$$

$$\begin{aligned}\int_{u_0}^{u} \frac{\widehat{\mathcal{A}}(0,u')}{r(\underline{u},u')}\,du' &\le \left(\int_{u_0}^{u} \widehat{\mathcal{A}}^2(0,u')\,du'\right)^{\frac{1}{2}} \left(\int_{u_0}^{u} \frac{1}{(\underline{u}-u')^2}\,du'\right)^{\frac{1}{2}} \\ &\le c\,\mathrm{Flux}^{\frac{1}{2}}(P_T)(u_0,u)\left(\frac{1}{\underline{u}-u} - \frac{1}{\underline{u}-u_0}\right)^{\frac{1}{2}} \\ &\le c\,\mathrm{Flux}^{\frac{1}{2}}(P_T)(u_0)\left(\frac{1}{\underline{u}-u}\right)^{\frac{1}{2}}. \qquad (5.41)\end{aligned}$$

We infer

$$\widehat{\mathcal{B}}(\underline{u},u) \le \widehat{\mathcal{B}}(\underline{u},u_0) + c\frac{\mathrm{Flux}^{\frac{1}{2}}(u_0)}{(\underline{u}-u)^{\frac{1}{2}}} + c\left(\int_{u_0}^{u}\int_{\underline{u}}^{0} \frac{\widehat{\mathcal{B}}(\underline{u}',u')}{r(\underline{u},u')r(\underline{u}',u')}\,d\underline{u}'\,du'\right). \qquad (5.42)$$

Let us define the function

$$\widehat{\mathcal{H}}(\underline{u},u) := \sup_{\underline{u}\le\underline{u}'\le 0} \sqrt{\underline{u}'-u}\,\widehat{\mathcal{B}}(\underline{u}',u).$$

We have,

$$\sqrt{\underline{u}'-u'}\,\widehat{\mathcal{B}}(\underline{u}',u') \le \sup_{\underline{u}\le\underline{u}'\le u} \sqrt{\underline{u}'-u'}\,\widehat{\mathcal{B}}(\underline{u}',u') = \widehat{\mathcal{H}}(\underline{u},u').$$

So,

$$\begin{aligned}(\underline{u}-u)^{\frac{1}{2}}\widehat{\mathcal{B}}(\underline{u},u) &\le \left(\frac{\underline{u}-u}{\underline{u}-u_0}\right)^{\frac{1}{2}}(\underline{u}-u_0)^{\frac{1}{2}}\widehat{\mathcal{B}}(\underline{u},u_0) + c\mathrm{Flux}^{\frac{1}{2}}(u_0) \\ &\quad + c\left(\int_{u_0}^{u}\int_{\underline{u}}^{0} \widehat{\mathcal{H}}(\underline{u},u')\frac{(\underline{u}-u)^{\frac{1}{2}}}{(\underline{u}-u')(\underline{u}'-u')^{3/2}}\,d\underline{u}'\,du'\right). \qquad (5.43)\end{aligned}$$

Now consider the function $p(\underline{u})$ defined as follows

$$p(\underline{u}) := \frac{\underline{u}-u}{\underline{u}-u_0}.$$

We have $\underline{u}-u \leq \underline{u}-u_0$ so $p \leq 1$. Also, p is increasing and hence

$$p(\underline{u}) \leq p(0). \tag{5.44}$$

Let us go back to the inequality (5.43) and use (5.44). We infer

$$\begin{aligned}(\underline{u}-u)^{\frac{1}{2}}\widehat{\mathcal{B}}(\underline{u},u) \leq \left(\frac{-u}{-u_0}\right)^{\frac{1}{2}}(\underline{u}-u_0)^{\frac{1}{2}}\widehat{\mathcal{B}}(\underline{u},u_0) + c\mathrm{Flux}^{\frac{1}{2}}(u_0)\\ + c\left(\int_{u_0}^{u}\widehat{\mathcal{H}}(\underline{u},u')\,\frac{(\underline{u}-u)^{\frac{1}{2}}}{(\underline{u}-u')}\left(\frac{1}{\sqrt{\underline{u}-u'}}-\frac{1}{\sqrt{-u'}}\right)d\,u'\right).\end{aligned} \tag{5.45}$$

Consequently,

$$\widehat{\mathcal{H}}(\underline{u},u) \leq \left(\frac{-u}{-u_0}\right)^{\frac{1}{2}}\widehat{\mathcal{H}}(\underline{u},u_0) + c\mathrm{Flux}^{\frac{1}{2}}(u_0) + c\int_{u_0}^{u}\widehat{\mathcal{H}}(\underline{u},u')\frac{\underline{u}}{u'(\underline{u}-u')}\,d\,u'. \tag{5.46}$$

Also, we have

$$\begin{aligned}\widehat{\mathcal{H}}(\underline{u},u_0) &= \sup_{\underline{u}\leq\underline{u}'\leq 0}\sqrt{\underline{u}'-u_0}\;\widehat{\mathcal{B}}(\underline{u}',u_0)\\ &\leq \sup_{\underline{u}\leq\underline{u}'\leq 0}\sqrt{\underline{u}'-u_0}\;\sup_{\underline{u}\leq\underline{u}'\leq 0}\widehat{\mathcal{B}}(\underline{u}',u_0)\\ &\leq c(u_0)\sqrt{-u_0},\end{aligned} \tag{5.47}$$

where we have used the fact that u is regular away from the axis so that $\widehat{\mathcal{B}}(\underline{u},u_0)$ is finite. So,

$$\widehat{\mathcal{H}}(\underline{u},u) \leq c(u_0)\sqrt{-u} + c\mathrm{Flux}^{\frac{1}{2}}(u_0) + c\int_{u_0}^{u}\widehat{\mathcal{H}}(\underline{u},u')\frac{\underline{u}}{u'(\underline{u}-u')}\,d\,u'. \tag{5.48}$$

Using Gronwall's lemma to obtain an estimate for $\widehat{\mathcal{H}}(\underline{u},u)$, we infer

$$\begin{aligned}\widehat{\mathcal{H}}(\underline{u},u) &\leq \sqrt{-u}c(u_0) + c\,\mathrm{Flux}^{\frac{1}{2}}(u_0)\\ &+ c\int_{u_0}^{u}\left(\sqrt{-u}c(u_0)+c\,\mathrm{Flux}^{\frac{1}{2}}(u_0)\right)\left(\frac{\underline{u}}{u'(\underline{u}-u')}\right)e^{\int_{u'}^{u}\frac{\underline{u}}{u''(\underline{u}-u'')}d\,u''}\,d\,u'.\end{aligned} \tag{5.49}$$

We have for $u_0 \leq u' \leq u$ and setting $\underline{u} = \lambda' u$ where $\lambda' := \frac{1-\lambda}{1+\lambda} < 1$

$$\begin{aligned}\int_{u'}^{u}\frac{\underline{u}}{u''(\underline{u}-u'')}d\,u'' &= \log\frac{u(\lambda' u-u')}{u'(\lambda' u-u)}\\ &\leq \log\frac{1}{1-\lambda'}.\end{aligned}$$

For any $\epsilon > 0$ we can choose an u_0 small enough such that

$$c\mathrm{Flux}^{\frac{1}{2}}(u_0) < \frac{\epsilon}{2}.$$

Furthermore, one can choose $u \in (u_0, 0)$ small enough such that

$$c(u_0)\sqrt{-u} < \frac{\epsilon}{2}.$$

So we have $\widehat{\mathcal{H}}(\underline{u},u) < \epsilon$ for $u_0 < u < 0$ small enough. Then,

$$\widehat{\mathcal{B}}(\underline{u},u) \le \frac{\widehat{\mathcal{H}}(\underline{u},u)}{\sqrt{\underline{u}-u}} \le \frac{\epsilon}{\sqrt{\underline{u}-u}}.$$

Now going back to the flux integrals in (5.33), we have

$$\begin{aligned}
\int_{\partial\mathcal{S}_1} e^{-\mathcal{G}}(\mathbf{e}+\mathbf{m})\bar{\mu}_u &\le c\int_{u}^{0} \widehat{\mathcal{B}}(\underline{u}',u)^2\, d\,\underline{u}' \\
&\le \epsilon \int_{u}^{0} \frac{1}{(\underline{u}'-u)}\, d\,\underline{u}' \\
&= \epsilon \int_{\lambda' u}^{0} \frac{1}{(\underline{u}'-u)}\, d\,\underline{u}' \\
&= \epsilon \log\frac{1}{1-\lambda'} \\
&< c\epsilon
\end{aligned} \tag{5.50}$$

and

$$\frac{1}{2}\int_{\partial\mathcal{S}_2} r\,\Omega^2\, e^{-\mathcal{F}}(\mathbf{e}-\mathbf{m})\; d\,u\wedge d\,\theta = \text{Flux}(P_X)(u_0,u) < \epsilon \tag{5.51}$$

for u_0, u small enough. Finally, in view of (5.33), (5.50) and (5.51), we conclude that $E^O_{\text{ext}}(\tau) \to 0$ as $\tau \to 0$. This concludes the proof of Lemma 5.12. □

5.6. **Local spacetime integral estimates.**

Lemma 5.13 (Non-concentration of integrated kinetic energy)**.** *Let the kinetic energy be defined as*

$$\mathbf{e}_{kin} := \frac{1}{2}e^{-2\alpha}\phi_t^2$$

then the spacetime integral of $\mathbf{e}_{kin}$ *does not concentrate in the past null cone of* O*, i.e.,*

$$\frac{1}{r_2(\tau)}\int_{K_\tau} \mathbf{e}_{kin}\,\bar{\mu}_g \to 0 \;\; as\;\; \tau\to 0$$

where $r_2(\tau)$ *is the radial function defined as in Lemma 5.12.*

Proof. Recall from (5.15) the computation of the vectorfield $P_{\mathcal{R}_1}$ and its divergence

$$\begin{aligned}
P_{\mathcal{R}_1} &= -\, re^{(1-k)\beta-\alpha}\,\mathbf{m}\,\partial_t + re^{-k\beta}(\mathbf{e}-\mathbf{f})\partial_r, \\
\nabla_\nu P^\nu_{\mathcal{R}_1} &= e^{-2\alpha}\phi_t^2.
\end{aligned}$$

Using the Stokes theorem as in (5.16) for $P_{\mathcal{R}_1}$

$$\int_{K(\tau,s)} \nabla_\nu P^\nu_{\mathcal{R}_1}\,\bar{\mu}_g = \int_{\Sigma^O_s} e^\alpha P^t_{\mathcal{R}_1}\,\bar{\mu}_q - \int_{\Sigma^O_\tau} e^\alpha P^t_{\mathcal{R}_1}\,\bar{\mu}_q + \text{Flux}(P_{\mathcal{R}_1})(\tau,s)$$

that is

$$\int_{K(\tau,s)} e^{-2\alpha}\phi_t^2\,\bar{\mu}_g = -\int_{\Sigma^O_s} re^\beta\mathbf{m}\,\bar{\mu}_q + \int_{\Sigma^O_\tau} re^\beta\mathbf{m}\,\bar{\mu}_q + \text{Flux}(P_{\mathcal{R}_1})(\tau,s) \tag{5.52}$$

where,

$$\begin{aligned}\mathrm{Flux}(P_{\mathcal{R}_1})(\tau,s) &= \int_{C(\tau,s)} d\,\underline{u}(P_{\mathcal{R}_1})\,\bar{\mu}_{\underline{u}}\\ &= \int_{C(\tau,s)} re^{\beta-\mathcal{F}}(\mathbf{e}-\mathbf{m}-\mathbf{f})\,\bar{\mu}_{\underline{u}}\\ &\le c\,r_2(\tau)\int_{C(\tau,s)}(\mathbf{e}-\mathbf{m})\,\bar{\mu}_{\underline{u}}\\ &= -\,c\,r_2(\tau)\mathrm{Flux}(P_T)(\tau,s).\end{aligned}$$

We infer

$$\begin{aligned}\int_{K(\tau,s)} e^{-2\alpha}\phi_t^2\,\bar{\mu}_g &\le \int_{\Sigma_s^O} re^{\beta}\mathbf{e}\,\bar{\mu}_q + \int_{\Sigma_\tau^O} re^{\beta}\mathbf{e}\;\bar{\mu}_q - c\,r_2(\tau)\mathrm{Flux}(P_T)(\tau,s)\\ &\le cr_2(s)\int_{\Sigma_s^O}\mathbf{e}\,\bar{\mu}_q + \int_{\Sigma_\tau^O} re^{\beta}\mathbf{e}\,\bar{\mu}_q - c\,r_2(\tau)\mathrm{Flux}(P_T)(\tau,s).\end{aligned}$$

Now let $s \to 0$. We get

$$\frac{1}{r_2(\tau)}\int_{K(\tau)} e^{-2\alpha}\phi_t^2\,\bar{\mu}_g \le \frac{1}{r_2(\tau)}\int_{\Sigma_\tau^O} re^{\beta}\mathbf{e}\,\bar{\mu}_q - c\,\mathrm{Flux}(P_T)(\tau).$$

Therefore,

$$\begin{aligned}\frac{1}{r_2(\tau)}\int_{K(\tau)} e^{-2\alpha}\phi_t^2\,\bar{\mu}_g &\le \frac{1}{r_2(\tau)}\int_{B_{r_2}(\tau)} re^{\beta}\mathbf{e}\,\bar{\mu}_q - c\;\mathrm{Flux}(P_T)(\tau)\\ &= \frac{1}{r_2(\tau)}\left(\int_{B_{r_1(\tau)}} re^{\beta}\mathbf{e}\,\bar{\mu}_q + \int_{B_{r_2(\tau)}\setminus B_{r_1(\tau)}} re^{\beta}\mathbf{e}\,\bar{\mu}_q\right)\\ &\quad - c\,\mathrm{Flux}(P_T)(\tau)\\ &\le c\,\lambda\,E_0 + c\,E^O_{\mathrm{ext}}(\tau) - c\,\mathrm{Flux}(P_T)(\tau).\end{aligned}$$

For any $\epsilon > 0$ we can choose λ small enough so that the first term $< \frac{\epsilon}{3}$, then we can make τ small enough so that $E^O_{\mathrm{ext}}(\tau) < \frac{\epsilon}{3}$ and $|\mathrm{Flux}(P_T)(\tau)| < \frac{\epsilon}{3}$ as discussed previously. This concludes the proof of Lemma 5.13. □

In Lemma 5.13 we proved that the spacetime integral of $e^{-2\alpha}\phi_t^2$ does not concentrate in the past null cone of O. In the following lemma we shall prove that the spacetime integral of rotational potential energy i.e.,

$$\int_{K_\tau}\frac{g^2(\phi)}{r^2}\bar{\mu}_g = \int_{K_\tau}\mathbf{f}\bar{\mu}_g$$

does not concentrate.

Lemma 5.14 (Non-concentration of integrated rotational potential energy)**.** *Let (N,h) be the target manifold satisfying the Grillakis condition* (1.5)*. Then the spacetime integral of rotational potential energy does not concentrate i.e.,*

$$\int_{K_\tau}\mathbf{f}\,\bar{\mu}_g \to 0 \;\; as \;\; \tau \to 0. \tag{5.53}$$

Proof. Recall from (5.14) the computation of the vectorfield $P_{\mathcal{R}_a}$ and its divergence

$$\begin{aligned} P_{\mathcal{R}_a} &= -e^{\beta-\alpha} r^a \, \mathbf{m}\partial_t + r^a \, (\mathbf{e}-\mathbf{f})\partial_r, \\ \nabla_\nu P^\nu_{\mathcal{R}_a} &= \frac{1}{2}\left((1+a)r^{a-1}\right) e^{-2\alpha}\phi_t^2 + \frac{1}{2}\left((a-1)r^{a-1}\right) e^{-2\beta}\phi_r^2 \\ &\quad + \frac{1}{2}\left((1-a)r^{a-1}\right)\frac{g^2(\phi)}{r^2}. \end{aligned}$$

Let now us construct the vector P^ν_ζ such that

$$P^\nu_\zeta := \zeta\phi^\nu\phi - \zeta^\nu\frac{\phi^2}{2},$$

where $\zeta := \frac{1-a}{2} r^{a-1}$ for $a \in (\frac{1}{2}, 1)$. Then the divergence is given by

$$\begin{aligned} \nabla_\nu P^\nu_\zeta &= \nabla_\nu(\zeta\phi^\nu\phi) - \nabla_\nu\left(\zeta^\nu\frac{\phi^2}{2}\right) \\ &= \zeta(\Box\phi)\phi + \zeta\phi^\nu\phi_\nu + \phi^\nu\zeta_\nu\phi - (\Box\zeta)\frac{\phi^2}{2} - \zeta^\nu\phi\phi_\nu \\ &= \zeta\frac{g(\phi)g'(\phi)\phi}{r^2} + \zeta\phi^\nu\phi_\nu - (\Box\zeta)\frac{\phi^2}{2} \end{aligned}$$

and

$$\begin{aligned} \Box\zeta &= e^{-2\beta}\left(\zeta_{rr} + \frac{\zeta_r}{r} + (\alpha_r - \beta_r)\zeta_r\right) \\ &= e^{-2\beta} r^{a-3}\frac{(1-a)^2}{2}\left(1 - a + r^2\kappa e^{2\beta}\,\mathbf{f}\right). \end{aligned}$$

Let us define a vector P^ν_{tot} such that

$$P^\nu_{\text{tot}} := P^\nu_{\mathcal{R}_a} + P^\nu_\zeta.$$

We have

$$\begin{aligned} \nabla_\nu P^\nu_{\text{tot}} &= \nabla_\nu P^\nu_{\mathcal{R}_a} + \nabla_\nu P^\nu_\zeta \\ &= \zeta\frac{g(\phi)g'(\phi)\phi}{r^2} + ar^{a-1}e^{-2\alpha}\phi_t^2 + \zeta\,\mathbf{f} - e^{-2\beta}\frac{(1-a)^2}{4} r^{a-1}\left(1 - a + r^2\kappa e^{2\beta}\mathbf{f}\right)\frac{\phi^2}{r^2} \\ &= \zeta\left[\frac{1}{r^2}B_1 + B_2\right], \end{aligned} \tag{5.54}$$

where

$$B_1 = g(\phi)g'(\phi)\phi + g^2(\phi) - e^{-2\beta}\frac{(1-a)^2}{2}\phi^2 - \frac{1-a}{2}\kappa g^2(\phi) \tag{5.55a}$$

$$B_2 = \frac{2a}{1-a}e^{-2\alpha}\phi_t^2 \tag{5.55b}$$

Applying the Stokes' theorem on $K(\tau, s)$,

$$\int_{K(\tau,s)} \nabla_\nu P^\nu_{\text{tot}}\,\bar{\mu}_g = \int_{\Sigma^O_s} e^\alpha\, P^t_{\text{tot}}\,\bar{\mu}_q - \int_{\Sigma^O_\tau} e^\alpha\, P^t_{\text{tot}}\,\bar{\mu}_q + \text{Flux}(P_{\text{tot}})(\tau, s). \tag{5.56}$$

We have

$$\begin{aligned} \int_{\Sigma^O_s} e^\alpha\, P^t_{\text{tot}}\,\bar{\mu}_q &= \int_{\Sigma^O_s} (-\mathbf{m}\, r^a e^\beta + e^\alpha\zeta\,\phi^t\,\phi)\,\bar{\mu}_q \\ &\leq \int_{\Sigma^O_s} \mathbf{e}\, r^a e^\beta + |\, e^{-\alpha}\,\phi_t|\,|\zeta\,\phi|\,\bar{\mu}_q, \end{aligned}$$

and applying the Cauchy-Schwarz inequality, we get

$$\begin{aligned}\int_{\Sigma_s^Q} e^{\alpha}\, P^t_{\text{tot}}\, \bar{\mu}_q &\leq c\, r_2^a(s) \int_{\Sigma_s^Q} \mathbf{e}\, \bar{\mu}_q + \frac{1-a}{2} \left(\int_{\Sigma_s^Q} e^{-2\alpha}\, \phi_t^2 r^{2a}\, \bar{\mu}_q \right)^{\frac{1}{2}} \left(\int_{\Sigma_s^Q} \frac{\phi^2}{r^2}\, \bar{\mu}_q \right)^{\frac{1}{2}} \\ &\leq c\, r_2^a(s). \end{aligned} \tag{5.57}$$

Similarly, the second term in (5.56) can be estimated as

$$-\int_{\Sigma_\tau^Q} e^{\alpha}\, P^t_{\text{tot}}\, \bar{\mu}_q \leq c \int_{\Sigma_\tau^Q} \mathbf{e}\, r^a\, \bar{\mu}_q + c \left(\int_{\Sigma_\tau^Q} \mathbf{e}\, r^{2a}\, \bar{\mu}_q \right)^{\frac{1}{2}}. \tag{5.58}$$

The flux of P_{tot} though the null surface $C(\tau, s)$ can be decomposed as

$$\text{Flux}(P_{\text{tot}})(t,s) = \text{Flux}(P_{\mathcal{R}_a})(t,s) + \text{Flux}(P_\zeta)(t,s). \tag{5.59}$$

Let us consider the terms in the right side of (5.59) individually. We have

$$\begin{aligned}\text{Flux}(P_{\mathcal{R}_a})(\tau,s) &= \int_{C(\tau,s)} d\,\underline{u}(P_{\mathcal{R}_a})\bar{\mu}_{\underline{u}} \\ &= \int_{C(\tau,s)} e^{\beta-\mathcal{F}} r^a(\mathbf{e}-\mathbf{m}-\mathbf{f})\bar{\mu}_{\underline{u}} \\ &\leq -c r_2^a(\tau)\, \text{Flux}(P_T)(\tau,s)\end{aligned}$$

and

$$\begin{aligned}\text{Flux}(P_\zeta)(\tau,s) &= \int_{C(\tau,s)} d\,\underline{u}(P_\zeta)\bar{\mu}_{\underline{u}} \\ &= \int_{C(\tau,s)} \left(\phi\,\zeta\, e^{-\mathcal{F}}\, (-T(\phi)+R(\phi)) + \frac{1}{2}\zeta e^{-(\beta+\mathcal{F})}(1-a) r^{-1}\phi^2) \right) \bar{\mu}_{\underline{u}} \\ &= \int_{C(\tau,s)} \left(\phi\,\zeta\, e^{-\mathcal{F}}\, (-T(\phi)+R(\phi)) + e^{-(\beta+\mathcal{F})} \frac{(1-a)^2}{4} \frac{\phi^2}{r^2} r^a \right) \bar{\mu}_{\underline{u}} \\ &\leq \int_{C(\tau,s)} \left(\phi\,\zeta\, e^{-\mathcal{F}}\, (-T(\phi)+R(\phi)) + c\, \frac{(1-a)^2}{4} \mathbf{f}\, r^a\, e^{-\mathcal{F}} \right) \bar{\mu}_{\underline{u}} \\ &\leq \int_{C(\tau,s)} \left(\phi\,\zeta\, e^{-\mathcal{F}}\, (-T(\phi)+R(\phi)) + c\, \frac{(1-a)^2}{2} (\mathbf{e}-\mathbf{m})\, r^a\, e^{-\mathcal{F}} \right) \bar{\mu}_{\underline{u}}.\end{aligned} \tag{5.60}$$

Using the Cauchy-Schwarz inequality, (5.62) can be estimated as

$$\begin{aligned}\text{Flux}(P_\zeta)(\tau,s) &\leq c r_2^a(\tau) \left(\int_{C\tau,s)} (\mathbf{e}-\mathbf{m})\bar{\mu}_{\underline{u}} \right) \\ &\leq -c r_2^a(\tau)\text{Flux}(P_T)(\tau,s).\end{aligned} \tag{5.61}$$

Therefore,

$$\text{Flux}(P_{\text{tot}})(t,s) \leq -c r_2^a(\tau)\text{Flux}(P_T)(\tau,s). \tag{5.62}$$

If follows from our previous estimates that exists a real constant c dependent only on the initial energy E_0 such that

$$\frac{1}{c}\phi^2 \leq g^2(\phi) \leq c\phi^2 \tag{5.63}$$

(see for example [30, Eq. (2.11)]). Using this fact and proceeding as in [31], we can use the Grillakis condition (1.5) to conclude that for a sufficiently close to 1 there is a small constant $c_a > 0$ such that

$$\frac{B_1}{r^2} + B_2 \geq c_a \left(e^{-2\beta} \frac{\phi^2}{r^2} + e^{-2\alpha} \phi_t^2 \right)$$

Now, if we go back to the Stokes' theorem (5.56) and use the estimates (5.57), (5.58) and (5.62), we get

$$c_a \left[\int_{K(\tau,s)} e^{-2\alpha} \phi_t^2 r^{a-1} d\bar{\mu}_g + \int_{K(\tau,s)} e^{-2\beta} \frac{\phi^2}{r^2} r^{a-1} \, \bar{\mu}_g \right]$$

$$\leq c\, r_2^a(s) + c \int_{\Sigma_\tau^O} \mathbf{e}\, r^a \, \bar{\mu}_q + c \left(\int_{\Sigma_\tau^O} \mathbf{e}\, r^{2a} \, \bar{\mu}_q \right)^{\frac{1}{2}} - c\, r_2^a(\tau) \mathrm{Flux}(P_T)(\tau, s).$$

As $s \to 0$ we get,

$$c_a \left[\int_{K(\tau)} e^{-2\alpha} \phi_t^2 r^{a-1} \, \bar{\mu}_g + \int_{K(\tau)} e^{-2\beta} \frac{\phi^2}{r^2} r^{a-1} \, \bar{\mu}_g \right]$$

$$\leq c \int_{\Sigma_\tau^O} \mathbf{e}\, r^a \, \bar{\mu}_q + c \left(\int_{\Sigma_\tau^O} \mathbf{e}\, r^{2a} \, \bar{\mu}_q \right)^{\frac{1}{2}} - c\, r_2^a(\tau) \mathrm{Flux}(P_T)(\tau). \tag{5.64}$$

In (5.64), we can estimate

$$\begin{aligned} r_2^{-a}(\tau) \int_{\Sigma_\tau^O} \mathbf{e}\, r^a \, \bar{\mu}_q &= r_2^{-a}(\tau) \left(\int_{B_{r_1(\tau)}} \mathbf{e}\, r^a \, \bar{\mu}_q + \int_{B_{r_2(\tau)} \setminus B_{r_1(\tau)}} \mathbf{e}\, r^a \, \bar{\mu}_q \right) \\ &\leq r_2^{-a}(\tau) \left(r_1^a(\tau) \int_{B_{r_1(\tau)}} \mathbf{e}\, \bar{\mu}_q + r_2^a(\tau) \int_{B_{r_2(\tau)} \setminus B_{r_1(\tau)}} \mathbf{e}\, \bar{\mu}_q \right) \\ &\leq \lambda^a E_0 + E^O_{\mathrm{ext}}(\tau) \end{aligned} \tag{5.65}$$

and

$$\begin{aligned} r_2^{-a}(\tau) \left(\int_{\Sigma_\tau^O} \mathbf{e}\, r^{2a} \, \bar{\mu}_q \right)^{\frac{1}{2}} &= r_2^{-a}(\tau) \left(\int_{B_{r_1(\tau)}} \mathbf{e}\, r^{2a} \, \bar{\mu}_q + \int_{B_{r_2(\tau)} \setminus B_{r_1(\tau)}} \mathbf{e}\, r^{2a} \, \bar{\mu}_q \right)^{\frac{1}{2}} \\ &\leq \left(\left(\frac{r_1(\tau)}{r_2(\tau)} \right)^{2a} \int_{B_{r_1(\tau)}} \mathbf{e}\, \bar{\mu}_q + \int_{B_{r_2(\tau)}} \mathbf{e}\, \bar{\mu}_q \right)^{\frac{1}{2}} \\ &\leq \left(\lambda^{2a} E_0 + E^O_{\mathrm{ext}}(\tau) \right)^{\frac{1}{2}}. \end{aligned} \tag{5.66}$$

Hence, in view of (5.65), (5.66), Corollary 5.6 and Lemma 5.12, we can choose λ and τ in (5.64) small enough so that

$$\frac{1}{r_2^a(\tau)} \int_{K(\tau)} \frac{\phi^2}{r^2} r^{a-1} \, \bar{\mu}_g < \epsilon$$

for any $\epsilon > 0$. In view of (5.63),

$$\mathbf{f} \leq \frac{c\phi^2}{r^2}.$$

Therefore it follows that

$$\frac{1}{r_2^a(\tau)} \int_{K_\tau} \mathbf{f}\; r^{a-1} \, \bar{\mu}_g \to 0 \;\text{ as }\; \tau \to 0.$$

This concludes the proof of Lemma 5.14. □

The remaining term in the energy is $e^{-2\beta}\phi_r^2$. We control it below.

Corollary 5.15. *Under the assumptions of Lemma 5.14, the spacetime integral of radial potential energy in the past null cone of O does not concentrate*

$$\frac{1}{r_2^a(\tau)}\int_{K_\tau} e^{-2\beta}\phi_r^2 r^{a-1}\,\bar{\mu}_g \to 0 \;\; as \;\; \tau \to 0. \tag{5.67}$$

Proof. Let us again apply the Stokes' theorem for the $\bar{\mu}_g$-divergence of $P_{\mathcal{R}_a}$

$$\int_{K(\tau,s)} \nabla_\mu P^\mu_{\mathcal{R}_a}\,\bar{\mu}_g = \int_{\Sigma^O_s} e^\alpha\, P^t_{\mathcal{R}_a}\,\bar{\mu}_q - \int_{\Sigma^O_\tau} e^\alpha\, P^t_{\mathcal{R}_a}\,\bar{\mu}_q + \mathrm{Flux}(P_{\mathcal{R}_a})(\tau, s)$$

therefore, as $s \to 0$

$$\int_{K(\tau)} e^{-2\beta}\phi_r^2\, r^{a-1}\,\bar{\mu}_g \le c\int_{K(\tau)}\left(e^{-2\alpha}\phi_t^2 + \frac{g^2(\phi)}{r^2}\right) r^{a-1}\,\bar{\mu}_g + \int_{\Sigma^O_\tau} \mathbf{e}\, r^a\,\bar{\mu}_q + r_2^a(\tau)\mathrm{Flux}(P_T)(\tau).$$

Hence,

$$\frac{1}{r_2^a(\tau)}\int_{K(\tau)} e^{-2\beta}\phi_r^2\, r^{a-1}\,\bar{\mu}_g < \epsilon$$

for τ small enough. This concludes the proof of the corollary. □

5.7. **Proof of non-concentration of energy.** We are now in position to conclude the proof of Theorem 5.1. If we collect the terms from Lemmas 5.13, 5.14 and Corollary 5.15, we get

$$\frac{1}{r_2^a(\tau)}\int_{K(\tau)} \mathbf{e}\, r^{a-1}\,\bar{\mu}_g \to 0$$

as $\tau \to 0$. But then,

$$\begin{aligned}\frac{1}{r_2^a(\tau)}\int_{K(\tau)} \mathbf{e}\, r^{a-1}\,\bar{\mu}_g &\ge c\,\frac{1}{r_2^a(\tau)}\int_{K(\tau)} \mathbf{e}\, r^{a-1}\,\bar{\mu}_q\, dt\\ &\ge c\frac{1}{r_2(\tau)}\int_{K(\tau)} \mathbf{e}\,\bar{\mu}_q\, dt\\ &\to 0\end{aligned} \tag{5.68}$$

as $\tau \to 0$. We claim that there exists a sequence $\{\tau_i\}_i$ such that

$$\int_{\Sigma^O_{\tau_i}} \mathbf{e}\,\bar{\mu}_q \to 0 \tag{5.69}$$

as $\{\tau_i\}_i \to 0$. Let us prove the claim by contradiction. Suppose there exists no sequence such that (5.69) holds true. Then there exists an $\epsilon > 0$ such that

$$\int_{\Sigma^O_\tau} \mathbf{e}\,\bar{\mu}_q > \epsilon$$

for all $\tau \in (-1, 0)$. Consequently,

$$\frac{1}{|\tau|}\int_{K(\tau)} \mathbf{e}\,\bar{\mu}_q\, dt > \epsilon.$$

This implies,

$$\frac{1}{r_2(\tau)}\int_{K(\tau)} \mathbf{e}\,\bar{\mu}_q\, dt > \epsilon \tag{5.70}$$

for all $\tau \in [-1, 0)$. This contradicts (5.68). Hence, there exists a $\{\tau_i\}_i$ such that

$$E^O(\tau_i) = \int_{\Sigma^O_{\tau_i}} \mathbf{e}\, \bar{\mu}_q \to 0. \tag{5.71}$$

But $E^O(\tau)$ is monotonic with respect to τ, therefore

$$E^O(\tau) \to 0$$

for all $\tau \to 0$ i.e., $E^O_{\text{conc}} = 0$. This concludes the proof of Theorem 5.1.

6. Global regularity of the $2+1$ Einstein-wave map problem

We now proceed to the proof of our main theorem, i.e. Theorem 1.8. Let $(M, \mathbf{g}, \phi)$ be the maximal Cauchy development of an asymptotically flat and regular Cauchy data set for the $2+1$ equivariant Einstein-wave map problem (1.8) with target (N, h). Assume that the metric h has the form

$$h = d\rho^2 + g^2(\rho) d\theta^2$$

for an odd function $g : \mathbb{R} \to \mathbb{R}$ with $g'(0) = 1$, such that g satisfies (1.11) and the Grillakis condition (1.5).

Our goal is to prove that $(M, \mathbf{g}, \phi)$ is regular. Assume by contradiction that this does not hold. Then, there exists a first singularity which in view of Theorem 3.5 occurs on the axis of symmetry Γ. Let us denote by O this first singularity which corresponds to $u = \underline{u} = 0$ and $t = r = 0$ in the $(u, \underline{u})$ and (t, r) coordinates systems constructed respectively in section 2 and section 4.

Let $\varepsilon > 0$ small to be chosen later. In view of Theorem 5.1, there exists a time $t_0 < 0$ such that

$$E^O(t_0) \leq \varepsilon. \tag{6.1}$$

Lemma 6.1. *Recall that $\tau = (u - \underline{u})/2$. There exists $\tau_0 < 0$ such that in the space-time region*

$$\{\tau_0 \leq \tau \leq 0\} \cap \{\underline{u} \leq 0\},$$

we have

$$\int_{2\tau_0 - \underline{u}}^{\underline{u}} \left((\partial_u \phi)^2 + \frac{g(\phi)^2}{r^2} \right) (u', \underline{u}) du' \lesssim \varepsilon$$

and

$$\int_{\max(u, 2\tau_0 - u)}^{0} \left((\partial_{\underline{u}} \phi)^2 + \frac{g(\phi)^2}{r^2} \right) (u, \underline{u}') d\underline{u}' \lesssim \varepsilon.$$

Proof. We choose $\tau_0 < 0$ small enough such that

$$\{\tau_0 \leq \tau \leq 0\} \cap \{\underline{u} \leq 0\} \subset \{t_0 \leq t \leq 0\} \cap J^-(O)$$

which is possible since on the one hand $J^-(O) = \{\underline{u} \leq 0\}$, and on the other hand t and τ are comparable in view of Corollary 5.11. In particular, together with (6.1), Stokes theorem, and the fact that the vectorfield P_T is divergence free, we infer

$$\left| \int_{2\tau_0 - \underline{u}}^{\underline{u}} \int_{\theta=0}^{2\pi} e^{-\mathcal{F}} (\mathbf{e} - \mathbf{m}) \bar{\mu}_{\underline{u}} \right| \leq \varepsilon$$

and

$$\left| \int_{\max(u, 2\tau_0 - u)}^{0} \int_{\theta=0}^{2\pi} e^{-\mathcal{G}} (\mathbf{e} + \mathbf{m}) \bar{\mu}_u \right| \leq \varepsilon$$

where we are relying on notations and computations introduced in section 5.4. In view of the definition of $\bar{\mu}_{\underline{u}}$ and $\bar{\mu}_u$, and the rotation invariance, we infer

$$\int_{2\tau_0-\underline{u}}^{\underline{u}} e^{-\mathcal{F}}(\mathbf{e}-\mathbf{m})r\Omega^2 du \lesssim \varepsilon,$$
$$\int_{\max(u,2\tau_0-u)}^{0} e^{-\mathcal{G}}(\mathbf{e}+\mathbf{m})r\Omega^2 d\underline{u} \lesssim \varepsilon.$$

In view of the definition of e and m and the identity (5.26), we deduce

$$\int_{2\tau_0-\underline{u}}^{\underline{u}} e^{\mathcal{G}}\left(e^{-2\mathcal{G}}(\partial_u\phi)^2+\frac{g(\phi)^2}{r^2}\right) rdu \lesssim \varepsilon,$$
$$\int_{\max(u,2\tau_0-u)}^{0} e^{\mathcal{F}}\left(e^{-2\mathcal{F}}(\partial_{\underline{u}}\phi)^2+\frac{g(\phi)^2}{r^2}\right) rd\underline{u} \lesssim \varepsilon.$$

Together with the estimates of Lemma 5.8 for $\mathcal{F}$ and $\mathcal{G}$, we obtain

$$\int_{2\tau_0-\underline{u}}^{\underline{u}} \left((\partial_u\phi)^2+\frac{g(\phi)^2}{r^2}\right) rdu \lesssim \varepsilon,$$
$$\int_{\max(u,2\tau_0-u)}^{0} \left((\partial_{\underline{u}}\phi)^2+\frac{g(\phi)^2}{r^2}\right) rd\underline{u} \lesssim \varepsilon.$$

This concludes the proof of the lemma. □

We now rescale the coordinates u and $\underline{u}$ by

$$u \to \frac{u}{|\tau_0|}, \quad \underline{u} \to \frac{\underline{u}}{|\tau_0|}.$$

The Einstein-wave map problem is invariant under this scaling so that in the rescaled $(u, \underline{u})$ coordinates, we have in view of Lemma 6.1

$$\int_{-2-\underline{u}}^{\underline{u}} \left((\partial_u\phi)^2+\frac{g(\phi)^2}{r^2}\right) r(u',\underline{u})du' \leq \varepsilon, \tag{6.2}$$
$$\int_{\max(u,-2-u)}^{0} \left((\partial_{\underline{u}}\phi)^2+\frac{g(\phi)^2}{r^2}\right) r(u,\underline{u}')d\underline{u}' \leq \varepsilon. \tag{6.3}$$

over the space-time region $\{-1 \leq \tau \leq 0\} \cap \{\underline{u} \leq 0\}$.

Theorem 6.2 (Small energy implies regularity)**.** *Let $(M, \mathbf{g}, \Phi)$ be a solution of the* 2+1 *equivariant Einstein-wave map problem* (1.8) *which is regular in the space-time region*

$$\{-1 \leq \tau < 0\} \cap \{\underline{u} \leq 0\}.$$

Assume furthermore the smallness condition (6.2) (6.3) *on the energy flux. Then, $(M, \mathbf{g}, \Phi)$ is regular on the closure of $\{-1 \leq \tau < 0\} \cap \{\underline{u} \leq 0\}$. In particular, there is no singularity at O.*

In view of Theorem 6.2, we infer that O can not be a first singularity of $(M, \mathbf{g}, \Phi)$, hence contradicting our assumption. Thus, global regularity holds for $(M, \mathbf{g}, \Phi)$. This concludes the proof of Theorem 1.8.

The rest of the paper is devoted to the proof of Theorem 6.2. In section 7 we derive a uniform weighted upper bound for ϕ. In section 8, we rely on the upper bound of section 7 to derive a uniform upper bound for $\partial\phi$. Finally, we rely on the upper bound of section 8 to conclude the proof of Theorem 6.2.

7. An improved uniform bound for ϕ

From now on, we will only work in the $(u, \underline{u})$ coordinate system. Recall from (2.1) our choice of normalization on Γ for the $(u, \underline{u})$ coordinates system

$$r = 0,\ \partial_{\underline{u}} r = \frac{1}{2},\ \partial_u r = -\frac{1}{2} \text{ and } \Omega = 1 \text{ on } \Gamma.$$

Also, recall that τ and ϱ are defined by

$$\tau = \frac{u + \underline{u}}{2},\ \varrho = \frac{\underline{u} - u}{2}.$$

We restrict our attention to the space-time region

$$\{-1 \leq \tau < 0\} \cap \{\underline{u} \leq 0\}$$

where our solution is regular, and we where intend to derive estimates which are uniform up to the origin O. We assume throughout the rest of the paper the smallness condition (6.2) (6.3) on the energy flux. Finally, recall from section 2.2 and section 2.3 that the 2+1 dimensional equivariant Einstein-wave map system is given in the $(u, \underline{u})$ coordinates by

$$\begin{cases} \partial_u(\Omega^{-2}\partial_u r) &= -\Omega^{-2} r\kappa(\partial_u\phi)^2, \\ \partial_{\underline{u}}(\Omega^{-2}\partial_{\underline{u}} r) &= -\Omega^{-2} r\kappa(\partial_{\underline{u}}\phi)^2, \\ \partial_u\partial_{\underline{u}} r &= r\kappa\frac{\Omega^2}{4}\frac{g(\phi)^2}{r^2}, \\ \Omega^{-2}(\partial_u\Omega\partial_{\underline{u}}\Omega - \Omega\partial_u\partial_{\underline{u}}\Omega) &= \frac{1}{8}\Omega^2\kappa\left(\frac{4}{\Omega^2}\partial_u\phi\partial_{\underline{u}}\phi + \frac{g(\phi)^2}{r^2}\right) \\ \frac{2}{r\Omega^2}\left(-\partial_u(r\partial_{\underline{u}}\phi) - \partial_u(r\partial_{\underline{u}}\phi)\right) &= \frac{f(\phi)}{r^2} \end{cases}$$

where $f(\phi) = g(\phi)g'(\phi)$. Since g is odd with $g'(0) = 1$, note that there exists a smooth function ζ such that

$$f(\phi) = \phi + \phi^3\zeta(\phi).$$

7.1. **Preliminary estimates.** We start with simple consequences of the smallness condition (6.2) (6.3) on the energy flux.

Lemma 7.1. *We have*

$$|\phi| \lesssim \sqrt{\varepsilon}.$$

Proof. The proof is in the same spirit as Lemma 5.4. Let

$$\wp(\phi) := \int_0^\phi g(s)\, ds.$$

Then, since ϕ vanishes on Γ, we have for $\underline{u} < 0$,

$$\begin{aligned} \wp(\phi(u, \underline{u})) &= \int_{\underline{u}}^u \partial_u(\wp(\phi(u', \underline{u})))\, du' \\ &= \int_{\underline{u}}^u g(\phi)\partial_u\phi(u', \underline{u})\, du'. \end{aligned}$$

Together with (6.2), we infer

$$\begin{aligned} |\wp(\phi(u, \underline{u}))| &\leq \left(\int_u^{\underline{u}} \frac{g(\phi)^2}{r^2} r(u', \underline{u})\, du'\right)^{\frac{1}{2}} \left(\int_u^{\underline{u}} (\partial_u\phi)^2 r(u', \underline{u})\, du'\right)^{\frac{1}{2}} \\ &\leq \varepsilon. \end{aligned}$$

Since $\wp(0) = 0$, $\wp'(0) = 0$ and $\wp''(0) = 1$, we have

$$\wp(\phi) = \frac{\phi^2}{2} + O(\phi^3)$$

and hence

$$|\phi| \lesssim \sqrt{\varepsilon}.$$

This concludes the proof of the lemma. □

Lemma 7.2. *We have*

$$\left|\partial_{\underline{u}} r - \frac{1}{2}\right| \lesssim \varepsilon, \quad \left|\partial_u r + \frac{1}{2}\right| \lesssim \varepsilon, \quad |r - \varrho| \lesssim \varepsilon\varrho, \quad |\Omega - 1| \lesssim \varepsilon.$$

Proof. Integrating from the axis of symmetry Γ the following equation

$$\partial_u \partial_{\underline{u}} r = r\kappa \frac{\Omega^2}{4} \frac{g(\phi)^2}{r^2}$$

and in view of the smallness condition (6.2) (6.3) on the energy flux, and the initialization on Γ, we deduce

$$\left|\partial_{\underline{u}} r - \frac{1}{2}\right| \lesssim \varepsilon, \left|\partial_u r + \frac{1}{2}\right| \lesssim \varepsilon.$$

Moreover, in view of the definition of ϱ and the initialization on Γ, we have

$$r - \varrho = \partial_u (r - \varrho) = \partial_{\underline{u}} (r - \varrho) = 0 \text{ on } \Gamma$$

which together with the smallness condition (6.2) (6.3) on the energy flux and the fact that

$$\partial_u \partial_{\underline{u}} \varrho = 0,$$

yields

$$|r - \varrho| \lesssim \varepsilon\varrho.$$

Finally, the control of ∂_u together with the integration from the axis of symmetry Γ of the equation

$$\partial_u (\Omega^{-2} \partial_u r) = -\Omega^{-2} r\kappa (\partial_u \phi)^2,$$

the initialization on Γ and the smallness condition (6.2) on the energy flux yields

$$|\Omega - 1| \lesssim \varepsilon.$$

This concludes the proof of the lemma. □

7.2. Reduction to a semilinear wave equation.

Lemma 7.3. *Let ϕ a function depending only on u and $\underline{u}$. Then, we have*

$$\begin{aligned} \square_{\mathbf{g}}(\phi) &= \frac{1}{\Omega^2}\Bigg(- 4\partial_u \partial_{\underline{u}} \phi + \frac{1}{\varrho}(\partial_{\underline{u}} \phi - \partial_u \phi) \\ &\quad + \frac{\varrho - r}{r\varrho}(\partial_{\underline{u}} \phi - \partial_u \phi) - \frac{2\partial_u r + 1}{r} \partial_{\underline{u}} \phi - \frac{2\partial_{\underline{u}} r - 1}{r} \partial_u \phi \Bigg). \end{aligned}$$

Proof. Recall that

$$\square_{\mathbf{g}}(\phi) = \frac{2}{r\Omega^2} \left(-\partial_u (r \partial_{\underline{u}} \phi) - \partial_u (r \partial_{\underline{u}} \phi) \right).$$

We infer

$$
\begin{aligned}
\square_{\mathbf{g}}(\phi) &= \frac{1}{\Omega^2}\left(-4\partial_u\partial_{\underline{u}}\phi - \frac{2\partial_u r}{r}\partial_{\underline{u}}\phi - \frac{2\partial_{\underline{u}} r}{r}\partial_u\phi\right)\\
&= \frac{1}{\Omega^2}\left(-4\partial_u\partial_{\underline{u}}\phi + \frac{1}{r}(\partial_{\underline{u}}\phi - \partial_u\phi) - \frac{2\partial_u r+1}{r}\partial_{\underline{u}}\phi - \frac{2\partial_{\underline{u}} r-1}{r}\partial_u\phi\right)\\
&= \frac{1}{\Omega^2}\Bigg(-4\partial_u\partial_{\underline{u}}\phi + \frac{1}{\varrho}(\partial_{\underline{u}}\phi - \partial_u\phi)\\
&\quad + \frac{\varrho - r}{r\varrho}(\partial_{\underline{u}}\phi - \partial_u\phi) - \frac{2\partial_u r+1}{r}\partial_{\underline{u}}\phi - \frac{2\partial_{\underline{u}} r-1}{r}\partial_u\phi\Bigg).
\end{aligned}
$$

This concludes the proof of the lemma. □

We deduce the following corollary.

Corollary 7.4. *We have*

$$
\begin{aligned}
&-4\partial_u\partial_{\underline{u}}\phi + \frac{1}{\varrho}(\partial_{\underline{u}}\phi - \partial_u\phi) - \frac{\phi}{\varrho^2}\\
= &-\frac{\varrho - r}{r\varrho}(\partial_{\underline{u}}\phi - \partial_u\phi) + \frac{2\partial_u r+1}{r}\partial_{\underline{u}}\phi + \frac{2\partial_{\underline{u}} r-1}{r}\partial_u\phi + \frac{\varrho^2 - r^2}{r^2\varrho^2}\phi + \frac{\phi^3\zeta(\phi)}{r^2}\\
&+(\Omega^2 - 1)\frac{f(\phi)}{r^2}.
\end{aligned}
$$

Proof. In view of the previous lemma, we have

$$
\begin{aligned}
&\frac{1}{\Omega^2}\Bigg(-4\partial_u\partial_{\underline{u}}\phi + \frac{1}{\varrho}(\partial_{\underline{u}}\phi - \partial_u\phi)\\
&+\frac{\varrho - r}{r\varrho}(\partial_{\underline{u}}\phi - \partial_u\phi) - \frac{2\partial_u r+1}{r}\partial_{\underline{u}}\phi - \frac{2\partial_{\underline{u}} r-1}{r}\partial_u\phi\Bigg) = \frac{f(\phi)}{r^2},
\end{aligned}
$$

which we rewrite

$$
\begin{aligned}
&-4\partial_u\partial_{\underline{u}}\phi + \frac{1}{\varrho}(\partial_{\underline{u}}\phi - \partial_u\phi)\\
= &-\frac{\varrho - r}{r\varrho}(\partial_{\underline{u}}\phi - \partial_u\phi) + \frac{2\partial_u r+1}{r}\partial_{\underline{u}}\phi + \frac{2\partial_{\underline{u}} r-1}{r}\partial_u\phi + \Omega^2\frac{f(\phi)}{r^2}.
\end{aligned}
$$

We have

$$
\begin{aligned}
\Omega^2\frac{f(\phi)}{r^2} &= \frac{f(\phi)}{r^2} + (\Omega^2 - 1)\frac{f(\phi)}{r^2}\\
&= \frac{\phi}{r^2} + \frac{\phi^3\zeta(\phi)}{r^2} + (\Omega^2 - 1)\frac{f(\phi)}{r^2}\\
&= \frac{\phi}{\varrho^2} + \frac{\varrho^2 - r^2}{r^2\varrho^2}\phi + \frac{\phi^3\zeta(\phi)}{r^2} + (\Omega^2 - 1)\frac{f(\phi)}{r^2}.
\end{aligned}
$$

We infer

$$
\begin{aligned}
&-4\partial_u\partial_{\underline{u}}\phi + \frac{1}{\varrho}(\partial_{\underline{u}}\phi - \partial_u\phi) - \frac{\phi}{\varrho^2}\\
= &-\frac{\varrho - r}{r\varrho}(\partial_{\underline{u}}\phi - \partial_u\phi) + \frac{2\partial_u r+1}{r}\partial_{\underline{u}}\phi + \frac{2\partial_{\underline{u}} r-1}{r}\partial_u\phi + \frac{\varrho^2 - r^2}{r^2\varrho^2}\phi + \frac{\phi^3\zeta(\phi)}{r^2}\\
&+(\Omega^2 - 1)\frac{f(\phi)}{r^2}.
\end{aligned}
$$

This conclues the proof of the corollary. □

Corollary 7.5. *We have*

$$\left(-\partial_\tau^2+\partial_\varrho^2+\frac{1}{\varrho}\partial_\varrho-\frac{1}{\varrho^2}\right)\phi=\frac{F}{\varrho^2}$$

where

$$\begin{aligned}F &= -\frac{\varrho(\varrho-r)}{r}(\partial_{\underline{u}}\phi-\partial_u\phi)+\frac{\varrho^2(2\partial_u r+1)}{r}\partial_{\underline{u}}\phi+\frac{\varrho^2(2\partial_{\underline{u}}r-1)}{r}\partial_u\phi+\frac{\varrho^2-r^2}{r^2}\phi+\frac{\varrho^2\phi^3\zeta(\phi)}{r^2}\\ &\quad+\varrho^2(\Omega^2-1)\frac{f(\phi)}{r^2}.\end{aligned}$$

Proof.

$$\partial_u=\frac{1}{2}(\partial_\tau-\partial_\varrho)\text{ and }\partial_{\underline{u}}=\frac{1}{2}(\partial_\tau+\partial_\varrho)$$

and hence

$$\partial_u\partial_{\underline{u}}=\frac{1}{4}(\partial_\tau^2-\partial_\varrho^2)\text{ and }\frac{1}{\varrho}(\partial_{\underline{u}}-\partial_u)=\frac{1}{\varrho}\partial_\varrho.$$

Thus, we have

$$-4\partial_u\partial_{\underline{u}}+\frac{1}{\varrho}(\partial_{\underline{u}}-\partial_u)-\frac{1}{\varrho^2}=-\partial_\tau^2+\partial_\varrho^2+\frac{1}{\varrho}\partial_\varrho-\frac{1}{\varrho^2}.$$

In view of the previous corollary, this concludes the proof of this corollary. □

7.3. **Set up of the bootstrap procedure.** Let

$$-1\le \underline{u}_b<0$$

and the space-time domain

$$Q_{\underline{u}_b}=\{-1\le\underline{u}<\underline{u}_b,\ -1\le u<0\}.$$

Let

$$0<\delta<\frac{1}{2}.$$

We make the following bootstrap assumption on $Q_{\underline{u}_b}$:

$$\sup_{Q_{\underline{u}_b}} r^{1-\delta}|\partial_{\underline{u}}\phi|\le C. \tag{7.1}$$

The goal of this section will be to prove that we can improve this bootstrap assumption.

Proposition 7.6. *Assume that*

$$0<\delta<\frac{1}{2}.$$

Then, there exists a universal constant $\underline{C}$ and a constant C_0 only depending on initial data such that for any $-1\le\underline{u}_b<0$, we have

$$\sup_{Q_{\underline{u}_b}} r^{1-\delta}|\partial_{\underline{u}}\phi|\le\underline{C}(C_0+\varepsilon C).$$

Remark 7.7. *The constant $\underline{C}$ in Proposition 7.6 depends on δ such that $0<\delta<1/2$. In particular, it degenerates as $\delta\to 0$ of $\delta\to 1/2$. We will use the improved estimate of Proposition 7.6 at two places in the proof of Theorem 6.2*

- *In Lemma 7.22, where we apply Proposition 7.6 with any fixed δ such that*

$$\frac{1}{6}<\delta<\frac{1}{2}.$$

- *In Proposition 8.13, where we apply Proposition 7.6 with any fixed δ such that*

$$\frac{1}{3} < \delta < \frac{1}{2}.$$

Thus, we could replace δ for instance with $5/12$ in the statement of Proposition 7.6. To make the proof easier to follow, we choose to do it with a general δ, but do not mention the dependence of various constants on δ since one should think of a δ fixed once and for all, e.g. $5/12$.

Remark 7.8. *The constant C_0 appearing in Proposition 7.6 denotes the supremum of the (finitely many) norms of the initial data appearing in the proof of Proposition 7.6 below. These norms are not controlled by the energy, and could be in particular arbitrary large compared to ε^{-1}. It is thus crucial that the constant in front of C in the statement of Proposition 7.6, i.e. $\underline{C}\varepsilon$, does not depend on C_0. This will allow us to improve on our bootstrap assumption (7.1) in Corollary 7.21 by choosing ε sufficiently small compared to the universal constant $\underline{C}$.*

Remark 7.9. *In order to prove Proposition 7.6, we will first obtain an improved bound for ϕ using a representation formula for the wave equation (see Lemma 7.18). Then, we infer an improved bound for $\partial_{\underline{u}}\phi$ using Lemma 7.19. Note that we can not infer a improved bound for $\partial_u\phi$ in this way (see Remark 7.20). This explains why we only have a bootstrap assumption for $\partial_{\underline{u}}\phi$ in Proposition 7.6, while the terms $\partial_u\phi$ will have to be integrated by parts (see Remark 7.13).*

Remark 7.10. *The non-concentration of energy is used in two crucial places in the proof of Theorem 6.2. One chooses $\varepsilon > 0$ small enough*

- *In Corollary 7.21 to improve the bootstrap assumption (7.1) thanks to Proposition 7.6.*
- *In Proposition 8.13 in order to exploit the estimate of Corollary 8.12.*

7.4. **First consequences of the bootstrap assumptions.**

Lemma 7.11. *We have*

$$\sup_{Q_{\underline{u}_b}} r^{-\delta}|\phi| \lesssim C.$$

and

$$\sup_{Q_{\underline{u}_b}} r^{-2\delta}\left(\frac{|r-\varrho|}{\varrho} + \left|\partial_u r + \frac{1}{2}\right| + \left|\partial_{\underline{u}} r - \frac{1}{2}\right| + |\Omega - 1|\right) \lesssim C^2.$$

Proof. We start with ϕ. We have

$$\phi(u,\underline{u}) = \int_u^{\underline{u}} \partial_{\underline{u}}\phi(u,\sigma)d\sigma,$$

and hence using the bootstrap assumption (7.1)

$$\begin{aligned} |\phi(u,\underline{u})| &\leq \int_u^{\underline{u}} \partial_{\underline{u}}|\phi(u,\sigma)|d\sigma \\ &\leq C\int_u^{\underline{u}} r(u,\sigma)^{\delta-1}d\sigma \\ &\lesssim C\int_u^{\underline{u}} (\sigma-u)^{\delta-1}d\sigma \\ &\lesssim C(\underline{u}-u)^{\delta} \\ &\lesssim Cr(u,\underline{u})^{\delta}, \end{aligned}$$

where we used the fact that $\delta > 0$.

Next, recall that

$$\partial_u\partial_{\underline{u}}r = r\kappa\frac{\Omega^2}{4}\frac{g(\phi)^2}{r^2}.$$

We infer

$$\begin{aligned}\left|\partial_u r + \frac{1}{2}\right| &\leq \int_u^{\underline{u}} |\partial_{\underline{u}}\partial_u r|(u,\sigma)d\sigma \\ &\lesssim \int_u^{\underline{u}} \frac{g(\phi)^2}{r}(u,\sigma)d\sigma \\ &\lesssim C^2\int_u^{\underline{u}} \frac{\phi^2}{r}(u,\sigma)d\sigma \\ &\lesssim C^2\int_u^{\underline{u}} r(u,\sigma)^{2\delta-1}d\sigma \\ &\lesssim C^2\int_u^{\underline{u}} (\sigma-u)^{2\delta-1}d\sigma \\ &\lesssim C^2(\underline{u}-u)^{2\delta} \\ &\lesssim C^2 r(u,\underline{u})^{2\delta},\end{aligned}$$

where we used the fact that $\delta > 0$ and the previous bound on ϕ.

Similarly, we have

$$\begin{aligned}\left|\partial_{\underline{u}}r - \frac{1}{2}\right| &\leq \int_u^{\underline{u}} |\partial_{\underline{u}}\partial_u r|(\sigma,\underline{u})d\sigma \\ &\lesssim C^2 r(u,\underline{u})^{2\delta}.\end{aligned}$$

Next, we consider the bound for $r-\varrho$. We have

$$\begin{aligned}|r-\varrho| &\leq \int_u^{\underline{u}} \left|\partial_{\underline{u}}r - \frac{1}{2}\right|(u,\sigma)d\sigma \\ &\lesssim C^2\int_u^{\underline{u}} r(u,\underline{u})^{2\delta}(u,\sigma)\sigma \\ &\lesssim C^2\int_u^{\underline{u}} (\sigma-u)^{2\delta}\sigma \\ &\lesssim C^2(\underline{u}-u)^{2\delta+1} \\ &\lesssim C^2 r(u,\underline{u})^{2\delta+1}.\end{aligned}$$

Finally, we consider Ω. We have

$$\partial_{\underline{u}}(\Omega^{-2}\partial_{\underline{u}}r) = -\Omega^{-2}r\kappa(\partial_{\underline{u}}\phi)^2.$$

This yields

$$\begin{aligned}\left|\Omega^{-2}\partial_{\underline{u}}r - \frac{1}{2}\right| &\leq \int_u^{\underline{u}} |\partial_{\underline{u}}\partial_{\underline{u}}ur|(u,\sigma)d\sigma \\ &\lesssim \int_u^{\underline{u}} r(\partial_{\underline{u}}\phi)^2(u,\sigma)d\sigma \\ &\lesssim C^2\int_u^{\underline{u}} r(u,\sigma)^{2\delta-1}d\sigma \\ &\lesssim C^2 r(u,\underline{u})^{2\delta}.\end{aligned}$$

We infer

$$
\begin{aligned}
|\Omega^{-2}-1| &\lesssim \left|\Omega^{-2}\partial_{\underline{u}} r - \partial_{\underline{u}} r\right| \\
&\lesssim \left|\Omega^{-2}\partial_{\underline{u}} r - \frac{1}{2}\right| + \left|\partial_{\underline{u}} r - \frac{1}{2}\right| \\
&\lesssim C^2 r(u,\underline{u})^{2\delta}.
\end{aligned}
$$

This concludes the proof of the lemma. □

7.5. **An improved uniform bound for ϕ.** Here we derive an improved uniform bound for $r^{-\delta}\phi$ relying on an explicit representation formula for the flat wave equation. Our approach is inspired by [9] (see also [30] for a similar approach).

Lemma 7.12. *We have*

$$
\left(-\partial_\tau^2 + \partial_\varrho^2 + \frac{1}{\varrho}\partial_\varrho - \frac{1}{\varrho^2}\right)\phi = \partial_u\left(\frac{F_1}{\varrho}\right) + \frac{F_2}{\varrho^2},
$$

where

$$
F_1 = \frac{\varrho - r + \varrho(2\partial_{\underline{u}} r - 1)}{r}\phi,
$$

and

$$
\begin{aligned}
F_2 = & -\frac{\varrho(\varrho - r + \varrho(2\partial_u r + 1))}{r}\partial_{\underline{u}}\phi + \frac{r(\varrho - r) + \varrho^2(2\partial_u r + 1) + \varrho^2(2\partial_{\underline{u}} r - 1)\partial_u r}{r^2}\phi \\
& -\frac{\kappa\varrho^2\Omega^2 g(\phi)^2\phi}{2r^2} + \frac{\varrho^2\phi^3\zeta(\phi)}{r^2} + \varrho^2(\Omega^2-1)\frac{f(\phi)}{r^2}.
\end{aligned}
$$

Remark 7.13. *Since we have no control over $\partial_u\phi$ (see Remark 7.9), we need to integrate the terms involving $\partial_u\phi$ by parts. This results in the term $\partial_u(F_1/\varrho)$ in the statement of Lemma 7.12. The fact that this integration by parts is possible is a consequence of the following two observations*

- *We are able to estimate F_2 (see Lemma 7.14), which itself is a consequence of the null structure of the problem.*
- *We are able to control the u derivative of the kernel K of the representation formula for the wave equation of Lemma 7.15. To achieve this, the crucial estimate is the one of Lemma 7.16.*

Proof. Recall that

$$
\left(-\partial_\tau^2 + \partial_\varrho^2 + \frac{1}{\varrho}\partial_\varrho - \frac{1}{\varrho^2}\right)\phi = \frac{F}{\varrho^2}
$$

where

$$
\begin{aligned}
F = & -\frac{\varrho(\varrho - r)}{r}\partial_\varrho\phi + \frac{\varrho^2(2\partial_u r + 1)}{r}\partial_{\underline{u}}\phi + \frac{\varrho^2(2\partial_{\underline{u}} r - 1)}{r}\partial_u\phi + \frac{\varrho^2 - r^2}{r^2}\phi + \frac{\varrho^2\phi^3\zeta(\phi)}{r^2} \\
& + \varrho^2(\Omega^2-1)\frac{f(\phi)}{r^2}.
\end{aligned}
$$

We rewrite F as

$$
\begin{aligned}
F \quad = \quad & -\frac{\varrho(\varrho-r)}{r}\partial_{\underline{u}}\phi+\partial_u\left(\frac{\varrho(\varrho-r)}{r}\phi\right)-\partial_u\left(\frac{\varrho(\varrho-r)}{r}\right)\phi+\frac{\varrho^2(2\partial_u r+1)}{r}\partial_{\underline{u}}\phi\\
&+\partial_u\left(\frac{\varrho^2(2\partial_{\underline{u}}r-1)}{r}\phi\right)-\partial_u\left(\frac{\varrho^2(2\partial_{\underline{u}}r-1)}{r}\right)\phi+\frac{\varrho^2-r^2}{r^2}\phi+\frac{\varrho^2\phi^3\zeta(\phi)}{r^2}\\
&+\varrho^2(\Omega^2-1)\frac{f(\phi)}{r^2}\\
= \quad & \partial_u\left(\frac{\varrho(\varrho-r)}{r}\phi+\frac{\varrho^2(2\partial_{\underline{u}}r-1)}{r}\phi\right)-\frac{\varrho(\varrho-r)}{r}\partial_{\underline{u}}\phi-\frac{(r-\varrho)^2-\varrho^2(2\partial_u r+1)}{r^2}\phi\\
&+\frac{\varrho^2(2\partial_u r+1)}{r}\partial_{\underline{u}}\phi+\frac{\varrho(2\partial_{\underline{u}}r-1)(\varrho\partial_u r+r)-2\varrho^2 r\partial_u\partial_{\underline{u}}r}{r^2}\phi+\frac{\varrho^2-r^2}{r^2}\phi+\frac{\varrho^2\phi^3\zeta(\phi)}{r^2}\\
&+\varrho^2(\Omega^2-1)\frac{f(\phi)}{r^2}\\
= \quad & \partial_u\left(\frac{\varrho(\varrho-r+\varrho(2\partial_{\underline{u}}r-1))}{r}\phi\right)-\frac{\varrho(\varrho-r+\varrho(2\partial_u r+1))}{r}\partial_{\underline{u}}\phi\\
&+\frac{2r(\varrho-r)+\varrho^2(2\partial_u r+1)+\varrho(2\partial_{\underline{u}}r-1)(\varrho\partial_u r+r)}{r^2}\phi-\frac{\kappa\varrho^2\Omega^2 g(\phi)^2\phi}{2r^2}+\frac{\varrho^2\phi^3\zeta(\phi)}{r^2}\\
&+\varrho^2(\Omega^2-1)\frac{f(\phi)}{r^2}.
\end{aligned}
$$

This yields

$$
\frac{F}{\varrho^2} \quad = \quad \partial_u\left(\frac{F_1}{\varrho}\right)+\frac{F_2}{\varrho^2},
$$

where

$$
F_1 \quad = \quad \frac{\varrho-r+\varrho(2\partial_{\underline{u}}r-1)}{r}\phi,
$$

and

$$
\begin{aligned}
F_2 \quad = \quad & -\frac{\varrho(\varrho-r+\varrho(2\partial_u r+1))}{r}\partial_{\underline{u}}\phi+\frac{r(\varrho-r)+\varrho^2(2\partial_u r+1)+\varrho^2(2\partial_{\underline{u}}r-1)\partial_u r}{r^2}\phi\\
&-\frac{\kappa\varrho^2\Omega^2 g(\phi)^2\phi}{2r^2}+\frac{\varrho^2\phi^3\zeta(\phi)}{r^2}+\varrho^2(\Omega^2-1)\frac{f(\phi)}{r^2}.
\end{aligned}
$$

This concludes the proof of the lemma. □

Lemma 7.14. *We have*

$$
\sup_{Q_{\underline{u}_b}} r^{-\delta}(|F_1|+|F_2|) \lesssim C\varepsilon.
$$

Proof. This is an immediate consequence of Lemma 7.1, Lemma 7.2, the bootstrap assumption (7.1) and Lemma 7.11, as well as the definition of F_1 and F_2. □

Lemma 7.15. *We have*

$$\begin{aligned}\phi(\tau,\varrho) &= \phi_0(\tau,\varrho)+\frac{K(-1)}{\sqrt{\varrho}}\int_{\tau-\varrho}^{\tau+\varrho}\frac{F_1\left(\frac{\tau-\varrho+\underline{u}'}{2},\frac{-\tau+\varrho+\underline{u}'}{2}\right)}{\sqrt{\frac{-\tau+\varrho+\underline{u}'}{2}}}d\underline{u}'-\frac{1}{2}\sqrt{\varrho}\int_0^{+\infty}\frac{F_1(-1,\lambda)}{\sqrt{\lambda}(\mu\varrho+\lambda)}K(\mu)d\mu\\ &\quad -\frac{1}{2\sqrt{\varrho}}\int_{\sqrt{(\tau+1)^2-\varrho^2}}^{\tau+\varrho+1}K(\mu)\frac{F_1(-1,\lambda)}{\sqrt{\lambda}}d\lambda\\ &\quad +\int_{-1}^{+\infty}\int_0^{\lambda^*}\frac{\sqrt{\varrho}}{\sqrt{\lambda}\sqrt{\varrho^2+\lambda^2+2\varrho\lambda\mu}}K(\mu)\left(\frac{1}{4}F_1(\sigma,\lambda)+F_2(\sigma,\lambda)\right)d\lambda d\mu\\ &\quad -\int_{-1}^{+\infty}\int_0^{\lambda^*}\frac{\sqrt{\varrho}}{\sqrt{\lambda}\sqrt{\varrho^2+\lambda^2+2\varrho\lambda\mu}}\lambda\partial_u\mu\, K'(\mu)F_1(\sigma,\lambda)d\lambda d\mu,\end{aligned}$$

where ϕ_0 denotes the solution to the homogeneous equation with the same initial conditions as ϕ, μ is given by

$$\mu=\frac{(\tau-\sigma)^2-\varrho^2-\lambda^2}{2\varrho\lambda},$$

λ^* is given by

$$\lambda^*=\sqrt{(1+\tau)^2+(\mu^2-1)\varrho^2}-\mu\varrho,$$

with an initialization at $\tau=-1$ and K is given by

$$K(\mu)=\int_{\max(-\mu,-1)}^{1}\frac{xdx}{\sqrt{1-x^2}\sqrt{\mu+x}}.$$

Proof. We recall the representation formula derived in [30] for the solution ϕ of

$$\left(-\partial_\tau^2+\partial_\varrho^2+\frac{1}{\varrho}\partial_\varrho-\frac{1}{\varrho^2}\right)\phi=h.$$

ϕ is given by (see [31] p. 960/961)

$$\phi(\tau,\varrho)=\phi_0(\tau,\varrho)+\int_R\frac{\sqrt{\lambda}}{\sqrt{\varrho}}K(\mu)h(\sigma,\lambda)d\lambda d\sigma,$$

where

$$R=\{(\sigma,\lambda)/\ -1\le\sigma\le\tau,\ \max(0,\varrho-\tau+\sigma)\le\lambda\le\varrho+\tau-\sigma\},$$

ϕ_0 denotes the solution to the homogeneous equation with the same initial conditions as ϕ, μ is given by

$$\mu=\frac{(\tau-\sigma)^2-\varrho^2-\lambda^2}{2\varrho\lambda},$$

with an initialization at $\tau=-1$ and K is given by

$$K(\mu)=\int_{\max(-\mu,-1)}^{1}\frac{xdx}{\sqrt{1-x^2}\sqrt{\mu+x}}.$$

In our case, we have

$$h=\partial_u\left(\frac{F_1}{\varrho}\right)+\frac{F_2}{\varrho^2}.$$

Hence, we have

$$\begin{aligned}
\phi(\tau,\varrho) &= \phi_0(\tau,\varrho)+\int_R \frac{\sqrt{\lambda}}{\sqrt{\varrho}}K(\mu)\partial_u\left(\frac{F_1(\sigma,\lambda)}{\lambda}\right)d\lambda d\sigma+\int_R \frac{\sqrt{\lambda}}{\sqrt{\varrho}}K(\mu)\frac{F_2(\sigma,\lambda)}{\lambda^2}d\lambda d\sigma\\
&= \phi_0(\tau,\varrho)+\int_{\partial R} \frac{\sqrt{\lambda}}{\sqrt{\varrho}}K(\mu)\frac{F_1(\sigma,\lambda)}{\lambda}\mathbf{g}(\partial_u,\nu_R)-\int_R\partial_u\left(\frac{\sqrt{\lambda}}{\sqrt{\varrho}}K(\mu)\right)\frac{F_1(\sigma,\lambda)}{\lambda}d\lambda d\sigma\\
&\quad+\int_R \frac{\sqrt{\lambda}}{\sqrt{\varrho}}K(\mu)\frac{F_2(\sigma,\lambda)}{\lambda^2}d\lambda d\sigma\\
&= \phi_0(\tau,\varrho)+\int_{\partial R} \frac{\sqrt{\lambda}}{\sqrt{\varrho}}K(\mu)\frac{F_1(\sigma,\lambda)}{\lambda}\mathbf{g}(\partial_u,\nu_R)+\int_R\frac{1}{4\sqrt{\lambda\varrho}}K(\mu)\frac{F_1(\sigma,\lambda)}{\lambda}d\lambda d\sigma\\
&\quad-\int_R \frac{\sqrt{\lambda}}{\sqrt{\varrho}}\partial_u\mu\, K'(\mu)\frac{F_1(\sigma,\lambda)}{\lambda}d\lambda d\sigma+\int_R \frac{\sqrt{\lambda}}{\sqrt{\varrho}}K(\mu)\frac{F_2(\sigma,\lambda)}{\lambda^2}d\lambda d\sigma.
\end{aligned}$$

Next, we compute the boundary term. We have

$$\begin{aligned}
&\int_{\partial R} f\mathbf{g}(\partial_u,\nu_R)\\
&= \int_{\tau-\varrho}^{\tau+\varrho} f(\tau-\varrho,\underline{u}')d\underline{u}'+\frac{1}{2}\int_{-1}^{\tau-\varrho}f(\sigma,0)d\sigma-\frac{1}{2}\int_0^{\tau+\varrho+1}f(-1,\lambda)d\lambda,
\end{aligned}$$

and

$$\mu=-1 \text{ on } u=\tau-\varrho.$$

Hence, we have

$$\begin{aligned}
\int_{\partial R} \frac{\sqrt{\lambda}}{\sqrt{\varrho}}K(\mu)\frac{F_1(\sigma,\lambda)}{\lambda}\mathbf{g}(\partial_u,\nu_R) &= \frac{K(-1)}{\sqrt{\varrho}}\int_{\tau-\varrho}^{\tau+\varrho}\frac{F_1\left(\frac{\tau-\varrho+\underline{u}'}{2},\frac{-\tau+\varrho+\underline{u}'}{2}\right)}{\sqrt{\frac{-\tau+\varrho+\underline{u}'}{2}}}d\underline{u}'\\
&\quad-\frac{1}{2}\int_0^{\tau+\varrho+1}\frac{\sqrt{\lambda}}{\sqrt{\varrho}}K(\mu)\frac{F_1(-1,\lambda)}{\lambda}d\lambda.
\end{aligned}$$

We compute

$$\begin{aligned}
\partial_\lambda\mu &= \frac{\varrho^2-\lambda^2-(\tau-\sigma)^2}{2\varrho\lambda^2}\\
&= \frac{-2\mu\lambda\varrho-2\lambda^2}{2\varrho\lambda}\\
&= -\frac{\mu\varrho+\lambda}{\varrho}.
\end{aligned}$$

We decompose and perform a change of variable

$$\begin{aligned}
&\int_0^{\tau-\varrho+1}\frac{\sqrt{\lambda}}{\sqrt{\varrho}}K(\mu)\frac{F_1(-1,\lambda)}{\lambda}d\lambda\\
&= \int_0^{+\infty}\frac{\sqrt{\lambda}}{\sqrt{\varrho}}K(\mu)\frac{F_1(-1,\lambda)}{\lambda}\frac{\varrho}{\mu\varrho+\lambda}d\mu+\int_{\sqrt{(\tau+1)^2-\varrho^2}}^{\tau+\varrho+1}\frac{\sqrt{\lambda}}{\sqrt{\varrho}}K(\mu)\frac{F_1(-1,\lambda)}{\lambda}d\lambda\\
&= \sqrt{\varrho}\int_0^{+\infty}\frac{F_1(-1,\lambda)}{\sqrt{\lambda}(\mu\varrho+\lambda)}K(\mu)d\mu+\frac{1}{\sqrt{\varrho}}\int_{\sqrt{(\tau+1)^2-\varrho^2}}^{\tau+\varrho+1}K(\mu)\frac{F_1(-1,\lambda)}{\sqrt{\lambda}}d\lambda
\end{aligned}$$

which yields

$$\begin{aligned}
&\int_{\partial R} \frac{\sqrt{\lambda}}{\sqrt{\varrho}} K(\mu) \frac{F_1(\sigma,\lambda)}{\lambda} \mathbf{g}(\partial_u, \nu_R) \\
= \;& \frac{K(-1)}{\sqrt{\varrho}} \int_{\tau-\varrho}^{\tau+\varrho} \frac{F_1\left(\frac{\tau-\varrho+\underline{u}'}{2}, \frac{-\tau+\varrho+\underline{u}'}{2}\right)}{\sqrt{\frac{-\tau+\varrho+\underline{u}'}{2}}} d\underline{u}' - \frac{1}{2}\sqrt{\varrho} \int_0^{+\infty} \frac{F_1(-1,\lambda)}{\sqrt{\lambda}(\mu\varrho+\lambda)} K(\mu) d\mu \\
& - \frac{1}{2\sqrt{\varrho}} \int_{\sqrt{(\tau+1)^2-\varrho^2}}^{\tau+\varrho+1} K(\mu) \frac{F_1(-1,\lambda)}{\sqrt{\lambda}} d\lambda.
\end{aligned}$$

We deduce

$$\begin{aligned}
\phi(\tau,\varrho) \;=\;& \phi_0(\tau,\varrho) + \frac{K(-1)}{\sqrt{\varrho}} \int_{\tau-\varrho}^{\tau+\varrho} \frac{F_1\left(\frac{\tau-\varrho+\underline{u}'}{2}, \frac{-\tau+\varrho+\underline{u}'}{2}\right)}{\sqrt{\frac{-\tau+\varrho+\underline{u}'}{2}}} d\underline{u}' - \frac{1}{2}\sqrt{\varrho} \int_0^{+\infty} \frac{F_1(-1,\lambda)}{\sqrt{\lambda}(\mu\varrho+\lambda)} K(\mu) d\mu \\
& - \frac{1}{2\sqrt{\varrho}} \int_{\sqrt{(\tau+1)^2-\varrho^2}}^{\tau+\varrho+1} K(\mu) \frac{F_1(-1,\lambda)}{\sqrt{\lambda}} d\lambda + \int_R \frac{1}{4\sqrt{\lambda\varrho}} K(\mu) \frac{F_1(\sigma,\lambda)}{\lambda} d\lambda d\sigma \\
& - \int_R \frac{\sqrt{\lambda}}{\sqrt{\varrho}} \partial_u \mu \, K'(\mu) \frac{F_1(\sigma,\lambda)}{\lambda} d\lambda d\sigma + \int_R \frac{\sqrt{\lambda}}{\sqrt{\varrho}} K(\mu) \frac{F_2(\sigma,\lambda)}{\lambda^2} d\lambda d\sigma.
\end{aligned}$$

In the space-time integral, we perform the change of variable $(\sigma,\lambda) \to (\mu,\lambda)$ which yields

$$\begin{aligned}
\phi(\tau,\varrho) \;=\;& \phi_0(\tau,\varrho) + \frac{K(-1)}{\sqrt{\varrho}} \int_{\tau-\varrho}^{\tau+\varrho} \frac{F_1\left(\frac{\tau-\varrho+\underline{u}'}{2}, \frac{-\tau+\varrho+\underline{u}'}{2}\right)}{\sqrt{\frac{-\tau+\varrho+\underline{u}'}{2}}} d\underline{u}' - \frac{1}{2}\sqrt{\varrho} \int_0^{+\infty} \frac{F_1(-1,\lambda)}{\sqrt{\lambda}(\mu\varrho+\lambda)} K(\mu) d\mu \\
& - \frac{1}{2\sqrt{\varrho}} \int_{\sqrt{(\tau+1)^2-\varrho^2}}^{\tau+\varrho+1} K(\mu) \frac{F_1(-1,\lambda)}{\sqrt{\lambda}} d\lambda + \int_{-1}^{+\infty} \int_0^{\lambda^*} \frac{1}{4\sqrt{\lambda\varrho}} K(\mu) \frac{F_1(\sigma,\lambda)}{\lambda} \frac{1}{\partial_\sigma \mu} d\lambda d\mu \\
& - \int_{-1}^{+\infty} \int_0^{\lambda^*} \frac{\sqrt{\lambda}}{\sqrt{\varrho}} \partial_u \mu \, K'(\mu) \frac{F_1(\sigma,\lambda)}{\lambda} \frac{1}{\partial_\sigma \mu} d\lambda d\mu + \int_{-1}^{+\infty} \int_0^{\lambda^*} \frac{\sqrt{\lambda}}{\sqrt{\varrho}} K(\mu) \frac{F_2(\sigma,\lambda)}{\lambda^2} \frac{1}{\partial_\sigma \mu} d\lambda d\mu,
\end{aligned}$$

where λ^* is given by

$$\lambda^* = \sqrt{(1+\tau)^2 + (\mu^2-1)\varrho^2} - \mu\varrho.$$

We compute

$$\begin{aligned}
\partial_\sigma \mu \;&=\; \frac{\sigma-\tau}{\varrho\lambda} \\
&=\; -\frac{\sqrt{\varrho^2+\lambda^2+2\varrho\lambda\mu}}{\varrho\lambda}.
\end{aligned}$$

We infer

$$\begin{aligned}\phi(\tau,\varrho) &= \phi_0(\tau,\varrho)+\frac{K(-1)}{\sqrt{\varrho}}\int_{\tau-\varrho}^{\tau+\varrho}\frac{F_1\left(\frac{\tau-\varrho+\underline{u}'}{2},\frac{-\tau+\varrho+\underline{u}'}{2}\right)}{\sqrt{\frac{-\tau+\varrho+\underline{u}'}{2}}}d\underline{u}'-\frac{1}{2}\sqrt{\varrho}\int_0^{+\infty}\frac{F_1(-1,\lambda)}{\sqrt{\lambda}(\mu\varrho+\lambda)}K(\mu)d\mu\\ &-\frac{1}{2\sqrt{\varrho}}\int_{\sqrt{(\tau+1)^2-\varrho^2}}^{\tau+\varrho+1}K(\mu)\frac{F_1(-1,\lambda)}{\sqrt{\lambda}}d\lambda\\ &+\int_{-1}^{+\infty}\int_0^{\lambda^*}\frac{\sqrt{\varrho}}{\sqrt{\lambda}\sqrt{\varrho^2+\lambda^2+2\varrho\lambda\mu}}K(\mu)\left(\frac{1}{4}F_1(\sigma,\lambda)+F_2(\sigma,\lambda)\right)d\lambda d\mu\\ &-\int_{-1}^{+\infty}\int_0^{\lambda^*}\frac{\sqrt{\varrho}}{\sqrt{\lambda}\sqrt{\varrho^2+\lambda^2+2\varrho\lambda\mu}}\lambda\partial_u\mu\, K'(\mu)F_1(\sigma,\lambda)d\lambda d\mu.\end{aligned}$$

This concludes the proof of the lemma. □

Lemma 7.16. *We have*

$$|\lambda\partial_u\mu|\lesssim|\mu-1|,\ \forall-1\leq\mu<+\infty.$$

Proof. We compute

$$\begin{aligned}\partial_u\mu &= \frac{\lambda((\sigma-\tau)+\lambda)+\frac{1}{2}((\tau-\sigma)^2-\varrho^2-\lambda^2)}{2\varrho\lambda^2}\\ &= \frac{\lambda(\sigma-\tau)+\frac{\lambda^2}{2}+\frac{1}{2}(\tau-\sigma)^2-\frac{1}{2}\varrho^2}{2\varrho\lambda^2}\\ &= \frac{(\lambda+\sigma-\tau)^2-\varrho^2}{4\varrho\lambda^2}\\ &= \frac{(\lambda-\sqrt{\varrho^2+\lambda^2+2\varrho\lambda\mu})^2-\varrho^2}{4\varrho\lambda^2}\\ &= \frac{\lambda^2-2\lambda\sqrt{\varrho^2+\lambda^2+2\varrho\lambda\mu}+\varrho^2+\lambda^2+2\varrho\lambda\mu-\varrho^2}{4\varrho\lambda^2}\\ &= \frac{\lambda+\varrho\mu-\sqrt{\varrho^2+\lambda^2+2\varrho\lambda\mu}}{2\varrho\lambda}.\end{aligned}$$

We infer

$$\lambda\partial_u\mu = \frac{\lambda+\varrho\mu-\sqrt{\varrho^2+\lambda^2+2\varrho\lambda\mu}}{2\varrho}.$$

We consider two cases. If $\mu\geq 0$, we have

$$\begin{aligned}\lambda\partial_u\mu &= \frac{\lambda^2+\varrho^2\mu^2+2\lambda\varrho\mu-\varrho^2-\lambda^2-2\varrho\lambda\mu}{2\varrho(\lambda+\varrho\mu+\sqrt{\varrho^2+\lambda^2+2\varrho\lambda\mu})}\\ &= \frac{\varrho^2(\mu^2-1)}{2\varrho(\lambda+\varrho\mu+\sqrt{\varrho^2+\lambda^2+2\varrho\lambda\mu})}\\ &= \frac{\varrho(\mu+1)(\mu-1)}{2(\lambda+\varrho\mu+\sqrt{\varrho^2+\lambda^2+2\varrho\lambda\mu})}.\end{aligned}$$

Since $\mu\geq 0$, $\lambda\geq 0$ and $\varrho\geq 0$, we have

$$\lambda+\varrho\mu+\sqrt{\varrho^2+\lambda^2+2\varrho\lambda\mu}\geq\varrho(1+\mu).$$

We infer

$$|\lambda\partial_u\mu| \leq \frac{|\mu-1|}{2}.$$

If $-1 \leq \mu < 0$, we have

$$\begin{aligned}
\lambda \partial_u \mu &= \frac{\lambda^2 - \varrho^2\mu^2 + 2\varrho\mu\sqrt{\varrho^2 + \lambda^2 + 2\varrho\lambda\mu} - \varrho^2 - \lambda^2 - 2\varrho\lambda\mu}{2\varrho(\lambda - \varrho\mu + \sqrt{\varrho^2 + \lambda^2 + 2\varrho\lambda\mu})} \\
&= \frac{-\varrho(\mu^2+1) + 2\mu(\sqrt{\varrho^2 + \lambda^2 + 2\varrho\lambda\mu} - \lambda)}{2(\lambda - \varrho\mu + \sqrt{\varrho^2 + \lambda^2 + 2\varrho\lambda\mu})}.
\end{aligned}$$

Since $-1 \leq \mu < 0$, this yields

$$\begin{aligned}
|\lambda \partial_u \mu| &= \frac{\left|-\varrho(\mu^2+1) + 2|\mu|(\lambda - \sqrt{\varrho^2 + \lambda^2 - 2|\mu|\varrho\lambda})\right|}{2(\lambda + |\mu|\varrho + \sqrt{\varrho^2 + \lambda^2 - 2|\mu|\varrho\lambda})} \\
&\lesssim \frac{\varrho + \lambda + \sqrt{\varrho^2 + \lambda^2}}{\lambda + \varrho} \\
&\lesssim 1.
\end{aligned}$$

Together with the case $\mu \geq 0$, we obtain for all $\mu \geq -1$

$$|\lambda \partial_u \mu| \lesssim |\mu - 1|.$$

This concludes the proof of the lemma. □

Lemma 7.17. *We have*

$$\begin{aligned}
|\phi(\tau, \varrho)| &\lesssim C_0\sqrt{\varrho} + \varepsilon C \varrho^\delta \\
&\quad + \varepsilon C \sqrt{\varrho} \int_{-1}^{+\infty} \int_0^{\lambda^*} \frac{\lambda^{\delta - \frac{1}{2}}}{\sqrt{\varrho^2 + \lambda^2 + 2\varrho\lambda\mu}} (|K(\mu)| + |\mu - 1||K'(\mu)|) d\lambda d\mu.
\end{aligned}$$

where the constant C_0 only depend on initial data.

Proof. Recall that

$$\begin{aligned}
\phi(\tau, \varrho) &= \phi_0(\tau, \varrho) + \frac{K(-1)}{\sqrt{\varrho}} \int_{\tau-\varrho}^{\tau+\varrho} \frac{F_1\left(\frac{\tau - \varrho + \underline{u}'}{2}, \frac{-\tau + \varrho + \underline{u}'}{2}\right)}{\sqrt{\frac{-\tau + \varrho + \underline{u}'}{2}}} d\underline{u}' - \frac{1}{2}\sqrt{\varrho} \int_0^{+\infty} \frac{F_1(-1, \lambda)}{\sqrt{\lambda}(\mu\varrho + \lambda)} K(\mu) d\mu \\
&\quad - \frac{1}{2\sqrt{\varrho}} \int_{\sqrt{(\tau+1)^2 - \varrho^2}}^{\tau + \varrho + 1} K(\mu) \frac{F_1(-1, \lambda)}{\sqrt{\lambda}} d\lambda \\
&\quad + \int_{-1}^{+\infty} \int_0^{\lambda^*} \frac{\sqrt{\varrho}}{\sqrt{\lambda}\sqrt{\varrho^2 + \lambda^2 + 2\varrho\lambda\mu}} K(\mu) \left(\frac{1}{4} F_1(\sigma, \lambda) + F_2(\sigma, \lambda)\right) d\lambda d\mu \\
&\quad - \int_{-1}^{+\infty} \int_0^{\lambda^*} \frac{\sqrt{\varrho}}{\sqrt{\lambda}\sqrt{\varrho^2 + \lambda^2 + 2\varrho\lambda\mu}} \lambda \partial_u \mu \, K'(\mu) F_1(\sigma, \lambda) d\lambda d\mu.
\end{aligned}$$

We have the following properties for K (see for example [9] p. 1061):

$$K(-1) = \frac{\pi}{\sqrt{2}}, \quad \sup_{-1 \leq \mu \leq 0} |K| \lesssim 1, \quad K \in L^1(-1, +\infty).$$

Also, we have

$$\{\sqrt{(\tau+1)^2 - \varrho^2} \leq \lambda \leq \tau + \varrho + 1\} \cap \{\sigma = -1\} = \{-1 \leq \mu \leq 0\} \cap \{\sigma = -1\}.$$

We deduce

$$
\begin{aligned}
|\phi(\tau,\varrho)| \lesssim\ & |\phi_0(\tau,\varrho)| + \frac{|K(-1)|}{\sqrt{\varrho}}\int_0^{\varrho}\frac{|F_1(\tau-\varrho+\lambda,\lambda)|}{\sqrt{\lambda}}d\lambda \\
& +\sqrt{\varrho}\left(\sup_{\lambda\geq 0}\frac{|F_1(-1,\lambda)|}{\lambda^{\frac{3}{2}}}\right)\int_0^{+\infty}|K(\mu)|d\mu \\
& +\frac{1}{\sqrt{\varrho}}\left(\sup_{-1\leq\mu\leq 0}|K|\right)\left(\sup_{\lambda\geq 0}\frac{|F_1(-1,\lambda)|}{\sqrt{\lambda}}\right)(\tau+\varrho+1-\sqrt{(\tau+1)^2-\varrho^2}) \\
& +\int_{-1}^{+\infty}\int_0^{\lambda^*}\frac{\sqrt{\varrho}}{\sqrt{\lambda}\sqrt{\varrho^2+\lambda^2+2\varrho\lambda\mu}}|K(\mu)|(|F_1(\sigma,\lambda)|+|F_2(\sigma,\lambda)|)d\lambda d\mu \\
& +\int_{-1}^{+\infty}\int_0^{\lambda^*}\frac{\sqrt{\varrho}}{\sqrt{\lambda}\sqrt{\varrho^2+\lambda^2+2\varrho\lambda\mu}}|\lambda\partial_u\mu||K'(\mu)||F_1(\sigma,\lambda)|d\lambda d\mu \\
\lesssim\ & |\phi_0(\tau,\varrho)| + \sup_{0\leq\lambda\leq\varrho}|F_1(\tau-\varrho+\lambda,\lambda)| + \sqrt{\varrho}\left(\sup_{\lambda\geq 0}\frac{|F_1(-1,\lambda)|}{\lambda^{\frac{3}{2}}}\right) \\
& +\sqrt{\varrho}\left(\sup_{\lambda\geq 0}\frac{|F_1(-1,\lambda)|}{\sqrt{\lambda}}\right)\frac{\varrho+\tau+1}{\tau+\varrho+1+\sqrt{(\tau+1)^2-\varrho^2}} \\
& +\int_{-1}^{+\infty}\int_0^{\lambda^*}\frac{\sqrt{\varrho}}{\sqrt{\lambda}\sqrt{\varrho^2+\lambda^2+2\varrho\lambda\mu}}|K(\mu)|(|F_1(\sigma,\lambda)|+|F_2(\sigma,\lambda)|)d\lambda d\mu \\
& +\int_{-1}^{+\infty}\int_0^{\lambda^*}\frac{\sqrt{\varrho}}{\sqrt{\lambda}\sqrt{\varrho^2+\lambda^2+2\varrho\lambda\mu}}|\lambda\partial_u\mu||K'(\mu)||F_1(\sigma,\lambda)|d\lambda d\mu.
\end{aligned}
$$

Assuming enough regularity on the initial data, we have

$$\sup_{\varrho>0}\varrho^{-\frac{1}{2}}|\phi_0(\tau,\varrho)| + \sup_{\lambda\geq 0}\frac{|F_1(-1,\lambda)|}{\sqrt{\lambda}} + \sup_{\lambda\geq 0}\frac{|F_1(-1,\lambda)|}{\lambda^{\frac{3}{2}}} \leq C_0$$

where the constant C_0 only depend on initial data. Hence, together with the previous lemma, we deduce

$$
\begin{aligned}
|\phi(\tau,\varrho)| \lesssim\ & C_0\sqrt{\varrho} + \sup_{0\leq\lambda\leq\varrho}|F_1(\tau-\varrho+\lambda,\lambda)| \\
& +\int_{-1}^{+\infty}\int_0^{\lambda^*}\frac{\sqrt{\varrho}}{\sqrt{\lambda}\sqrt{\varrho^2+\lambda^2+2\varrho\lambda\mu}}|K(\mu)|(|F_1(\sigma,\lambda)|+|F_2(\sigma,\lambda)|)d\lambda d\mu \\
& +\int_{-1}^{+\infty}\int_0^{\lambda^*}\frac{\sqrt{\varrho}}{\sqrt{\lambda}\sqrt{\varrho^2+\lambda^2+2\varrho\lambda\mu}}|\mu-1||K'(\mu)||F_1(\sigma,\lambda)|d\lambda d\mu.
\end{aligned}
$$

Finally, recall that we have

$$\sup_{Q_{\underline{u}_b}} r^{-\delta}(|F_1|+|F_2|) \lesssim C\varepsilon.$$

We infer

$$
\begin{aligned}
|\phi(\tau,\varrho)| \lesssim\ & C_0\sqrt{\varrho} + \varepsilon C\varrho^{\delta} \\
& +\varepsilon C\sqrt{\varrho}\int_{-1}^{+\infty}\int_0^{\lambda^*}\frac{\lambda^{\delta-\frac{1}{2}}}{\sqrt{\varrho^2+\lambda^2+2\varrho\lambda\mu}}(|K(\mu)|+|\mu-1||K'(\mu)|)d\lambda d\mu.
\end{aligned}
$$

This concludes the proof of the lemma. □

Lemma 7.18. *We have*

$$|\phi(\tau,\varrho)| \lesssim C_0\sqrt{\varrho}+\varepsilon C\varrho^{\delta}.$$

Proof. Recall that

$$\begin{aligned}|\phi(\tau,\varrho)| \lesssim\ & C_0\sqrt{\varrho}+\varepsilon C\varrho^{\delta}\\ &+\varepsilon C\sqrt{\varrho}\int_{-1}^{+\infty}\int_0^{\lambda^*}\frac{\lambda^{\delta-\frac{1}{2}}}{\sqrt{\varrho^2+\lambda^2+2\varrho\lambda\mu}}(|K(\mu)|+|\mu-1||K'(\mu)|)d\lambda d\mu.\end{aligned}$$

We evaluate the integral in the right-hand side. We have

$$\begin{aligned}&\int_{-1}^{+\infty}\int_0^{\lambda^*}\frac{\lambda^{\delta-\frac{1}{2}}}{\sqrt{\varrho^2+\lambda^2+2\varrho\lambda\mu}}(|K(\mu)|+|\mu-1||K'(\mu)|)d\lambda d\mu\\ \lesssim\ & \int_{-1}^{0}\int_0^{\lambda^*}\frac{\lambda^{\delta-\frac{1}{2}}}{\sqrt{\varrho^2+\lambda^2+2\varrho\lambda\mu}}(|K(\mu)|+|K'(\mu)|)d\lambda d\mu\\ &+\int_{0}^{+\infty}\int_0^{\lambda^*}\frac{\lambda^{\delta-\frac{1}{2}}}{\sqrt{\varrho^2+\lambda^2+2\varrho\lambda\mu}}(|K(\mu)|+|\mu-1||K'(\mu)|)d\lambda d\mu.\end{aligned}$$

We estimate the two integral on the right-hand side starting with the first one. We have

$$\begin{aligned}&\int_{-1}^{0}\int_0^{\lambda^*}\frac{\lambda^{\delta-\frac{1}{2}}}{\sqrt{\varrho^2+\lambda^2+2\varrho\lambda\mu}}(|K(\mu)|+|K'(\mu)|)d\lambda d\mu\\ \lesssim\ & \sup_{-1\mu\leq 0}(|K(\mu)|+|K'(\mu)|)\int_{-1}^{0}\int_0^{\lambda^*}\frac{\lambda^{\delta-\frac{1}{2}}}{\sqrt{\varrho^2+\lambda^2+2\varrho\lambda\mu}}d\lambda d\mu\\ \lesssim\ & \int_0^{+\infty}\int_{\tau-\sqrt{\lambda^2+\varrho^2}}^{\tau-|\lambda-\varrho|}\frac{\lambda^{\delta-\frac{1}{2}}}{\varrho\lambda}d\sigma d\lambda,\end{aligned}$$

where we used the fact that

$$\sup_{-1\mu\leq 0}(|K(\mu)|+|K'(\mu)|)<+\infty,$$

which is a consequence of the estimates for K and K' on p. 1061 [9],

$$\{-1\leq\mu\leq 0\}=\{|\lambda-\varrho|\leq\tau-\sigma\leq\sqrt{\varrho^2+\lambda^2}\},$$

and

$$\partial_\sigma\mu=-\frac{\sqrt{\varrho^2+\lambda^2+2\varrho\lambda\mu}}{\varrho\lambda}.$$

We infer

$$\begin{aligned}&\int_{-1}^{0}\int_0^{\lambda^*}\frac{\lambda^{\delta-\frac{1}{2}}}{\sqrt{\varrho^2+\lambda^2+2\varrho\lambda\mu}}(|K(\mu)|+|K'(\mu)|)d\lambda d\mu\\ \lesssim\ & \int_0^{+\infty}(\sqrt{\lambda^2+\varrho^2}-|\lambda-\varrho|)\frac{\lambda^{\delta-\frac{1}{2}}}{\varrho\lambda}d\lambda\\ \lesssim\ & \int_0^{+\infty}\frac{\lambda^{\delta-\frac{1}{2}}}{\sqrt{\lambda^2+\varrho^2}+|\lambda-\varrho|}d\lambda\\ \lesssim\ & \varrho^{\delta-\frac{1}{2}}\int_0^{+\infty}\frac{\lambda^{\delta-\frac{1}{2}}}{\sqrt{\lambda^2+1}+|\lambda-1|}d\lambda\\ \lesssim\ & \varrho^{\delta-\frac{1}{2}},\end{aligned}$$

where we used in the last inequality the fact that $0 < \delta < 1/2$.

Next, we estimate the second integral on the right-hand side. We have

$$\begin{aligned}
&\int_0^{+\infty}\int_0^{\lambda^*}\frac{\lambda^{\delta-\frac{1}{2}}}{\sqrt{\varrho^2+\lambda^2+2\varrho\lambda\mu}}(|K(\mu)|+|\mu-1||K'(\mu)|)d\lambda d\mu\\
\lesssim\ & \left(\int_0^{+\infty}(|K(\mu)|+|\mu-1||K'(\mu)|)d\mu\right)\int_0^{+\infty}\frac{\lambda^{\delta-\frac{1}{2}}}{\varrho+\lambda}d\lambda\\
\lesssim\ & \varrho^\delta\int_0^{+\infty}\frac{\lambda^{\delta-\frac{1}{2}}}{1+\lambda}d\lambda\\
\lesssim\ & \varrho^{\delta-\frac{1}{2}},
\end{aligned}$$

where we used the fact that $0 < \delta < 1/2$ and

$$\int_0^{+\infty}(|K(\mu)|+|\mu-1||K'(\mu)|)d\mu < +\infty$$

which is a consequence of the estimates for K and K' on p. 1061 [9]. We deduce

$$\int_{-1}^{+\infty}\int_0^{\lambda^*}\frac{\lambda^{\delta-\frac{1}{2}}}{\sqrt{\varrho^2+\lambda^2+2\varrho\lambda\mu}}(|K(\mu)|+|\mu-1||K'(\mu)|)d\lambda d\mu \lesssim \varrho^{\delta-\frac{1}{2}} \tag{7.2}$$

which yields

$$|\phi(\tau,\varrho)| \lesssim C_0\sqrt{\varrho}+\varepsilon C\varrho^\delta.$$

This concludes the proof of the lemma. □

7.6. **Proof of Proposition 7.6.**

Lemma 7.19. *Let*

$$\Theta = r\partial_{\underline{u}}\phi \text{ and } \Xi = r\partial_u\phi.$$

We have

$$\partial_u\left(\frac{\Theta}{\sqrt{r}}\right) = -\frac{1}{2\sqrt{r}}\partial_{\underline{u}}r\partial_u\phi-\frac{\Omega^2 f(\phi)}{4r^{\frac{3}{2}}},$$

and

$$\partial_{\underline{u}}\left(\frac{\Xi}{\sqrt{r}}\right) = -\frac{1}{2\sqrt{r}}\partial_u r\partial_{\underline{u}}\phi-\frac{\Omega^2 f(\phi)}{4r^{\frac{3}{2}}}.$$

Proof. From

$$\partial_u(r\partial_{\underline{u}}\phi)+\partial_{\underline{u}}(r\partial_u\phi) = -\frac{\Omega^2}{2}\frac{f(\phi)}{r},$$

we infer

$$\partial_u\Theta = \frac{1}{2}\frac{\partial_u r}{r}\Theta-\frac{1}{2}\partial_{\underline{u}}r\partial_u\phi-\frac{\Omega^2 f(\phi)}{4r},$$

and

$$\partial_{\underline{u}}\Xi = \frac{1}{2}\frac{\partial_{\underline{u}}r}{r}\Xi-\frac{1}{2}\partial_u r\partial_{\underline{u}}\phi-\frac{\Omega^2 f(\phi)}{4r}.$$

We rewrite the system as

$$\partial_u\left(\frac{\Theta}{\sqrt{r}}\right) = -\frac{1}{2\sqrt{r}}\partial_{\underline{u}}r\partial_u\phi-\frac{\Omega^2 f(\phi)}{4r^{\frac{3}{2}}},$$

and

$$\partial_{\underline{u}}\left(\frac{\Xi}{\sqrt{r}}\right) = -\frac{1}{2\sqrt{r}}\partial_u r\partial_{\underline{u}}\phi-\frac{\Omega^2 f(\phi)}{4r^{\frac{3}{2}}}.$$

This concludes the proof of the lemma. □

We are now in position to prove Proposition 7.6. Recall that

$$\partial_u\left(\frac{\Theta}{\sqrt{r}}\right) = -\frac{1}{2\sqrt{r}}\partial_{\underline{u}} r\partial_u\phi - \frac{\Omega^2 f(\phi)}{4r^{\frac{3}{2}}}.$$

We integrate between $(u,\underline{u})$ and $(u_0,\underline{u})$ where $(u_0,\underline{u})$ is on the initial hyper surface. We deduce

$$\begin{aligned}\frac{\Theta(u,\underline{u})}{\sqrt{r(u,\underline{u})}} &= \frac{\Theta(u_0,\underline{u})}{\sqrt{r(u_0,\underline{u})}} - \int_{u_0}^{u}\frac{1}{2\sqrt{r}}\partial_{\underline{u}} r\partial_u\phi(\sigma,\underline{u})d\sigma - \int_{u_0}^{u}\frac{\Omega^2 f(\phi)}{4r^{\frac{3}{2}}}(\sigma,\underline{u})d\sigma\\ &= \frac{\Theta(u_0,\underline{u})}{\sqrt{r(u_0,\underline{u})}} - \left[\frac{1}{2\sqrt{r}}\partial_{\underline{u}} r\phi(\sigma,\underline{u})\right]_{u_0}^{u} + \int_{u_0}^{u}\frac{1}{2\sqrt{r}}\partial_u\partial_{\underline{u}} r\phi(\sigma,\underline{u})d\sigma\\ &\quad - \int_{u_0}^{u}\frac{1}{4r^{\frac{3}{2}}}\partial_u r\partial_{\underline{u}} r\phi(\sigma,\underline{u})d\sigma - \int_{u_0}^{u}\frac{\Omega^2 f(\phi)}{4r^{\frac{3}{2}}}(\sigma,\underline{u})d\sigma.\end{aligned}$$

We infer

$$\begin{aligned}\sqrt{r(u,\underline{u})}\partial_{\underline{u}}\phi(u,\underline{u}) &= \sqrt{r(u_0,\underline{u})}\partial_{\underline{u}}\phi(u_0,\underline{u}) - \left[\frac{1}{2\sqrt{r}}\partial_{\underline{u}} r\phi(\sigma,\underline{u})\right]_{u_0}^{u} + \int_{u_0}^{u}\frac{1}{2\sqrt{r}}\kappa\frac{f(\phi)^2}{r}\phi(\sigma,\underline{u})d\sigma\\ &\quad - \int_{u_0}^{u}\frac{1}{4r^{\frac{3}{2}}}\partial_u r\partial_{\underline{u}} r\phi(\sigma,\underline{u})d\sigma - \int_{u_0}^{u}\frac{\Omega^2 f(\phi)}{4r^{\frac{3}{2}}}(\sigma,\underline{u})d\sigma.\end{aligned}$$

We deduce

$$\sqrt{r(u,\underline{u})}|\partial_{\underline{u}}\phi(u,\underline{u})| \lesssim \sqrt{r(u_0,\underline{u})}|\partial_{\underline{u}}\phi(u_0,\underline{u})| + \frac{|\phi(u_0,\underline{u})|}{\sqrt{r(u_0,\underline{u})}} + \frac{|\phi(u,\underline{u})|}{\sqrt{r(u,\underline{u})}} + \int_{u_0}^{u}\frac{|\phi(\sigma,\underline{u})|}{r^{\frac{3}{2}}}d\sigma$$

and hence

$$\begin{aligned}r(u,\underline{u})^{1-\delta}|\partial_{\underline{u}}\phi(u,\underline{u})| &\lesssim r(u,\underline{u})^{\frac{1}{2}-\delta}\sqrt{r(u_0,\underline{u})}|\partial_{\underline{u}}\phi(u_0,\underline{u})| + r(u,\underline{u})^{\frac{1}{2}-\delta}\frac{|\phi(u_0,\underline{u})|}{\sqrt{r(u_0,\underline{u})}}\\ &\quad + r(u,\underline{u})^{-\delta}|\phi(u,\underline{u})| + r(u,\underline{u})^{\frac{1}{2}-\delta}\int_{u_0}^{u}\frac{|\phi(\sigma,\underline{u})|}{r(\sigma,\underline{u})^{\frac{3}{2}}}d\sigma\\ &\lesssim C_0 r(u,\underline{u})^{\frac{1}{2}-\delta} + r(u,\underline{u})^{-\delta}|\phi(u,\underline{u})| + r(u,\underline{u})^{\frac{1}{2}-\delta}\int_{u_0}^{u}\frac{|\phi(\sigma,\underline{u})|}{r(\sigma,\underline{u})^{\frac{3}{2}}}d\sigma\\ &\lesssim C_0 + r(u,\underline{u})^{-\delta}|\phi(u,\underline{u})| + r(u,\underline{u})^{\frac{1}{2}-\delta}\int_{u_0}^{u}\frac{|\phi(\sigma,\underline{u})|}{r(\sigma,\underline{u})^{\frac{3}{2}}}d\sigma\end{aligned}$$

where C_0 only depends on initial data.

Using the improved uniform bound for ϕ of lemma 7.18, we infer

$$\begin{aligned}r(u,\underline{u})^{1-\delta}|\partial_{\underline{u}}\phi(u,\underline{u})| &\lesssim C_0 + r(u,\underline{u})^{-\delta}(C_0\sqrt{\varrho(u,\underline{u})} + \varepsilon C\varrho(u,\underline{u})^{\delta})\\ &\quad + r(u,\underline{u})^{\frac{1}{2}-\delta}\int_{u_0}^{u}\frac{C_0\sqrt{\varrho(\sigma,\underline{u})} + \varepsilon C\varrho(\sigma,\underline{u})^{\delta}}{r(\sigma,\underline{u})^{\frac{3}{2}}}d\sigma\\ &\lesssim C_0 + \varepsilon C\end{aligned}$$

where we used the fact that $r \sim \varrho$ and $\delta < 1/2$. Finally, we have obtained the existence of a universal constant $0 < \underline{C} < +\infty$ such that

$$r(u,\underline{u})^{1-\delta}|\partial_{\underline{u}}\phi(u,\underline{u})| \leq \underline{C}(C_0 + \varepsilon C).$$

This concludes the proof of Proposition 7.6.

Remark 7.20. *Lemma 7.19 is used as follows. The equation for* Θ *is always integrated from the initial data, while the equation for* Ξ *is always integrated from the axis of symmetry* Γ*. We have the following three cases*

- *If* $|\phi| \leq Cr^\delta$ *with* $\delta < 1/2$*, we deduce* $|\partial_{\underline{u}}\phi| \lesssim Cr^{\delta-1}$*, but no estimate for* $\partial_u\phi$*. We used this case above for the proof of Proposition 7.6.*
- *If* $|\phi| \leq Cr^\delta$ *with* $\delta > 1/2$*, then we deduce* $|\partial_u\phi| \lesssim Cr^{\delta-1}$*, and only* $\partial_{\underline{u}}\phi \lesssim C\sqrt{r}$*. This case will be used in the proof Lemma 8.14.*
- *If* $|\phi| \leq C\sqrt{r}$*, then we have the log loss estimate* $|\partial_{\underline{u}}\phi| \lesssim C|\log(r)|\sqrt{r}$*, and no estimate for* $\partial_u\phi$*. Due to the log loss for* $\partial_{\underline{u}}\phi$*, this case is never used.*

7.7. A more refined bound for ϕ.

Corollary 7.21. *For any*

$$0 < \delta < \frac{1}{2},$$

there exists a constant C_0 *only depending on initial data such that we have*

$$r^{1-\delta}|\partial_{\underline{u}}\phi| + r^{-\delta}|\phi| \leq C_0$$

and

$$r^{-2\delta}\left(\frac{|r-\varrho|}{\varrho} + \left|\partial_u r + \frac{1}{2}\right| + \left|\partial_{\underline{u}} r - \frac{1}{2}\right| + |\Omega - 1|\right) \leq C_0.$$

Proof. Choosing $\varepsilon > 0$ sufficiently small in Proposition 7.6 yields

$$\sup_{Q_{\underline{u}_b}} r^{1-\delta}|\partial_{\underline{u}}\phi| \leq \underline{C}C_0 \tag{7.3}$$

which is an improvement of the bootstrap assumption (7.1). This implies that (7.3) holds for all $-1 \leq \underline{u}_b < 0$. Hence, we have

$$r^{1-\delta}|\partial_{\underline{u}}\phi| \leq C_0$$

for a constant C_0 only depending on initial data. Together with Lemma 7.11, we infer

$$r^{-\delta}|\phi| \leq C_0$$

and

$$r^{-2\delta}\left(\frac{|r-\varrho|}{\varrho} + \left|\partial_u r + \frac{1}{2}\right| + \left|\partial_{\underline{u}} r - \frac{1}{2}\right| + |\Omega - 1|\right) \leq C_0.$$

This concludes the proof of the corollary. □

In this section, we would like to prove the following refined bounds.

Lemma 7.22. *There exists a constant* C_0 *only depending on initial data such that we have*

$$r^{-\frac{1}{2}}|\phi| \leq C_0$$

and

$$r^{-1}\left(\frac{|r-\varrho|}{\varrho} + \left|\partial_u r + \frac{1}{2}\right| + \left|\partial_{\underline{u}} r - \frac{1}{2}\right|\right) \leq C_0.$$

Remark 7.23. *The above improvement can not be true for* $r\partial_{\underline{u}}\phi$ *due to a log loss when integrating the improved estimate for* ϕ *using the equation for* Θ *(see Remark 7.20). In turn, this improvement can also not be true for* $\Omega - 1$*.*

Proof. Recall that

$$\begin{aligned}
|\phi(\tau,\varrho)| &\lesssim C_0\sqrt{\varrho}+\sup_{0\leq\lambda\leq\varrho}|F_1(\tau-\varrho+\lambda,\lambda)|\\
&+\int_{-1}^{+\infty}\int_0^{\lambda^*}\frac{\sqrt{\varrho}}{\sqrt{\lambda}\sqrt{\varrho^2+\lambda^2+2\varrho\lambda\mu}}|K(\mu)|(|F_1(\sigma,\lambda)|+|F_2(\sigma,\lambda)|)d\lambda d\mu\\
&+\int_{-1}^{+\infty}\int_0^{\lambda^*}\frac{\sqrt{\varrho}}{\sqrt{\lambda}\sqrt{\varrho^2+\lambda^2+2\varrho\lambda\mu}}|\mu-1||K'(\mu)||F_1(\sigma,\lambda)|d\lambda d\mu.
\end{aligned}$$

Also, recall that we have

$$r^{-3\delta}(|F_1|+|F_2|)\lesssim C_0,$$

where we choose from now on δ such that

$$\frac{1}{6}<\delta<\frac{1}{2}.$$

We infer

$$\begin{aligned}
|\phi(\tau,\varrho)| &\lesssim C_0\sqrt{\varrho}+C_0\varrho^{3\delta}\\
&+C_0\sqrt{\varrho}\int_{-1}^{+\infty}\int_0^{\lambda^*}\frac{\lambda^{3\delta-\frac{1}{2}}}{\sqrt{\varrho^2+\lambda^2+2\varrho\lambda\mu}}(|K(\mu)|+|\mu-1||K'(\mu)|)d\lambda d\mu.
\end{aligned}$$

We have

$$\begin{aligned}
&\int_0^{\lambda^*}\frac{\lambda^{3\delta-\frac{1}{2}}}{\sqrt{\varrho^2+\lambda^2+2\varrho\lambda\mu}}d\lambda\\
=&\int_0^{2\varrho}\frac{\lambda^{3\delta-\frac{1}{2}}}{\sqrt{\varrho^2+\lambda^2+2\varrho\lambda\mu}}d\lambda+\int_{2\varrho}^{\lambda^*}\frac{\lambda^{3\delta-\frac{1}{2}}}{\sqrt{\varrho^2+\lambda^2+2\varrho\lambda\mu}}d\lambda\\
\lesssim&\ \varrho^{2\delta}\int_0^{2\varrho}\frac{\lambda^{\delta-\frac{1}{2}}}{\sqrt{\varrho^2+\lambda^2+2\varrho\lambda\mu}}d\lambda+\int_{2\varrho}^{\lambda^*}\frac{\lambda^{3\delta-\frac{1}{2}}}{\sqrt{(\varrho-\lambda)^2+2\varrho\lambda(1+\mu)}}d\lambda.
\end{aligned}$$

Since $\mu\geq-1$, we infer

$$\begin{aligned}
&\int_0^{\lambda^*}\frac{\lambda^{3\delta-\frac{1}{2}}}{\sqrt{\varrho^2+\lambda^2+2\varrho\lambda\mu}}d\lambda\\
\lesssim&\ \varrho^{2\delta}\int_0^{2\varrho}\frac{\lambda^{\delta-\frac{1}{2}}}{\sqrt{\varrho^2+\lambda^2+2\varrho\lambda\mu}}d\lambda+\int_{2\varrho}^{\lambda^*}\frac{\lambda^{3\delta-\frac{1}{2}}}{\lambda}d\lambda\\
\lesssim&\ \varrho^{2\delta}\int_0^{\lambda^*}\frac{\lambda^{\delta-\frac{1}{2}}}{\sqrt{\varrho^2+\lambda^2+2\varrho\lambda\mu}}d\lambda+\left[\lambda^{3\delta-\frac{1}{2}}\right]_{2\varrho}^{\lambda^*}\\
\lesssim&\ 1+\varrho^{2\delta}\int_0^{\lambda^*}\frac{\lambda^{\delta-\frac{1}{2}}}{\sqrt{\varrho^2+\lambda^2+2\varrho\lambda\mu}}d\lambda
\end{aligned}$$

where used the fact that we chose $\delta>1/6$.

We have obtained

$$\begin{aligned}
|\phi(\tau,\varrho)| &\lesssim C_0\sqrt{\varrho}\\
&+C_0\varrho^{2\delta}\sqrt{\varrho}\int_{-1}^{+\infty}\int_0^{\lambda^*}\frac{\lambda^{\delta-\frac{1}{2}}}{\sqrt{\varrho^2+\lambda^2+2\varrho\lambda\mu}}(|K(\mu)|+|\mu-1||K'(\mu)|)d\lambda d\mu,
\end{aligned}$$

where we used the fact that we chose δ such that $\delta>1/6$. Recall estimate (7.2):

$$\sqrt{\varrho}\int_{-1}^{+\infty}\int_0^{\lambda^*}\frac{\lambda^{\delta-\frac{1}{2}}}{\sqrt{\varrho^2+\lambda^2+2\varrho\lambda\mu}}(|K(\mu)|+|\mu-1||K'(\mu)|)d\lambda d\mu\lesssim\varrho^{\delta}.$$

We infer

$$\begin{aligned}|\phi(\tau, \varrho)| &\lesssim C_0\sqrt{\varrho} + C_0\varrho^{3\delta}\\ &\lesssim C_0\sqrt{\varrho},\end{aligned}$$

where we used the fact that we chose δ such that $\delta > 1/6$. This concludes the improvement for ϕ.

Then, one obtains the improvements[11] for $\partial_{\underline{u}} r - 1/2$, $\partial_u r + 1/2$ and $r - \varrho$ as in the proof of Lemma 7.11. This concludes the proof of the lemma. ☐

8. An improved uniform bound for $\partial\phi$

Here, we differentiate equation for ϕ once and we obtain a uniform bound for $\partial\phi$ and ϕ/r again relying on an explicit representation formula for the flat wave equation by adapting the approach in [30].

8.1. **Upper bounds for higher order derivatives.** We start by deriving an upper bound for $\partial_{\underline{u}}\Omega$.

Lemma 8.1. *There exists a constant C_0 only depending on initial data such that we have for any $0 < \delta < 1/2$*

$$r^{1-2\delta}|\partial_{\underline{u}}\Omega| \lesssim C_0.$$

Proof. Recall that

$$\Omega^{-2}(\partial_u\Omega\partial_{\underline{u}}\Omega - \Omega\partial_u\partial_{\underline{u}}\Omega) = \frac{1}{8}\Omega^2\kappa\left(\frac{4}{\Omega^2}\partial_u\phi\partial_{\underline{u}}\phi + \frac{g(\phi)^2}{r^2}\right).$$

We deduce

$$\begin{aligned}\partial_u(\partial_{\underline{u}}\log(\Omega)) &= \partial_u\left(\frac{\partial_{\underline{u}}\Omega}{\Omega}\right)\\ &= \frac{\partial_u\partial_{\underline{u}}\Omega}{\Omega} - \frac{\partial_u\Omega\partial_{\underline{u}}\Omega}{\Omega^2}\\ &= -\frac{\kappa}{2}\partial_u\phi\partial_{\underline{u}}\phi - \frac{\kappa\Omega^2}{8}\frac{g(\phi)^2}{r^2}.\end{aligned}$$

Also, recall that

$$\Box_{\mathbf{g}}(\phi) = \frac{1}{\Omega^2}\left(-4\partial_u\partial_{\underline{u}}\phi - \frac{2\partial_u r}{r}\partial_{\underline{u}}\phi - \frac{2\partial_{\underline{u}} r}{r}\partial_u\phi\right)$$

and

$$\Box_{\mathbf{g}}\phi = \frac{f(\phi)}{r^2}.$$

We infer

$$\partial_u\partial_{\underline{u}}\phi = -\frac{\partial_u r}{2r}\partial_{\underline{u}}\phi - \frac{\partial_{\underline{u}} r}{2r}\partial_u\phi - \frac{\Omega^2 f(\phi)}{4r^2}.$$

[11] Note that the proof of these improvements only relies on the equation $\partial_u\partial_{\underline{u}} r = \kappa\Omega^2 g(\phi)^2/(4r)$. Therefore, the proof requires the improved estimate for ϕ. The important point is that it does not require the corresponding improvement for $\partial_{\underline{u}}\phi$ which does not hold due to a log loss (see Remark 7.23).

Hence, we deduce

$$
\begin{aligned}
\partial_u(\partial_{\underline{u}}\log(\Omega)) &= -\frac{\kappa}{2}\partial_u(\phi\partial_{\underline{u}}\phi)+\frac{\kappa}{2}\phi\partial_u\partial_{\underline{u}}\phi-\frac{\kappa\Omega^2}{8}\frac{g(\phi)^2}{r^2}\\
&= -\frac{\kappa}{2}\partial_u(\phi\partial_{\underline{u}}\phi)+\frac{\kappa}{2}\phi\left(-\frac{\partial_u r}{2r}\partial_{\underline{u}}\phi-\frac{\partial_{\underline{u}} r}{2r}\partial_u\phi-\frac{\Omega^2 f(\phi)}{4r^2}\right)-\frac{\kappa\Omega^2}{8}\frac{g(\phi)^2}{r^2}\\
&= -\frac{\kappa}{2}\partial_u(\phi\partial_{\underline{u}}\phi)-\frac{\kappa}{8}\partial_u\left(\frac{\partial_{\underline{u}} r}{r}\phi^2\right)+\frac{\kappa\partial_u\partial_{\underline{u}} r}{8r}\phi^2-\frac{\kappa\partial_u r\partial_{\underline{u}} r}{8r^2}\phi^2-\frac{\kappa\partial_u r}{4r}\phi\partial_{\underline{u}}\phi\\
&\quad -\frac{\kappa\Omega^2\phi f(\phi)}{8r^2}-\frac{\kappa\Omega^2}{8}\frac{g(\phi)^2}{r^2}\\
&= -\frac{\kappa}{2}\partial_u(\phi\partial_{\underline{u}}\phi)-\frac{\kappa}{8}\partial_u\left(\frac{\partial_{\underline{u}} r}{r}\phi^2\right)+\frac{\kappa^2\Omega^2}{32r^2}g(\phi)^2\phi^2-\frac{\kappa\partial_u r\partial_{\underline{u}} r}{8r^2}\phi^2-\frac{\kappa\partial_u r}{4r}\phi\partial_{\underline{u}}\phi\\
&\quad -\frac{\kappa\Omega^2\phi f(\phi)}{8r^2}-\frac{\kappa\Omega^2}{8}\frac{g(\phi)^2}{r^2}.
\end{aligned}
$$

We integrate between $(u,\underline{u})$ and $(u_0,\underline{u})$ where $(u_0,\underline{u})$ is on the initial hyper surface. We deduce

$$
\begin{aligned}
\left[\partial_{\underline{u}}\log(\Omega)(\sigma,\underline{u})\right]_{u_0}^{u} &= -\frac{\kappa}{2}\left[\phi\partial_{\underline{u}}\phi(\sigma,\underline{u})\right]_{u_0}^{u}-\frac{\kappa}{8}\left[\frac{\partial_{\underline{u}} r}{r}\phi^2(\sigma,\underline{u})\right]_{u_0}^{u}+\int_{u_0}^{u}\frac{\kappa^2\Omega^2}{32r^2}g(\phi)^2\phi^2(\sigma,\underline{u})d\sigma\\
&\quad -\int_{u_0}^{u}\frac{\kappa\partial_u r\partial_{\underline{u}} r}{8r^2}\phi^2(\sigma,\underline{u})d\sigma-\int_{u_0}^{u}\frac{\kappa\partial_u r}{4r}\phi\partial_{\underline{u}}\phi(\sigma,\underline{u})d\sigma\\
&\quad -\int_{u_0}^{u}\frac{\kappa\Omega^2\phi f(\phi)}{8r^2}(\sigma,\underline{u})d\sigma-\int_{u_0}^{u}\frac{\kappa\Omega^2}{8}\frac{g(\phi)^2}{r^2}(\sigma,\underline{u})d\sigma.
\end{aligned}
$$

This yields

$$
\begin{aligned}
|\partial_{\underline{u}}\log(\Omega)(u,\underline{u})| &\lesssim C_0+|\phi|\left(|\partial_{\underline{u}}\phi|+\frac{|\partial_{\underline{u}} r|}{r}|\phi|\right)(u,\underline{u})\\
&\quad +\int_{u_0}^{u}\frac{|\partial_u r|}{r}|\phi|\left(|\partial_{\underline{u}}\phi|+\frac{|\partial_{\underline{u}} r|}{r}|\phi|\right)(\sigma,\underline{u})d\sigma+\int_{u_0}^{u}\frac{\Omega^2|\phi|}{r^2}(\sigma,\underline{u})d\sigma\\
&\lesssim C_0+r^{2\delta-1}C_0\\
&\lesssim r^{2\delta-1}C_0,
\end{aligned}
$$

where we used the fact that $0<\delta<1/2$, the upper bounds on ϕ, $\partial_{\underline{u}}\phi$ of Corollary 7.21, the uniform bounds for $\partial_u r$, $\partial_{\underline{u}} r$ and Ω, and the fact that $\varrho\sim r$. In view of the fact that $|\Omega|\sim 1$, we finally obtain

$$r^{1-2\delta}|\partial_{\underline{u}}\Omega| \lesssim C_0.$$

This conclues the proof of the lemma. □

Next, we derive an upper bound for $\partial_{\underline{u}}^2 r$.

Lemma 8.2. *There exists a constant C_0 only depending on initial data such that we have for any $0<\delta<1/2$*

$$|\partial_{\underline{u}}^2 r(u,\underline{u})| \lesssim r^{2\delta-1}C_0.$$

Proof. Recall that

$$\partial_u\partial_{\underline{u}} r=\kappa\frac{\Omega^2 g(\phi)^2}{4r}.$$

We deduce

$$\partial_u\partial_{\underline{u}}^2 r = \kappa\frac{\Omega(\partial_{\underline{u}}\Omega)g(\phi)^2}{2r}+\kappa\frac{\Omega^2 g(\phi)g'(\phi)\partial_{\underline{u}}\phi}{2r}-\kappa\frac{\Omega^2 g(\phi)^2\partial_{\underline{u}} r}{4r^2}.$$

We integrate between $(u, \underline{u})$ and $(u_0, \underline{u})$ where $(u_0, \underline{u})$ is on the initial hyper surface. We deduce

$$\begin{aligned}
[\partial_{\underline{u}}^2 r(\sigma, \underline{u})]_{u_0}^u &= \kappa \int_{u_0}^u \frac{\Omega(\partial_{\underline{u}}\Omega) g(\phi)^2}{2r}(\sigma, \underline{u}) d\sigma + \kappa \int_{u_0}^u \frac{\Omega^2 g(\phi) g'(\phi) \partial_{\underline{u}}\phi}{2r}(\sigma, \underline{u}) d\sigma \\
&\quad -\kappa \int_{u_0}^u \frac{\Omega^2 g(\phi)^2 \partial_{\underline{u}} r}{4r^2}(\sigma, \underline{u}) d\sigma
\end{aligned}$$

and hence

$$\begin{aligned}
|\partial_{\underline{u}}^2 r(u, \underline{u})| &\lesssim C_0 + \int_{u_0}^u \left(\frac{|\partial_{\underline{u}}\Omega||\phi|^2}{r} + \frac{|\phi||\partial_{\underline{u}}\phi|}{r} + \frac{|\phi|^2 |\partial_{\underline{u}} r|}{r^2} \right)(\sigma, \underline{u}) d\sigma \\
&\lesssim C_0 + r^{2\delta - 1} C_0 \\
&\lesssim r^{2\delta - 1} C_0,
\end{aligned}$$

where we used the fact that $0 < \delta < 1/2$, the upper bounds on ϕ, $\partial_{\underline{u}}\phi$ of Corollary 7.21, the uniform bounds for $\partial_{\underline{u}} r$ and Ω, the upper bounds on $\partial_{\underline{u}}\Omega$ of Lemma 8.1, and the fact that $\varrho \sim r$. This concludes the proof of the lemma. □

Next, we derive an upper bound for $\partial_{\underline{u}}^2\phi$.

Lemma 8.3. *There exists a constant C_0 only depending on initial data such that we have for any $0 < \delta < 1/2$*

$$|\partial_{\underline{u}}^2 \phi| \lesssim r^{\delta - 2} C_0.$$

Proof. Recall that

$$\Theta = r \partial_{\underline{u}} \phi$$

satisfies

$$\partial_u \left(\frac{\Theta}{\sqrt{r}} \right) = -\frac{1}{2\sqrt{r}} \partial_{\underline{u}} r \partial_u \phi - \frac{\Omega^2 f(\phi)}{4r^{\frac{3}{2}}}.$$

Differentiating with respect to $\underline{u}$, we obtain

$$\begin{aligned}
\partial_u \partial_{\underline{u}} \left(\frac{\Theta}{\sqrt{r}} \right) &= -\frac{1}{2\sqrt{r}} \partial_{\underline{u}} r \partial_u \partial_{\underline{u}} \phi - \frac{1}{2\sqrt{r}} \partial_{\underline{u}}^2 r \partial_u \phi + \frac{1}{4r^{\frac{3}{2}}} (\partial_{\underline{u}} r)^2 \partial_u \phi - \frac{\Omega^2 f'(\phi) \partial_{\underline{u}} \phi}{4r^{\frac{3}{2}}} \\
&\quad - \frac{2\Omega(\partial_{\underline{u}}\Omega) f(\phi)}{4r^{\frac{3}{2}}} + \frac{3\Omega^2 f(\phi) \partial_{\underline{u}} r}{8r^{\frac{5}{2}}}.
\end{aligned}$$

In view of

$$\partial_u \partial_{\underline{u}} \phi = -\frac{\partial_u r}{2r} \partial_{\underline{u}} \phi - \frac{\partial_{\underline{u}} r}{2r} \partial_u \phi - \frac{\Omega^2 f(\phi)}{4r^2}.$$

we deduce

$$
\begin{aligned}
\partial_u \partial_{\underline{u}} \left(\frac{\Theta}{\sqrt{r}} \right) &= -\frac{1}{2\sqrt{r}} \partial_{\underline{u}} r \left(-\frac{\partial_u r \partial_{\underline{u}} \phi}{2r} - \frac{\partial_{\underline{u}} r \partial_u \phi}{2r} - \frac{\Omega^2 f(\phi)}{4r^2} \right) - \frac{1}{2\sqrt{r}} \partial_{\underline{u}}^2 r \partial_u \phi + \frac{1}{4r^{\frac{3}{2}}} (\partial_{\underline{u}} r)^2 \partial_u \phi \\
&\quad - \frac{\Omega^2 f'(\phi) \partial_{\underline{u}} \phi}{4r^{\frac{3}{2}}} - \frac{2\Omega(\partial_{\underline{u}} \Omega) f(\phi)}{4r^{\frac{3}{2}}} + \frac{3\Omega^2 f(\phi) \partial_{\underline{u}} r}{8r^{\frac{5}{2}}} \\
&= \left(-\frac{1}{2\sqrt{r}} \partial_{\underline{u}}^2 r + \frac{1}{2r^{\frac{3}{2}}} (\partial_{\underline{u}} r)^2 \right) \partial_u \phi + \frac{\partial_{\underline{u}} r \partial_u r \partial_{\underline{u}} \phi}{4r^{\frac{3}{2}}} \\
&\quad - \frac{\Omega^2 f'(\phi) \partial_{\underline{u}} \phi}{4r^{\frac{3}{2}}} - \frac{2\Omega(\partial_{\underline{u}} \Omega) f(\phi)}{4r^{\frac{3}{2}}} + \frac{\Omega^2 f(\phi) \partial_{\underline{u}} r}{2r^{\frac{5}{2}}} \\
&= \partial_u \left[-\frac{1}{2\sqrt{r}} \partial_{\underline{u}}^2 r \phi + \frac{1}{2r^{\frac{3}{2}}} (\partial_{\underline{u}} r)^2 \phi \right] + \frac{1}{2\sqrt{r}} \partial_u \partial_{\underline{u}}^2 r \phi - \frac{\partial_u r}{4r^{\frac{3}{2}}} \partial_{\underline{u}}^2 r \phi - \frac{1}{r^{\frac{3}{2}}} \partial_{\underline{u}} r \partial_u \partial_{\underline{u}} r \phi \\
&\quad + \frac{3}{4r^{\frac{3}{2}}} \partial_u r (\partial_{\underline{u}} r)^2 \phi + \frac{\partial_{\underline{u}} r \partial_u r \partial_{\underline{u}} \phi}{4r^{\frac{3}{2}}} - \frac{\Omega^2 f'(\phi) \partial_{\underline{u}} \phi}{4r^{\frac{3}{2}}} - \frac{2\Omega(\partial_{\underline{u}} \Omega) f(\phi)}{4r^{\frac{3}{2}}} + \frac{\Omega^2 f(\phi) \partial_{\underline{u}} r}{2r^{\frac{5}{2}}}.
\end{aligned}
$$

Recall that

$$
\partial_u \partial_{\underline{u}} r = \kappa \frac{\Omega^2 g(\phi)^2}{4r},
$$

and

$$
\partial_u \partial_{\underline{u}}^2 r = \kappa \frac{\Omega(\partial_{\underline{u}} \Omega) g(\phi)^2}{2r} + \kappa \frac{\Omega^2 g(\phi) g'(\phi) \partial_{\underline{u}} \phi}{2r} - \kappa \frac{\Omega^2 g(\phi)^2 \partial_{\underline{u}} r}{4r^2}.
$$

We deduce

$$
\begin{aligned}
\partial_u \partial_{\underline{u}} \left(\frac{\Theta}{\sqrt{r}} \right) &= \partial_u \left[-\frac{1}{2\sqrt{r}} \partial_{\underline{u}}^2 r \phi + \frac{1}{2r^{\frac{3}{2}}} (\partial_{\underline{u}} r)^2 \phi \right] \\
&\quad + \frac{\kappa \Omega(\partial_{\underline{u}} \Omega) g(\phi)^2}{4r^{\frac{3}{2}}} \phi + \frac{\kappa \Omega^2 g(\phi) g'(\phi) \partial_{\underline{u}} \phi}{4r^{\frac{3}{2}}} \phi - \frac{3\kappa \Omega^2 g(\phi)^2 \partial_{\underline{u}} r}{8r^{\frac{5}{2}}} \phi \\
&\quad - \frac{\partial_u r}{4r^{\frac{3}{2}}} \partial_{\underline{u}}^2 r \phi + \frac{3}{4r^{\frac{3}{2}}} \partial_u r (\partial_{\underline{u}} r)^2 \phi + \frac{\partial_{\underline{u}} r \partial_u r \partial_{\underline{u}} \phi}{4r^{\frac{3}{2}}} - \frac{\Omega^2 f'(\phi) \partial_{\underline{u}} \phi}{4r^{\frac{3}{2}}} \\
&\quad - \frac{\Omega(\partial_{\underline{u}} \Omega) f(\phi)}{2r^{\frac{3}{2}}} + \frac{\Omega^2 f(\phi) \partial_{\underline{u}} r}{2r^{\frac{5}{2}}}.
\end{aligned}
$$

We integrate between $(u, \underline{u})$ and $(u_0, \underline{u})$ where $(u_0, \underline{u})$ is on the initial hyper surface

$$
\begin{aligned}
\left[\partial_{\underline{u}} \left(\frac{\Theta}{\sqrt{r}} \right) (\sigma, \underline{u}) \right]_{u_0}^{u} &= \left[-\frac{1}{2\sqrt{r}} \partial_{\underline{u}}^2 r \phi + \frac{1}{2r^{\frac{3}{2}}} (\partial_{\underline{u}} r)^2 \phi(\sigma, \underline{u}) \right]_{u_0}^{u} \\
&\quad + \int_{u_0}^{u} \left\{ \frac{\kappa \Omega(\partial_{\underline{u}} \Omega) g(\phi)^2}{4r^{\frac{3}{2}}} \phi + \frac{\kappa \Omega^2 g(\phi) g'(\phi) \partial_{\underline{u}} \phi}{4r^{\frac{3}{2}}} \phi - \frac{3\kappa \Omega^2 g(\phi)^2 \partial_{\underline{u}} r}{8r^{\frac{5}{2}}} \phi \right. \\
&\quad - \frac{\partial_u r}{4r^{\frac{3}{2}}} \partial_{\underline{u}}^2 r \phi + \frac{3}{4r^{\frac{3}{2}}} \partial_u r (\partial_{\underline{u}} r)^2 \phi + \frac{\partial_{\underline{u}} r \partial_u r \partial_{\underline{u}} \phi}{4r^{\frac{3}{2}}} - \frac{\Omega^2 f'(\phi) \partial_{\underline{u}} \phi}{4r^{\frac{3}{2}}} \\
&\quad \left. - \frac{\Omega(\partial_{\underline{u}} \Omega) f(\phi)}{2r^{\frac{3}{2}}} + \frac{\Omega^2 f(\phi) \partial_{\underline{u}} r}{2r^{\frac{5}{2}}} \right\} (\sigma, \underline{u}) d\sigma
\end{aligned}
$$

and hence

$$\begin{aligned}
\left|\partial_{\underline{u}}\left(\frac{\Theta}{\sqrt{r}}\right)(u,\underline{u})\right| &\lesssim C_0+\left(\frac{|\phi||\partial_{\underline{u}}^2 r|}{\sqrt{r}}+\frac{(\partial_{\underline{u}}r)^2|\phi|}{r^{\frac{3}{2}}}\right)(u,\underline{u})\\
&\quad+\int_{u_0}^{u}\frac{|\partial_{\underline{u}}\phi||\phi|+|\partial_{\underline{u}}^2 r||\phi|+|\phi|+|\partial_{\underline{u}}\phi|+|\partial_{\underline{u}}\Omega||\phi|}{r^{\frac{3}{2}}}(\sigma,\underline{u})d\sigma\\
&\quad+\int_{u_0}^{u}\frac{|\phi|}{r^{\frac{5}{2}}}(\sigma,\underline{u})d\sigma\\
&\lesssim C_0+r^{\delta-\frac{3}{2}}C_0\\
&\lesssim r^{\delta-\frac{3}{2}}C_0,
\end{aligned}$$

where we used the fact that $0<\delta<1/2$, the upper bounds on ϕ, $\partial_{\underline{u}}\phi$ of Corollary 7.21, the uniform bounds for $\partial_u r$, $\partial_{\underline{u}}r$ and Ω, the upper bounds on $\partial_{\underline{u}}\Omega$ of Lemma 8.1, the upper bounds on $\partial_{\underline{u}}^2 r$ of Lemma 8.2 and the fact that $\varrho\sim r$. Since

$$\Theta=r\partial_{\underline{u}}\phi,$$

we infer

$$\partial_{\underline{u}}\left(\frac{\Theta}{\sqrt{r}}\right)=\sqrt{r}\partial_{\underline{u}}^2\phi+\frac{1}{2\sqrt{r}}\partial_{\underline{u}}\phi$$

and hence

$$|\partial_{\underline{u}}^2\phi| \lesssim r^{\delta-2}C_0.$$

This concludes the proof of the lemma. □

8.2. **A wave equation satisfied by v.**

Lemma 8.4. *Let*

$$v=D\phi=\left(\partial_\varrho+\frac{1}{\varrho}\right)\phi.$$

We have

$$\left(-\partial_\tau^2+\partial_\varrho^2+\frac{1}{\varrho}\partial_\varrho\right)\phi = \partial_u\left(\frac{B_1}{\varrho}v+\frac{B_2}{\varrho^2}\right)+\frac{B_3}{\varrho^2}v+\frac{B_4}{\varrho^3},$$

where

$$B_1 = \frac{\varrho-r+\varrho(2\partial_{\underline{u}}r-1)}{r},$$

$$B_2 = \frac{2\varrho^2(\partial_\varrho r-1)}{r}\partial_{\underline{u}}\phi-\frac{\varrho-r+\varrho(2\partial_{\underline{u}}r-1)+2\varrho^2\partial_{\underline{u}}^2 r}{r}\phi-\frac{\varrho^2(\Omega^2-1)f(\phi)}{r^2},$$

$$\begin{aligned}
B_3 &= \frac{\varrho-r}{r}\left(\frac{5}{2}+\frac{3\varrho}{2r}\partial_{\underline{u}}r+\frac{\varrho}{r}\right)+\frac{\varrho(2\partial_{\underline{u}}r-1)}{r}\left(1+\frac{\varrho}{r}\partial_r u+\frac{\varrho}{r}\partial_\varrho r\right)+\frac{\varrho(2\partial_u r+1)}{2r}\left(1-\frac{\varrho}{r}\partial_{\underline{u}}r\right)\\
&\quad\frac{\varrho(\partial_\varrho r-1)}{r}\left(1-\frac{\varrho}{r}\partial_{\underline{u}}r\right)+\frac{\varrho^2(3\phi^2\zeta(\phi)+\phi^3\zeta'(\phi))}{r^2},
\end{aligned}$$

and

$$\begin{aligned}
B_4 &= \Bigg[\left(-\frac{7}{2}-\frac{\varrho}{r}\partial_u r-\frac{2\varrho^2+3\varrho r}{r^2}\partial_\varrho r-\frac{\varrho}{2r}\partial_{\underline{u}} r\right)\frac{\varrho-r}{r}+\left(-\frac{\varrho\partial_{\underline{u}} r}{r}-1\right)\frac{\varrho(2\partial_{\underline{u}} r-1)}{r}\\
&\quad+\left(-3+\frac{\varrho\partial_{\underline{u}} r}{r}\right)\frac{\varrho(\partial_\varrho r-1)}{r}+\left(-\frac{1}{2}+\frac{\varrho\partial_{\underline{u}} r}{2r}\right)\frac{\varrho\,(2\partial_u r+1)}{r}+\frac{\kappa\varrho^3\Omega^2 g(\phi)^2\partial_{\underline{u}} r}{2r^3}\\
&\quad+\frac{2\varrho(\varrho r+\partial_u r\varrho^2)\partial_{\underline{u}}^2 r}{r^2}-\frac{\varrho^2(2\phi^2\zeta(\phi)+\phi^3\zeta'(\phi))}{r^2}-\frac{2\varrho^3\phi^2\zeta(\phi)\partial_\varrho r}{r^3}\Bigg]\phi\\
&\quad+\Bigg[\left(4+\frac{\varrho(3\partial_u r+\partial_{\underline{u}} r)}{r}\right)\frac{\varrho^2(\partial_\varrho r-1)}{r}\\
&\quad+\left(-\frac{5\varrho^2(\varrho-r)}{2r}+\frac{\varrho^3(2\partial_u r+1)}{2r}-\frac{2\varrho^3(\partial_\varrho r-1)}{r}\right)\frac{\partial_u r+\partial_{\underline{u}} r}{r}\\
&\quad-\frac{\kappa\varrho^3\Omega^2 g(\phi)g'(\phi)\phi}{r^2}-\frac{\kappa\varrho^3\Omega^2 g(\phi)^2}{2r^2}\Bigg]\partial_{\underline{u}}\phi+\left[-\frac{\varrho^2(\varrho-r)}{r}+\frac{\varrho^3(2\partial_u r+1)}{r}\right]\partial_{\underline{u}}^2\phi\\
&\quad+\left[\left(-\frac{2\varrho\partial_{\underline{u}} r}{r}-1\right)\frac{\varrho^2 f(\phi)}{r^2}+\frac{\varrho^2 f'(\phi)\varrho\partial_{\underline{u}}\phi}{r^2}\right](\Omega^2-1)\\
&\quad+\left[-\frac{\varrho(\varrho-r)}{r}\frac{\varrho\Omega^2}{4r^2}+\frac{\varrho^3(2\partial_{\underline{u}} r-1)\Omega^2}{4r^3}\right]f(\phi)+(2f(\phi)-\kappa g(\phi)^2\phi)\frac{\varrho^3\Omega\partial_{\underline{u}}\Omega}{r^2}.
\end{aligned}$$

Remark 8.5. *Again, we need to integrate by parts the terms involving derivatives with respect to u due to a better behavior with respect to $\underline{u}$ derivatives (see for example the estimates for $\partial_{\underline{u}}^2 r$, $\partial_{\underline{u}}\Omega$ and $\partial_{\underline{u}}^2\phi$ of section 8.1 which do not have a corresponding u counterpart). This results in the term $\partial_u(B_1 v/\varrho + B_2\varrho^2)$ in the statement of Lemma 8.4. As emphasized in Remark 7.13, the fact that this integration by parts is possible is a consequence on the one hand of the null structure of the problem, and on the other hand of the nice behavior of the kernel of the representation formula for the wave equation with respect to u derivatives.*

Remark 8.6. *The crucial point of the decomposition of the right-hand side of the wave equation for v in the statement of Lemma 8.4 is the fact that both B_1 and B_3 include neither $\partial_{\underline{u}}\phi$ nor $\partial_{\underline{u}}\Omega$ in their definition, and hence will satisfy better estimates than B_2 and B_4 (see Lemma 8.7 and Remark 8.8).*

Proof. We have

$$D\circ\left[-\partial_\tau^2+\partial_\varrho^2+\frac{1}{\varrho}\partial_\varrho-\frac{1}{\varrho^2}\right]=\left[-\partial_\tau^2+\partial_\varrho^2+\frac{1}{\varrho}\partial_\varrho\right]\circ D.$$

Recall that

$$\left(-\partial_\tau^2+\partial_\varrho^2+\frac{1}{\varrho}\partial_\varrho-\frac{1}{\varrho^2}\right)\phi=\frac{F}{\varrho^2}$$

where

$$\begin{aligned}
F &= -\frac{\varrho(\varrho-r)}{r}(\partial_{\underline{u}}\phi-\partial_u\phi)+\frac{\varrho^2(2\partial_u r+1)}{r}\partial_{\underline{u}}\phi+\frac{\varrho^2(2\partial_{\underline{u}} r-1)}{r}\partial_u\phi+\frac{\varrho^2-r^2}{r^2}\phi+\frac{\varrho^2\phi^3\zeta(\phi)}{r^2}\\
&\quad+\varrho^2(\Omega^2-1)\frac{f(\phi)}{r^2}.
\end{aligned}$$

We infer

$$\left(-\partial_\tau^2+\partial_\varrho^2+\frac{1}{\varrho}\partial_\varrho\right)\phi=\frac{\partial_\varrho F}{\varrho^2}-\frac{F}{\varrho^3}.$$

Next, we compute $\partial_\varrho F$. We have

$$
\begin{aligned}
\partial_\varrho F \;=\;& -\frac{\varrho(\varrho-r)}{r}(\partial_{\underline{u}}\partial_\varrho\phi-\partial_u\partial_\varrho\phi)-\frac{r(2\varrho-r-\varrho\partial_\varrho r)-\varrho(\varrho-r)\partial_\varrho r}{r^2}(\partial_{\underline{u}}\phi-\partial_u\phi)\\
&+\frac{\varrho^2(2\partial_u r+1)}{r}\partial_{\underline{u}}\partial_\varrho\phi+\frac{r(2\varrho(2\partial_u r+1)+2\varrho^2\partial_u\partial_\varrho r)-\varrho^2(2\partial_u r+1)\partial_\varrho r}{r^2}\partial_{\underline{u}}\phi\\
&+\frac{\varrho^2(2\partial_{\underline{u}} r-1)}{r}\partial_u\partial_\varrho\phi+\frac{r(2\varrho(2\partial_{\underline{u}} r-1)+2\varrho^2\partial_{\underline{u}}\partial_\varrho r)-\varrho^2(2\partial_{\underline{u}} r-1)\partial_\varrho r}{r^2}\partial_u\phi\\
&+\frac{\varrho^2-r^2}{r^2}\partial_\varrho\phi+\frac{r^2(2\varrho-2r\partial_\varrho r)-2(\varrho^2-r^2)r\partial_\varrho r}{r^4}\phi\\
&+\frac{r^2(2\varrho\phi^3\zeta(\phi)+\varrho^2(3\phi^2\zeta(\phi)+\phi^3\zeta'(\phi))\partial_\varrho\phi)-2\varrho^2\phi^3\zeta(\phi)r\partial_\varrho r}{r^4}\\
&+\frac{r^2(2\varrho(\Omega^2-1)f(\phi)+2\varrho^2\Omega\partial_\varrho(\Omega)f(\phi)+\varrho^2(\Omega^2-1)f'(\phi)\partial_\varrho\phi)-2\varrho^2(\Omega^2-1)f(\phi)r\partial_\varrho r}{r^4}\\
=\;& \partial_u\left(\frac{\varrho(\varrho-r)}{r}\partial_\varrho\phi+\frac{2\varrho^2(\partial_\varrho r-1)}{r}\partial_{\underline{u}}\phi+\frac{\varrho^2(2\partial_{\underline{u}} r-1)}{r}\partial_\varrho\phi-\frac{\varrho^2(\Omega^2-1)f(\phi)}{r^2}\right)\\
&-\partial_u\left(\frac{\varrho(\varrho-r)}{r}\right)\partial_\varrho\phi-\partial_u\left(\frac{2\varrho^2}{r}\partial_{\underline{u}}\phi\right)(\partial_\varrho r-1)-\partial_u\left(\frac{\varrho^2(2\partial_{\underline{u}} r-1)}{r}\right)\partial_\varrho\phi\\
&+\partial_u\left(\frac{\varrho^2 f(\phi)}{r^2}\right)(\Omega^2-1)-\frac{\varrho(\varrho-r)}{r}\partial_{\underline{u}}\partial_\varrho\phi-\frac{r(2\varrho-r-\varrho\partial_\varrho r)-\varrho(\varrho-r)\partial_\varrho r}{r^2}\partial_\varrho\phi\\
&+\frac{\varrho^2(2\partial_u r+1)}{r}\partial_{\underline{u}}\partial_\varrho\phi+\frac{2r\varrho(2\partial_u r+1)-\varrho^2(2\partial_u r+1)\partial_\varrho r}{r^2}\partial_{\underline{u}}\phi\\
&+\frac{r(2\varrho(2\partial_{\underline{u}} r-1)-2\varrho^2\partial_{\underline{u}}\partial_u r)-\varrho^2(2\partial_{\underline{u}} r-1)\partial_\varrho r}{r^2}(\partial_{\underline{u}}\phi-\partial_\varrho\phi)+\frac{2\varrho^2\partial_{\underline{u}}^2 r}{r}\partial_u\phi\\
&+\frac{\varrho^2-r^2}{r^2}\partial_\varrho\phi+\frac{r^2(2\varrho-2r\partial_\varrho r)-2(\varrho^2-r^2)r\partial_\varrho r}{r^4}\phi\\
&+\frac{r^2(2\varrho\phi^3\zeta(\phi)+\varrho^2(3\phi^2\zeta(\phi)+\phi^3\zeta'(\phi))\partial_\varrho\phi)-2\varrho^2\phi^3\zeta(\phi)r\partial_\varrho r}{r^4}\\
&+\frac{r^2(2\varrho(\Omega^2-1)f(\phi)+2\varrho^2\Omega\partial_{\underline{u}}(\Omega)f(\phi)+\varrho^2(\Omega^2-1)f'(\phi)\partial_\varrho\phi)-2\varrho^2(\Omega^2-1)f(\phi)r\partial_\varrho r}{r^4}.
\end{aligned}
$$

We rewrite the term $\varrho^2\partial_{\underline{u}}^2 r\partial_u\phi/r$ as[12]

$$
\frac{\varrho^2\partial_{\underline{u}}^2 r}{r}\partial_u\phi \;=\; \partial_u\left(\frac{\varrho^2\partial_{\underline{u}}^2 r}{r}\phi\right)-\frac{\varrho^2\partial_u\partial_{\underline{u}}^2 r}{r}\phi+\frac{(\varrho r+\partial_u r\varrho^2)\partial_{\underline{u}}^2 r\phi}{r^2}.
$$

Recall that

$$
\partial_u\partial_{\underline{u}}^2 r \;=\; \kappa\frac{\Omega(\partial_{\underline{u}}\Omega)g(\phi)^2}{2r}+\kappa\frac{\Omega^2 g(\phi)g'(\phi)\partial_{\underline{u}}\phi}{2r}-\kappa\frac{\Omega^2 g(\phi)^2\partial_{\underline{u}} r}{4r^2}.
$$

We infer

$$
\begin{aligned}
\frac{\varrho^2\partial_{\underline{u}}^2 r}{r}\partial_u\phi \;=\;& \partial_u\left(\frac{\varrho^2\partial_{\underline{u}}^2 r}{r}\phi\right)-\frac{\kappa\varrho^2\Omega(\partial_{\underline{u}}\Omega)g(\phi)^2\phi}{2r^2}-\frac{\kappa\varrho^2\Omega^2 g(\phi)g'(\phi)\phi\partial_{\underline{u}}\phi}{2r^2}\\
&+\frac{\kappa\varrho^2\Omega^2 g(\phi)^2\phi\partial_{\underline{u}} r}{4r^3}+\frac{(\varrho r+\partial_u r\varrho^2)\partial_{\underline{u}}^2 r\phi}{r^2}.
\end{aligned}
$$

[12]This term needs to be integrated by parts as it would otherwise lead to a dangerous term of the type $\partial_{\underline{u}}^2 rv$.

We obtain

$$
\begin{aligned}
\partial_\varrho F \quad = \quad & \partial_u\Bigg(\frac{\varrho(\varrho-r)}{r}\left(v-\frac{\phi}{\varrho}\right)+\frac{2r\varrho^2(\partial_\varrho r-1)}{r^2}\partial_{\underline{u}}\phi+\frac{\varrho^2(2\partial_{\underline{u}}r-1)}{r}\left(v-\frac{\phi}{\varrho}\right)-\frac{\varrho^2(\Omega^2-1)f(\phi)}{r^2} \\
& +\frac{2\varrho^2\partial_{\underline{u}}^2 r}{r}\phi\Bigg) \\
& -\frac{r(-\varrho+\frac{1}{2}r-\varrho\partial_u r)-\varrho(\varrho-r)\partial_u r}{r^2}\left(v-\frac{\phi}{\varrho}\right)-2\frac{r(-\varrho\partial_{\underline{u}}\phi+\varrho^2\partial_u\partial_{\underline{u}}\phi)-\varrho^2\partial_{\underline{u}}\phi\partial_u r}{r^2}(\partial_\varrho r-1) \\
& -\frac{r(-\varrho(2\partial_{\underline{u}}r-1)+2\varrho^2\partial_u\partial_{\underline{u}}r)-\varrho^2(2\partial_{\underline{u}}r-1)\partial_u r}{r^2}\left(v-\frac{\phi}{\varrho}\right) \\
& +\frac{r^2\left(-\varrho f(\phi)+\varrho^2 f'(\phi)\left(\partial_{\underline{u}}\phi-v+\frac{\phi}{\varrho}\right)\right)-2\varrho^2 f(\phi)r\partial_u r}{r^4}(\Omega^2-1) \\
& -\frac{\varrho(\varrho-r)}{r}(\partial_{\underline{u}}^2\phi-\partial_{\underline{u}}\partial_u\phi)-\frac{r(2\varrho-r-\varrho\partial_\varrho r)-\varrho(\varrho-r)\partial_\varrho r}{r^2}\left(v-\frac{\phi}{\varrho}\right) \\
& +\frac{\varrho^2(2\partial_u r+1)}{r}(\partial_{\underline{u}}^2\phi-\partial_{\underline{u}}\partial_u\phi)+\frac{2r\varrho(2\partial_u r+1)-\varrho^2(2\partial_u r+1)\partial_\varrho r}{r^2}\partial_{\underline{u}}\phi \\
& +\frac{r(2\varrho(2\partial_{\underline{u}}r-1)-2\varrho^2\partial_{\underline{u}}\partial_u r)-\varrho^2(2\partial_{\underline{u}}r-1)\partial_\varrho r}{r^2}\left(\partial_{\underline{u}}\phi-v+\frac{\phi}{\varrho}\right) \\
& -\frac{\kappa\varrho^2\Omega(\partial_{\underline{u}}\Omega)g(\phi)^2\phi}{r^2}-\frac{\kappa\varrho^2\Omega^2 g(\phi)g'(\phi)\phi\partial_{\underline{u}}\phi}{r^2}+\frac{\kappa\varrho^2\Omega^2 g(\phi)^2\phi\partial_{\underline{u}}r}{2r^3}+\frac{2(\varrho r+\partial_u r\varrho^2)\partial_{\underline{u}}^2 r\phi}{r^2} \\
& +\frac{\varrho^2-r^2}{r^2}\left(v-\frac{\phi}{\varrho}\right)+\frac{r^2(2\varrho-2r\partial_\varrho r)-2(\varrho^2-r^2)r\partial_\varrho r}{r^4}\phi \\
& +\frac{r^2\left(2\varrho\phi^3\zeta(\phi)+\varrho^2(3\phi^2\zeta(\phi)+\phi^3\zeta'(\phi))\left(v-\frac{\phi}{\varrho}\right)\right)-2\varrho^2\phi^3\zeta(\phi)r\partial_\varrho r}{r^4} \\
& +\frac{r^2\left(2\varrho(\Omega^2-1)f(\phi)+2\varrho^2\Omega\partial_{\underline{u}}(\Omega)f(\phi)+\varrho^2(\Omega^2-1)f'(\phi)\left(v-\frac{\phi}{\varrho}\right)\right)-2\varrho^2(\Omega^2-1)f(\phi)r\partial_\varrho r}{r^4}.
\end{aligned}
$$

Recall that

$$\partial_u\partial_{\underline{u}}r=r\kappa\frac{\Omega^2}{4}\frac{g(\phi)^2}{r^2}.$$

Also, recall that

$$
\begin{aligned}
& -4\partial_u\partial_{\underline{u}}\phi+\frac{1}{\varrho}(\partial_{\underline{u}}\phi-\partial_u\phi)-\frac{\phi}{\varrho^2} \\
= \quad & -\frac{\varrho-r}{r\varrho}(\partial_{\underline{u}}\phi-\partial_u\phi)+\frac{2\partial_u r+1}{r}\partial_{\underline{u}}\phi+\frac{2\partial_{\underline{u}}r-1}{r}\partial_u\phi+\frac{\varrho^2-r^2}{r^2\varrho^2}\phi+\frac{\phi^3\zeta(\phi)}{r^2} \\
& +(\Omega^2-1)\frac{f(\phi)}{r^2},
\end{aligned}
$$

which yields

$$-\partial_u\partial_{\underline{u}}\phi \quad = \quad -\frac{\partial_{\underline{u}}r}{2r}\left(v-\frac{\phi}{\varrho}\right)+\frac{\partial_u r+\partial_{\underline{u}}r}{2r}\partial_{\underline{u}}\phi+\frac{\Omega^2 f(\phi)}{4r^2}.$$

This allows us to rewrite $\partial_\varrho F$ as

$$
\begin{aligned}
\partial_\varrho F \;=\; & \partial_u\Bigg(\frac{\varrho(\varrho-r)}{r}\left(v-\frac{\phi}{\varrho}\right)+\frac{2r\varrho^2(\partial_\varrho r-1)}{r^2}\partial_{\underline{u}}\phi+\frac{\varrho^2(2\partial_{\underline{u}}r-1)}{r}\left(v-\frac{\phi}{\varrho}\right)-\frac{\varrho^2(\Omega^2-1)f(\phi)}{r^2}\\
&+\frac{2\varrho^2\partial_{\underline{u}}^2 r}{r}\phi\Bigg)-\frac{r(-\varrho+\frac{1}{2}r-\varrho\partial_u r)-\varrho(\varrho-r)\partial_u r}{r^2}\left(v-\frac{\phi}{\varrho}\right)\\
&-2\frac{r\left(-\varrho\partial_{\underline{u}}\phi+\varrho^2\left(\frac{\partial_{\underline{u}}r}{2r}\left(v-\frac{\phi}{\varrho}\right)-\frac{\partial_u r+\partial_{\underline{u}}r}{2r}\partial_{\underline{u}}\phi-\frac{\Omega^2 f(\phi)}{4r^2}\right)\right)-\varrho^2\partial_{\underline{u}}\phi\partial_u r}{r^2}(\partial_\varrho r-1)\\
&-\frac{r\left(-\varrho(2\partial_{\underline{u}}r-1)+2\varrho^2\kappa\frac{\Omega^2}{4}\frac{g(\phi)^2}{r}\right)-\varrho^2(2\partial_{\underline{u}}r-1)\partial_u r}{r^2}\left(v-\frac{\phi}{\varrho}\right)\\
&+\frac{r^2\left(-\varrho f(\phi)+\varrho^2 f'(\phi)\left(\partial_{\underline{u}}\phi-v+\frac{\phi}{\varrho}\right)\right)-2\varrho^2 f(\phi)r\partial_u r}{r^4}(\Omega^2-1)\\
&-\frac{\varrho(\varrho-r)}{r}\left(\partial_{\underline{u}}^2\phi-\frac{\partial_{\underline{u}}r}{2r}\left(v-\frac{\phi}{\varrho}\right)+\frac{\partial_u r+\partial_{\underline{u}}r}{2r}\partial_{\underline{u}}\phi+\frac{\Omega^2 f(\phi)}{4r^2}\right)\\
&-\frac{r(2\varrho-r-\varrho\partial_\varrho r)-\varrho(\varrho-r)\partial_\varrho r}{r^2}\left(v-\frac{\phi}{\varrho}\right)\\
&+\frac{\varrho^2(2\partial_u r+1)}{r}\left(\partial_{\underline{u}}^2\phi-\frac{\partial_{\underline{u}}r}{2r}\left(v-\frac{\phi}{\varrho}\right)+\frac{\partial_u r+\partial_{\underline{u}}r}{2r}\partial_{\underline{u}}\phi+\frac{\Omega^2 f(\phi)}{4r^2}\right)\\
&+\frac{2r\varrho(2\partial_u r+1)-\varrho^2(2\partial_u r+1)\partial_\varrho r}{r^2}\partial_{\underline{u}}\phi\\
&+\frac{r\left(2\varrho(2\partial_{\underline{u}}r-1)-\varrho^2\kappa\frac{\Omega^2}{2}\frac{g(\phi)^2}{r}\right)-\varrho^2(2\partial_{\underline{u}}r-1)\partial_\varrho r}{r^2}\left(\partial_{\underline{u}}\phi-v+\frac{\phi}{\varrho}\right)\\
&-\frac{\kappa\varrho^2\Omega(\partial_{\underline{u}}\Omega)g(\phi)^2\phi}{r^2}-\frac{\kappa\varrho^2\Omega^2 g(\phi)g'(\phi)\phi\partial_{\underline{u}}\phi}{r^2}+\frac{\kappa\varrho^2\Omega^2 g(\phi)^2\phi\partial_{\underline{u}}r}{2r^3}+\frac{2(\varrho r+\partial_u r\varrho^2)\partial_{\underline{u}}^2 r\phi}{r^2}\\
&+\frac{\varrho^2-r^2}{r^2}\left(v-\frac{\phi}{\varrho}\right)+\frac{r^2(2\varrho-2r\partial_\varrho r)-2(\varrho^2-r^2)r\partial_\varrho r}{r^4}\phi\\
&+\frac{r^2\left(2\varrho\phi^3\zeta(\phi)+\varrho^2(3\phi^2\zeta(\phi)+\phi^3\zeta'(\phi))\left(v-\frac{\phi}{\varrho}\right)\right)-2\varrho^2\phi^3\zeta(\phi)r\partial_\varrho r}{r^4}\\
&+\frac{r^2\left(2\varrho(\Omega^2-1)f(\phi)+2\varrho^2\Omega\partial_{\underline{u}}(\Omega)f(\phi)+\varrho^2(\Omega^2-1)f'(\phi)\left(v-\frac{\phi}{\varrho}\right)\right)-2\varrho^2(\Omega^2-1)f(\phi)r\partial_\varrho r}{r^4}.
\end{aligned}
$$

We infer

$$\partial_\varrho F \;=\; \partial_u(A_1 v+A_2)+A_3 v+A_4,$$

where

$$A_1 \;=\; \frac{\varrho(\varrho-r)}{r}+\frac{\varrho^2(2\partial_{\underline{u}}r-1)}{r},$$

$$A_2 \;=\; -\frac{\varrho(\varrho-r)}{r}\frac{\phi}{\varrho}+\frac{2r\varrho^2(\partial_\varrho r-1)}{r^2}\partial_{\underline{u}}\phi-\frac{\varrho^2(2\partial_{\underline{u}}r-1)}{r}\frac{\phi}{\varrho}-\frac{\varrho^2(\Omega^2-1)f(\phi)}{r^2}+\frac{2\varrho^2\partial_{\underline{u}}^2 r}{r}\phi,$$

$$
\begin{aligned}
A_3 \quad = \quad & -\frac{r(-\varrho+\frac{1}{2}r-\varrho\partial_u r)-\varrho(\varrho-r)\partial_u r}{r^2} - 2\frac{r\varrho^2\frac{\partial_{\underline{u}} r}{2r}}{r^2}(\partial_\varrho r - 1) \\
& -\frac{r\left(-\varrho(2\partial_{\underline{u}}r-1)+2\varrho^2\kappa\frac{\Omega^2}{4}\frac{g(\phi)^2}{r}\right)-\varrho^2(2\partial_{\underline{u}}r-1)\partial_u r}{r^2} \\
& -\frac{r^2\varrho^2 f'(\phi)}{r^4}(\Omega^2-1)+\frac{\varrho(\varrho-r)}{r}\frac{\partial_{\underline{u}}r}{2r}-\frac{r(2\varrho-r-\varrho\partial_\varrho r)-\varrho(\varrho-r)\partial_\varrho r}{r^2} \\
& -\frac{\varrho^2(2\partial_u r+1)}{r}\frac{\partial_{\underline{u}}r}{2r} \\
& -\frac{r\left(2\varrho(2\partial_{\underline{u}}r-1)-\varrho^2\kappa\frac{\Omega^2}{2}\frac{g(\phi)^2}{r}\right)-\varrho^2(2\partial_{\underline{u}}r-1)\partial_\varrho r}{r^2} \\
& +\frac{\varrho^2-r^2}{r^2}+\frac{r^2\varrho^2(3\phi^2\zeta(\phi)+\phi^3\zeta'(\phi))}{r^4}+\frac{r^2\varrho^2(\Omega^2-1)f'(\phi)}{r^4},
\end{aligned}
$$

and

$$
\begin{aligned}
A_4 \quad = \quad & \frac{r(-\varrho+\frac{1}{2}r-\varrho\partial_u r)-\varrho(\varrho-r)\partial_u r}{r^2}\frac{\phi}{\varrho} \\
& -2\frac{r\left(-\varrho\partial_{\underline{u}}\phi+\varrho^2\left(-\frac{\partial_{\underline{u}}r}{2r}\frac{\phi}{\varrho}-\frac{\partial_u r+\partial_{\underline{u}}r}{2r}\partial_{\underline{u}}\phi-\frac{\Omega^2 f(\phi)}{4r^2}\right)\right)-\varrho^2\partial_{\underline{u}}\phi\partial_u r}{r^2}(\partial_\varrho r-1) \\
& +\frac{r\left(-\varrho(2\partial_{\underline{u}}r-1)+2\varrho^2\kappa\frac{\Omega^2}{4}\frac{g(\phi)^2}{r}\right)-\varrho^2(2\partial_{\underline{u}}r-1)\partial_u r}{r^2}\frac{\phi}{\varrho} \\
& +\frac{r^2\left(-\varrho f(\phi)+\varrho^2 f'(\phi)\left(\partial_{\underline{u}}\phi+\frac{\phi}{\varrho}\right)\right)-2\varrho^2 f(\phi)r\partial_u r}{r^4}(\Omega^2-1) \\
& -\frac{\varrho(\varrho-r)}{r}\left(\partial_{\underline{u}}^2\phi+\frac{\partial_{\underline{u}}r}{2r}\frac{\phi}{\varrho}+\frac{\partial_u r+\partial_{\underline{u}}r}{2r}\partial_{\underline{u}}\phi+\frac{\Omega^2 f(\phi)}{4r^2}\right) \\
& +\frac{r(2\varrho-r-\varrho\partial_\varrho r)-\varrho(\varrho-r)\partial_\varrho r}{r^2}\frac{\phi}{\varrho} \\
& +\frac{\varrho^2(2\partial_u r+1)}{r}\left(\partial_{\underline{u}}^2\phi+\frac{\partial_{\underline{u}}r}{2r}\frac{\phi}{\varrho}+\frac{\partial_u r+\partial_{\underline{u}}r}{2r}\partial_{\underline{u}}\phi+\frac{\Omega^2 f(\phi)}{4r^2}\right) \\
& +\frac{2r\varrho(2\partial_u r+1)-\varrho^2(2\partial_u r+1)\partial_\varrho r}{r^2}\partial_{\underline{u}}\phi \\
& +\frac{r\left(2\varrho(2\partial_{\underline{u}}r-1)-\varrho^2\kappa\frac{\Omega^2}{2}\frac{g(\phi)^2}{r}\right)-\varrho^2(2\partial_{\underline{u}}r-1)\partial_\varrho r}{r^2}\left(\partial_{\underline{u}}\phi+\frac{\phi}{\varrho}\right) \\
& -\frac{\kappa\varrho^2\Omega(\partial_{\underline{u}}\Omega)g(\phi)^2\phi}{r^2}-\frac{\kappa\varrho^2\Omega^2 g(\phi)g'(\phi)\phi\partial_{\underline{u}}\phi}{r^2}+\frac{\kappa\varrho^2\Omega^2 g(\phi)^2\phi\partial_{\underline{u}}r}{2r^3}+\frac{2(\varrho r+\partial_u r\varrho^2)\partial_{\underline{u}}^2 r\phi}{r^2} \\
& -\frac{\varrho^2-r^2}{r^2}\frac{\phi}{\varrho}+\frac{r^2(2\varrho-2r\partial_\varrho r)-2(\varrho^2-r^2)r\partial_\varrho r}{r^4}\phi \\
& +\frac{r^2\left(2\varrho\phi^3\zeta(\phi)-\varrho^2(3\phi^2\zeta(\phi)+\phi^3\zeta'(\phi))\frac{\phi}{\varrho}\right)-2\varrho^2\phi^3\zeta(\phi)r\partial_\varrho r}{r^4} \\
& +\frac{r^2\left(2\varrho(\Omega^2-1)f(\phi)+2\varrho^2\Omega\partial_{\underline{u}}(\Omega)f(\phi)-\varrho^2(\Omega^2-1)f'(\phi)\frac{\phi}{\varrho}\right)-2\varrho^2(\Omega^2-1)f(\phi)r\partial_\varrho r}{r^4}.
\end{aligned}
$$

We also rewrite F. We have

$$\begin{aligned} F &= -\frac{\varrho(\varrho-r)}{r}\left(v-\frac{\phi}{\varrho}\right)+\frac{\varrho^2(2\partial_u r+1)}{r}\partial_{\underline{u}}\phi+\frac{\varrho^2(2\partial_{\underline{u}}r-1)}{r}\left(\partial_{\underline{u}}\phi-v+\frac{\phi}{\varrho}\right)\\ &\quad+\frac{\varrho^2-r^2}{r^2}\phi+\frac{\varrho^2\phi^3\zeta(\phi)}{r^2}+\varrho^2(\Omega^2-1)\frac{f(\phi)}{r^2}\\ &= A_5v+A_6, \end{aligned}$$

where

$$A_5 = -\frac{\varrho(\varrho-r)}{r}-\frac{\varrho^2(2\partial_{\underline{u}}r-1)}{r},$$

and

$$\begin{aligned} A_6 &= \frac{\varrho(\varrho-r)}{r}\frac{\phi}{\varrho}+\frac{\varrho^2(2\partial_u r+1)}{r}\partial_{\underline{u}}\phi+\frac{\varrho^2(2\partial_{\underline{u}}r-1)}{r}\left(\partial_{\underline{u}}\phi+\frac{\phi}{\varrho}\right)+\frac{\varrho^2-r^2}{r^2}\phi\\ &\quad+\frac{\varrho^2\phi^3\zeta(\phi)}{r^2}+\varrho^2(\Omega^2-1)\frac{f(\phi)}{r^2}. \end{aligned}$$

Finally, we have obtained

$$\begin{aligned} \frac{\partial_\varrho F}{\varrho^2}-\frac{F}{\varrho^3} &= \frac{\partial_u(A_1v+A_2)+A_3v+A_4}{\varrho^2}-\frac{A_5v+A_6}{\varrho^3}\\ &= \partial_u\left(\frac{A_1v+A_2}{\varrho^2}\right)+\frac{A_1v+A_2}{\varrho^3}+\frac{A_3v+A_4}{\varrho^2}-\frac{A_5v+A_6}{\varrho^3}\\ &= \partial_u\left(\frac{B_1}{\varrho}v+\frac{B_2}{\varrho^2}\right)+\frac{B_3}{\varrho^2}v+\frac{B_4}{\varrho^3}, \end{aligned}$$

where

$$\begin{aligned} B_1 &= \frac{A_1}{\varrho}\\ &= \frac{\varrho-r+\varrho(2\partial_{\underline{u}}r-1)}{r}, \end{aligned}$$

$$\begin{aligned} B_2 &= A_2\\ &= \frac{2\varrho^2(\partial_\varrho r-1)}{r}\partial_{\underline{u}}\phi-\frac{\varrho-r+\varrho(2\partial_{\underline{u}}r-1)+2\varrho^2\partial_{\underline{u}}^2 r}{r}\phi-\frac{\varrho^2(\Omega^2-1)f(\phi)}{r^2}, \end{aligned}$$

$$
\begin{aligned}
B_3 &= \frac{A_1}{\varrho} + A_3 - \frac{A_5}{\varrho} \\
&= 2\frac{\varrho - r + \varrho(2\partial_{\underline{u}} r - 1)}{r} \\
&\quad + \frac{\varrho - r + \varrho(2\partial_u r + 1)}{2r} + \frac{\varrho(\varrho - r)\partial_u r}{r^2} - \frac{\varrho^2 \partial_{\underline{u}} r}{r^2}(\partial_\varrho r - 1) \\
&\quad + \frac{\varrho(2\partial_{\underline{u}} r - 1)}{r} - \frac{\kappa\varrho^2\Omega^2 g(\phi)^2}{2r^2} + \frac{\varrho^2(2\partial_{\underline{u}} r - 1)\partial_u r}{r^2} - \frac{\varrho^2 f'(\phi)}{r^2}(\Omega^2 - 1) \\
&\quad + \frac{\varrho(\varrho - r)}{2r^2}\partial_{\underline{u}} r - \frac{\varrho - r + \varrho(1 - \partial_\varrho r)}{r} + \frac{\varrho(\varrho - r)\partial_\varrho r}{r^2} - \frac{\varrho^2(2\partial_u r + 1)}{2r^2}\partial_{\underline{u}} r \\
&\quad - \frac{2\varrho(2\partial_{\underline{u}} r - 1)}{r} + \frac{\kappa\varrho^2\Omega^2 g(\phi)^2}{2r^2} + \frac{\varrho^2(2\partial_{\underline{u}} r - 1)\partial_\varrho r}{r^2} \\
&\quad + \frac{\varrho^2 - r^2}{r^2} + \frac{\varrho^2(3\phi^2\zeta(\phi) + \phi^3\zeta'(\phi))}{r^2} + \frac{\varrho^2(\Omega^2 - 1)f'(\phi)}{r^2} \\
&= \frac{\varrho - r}{r}\left(\frac{5}{2} + \frac{3\varrho}{2r}\partial_{\underline{u}} r + \frac{\varrho}{r}\right) + \frac{\varrho(2\partial_{\underline{u}} r - 1)}{r}\left(1 + \frac{\varrho}{r}\partial_r u + \frac{\varrho}{r}\partial_\varrho r\right) + \frac{\varrho(2\partial_u r + 1)}{2r}\left(1 - \frac{\varrho}{r}\partial_{\underline{u}} r\right) \\
&\quad \frac{\varrho(\partial_\varrho r - 1)}{r}\left(1 - \frac{\varrho}{r}\partial_{\underline{u}} r\right) + \frac{\varrho^2(3\phi^2\zeta(\phi) + \phi^3\zeta'(\phi))}{r^2},
\end{aligned}
$$

and

$$
\begin{aligned}
B_4 &= A_2 + \varrho A_4 - A_6 \\
&= -\frac{\varrho - r}{r}\phi + \frac{2\varrho^2(\partial_\varrho r - 1)}{r}\partial_{\underline{u}}\phi - \frac{\varrho(2\partial_{\underline{u}} r - 1)}{r}\phi - \frac{\varrho^2(\Omega^2-1)f(\phi)}{r^2} \\
&\quad + \left(-\frac{\varrho - r + \varrho\,(2\partial_u r + 1)}{2r} - \frac{\varrho(\varrho - r)\partial_u r}{r^2}\right)\phi \\
&\quad + 2\left(\frac{\varrho^2\partial_{\underline{u}}\phi}{r} + \frac{\varrho^2\partial_{\underline{u}} r\phi}{2r^2} + \frac{\varrho^3(\partial_u r + \partial_{\underline{u}} r)\partial_{\underline{u}}\phi}{2r^2} + \frac{\varrho^3\Omega^2 f(\phi)}{4r^3} + \frac{\varrho^3\partial_{\underline{u}}\phi\partial_u r}{r^2}\right)(\partial_\varrho r - 1) \\
&\quad - \frac{\varrho(2\partial_{\underline{u}} r - 1)}{r}\phi + \frac{\kappa\varrho^2\Omega^2 g(\phi)^2}{2r^2}\phi - \frac{\varrho^2(2\partial_{\underline{u}} r - 1)\partial_u r}{r^2}\phi \\
&\quad + \left(\frac{-\varrho^2 f(\phi) + \varrho^2 f'(\phi)\left(\varrho\partial_{\underline{u}}\phi + \phi\right)}{r^2} - \frac{2\varrho^3 f(\phi)\partial_u r}{r^3}\right)(\Omega^2 - 1) \\
&\quad - \frac{\varrho(\varrho - r)}{r}\left(\varrho\partial_{\underline{u}}^2\phi + \frac{\partial_{\underline{u}} r}{2r}\phi + \frac{\partial_u r + \partial_{\underline{u}} r}{2r}\varrho\partial_{\underline{u}}\phi + \frac{\varrho\Omega^2 f(\phi)}{4r^2}\right) \\
&\quad + \left(\frac{\varrho - r - \varrho(\partial_\varrho r - 1)}{r} - \frac{\varrho(\varrho - r)\partial_\varrho r}{r^2}\right)\phi \\
&\quad + \frac{\varrho^2(2\partial_u r + 1)}{r}\left(\varrho\partial_{\underline{u}}^2\phi + \frac{\partial_{\underline{u}} r}{2r}\phi + \frac{\partial_u r + \partial_{\underline{u}} r}{2r}\varrho\partial_{\underline{u}}\phi + \frac{\varrho\Omega^2 f(\phi)}{4r^2}\right) \\
&\quad + \left(\frac{2\varrho^2(2\partial_u r + 1)}{r} - \frac{\varrho^3(2\partial_u r + 1)\partial_\varrho r}{r^2}\right)\partial_{\underline{u}}\phi \\
&\quad + \left(\frac{2\varrho(2\partial_{\underline{u}} r - 1)}{r} - \frac{\kappa\varrho^2\Omega^2 g(\phi)^2}{2r^2} - \frac{\varrho^2(2\partial_{\underline{u}} r - 1)\partial_\varrho r}{r^2}\right)\left(\varrho\partial_{\underline{u}}\phi + \phi\right) \\
&\quad - \frac{\kappa\varrho^3\Omega(\partial_{\underline{u}}\Omega)g(\phi)^2\phi}{r^2} - \frac{\kappa\varrho^3\Omega^2 g(\phi)g'(\phi)\phi\partial_{\underline{u}}\phi}{r^2} + \frac{\kappa\varrho^3\Omega^2 g(\phi)^2\phi\partial_{\underline{u}} r}{2r^3} + \frac{2\varrho(\varrho r + \partial_u r\varrho^2)\partial_{\underline{u}}^2 r\phi}{r^2} \\
&\quad - \frac{\varrho^2 - r^2}{r^2}\phi + \frac{\varrho(2\varrho - 2r\partial_\varrho r)}{r^2}\phi - \frac{2\varrho(\varrho^2 - r^2)\partial_\varrho r}{r^3}\phi \\
&\quad - \frac{\varrho^2(\phi^2\zeta(\phi) + \phi^3\zeta'(\phi))\phi}{r^2} - \frac{2\varrho^3\phi^3\zeta(\phi)\partial_\varrho r}{r^3} \\
&\quad + \frac{2\varrho^2(\Omega^2-1)f(\phi) + 2\varrho^3\Omega\partial_{\underline{u}}(\Omega)f(\phi) - \varrho^2(\Omega^2-1)f'(\phi)\phi}{r^2} - \frac{2\varrho^3(\Omega^2-1)f(\phi)\partial_\varrho r}{r^3} \\
&\quad - \frac{\varrho - r}{r}\phi - \frac{\varrho^2(2\partial_u r + 1)}{r}\partial_{\underline{u}}\phi - \frac{\varrho(2\partial_{\underline{u}} r - 1)}{r}\left(\varrho\partial_{\underline{u}}\phi + \phi\right) - \frac{\varrho^2 - r^2}{r^2}\phi \\
&\quad - \frac{\varrho^2\phi^3\zeta(\phi)}{r^2} - \varrho^2(\Omega^2 - 1)\frac{f(\phi)}{r^2}.
\end{aligned}
$$

We rewrite B_4 as follows

$$
\begin{aligned}
B_4 \;=\; & \Bigg[-\frac{\varrho - r}{r} - \frac{\varrho(2\partial_{\underline{u}} r - 1)}{r} - \frac{\varrho - r + \varrho\,(2\partial_u r + 1)}{2r} - \frac{\varrho(\varrho - r)\partial_u r}{r^2} \\
& - \frac{\varrho(2\partial_{\underline{u}} r - 1)}{r} + \frac{\kappa\varrho^2\Omega^2 g(\phi)^2}{2r^2} - \frac{\varrho^2(2\partial_{\underline{u}} r - 1)\partial_u r}{r^2} + \frac{\varrho - r - \varrho(\partial_\varrho r - 1)}{r} \\
& - \frac{\varrho(\varrho - r)\partial_\varrho r}{r^2} + \frac{2\varrho(2\partial_{\underline{u}} r - 1)}{r} - \frac{\kappa\varrho^2\Omega^2 g(\phi)^2}{2r^2} - \frac{\varrho^2(2\partial_{\underline{u}} r - 1)\partial_\varrho r}{r^2} \\
& + \frac{\kappa\varrho^3\Omega^2 g(\phi)^2\partial_{\underline{u}} r}{2r^3} + \frac{2\varrho(\varrho r + \partial_u r\varrho^2)\partial_{\underline{u}}^2 r}{r^2} \\
& - \frac{\varrho^2 - r^2}{r^2} + \frac{\varrho(2\varrho - 2r\partial_\varrho r)}{r^2} - \frac{2\varrho(\varrho^2 - r^2)\partial_\varrho r}{r^3} - \frac{\varrho - r}{r} - \frac{\varrho(2\partial_{\underline{u}} r - 1)}{r} - \frac{\varrho^2 - r^2}{r^2} \\
& + 2\frac{\varrho^2\partial_{\underline{u}} r}{2r^2}(\partial_\varrho r - 1) - \frac{\varrho(\varrho - r)}{r}\frac{\partial_{\underline{u}} r}{2r} + \frac{\varrho^2(2\partial_u r + 1)}{r}\frac{\partial_{\underline{u}} r}{2r} - \frac{\varrho^2(\phi^2\zeta(\phi) + \phi^3\zeta'(\phi))}{r^2} \\
& - \frac{2\varrho^3\phi^2\zeta(\phi)\partial_\varrho r}{r^3} - \frac{\varrho^2\phi^2\zeta(\phi)}{r^2} \Bigg]\phi \\
& + \Bigg[\frac{2\varrho^2(\partial_\varrho r - 1)}{r} + 2\left(\frac{\varrho^2}{r} + \frac{\varrho^3(\partial_u r + \partial_{\underline{u}} r)}{2r^2} + \frac{\varrho^3\partial_u r}{r^2} \right)(\partial_\varrho r - 1) \\
& - \frac{\varrho(\varrho - r)}{r}\frac{\partial_u r + \partial_{\underline{u}} r}{2r}\varrho + \frac{\varrho^2(2\partial_u r + 1)}{r}\frac{\partial_u r + \partial_{\underline{u}} r}{2r}\varrho + \frac{2\varrho^2(2\partial_u r + 1)}{r} - \frac{\varrho^3(2\partial_u r + 1)\partial_\varrho r}{r^2} \\
& + \left(\frac{2\varrho(2\partial_{\underline{u}} r - 1)}{r} - \frac{\kappa\varrho^2\Omega^2 g(\phi)^2}{2r^2} - \frac{\varrho^2(2\partial_{\underline{u}} r - 1)\partial_\varrho r}{r^2} \right)\varrho - \frac{\kappa\varrho^3\Omega^2 g(\phi) g'(\phi)\phi}{r^2} - \frac{\varrho^2(2\partial_u r + 1)}{r} \\
& - \frac{\varrho(2\partial_{\underline{u}} r - 1)}{r}\varrho \Bigg]\partial_{\underline{u}}\phi + \Bigg[-\frac{\varrho(\varrho - r)}{r}\varrho + \frac{\varrho^2(2\partial_u r + 1)}{r}\varrho \Bigg]\partial_{\underline{u}}^2\phi \\
& + \Bigg[-\frac{\varrho^2 f(\phi)}{r^2} + \frac{-\varrho^2 f(\phi) + \varrho^2 f'(\phi)\left(\varrho\partial_{\underline{u}}\phi + \phi\right)}{r^2} - \frac{2\varrho^3 f(\phi)\partial_u r}{r^3} \\
& + \frac{2\varrho^2 f(\phi) - \varrho^2 f'(\phi)\phi}{r^2} - \frac{2\varrho^3 f(\phi)\partial_\varrho r}{r^3} - \varrho^2\frac{f(\phi)}{r^2} \Bigg](\Omega^2 - 1) \\
& + \Bigg[2\frac{\varrho^3\Omega^2}{4r^3}(\partial_\varrho r - 1) - \frac{\varrho(\varrho - r)}{r}\frac{\varrho\Omega^2}{4r^2} + \frac{\varrho^2(2\partial_u r + 1)}{r}\frac{\varrho\Omega^2}{4r^2} \Bigg] f(\phi) + (2f(\phi) - \kappa g(\phi)^2\phi)\frac{\varrho^3\Omega\partial_{\underline{u}}\Omega}{r^2},
\end{aligned}
$$

and we finally obtain

$$
\begin{aligned}
B_4 \;=\; & \Bigg[\left(-\frac{7}{2}-\frac{\varrho}{r}\partial_u r-\frac{2\varrho^2+3\varrho r}{r^2}\partial_\varrho r-\frac{\varrho}{2r}\partial_{\underline{u}}r\right)\frac{\varrho-r}{r}+\left(-\frac{\varrho\partial_{\underline{u}}r}{r}-1\right)\frac{\varrho(2\partial_{\underline{u}}r-1)}{r}\\
&+\left(-3+\frac{\varrho\partial_{\underline{u}}r}{r}\right)\frac{\varrho(\partial_\varrho r-1)}{r}+\left(-\frac{1}{2}+\frac{\varrho\partial_{\underline{u}}r}{2r}\right)\frac{\varrho\,(2\partial_u r+1)}{r}+\frac{\kappa\varrho^3\Omega^2 g(\phi)^2\partial_{\underline{u}}r}{2r^3}\\
&+\frac{2\varrho(\varrho r+\partial_u r\varrho^2)\partial_{\underline{u}}^2 r}{r^2}-\frac{\varrho^2(2\phi^2\zeta(\phi)+\phi^3\zeta'(\phi))}{r^2}-\frac{2\varrho^3\phi^2\zeta(\phi)\partial_\varrho r}{r^3}\Bigg]\phi\\
&+\Bigg[\left(4+\frac{\varrho(3\partial_u r+\partial_{\underline{u}}r)}{r}\right)\frac{\varrho^2(\partial_\varrho r-1)}{r}\\
&+\left(-\frac{5\varrho^2(\varrho-r)}{2r}+\frac{\varrho^3(2\partial_u r+1)}{2r}-\frac{2\varrho^3(\partial_\varrho r-1)}{r}\right)\frac{\partial_u r+\partial_{\underline{u}}r}{r}\\
&-\frac{\kappa\varrho^3\Omega^2 g(\phi)g'(\phi)\phi}{r^2}-\frac{\kappa\varrho^3\Omega^2 g(\phi)^2}{2r^2}\Bigg]\partial_{\underline{u}}\phi+\left[-\frac{\varrho^2(\varrho-r)}{r}+\frac{\varrho^3(2\partial_u r+1)}{r}\right]\partial_{\underline{u}}^2\phi\\
&+\left[\left(-\frac{2\varrho\partial_{\underline{u}}r}{r}-1\right)\frac{\varrho^2 f(\phi)}{r^2}+\frac{\varrho^2 f'(\phi)\varrho\partial_{\underline{u}}\phi}{r^2}\right](\Omega^2-1)\\
&+\left[-\frac{\varrho(\varrho-r)}{r}\frac{\varrho\Omega^2}{4r^2}+\frac{\varrho^3(2\partial_{\underline{u}}r-1)\Omega^2}{4r^3}\right]f(\phi)+(2f(\phi)-\kappa g(\phi)^2\phi)\frac{\varrho^3\Omega\partial_{\underline{u}}\Omega}{r^2}.
\end{aligned}
$$

This concludes the proof of the lemma. □

Next, we derive upper bounds for B_1, B_2, B_3 and B_4.

Lemma 8.7. *We have*

$$|B_1|\lesssim\varepsilon \text{ and } |B_1|\lesssim C_0 r,\quad |B_2|\lesssim C_0 r^{3\delta},\quad |B_3|\lesssim C_0 r,\ \text{ and } |B_4|\lesssim C_0 r^{3\delta}.$$

Remark 8.8. *B_1 and B_3 behave better*[13] *than B_2 and B_4. This is due to the fact that both B_1 and B_3 include neither $\partial_{\underline{u}}\phi$ nor $\partial_{\underline{u}}\Omega$ in their definition, so that we can estimate them using Lemma 7.22 which has a $1/2-\delta$ gain with respect to Corollary 7.21.*

Proof. In view of the definition of B_1 and the Lemma 7.2, we have

$$|B_1| \;\lesssim\; \varepsilon.$$

Also, in view of Lemma 7.22, we have

$$
\begin{aligned}
|B_1| &\lesssim \frac{|\varrho-r|}{r}+\left|2\partial_{\underline{u}}r-1\right|\\
&\lesssim C_0 r.
\end{aligned}
$$

In view of the definition of B_2 and Corollary 7.21, we have

$$
\begin{aligned}
|B_2| &\lesssim \varrho\,|\partial_\varrho r-1|\,|\partial_{\underline{u}}\phi|+\frac{|\varrho-r|}{r}|\phi|+\left|2\partial_{\underline{u}}r-1\right|\,|\phi|+|\Omega^2-1||f(\phi)|+\varrho|\partial_{\underline{u}}^2 r||\phi|\\
&\lesssim C_0 r^{3\delta}.
\end{aligned}
$$

[13]The estimates for B_1 and B_3 correspond to the case $\delta=1/2$, while we have $\delta<1/2$ for B_2 and B_4.

In view of the definition of B_3 and Lemma 7.22, we have

$$\begin{aligned}|B_3| &\lesssim \frac{|\varrho - r|}{r} + \left|\partial_{\underline{u}} r - \frac{1}{2}\right| + \left|\partial_u r + \frac{1}{2}\right| + |\phi|^2 \\ &\lesssim C_0 r.\end{aligned}$$

Finally, in view of the definition of B_4, Corollary 7.21, Lemma 8.1, Lemma 8.2 and Lemma 8.3, we have

$$\begin{aligned}|B_4| &\lesssim \left(\frac{|\varrho - r|}{r} + \left|\partial_{\underline{u}} r - \frac{1}{2}\right| + \left|\partial_u r + \frac{1}{2}\right| + \varrho|\partial_{\underline{u}}^2 r| + |\phi|^2 + |\Omega^2 - 1| + \varrho|\partial_{\underline{u}}\Omega|\right) \\ &\quad \times(|\phi| + \varrho|\partial_{\underline{u}}\phi| + \varrho^2|\partial_{\underline{u}}^2\phi|) \\ &\lesssim C_0 r^{3\delta}.\end{aligned}$$

This concludes the proof of the lemma. □

8.3. **An upper bound for v.**

Lemma 8.9. *We have*

$$\begin{aligned}& v(\tau, \varrho) \\ = \; & v_0(\tau, \varrho) \\ + \; & \frac{J(-1)}{\sqrt{\varrho}} \int_{\tau-\varrho}^{\tau+\varrho} \left(\frac{B_1\left(\frac{\tau-\varrho+\underline{u}'}{2}, \frac{-\tau+\varrho+\underline{u}'}{2}\right)}{\sqrt{\frac{-\tau+\varrho+\underline{u}'}{2}}} v\left(\frac{\tau-\varrho+\underline{u}'}{2}, \frac{-\tau+\varrho+\underline{u}'}{2}\right) + \frac{B_2\left(\frac{\tau-\varrho+\underline{u}'}{2}, \frac{-\tau+\varrho+\underline{u}'}{2}\right)}{\left(\frac{-\tau+\varrho+\underline{u}'}{2}\right)^{\frac{3}{2}}} \right) d\underline{u}' \\ & -\frac{1}{2}\sqrt{\varrho} \int_0^{+\infty} J(\mu) \left(\frac{B_1(-1,\lambda)}{\sqrt{\lambda}} v(-1,\lambda) + \frac{B_2(-1,\lambda)}{\lambda^{\frac{3}{2}}}\right) \frac{1}{\mu\varrho+\lambda} d\mu \\ & -\frac{1}{2\sqrt{\varrho}} \int_{\sqrt{(\tau+1)^2-\varrho^2}}^{\tau+\varrho+1} J(\mu) \left(\frac{B_1(-1,\lambda)}{\sqrt{\lambda}} v(-1,\lambda) + \frac{B_2(-1,\lambda)}{\lambda^{\frac{3}{2}}}\right) d\lambda \\ & + \int_R \frac{1}{4\sqrt{\lambda\varrho}} J(\mu) \left(\frac{B_1(\sigma,\lambda)}{\lambda} v(\sigma,\lambda) + \frac{B_2(\sigma,\lambda)}{\lambda^2}\right) d\lambda d\sigma \\ & - \int_R \frac{\sqrt{\lambda}}{\sqrt{\varrho}} \partial_u \mu \, J'(\mu) \left(\frac{B_1(\sigma,\lambda)}{\lambda} v(\sigma,\lambda) + \frac{B_2(\sigma,\lambda)}{\lambda^2}\right) d\lambda d\sigma \\ & + \int_R \frac{\sqrt{\lambda}}{\sqrt{\varrho}} J(\mu) \left(\frac{B_3(\sigma,\lambda)}{\lambda^2} v(\sigma,\lambda) + \frac{B_4(\sigma,\lambda)}{\lambda^3}\right) d\lambda d\sigma,\end{aligned}$$

where v_0 denotes the solution to the homogeneous equation with the same initial conditions as v, μ is given by

$$\mu = \frac{(\tau-\sigma)^2 - \varrho^2 - \lambda^2}{2\varrho\lambda},$$

R is the space-time region given by

$$R = \{(\sigma,\lambda) / \; -1 \le \sigma \le \tau, \; \max(0, \varrho - \tau + \sigma) \le \lambda \le \varrho + \tau - \sigma\},$$

and J is given by

$$J(\mu) = \int_{\max(-\mu,-1)}^1 \frac{dx}{\sqrt{1-x^2}\sqrt{\mu+x}}.$$

Proof. We recall the representation formula derived in [9] for the solution v of

$$\left(-\partial_\tau^2 + \partial_\varrho^2 + \frac{1}{\varrho}\partial_\varrho\right) v = h.$$

v is given by (see [9] p. 1060)

$$v(\tau,\varrho) = v_0(\tau,\varrho) + \int_R \frac{\sqrt{\lambda}}{\sqrt{\varrho}} J(\mu) h(\sigma,\lambda) d\lambda d\sigma,$$

where

$$R = \{(\sigma,\lambda) / \ -1 \leq \sigma \leq \tau, \ \max(0, \varrho - \tau + \sigma) \leq \lambda \leq \varrho + \tau - \sigma\},$$

v_0 denotes the solution to the homogeneous equation with the same initial conditions as v, μ is given as before by

$$\mu = \frac{(\tau-\sigma)^2 - \varrho^2 - \lambda^2}{2\varrho\lambda},$$

with an initialization at $\tau = -1$ and J is given by

$$J(\mu) = \int_{\max(-\mu,-1)}^1 \frac{dx}{\sqrt{1-x^2}\sqrt{\mu+x}}.$$

In our case, we have

$$h = \partial_u \left(\frac{B_1}{\varrho} v + \frac{B_2}{\varrho^2} \right) + \frac{B_3}{\varrho^2} v + \frac{B_4}{\varrho^3}.$$

Hence, we have

$$\begin{aligned}
v(\tau,\varrho) &= v_0(\tau,\varrho) + \int_R \frac{\sqrt{\lambda}}{\sqrt{\varrho}} J(\mu) \partial_u \left(\frac{B_1(\sigma,\lambda)}{\lambda} v(\sigma,\lambda) + \frac{B_2(\sigma,\lambda)}{\lambda^2} \right) d\lambda d\sigma \\
&\quad + \int_R \frac{\sqrt{\lambda}}{\sqrt{\varrho}} J(\mu) \left(\frac{B_3(\sigma,\lambda)}{\lambda^2} v(\sigma,\lambda) + \frac{B_4(\sigma,\lambda)}{\lambda^3} \right) d\lambda d\sigma \\
&= v_0(\tau,\varrho) + \int_{\partial R} \frac{\sqrt{\lambda}}{\sqrt{\varrho}} J(\mu) \left(\frac{B_1(\sigma,\lambda)}{\lambda} v(\sigma,\lambda) + \frac{B_2(\sigma,\lambda)}{\lambda^2} \right) \mathbf{g}(\partial_u, \nu_R) \\
&\quad - \int_R \partial_u \left(\frac{\sqrt{\lambda}}{\sqrt{\varrho}} J(\mu) \right) \left(\frac{B_1(\sigma,\lambda)}{\lambda} v(\sigma,\lambda) + \frac{B_2(\sigma,\lambda)}{\lambda^2} \right) d\lambda d\sigma \\
&\quad + \int_R \frac{\sqrt{\lambda}}{\sqrt{\varrho}} J(\mu) \left(\frac{B_3(\sigma,\lambda)}{\lambda^2} v(\sigma,\lambda) + \frac{B_4(\sigma,\lambda)}{\lambda^3} \right) d\lambda d\sigma \\
&= v_0(\tau,\varrho) + \int_{\partial R} \frac{\sqrt{\lambda}}{\sqrt{\varrho}} J(\mu) \left(\frac{B_1(\sigma,\lambda)}{\lambda} v(\sigma,\lambda) + \frac{B_2(\sigma,\lambda)}{\lambda^2} \right) \mathbf{g}(\partial_u, \nu_R) \\
&\quad + \int_R \frac{1}{4\sqrt{\lambda\varrho}} J(\mu) \left(\frac{B_1(\sigma,\lambda)}{\lambda} v(\sigma,\lambda) + \frac{B_2(\sigma,\lambda)}{\lambda^2} \right) d\lambda d\sigma \\
&\quad - \int_R \frac{\sqrt{\lambda}}{\sqrt{\varrho}} \partial_u \mu \, J'(\mu) \left(\frac{B_1(\sigma,\lambda)}{\lambda} v(\sigma,\lambda) + \frac{B_2(\sigma,\lambda)}{\lambda^2} \right) d\lambda d\sigma \\
&\quad + \int_R \frac{\sqrt{\lambda}}{\sqrt{\varrho}} J(\mu) \left(\frac{B_3(\sigma,\lambda)}{\lambda^2} v(\sigma,\lambda) + \frac{B_4(\sigma,\lambda)}{\lambda^3} \right) d\lambda d\sigma.
\end{aligned}$$

Next, we compute the boundary term. Recall that we have

$$\begin{aligned}
&\int_{\partial R} f \mathbf{g}(\partial_u, \nu_R) \\
&= \int_{\tau-\varrho}^{\tau+\varrho} f(\tau-\varrho, \underline{u}') d\underline{u}' + \frac{1}{2} \int_{-1}^{\tau-\varrho} f(\sigma, 0) d\sigma - \frac{1}{2} \int_0^{\tau+\varrho+1} f(-1,\lambda) d\lambda,
\end{aligned}$$

and

$$\mu = -1 \text{ on } u = \tau - \varrho.$$

Hence, we have

$$
\begin{aligned}
&\int_{\partial R}\frac{\sqrt{\lambda}}{\sqrt{\varrho}}J(\mu)\left(\frac{B_1(\sigma,\lambda)}{\lambda}v(\sigma,\lambda)+\frac{B_2(\sigma,\lambda)}{\lambda^2}\right)\mathbf{g}(\partial_u,\nu_R)\\
=\;&\frac{J(-1)}{\sqrt{\varrho}}\int_{\tau-\varrho}^{\tau+\varrho}\left(\frac{B_1\left(\frac{\tau-\varrho+\underline{u}'}{2},\frac{-\tau+\varrho+\underline{u}'}{2}\right)}{\sqrt{\frac{-\tau+\varrho+\underline{u}'}{2}}}v\left(\frac{\tau-\varrho+\underline{u}'}{2},\frac{-\tau+\varrho+\underline{u}'}{2}\right)+\frac{B_2\left(\frac{\tau-\varrho+\underline{u}'}{2},\frac{-\tau+\varrho+\underline{u}'}{2}\right)}{\left(\frac{-\tau+\varrho+\underline{u}'}{2}\right)^{\frac{3}{2}}}\right)d\underline{u}'\\
&-\frac{1}{2}\int_0^{\tau+\varrho+1}\frac{\sqrt{\lambda}}{\sqrt{\varrho}}J(\mu)\left(\frac{B_1(-1,\lambda)}{\lambda}v(-1,\lambda)+\frac{B_2(-1,\lambda)}{\lambda^2}\right)d\lambda
\end{aligned}
$$

Recall that

$$
\partial_\lambda\mu \;=\; -\frac{\mu\varrho+\lambda}{\varrho}.
$$

We decompose and perform a change of variable

$$
\begin{aligned}
&\int_0^{\tau+\varrho+1}\frac{\sqrt{\lambda}}{\sqrt{\varrho}}J(\mu)\left(\frac{B_1(-1,\lambda)}{\lambda}v(-1,\lambda)+\frac{B_2(-1,\lambda)}{\lambda^2}\right)d\lambda\\
=\;&\int_0^{+\infty}\frac{\sqrt{\lambda}}{\sqrt{\varrho}}J(\mu)\left(\frac{B_1(-1,\lambda)}{\lambda}v(-1,\lambda)+\frac{B_2(-1,\lambda)}{\lambda^2}\right)\frac{\varrho}{\mu\varrho+\lambda}d\mu\\
&+\int_{\sqrt{(\tau+1)^2-\varrho^2}}^{\tau+\varrho+1}\frac{\sqrt{\lambda}}{\sqrt{\varrho}}J(\mu)\left(\frac{B_1(-1,\lambda)}{\lambda}v(-1,\lambda)+\frac{B_2(-1,\lambda)}{\lambda^2}\right)d\lambda\\
=\;&\sqrt{\varrho}\int_0^{+\infty}J(\mu)\left(\frac{B_1(-1,\lambda)}{\sqrt{\lambda}}v(-1,\lambda)+\frac{B_2(-1,\lambda)}{\lambda^{\frac{3}{2}}}\right)\frac{1}{\mu\varrho+\lambda}d\mu\\
&+\frac{1}{\sqrt{\varrho}}\int_{\sqrt{(\tau+1)^2-\varrho^2}}^{\tau+\varrho+1}J(\mu)\left(\frac{B_1(-1,\lambda)}{\sqrt{\lambda}}v(-1,\lambda)+\frac{B_2(-1,\lambda)}{\lambda^{\frac{3}{2}}}\right)d\lambda
\end{aligned}
$$

which yields

$$
\begin{aligned}
&\int_{\partial R}\frac{\sqrt{\lambda}}{\sqrt{\varrho}}J(\mu)\left(\frac{B_1(\sigma,\lambda)}{\lambda}v(\sigma,\lambda)+\frac{B_2(\sigma,\lambda)}{\lambda^2}\right)\mathbf{g}(\partial_u,\nu_R)\\
=\;&\frac{J(-1)}{\sqrt{\varrho}}\int_{\tau-\varrho}^{\tau+\varrho}\left(\frac{B_1\left(\frac{\tau-\varrho+\underline{u}'}{2},\frac{-\tau+\varrho+\underline{u}'}{2}\right)}{\sqrt{\frac{-\tau+\varrho+\underline{u}'}{2}}}v\left(\frac{\tau-\varrho+\underline{u}'}{2},\frac{-\tau+\varrho+\underline{u}'}{2}\right)+\frac{B_2\left(\frac{\tau-\varrho+\underline{u}'}{2},\frac{-\tau+\varrho+\underline{u}'}{2}\right)}{\left(\frac{-\tau+\varrho+\underline{u}'}{2}\right)^{\frac{3}{2}}}\right)d\underline{u}'\\
&-\frac{1}{2}\sqrt{\varrho}\int_0^{+\infty}J(\mu)\left(\frac{B_1(-1,\lambda)}{\sqrt{\lambda}}v(-1,\lambda)+\frac{B_2(-1,\lambda)}{\lambda^{\frac{3}{2}}}\right)\frac{1}{\mu\varrho+\lambda}d\mu\\
&-\frac{1}{2\sqrt{\varrho}}\int_{\sqrt{(\tau+1)^2-\varrho^2}}^{\tau+\varrho+1}J(\mu)\left(\frac{B_1(-1,\lambda)}{\sqrt{\lambda}}v(-1,\lambda)+\frac{B_2(-1,\lambda)}{\lambda^{\frac{3}{2}}}\right)d\lambda.
\end{aligned}
$$

Finally, we deduce

$$\begin{aligned}
& v(\tau,\varrho)\\
= \;& v_0(\tau,\varrho)\\
+ \;& \frac{J(-1)}{\sqrt{\varrho}}\int_{\tau-\varrho}^{\tau+\varrho}\left(\frac{B_1\left(\frac{\tau-\varrho+\underline{u}'}{2},\frac{-\tau+\varrho+\underline{u}'}{2}\right)}{\sqrt{\frac{-\tau+\varrho+\underline{u}'}{2}}}v\left(\frac{\tau-\varrho+\underline{u}'}{2},\frac{-\tau+\varrho+\underline{u}'}{2}\right)+\frac{B_2\left(\frac{\tau-\varrho+\underline{u}'}{2},\frac{-\tau+\varrho+\underline{u}'}{2}\right)}{\left(\frac{-\tau+\varrho+\underline{u}'}{2}\right)^{\frac{3}{2}}}\right)d\underline{u}'\\
& -\frac{1}{2}\sqrt{\varrho}\int_0^{+\infty}J(\mu)\left(\frac{B_1(-1,\lambda)}{\sqrt{\lambda}}v(-1,\lambda)+\frac{B_2(-1,\lambda)}{\lambda^{\frac{3}{2}}}\right)\frac{1}{\mu\varrho+\lambda}d\mu\\
& -\frac{1}{2\sqrt{\varrho}}\int_{\sqrt{(\tau+1)^2-\varrho^2}}^{\tau+\varrho+1}J(\mu)\left(\frac{B_1(-1,\lambda)}{\sqrt{\lambda}}v(-1,\lambda)+\frac{B_2(-1,\lambda)}{\lambda^{\frac{3}{2}}}\right)d\lambda\\
& +\int_R\frac{1}{4\sqrt{\lambda\varrho}}J(\mu)\left(\frac{B_1(\sigma,\lambda)}{\lambda}v(\sigma,\lambda)+\frac{B_2(\sigma,\lambda)}{\lambda^2}\right)d\lambda d\sigma\\
& -\int_R\frac{\sqrt{\lambda}}{\sqrt{\varrho}}\partial_u\mu\, J'(\mu)\left(\frac{B_1(\sigma,\lambda)}{\lambda}v(\sigma,\lambda)+\frac{B_2(\sigma,\lambda)}{\lambda^2}\right)d\lambda d\sigma\\
& +\int_R\frac{\sqrt{\lambda}}{\sqrt{\varrho}}J(\mu)\left(\frac{B_3(\sigma,\lambda)}{\lambda^2}v(\sigma,\lambda)+\frac{B_4(\sigma,\lambda)}{\lambda^3}\right)d\lambda d\sigma.
\end{aligned}$$

This concludes the proof of the lemma. □

Lemma 8.10. *We have*

$$\begin{aligned}
|v(\tau,\varrho)| \;\lesssim\; & C_0+C_0r^{3\delta-1}+\varepsilon\sup_{0\le\lambda\le\varrho}|v(\tau-\varrho+\lambda,\lambda)|\\
& +C_0\int_R\frac{\sqrt{\lambda}}{\sqrt{\varrho}}(J(\mu)+|\mu-1||J'(\mu)|)\left(\frac{|v(\sigma,\lambda)|}{\lambda}+\frac{\lambda^{3\delta}}{\lambda^3}\right)d\lambda d\sigma.
\end{aligned}$$

where the constant C_0 only depends on initial data.

Proof. Recall that

$$\begin{aligned}
& v(\tau,\varrho)\\
= \;& v_0(\tau,\varrho)\\
+ \;& \frac{J(-1)}{\sqrt{\varrho}}\int_{\tau-\varrho}^{\tau+\varrho}\left(\frac{B_1\left(\frac{\tau-\varrho+\underline{u}'}{2},\frac{-\tau+\varrho+\underline{u}'}{2}\right)}{\sqrt{\frac{-\tau+\varrho+\underline{u}'}{2}}}v\left(\frac{\tau-\varrho+\underline{u}'}{2},\frac{-\tau+\varrho+\underline{u}'}{2}\right)+\frac{B_2\left(\frac{\tau-\varrho+\underline{u}'}{2},\frac{-\tau+\varrho+\underline{u}'}{2}\right)}{\left(\frac{-\tau+\varrho+\underline{u}'}{2}\right)^{\frac{3}{2}}}\right)d\underline{u}'\\
& -\frac{1}{2}\sqrt{\varrho}\int_0^{+\infty}J(\mu)\left(\frac{B_1(-1,\lambda)}{\sqrt{\lambda}}v(-1,\lambda)+\frac{B_2(-1,\lambda)}{\lambda^{\frac{3}{2}}}\right)\frac{1}{\mu\varrho+\lambda}d\mu\\
& -\frac{1}{2\sqrt{\varrho}}\int_{\sqrt{(\tau+1)^2-\varrho^2}}^{\tau+\varrho+1}J(\mu)\left(\frac{B_1(-1,\lambda)}{\sqrt{\lambda}}v(-1,\lambda)+\frac{B_2(-1,\lambda)}{\lambda^{\frac{3}{2}}}\right)d\lambda\\
& +\int_R\frac{1}{4\sqrt{\lambda\varrho}}J(\mu)\left(\frac{B_1(\sigma,\lambda)}{\lambda}v(\sigma,\lambda)+\frac{B_2(\sigma,\lambda)}{\lambda^2}\right)d\lambda d\sigma\\
& -\int_R\frac{\sqrt{\lambda}}{\sqrt{\varrho}}\partial_u\mu\, J'(\mu)\left(\frac{B_1(\sigma,\lambda)}{\lambda}v(\sigma,\lambda)+\frac{B_2(\sigma,\lambda)}{\lambda^2}\right)d\lambda d\sigma\\
& +\int_R\frac{\sqrt{\lambda}}{\sqrt{\varrho}}J(\mu)\left(\frac{B_3(\sigma,\lambda)}{\lambda^2}v(\sigma,\lambda)+\frac{B_4(\sigma,\lambda)}{\lambda^3}\right)d\lambda d\sigma.
\end{aligned}$$

Noticing that $J(\mu) \geq 0$ for all $\mu \geq -1$, this yields

$$\begin{aligned}
& |v(\tau,\varrho)| \\
\lesssim\ & |v_0(\tau,\varrho)| + \frac{J(-1)}{\sqrt{\varrho}} \int_0^{\varrho} \left(\frac{B_1(\tau-\varrho+\lambda,\lambda)}{\sqrt{\lambda}} v(\tau-\varrho+\lambda,\lambda) + \frac{B_2(\tau-\varrho+\lambda,\lambda)}{\lambda^{\frac{3}{2}}} \right) d\lambda \\
& + \sqrt{\varrho} \left(\sup_{\lambda \geq 0} \left(\frac{B_1(-1,\lambda)}{\lambda^{\frac{3}{2}}} v(-1,\lambda) + \frac{B_2(-1,\lambda)}{\lambda^{\frac{5}{2}}} \right) \right) \int_0^2 J(\mu) d\mu \\
& + \sqrt{\varrho} \left(\sup_{\lambda \geq 0} \left(\frac{B_1(-1,\lambda)}{\lambda^2} v(-1,\lambda) + \frac{B_2(-1,\lambda)}{\lambda^3} \right) \right) \int_2^{+\infty} \sqrt{\lambda} J(\mu) d\mu \\
& - \frac{1}{\sqrt{\varrho}} \left(\sup_{-1 \leq \mu \leq 0} J(\mu) \right) \left(\sup_{\lambda \geq 0} \left(\frac{B_1(-1,\lambda)}{\sqrt{\lambda}} v(-1,\lambda) + \frac{B_2(-1,\lambda)}{\lambda^{\frac{3}{2}}} \right) \right) (\tau+\varrho+1-\sqrt{(\tau+1)^2-\varrho^2}) \\
& + \int_R \frac{1}{4\sqrt{\lambda\varrho}} J(\mu) \left(\frac{B_1(\sigma,\lambda)}{\lambda} v(\sigma,\lambda) + \frac{B_2(\sigma,\lambda)}{\lambda^2} \right) d\lambda d\sigma \\
& - \int_R \frac{\sqrt{\lambda}}{\sqrt{\varrho}} \partial_u \mu\, J'(\mu) \left(\frac{B_1(\sigma,\lambda)}{\lambda} v(\sigma,\lambda) + \frac{B_2(\sigma,\lambda)}{\lambda^2} \right) d\lambda d\sigma \\
& + \int_R \frac{\sqrt{\lambda}}{\sqrt{\varrho}} J(\mu) \left(\frac{B_3(\sigma,\lambda)}{\lambda^2} v(\sigma,\lambda) + \frac{B_4(\sigma,\lambda)}{\lambda^3} \right) d\lambda d\sigma.
\end{aligned}$$

We have the following properties for J (see for example [9] p. 1061):

$$J(-1) = \frac{\pi}{\sqrt{2}}, \quad \sup_{-1 \leq \mu \leq 0} J(\mu) \lesssim 1, \;\; J \in L^1(0,2).$$

Also, we have

$$\sup_{2 \leq \mu < +\infty} \sqrt{\mu} J(\mu) \lesssim 1$$

and since

$$\lambda\mu = \frac{(\tau-\sigma)^2 - \varrho^2 - \lambda^2}{2\varrho} \leq \frac{1}{\varrho}$$

we infer

$$\begin{aligned}
\int_2^{+\infty} \sqrt{\lambda} J(\mu) d\mu &\lesssim \int_2^{+\infty} \sqrt{\frac{\lambda}{\mu}} d\mu \\
&\lesssim \int_0^{\frac{1}{\varrho}} \frac{1}{\sqrt{s}} ds \\
&\lesssim \frac{1}{\sqrt{\varrho}}.
\end{aligned}$$

Moreover, we have

$$\{\sqrt{(\tau+1)^2-\varrho^2} \leq \lambda \leq \tau+\varrho+1\} \cap \{\sigma = -1\} = \{-1 \leq \mu \leq 0\} \cap \{\sigma = -1\}.$$

We deduce

$$
\begin{aligned}
&|v(\tau,\varrho)|\\
\lesssim\ & |v_0(\tau,\varrho)| + \sup_{0\leq\lambda\leq\varrho}\left(|B_1(\tau-\varrho+\lambda,\lambda)||v(\tau-\varrho+\lambda,\lambda)| + \frac{|B_2(\tau-\varrho+\lambda,\lambda)|}{\lambda}\right)\\
&+\sqrt{\varrho}\left(\sup_{\lambda\geq 0}\left(\frac{|B_1(-1,\lambda)|}{\lambda^{\frac{3}{2}}}|v(-1,\lambda)| + \frac{|B_2(-1,\lambda)|}{\lambda^{\frac{5}{2}}}\right)\right)\\
&+\sup_{\lambda\geq 0}\left(\frac{|B_1(-1,\lambda)|}{\lambda^2}|v(-1,\lambda)| + \frac{|B_2(-1,\lambda)|}{\lambda^3}\right)\\
&+\sqrt{\varrho}\left(\sup_{\lambda\geq 0}\left(\frac{|B_1(-1,\lambda)|}{\sqrt{\lambda}}|v(-1,\lambda)| + \frac{|B_2(-1,\lambda)|}{\lambda^{\frac{3}{2}}}\right)\right)\frac{\tau+\varrho+1}{\tau+\varrho+1+\sqrt{(\tau+1)^2-\varrho^2}}\\
&+\int_R\frac{\sqrt{\lambda}}{\sqrt{\varrho}}J(\mu)\left(\frac{|B_1(\sigma,\lambda)|+|B_3(\sigma,\lambda)|}{\lambda^2}|v(\sigma,\lambda)| + \frac{|B_2(\sigma,\lambda)|+|B_4(\sigma,\lambda)|}{\lambda^3}\right)d\lambda d\sigma\\
&+\int_R\frac{\sqrt{\lambda}}{\sqrt{\varrho}}|\partial_u\mu|\,|J'(\mu)|\left(\frac{|B_1(\sigma,\lambda)|}{\lambda}|v(\sigma,\lambda)| + \frac{|B_2(\sigma,\lambda)|}{\lambda^2}\right)d\lambda d\sigma.
\end{aligned}
$$

Assuming enough regularity on the initial data, we have

$$
\begin{aligned}
&\sup_{\varrho\geq 0}|v_0(\tau,\varrho)| + \sup_{\lambda\geq 0}\left(\frac{|B_1(-1,\lambda)|}{\lambda^{\frac{3}{2}}}|v(-1,\lambda)| + \frac{|B_2(-1,\lambda)|}{\lambda^{\frac{5}{2}}}\right)\\
&+\sup_{\lambda\geq 0}\left(\frac{|B_1(-1,\lambda)|}{\lambda^2}|v(-1,\lambda)| + \frac{|B_2(-1,\lambda)|}{\lambda^3}\right)\\
\leq\ & C_0,
\end{aligned}
$$

where the constant C_0 only depends on initial data. Hence, together with Lemma 7.16, we deduce

$$
\begin{aligned}
&|v(\tau,\varrho)|\\
\lesssim\ & C_0 + \sup_{0\leq\lambda\leq\varrho}\left(|B_1(\tau-\varrho+\lambda,\lambda)||v(\tau-\varrho+\lambda,\lambda)| + \frac{|B_2(\tau-\varrho+\lambda,\lambda)|}{\lambda}\right)\\
&+\int_R\frac{\sqrt{\lambda}}{\sqrt{\varrho}}(J(\mu)+|\mu-1||J'(\mu)|)\left(\frac{|B_1(\sigma,\lambda)|+|B_3(\sigma,\lambda)|}{\lambda^2}|v(\sigma,\lambda)| + \frac{|B_2(\sigma,\lambda)|+|B_4(\sigma,\lambda)|}{\lambda^3}\right)d\lambda d\sigma.
\end{aligned}
$$

Together with Lemma 8.7, we infer

$$
\begin{aligned}
|v(\tau,\varrho)| \lesssim\ & C_0 + C_0 r^{3\delta-1} + \varepsilon\sup_{0\leq\lambda\leq\varrho}|v(\tau-\varrho+\lambda,\lambda)|\\
&+C_0\int_R\frac{\sqrt{\lambda}}{\sqrt{\varrho}}(J(\mu)+|\mu-1||J'(\mu)|)\left(\frac{|v(\sigma,\lambda)|}{\lambda} + \frac{\lambda^{3\delta}}{\lambda^3}\right)d\lambda d\sigma.
\end{aligned}
$$

This concludes the proof of the lemma. □

Lemma 8.11. *We have for all* $\mu \geq -1$

$$|\mu-1||J'(\mu)| \lesssim J(\mu).$$

Proof. For $-1\leq\mu<1$, we have (see for example [9] p. 1087)

$$J'(\mu) = \frac{1}{4(1+\mu)}\int_{-\mu}^1\frac{\sqrt{1-x}dx}{(1+x)^{\frac{3}{2}}\sqrt{\mu+x}}.$$

We see in particular that $J'(\mu)\geq 0$ for $-1\leq\mu<1$, and hence

$$J(\mu)\geq J(-1) \text{ on } -1\leq\mu<1.$$

Since we have

$$J(-1) = \frac{\pi}{\sqrt{2}},$$

we deduce

$$J(\mu) \gtrsim 1 \text{ on } -1 \leq \mu < 1.$$

Also, we have for all $\mu \geq 1$ (see for example [9] p. 1061)

$$|\mu - 1||J'(\mu)| \lesssim 1.$$

We infer

$$|\mu - 1||J'(\mu)| \lesssim J(\mu) \text{ on } -1 \leq \mu < 1.$$

Next, we consider the case $\mu \geq 1$, Then, we have

$$J'(\mu) = -\frac{1}{2}\int_{-1}^{1} \frac{dx}{\sqrt{1-x^2}(\mu+x)^{\frac{3}{2}}}.$$

We infer

$$\begin{aligned} |\mu - 1||J'(\mu)| &\lesssim \int_{-1}^{1} \frac{|\mu-1|dx}{\sqrt{1-x^2}(\mu+x)^{\frac{3}{2}}} \\ &\lesssim \int_{-1}^{1} \frac{dx}{\sqrt{1-x^2}\sqrt{\mu+x}} \\ &\lesssim J(\mu). \end{aligned}$$

This concludes the proof of the lemma. □

We deduce the following corollary.

Corollary 8.12. *We have*

$$|v(\tau,\varrho)| \lesssim C_0 + C_0 r^{3\delta-1} + \varepsilon \sup_{0\leq\lambda\leq\varrho} |v(\tau-\varrho+\lambda,\lambda)| + C_0 \int_R \frac{\sqrt{\lambda}}{\sqrt{\varrho}} J(\mu) \left(\frac{|v(\sigma,\lambda)|}{\lambda} + \frac{\lambda^{3\delta}}{\lambda^3}\right) d\lambda d\sigma.$$

where the constant C_0 only depends on initial data.

The following proposition is the core of this section.

Proposition 8.13. *We have*

$$|v| \leq C_0,$$

where C_0 is a constant only depending on initial data.

Proof. We choose δ such that

$$\frac{1}{3} < \delta < \frac{1}{2}.$$

Let

$$\varpi = \varrho^{3\delta-1}.$$

Then, ϖ satisfies

$$\left(-\partial_\tau^2 + \partial_\varrho^2 + \frac{1}{\varrho}\partial_\varrho\right)\varpi = (3\delta-1)^2 \varrho^{3\delta-3},$$

and hence

$$\varpi = \varpi_0(\tau,\varrho) + (3\delta-1)^2 \int_R \frac{\sqrt{\lambda}}{\sqrt{\varrho}} J(\mu) \frac{\lambda^{3\delta}}{\lambda^3} d\lambda d\sigma,$$

where ϖ_0 denotes the solution to the homogeneous wave equation with the same initial conditions as ϖ. This yields

$$\left|\int_R \frac{\sqrt{\lambda}}{\sqrt{\varrho}} J(\mu) \frac{\lambda^{3\delta}}{\lambda^3} d\lambda d\sigma\right| \lesssim \varrho^{3\delta-1} + |\varpi_0(\tau,\varrho)| \lesssim 1,$$

where we used the fact that $\delta > 1/3$ and $0 \leq \varrho \leq 1$.

Recall that

$$|v(\tau,\varrho)| \lesssim C_0 + C_0 r^{3\delta-1} + \varepsilon \sup_{0\leq\lambda\leq\varrho} |v(\tau-\varrho+\lambda,\lambda)| + C_0 \int_R \frac{\sqrt{\lambda}}{\sqrt{\varrho}} J(\mu) \left(\frac{|v(\sigma,\lambda)|}{\lambda} + \frac{\lambda^{3\delta}}{\lambda^3} \right) d\lambda d\sigma.$$

We infer

$$|v(\tau,\varrho)| \lesssim C_0 + \varepsilon \sup_{0\leq\lambda\leq\varrho} |v(\tau-\varrho+\lambda,\lambda)| + C_0 \int_R \frac{\sqrt{\lambda}}{\sqrt{\varrho}} J(\mu) \frac{|v(\sigma,\lambda)|}{\lambda} d\lambda d\sigma,$$

where we used again the fact that $\delta > 1/3$ and $0 \leq \varrho \leq 1$.

We introduce the notation

$$\beta(u) = \sup_{u\leq\underline{u}\leq 0} |v(u,\underline{u})|.$$

We obtain

$$|v(u,\underline{u})| \lesssim C_0 + \varepsilon\beta(u) + C_0 \int_{-2}^{u} \Phi(u,\underline{u},u')\beta(u')du',$$

where Φ is given by

$$\Phi(u,\underline{u},u') = \int_{u'}^{\underline{u}} \frac{\sqrt{\lambda}}{\sqrt{\varrho}} J(\mu) \frac{1}{\lambda} d\underline{u}'.$$

Next, we compute Φ. We have

$$\Phi(u,\underline{u},u') = \int_{-1}^{+\infty} \frac{\sqrt{\lambda}}{\sqrt{\varrho}} J(\mu) \frac{1}{\lambda} \frac{1}{\partial_{\underline{u}}\mu} d\mu$$

Since

$$\begin{aligned}
\partial_{\underline{u}}\mu &= \frac{\lambda(-(\tau-\sigma)-\lambda) - \frac{1}{2}((\tau-\sigma)^2 - \varrho^2 - \lambda^2)}{2\varrho\lambda^2} \\
&= \frac{\lambda(-(\tau-\sigma)-\lambda) - \frac{1}{2}((\tau-\sigma)^2 - \varrho^2 - \lambda^2)}{2\varrho\lambda^2} \\
&= \frac{\varrho^2 - (\tau-\sigma+\lambda)^2}{4\varrho\lambda^2} \\
&= \frac{\varrho^2 - (\tau-u')^2}{4\varrho\lambda^2},
\end{aligned}$$

we infer

$$\Phi(u,\underline{u},u') = \frac{4\sqrt{\varrho}}{(\tau-u')^2 - \varrho^2} \int_{-1}^{+\infty} J(\mu)\lambda^{\frac{3}{2}} d\mu.$$

Also, we have

$$\sigma = u' + \lambda$$

and hence

$$\mu = \frac{(\tau-u'-\lambda)^2 - \varrho^2 - \lambda^2}{2\varrho\lambda} = \frac{(\tau-u')^2 - 2\lambda(\tau-u') - \varrho^2}{2\varrho\lambda},$$

which yields

$$\lambda = \frac{(\tau-u')^2 - \varrho^2}{2(\varrho\mu + \tau - u')}.$$

Hence, we obtain

$$\Phi(u,\underline{u},u') = \sqrt{2}\sqrt{\varrho}\sqrt{(\tau-u')^2 - \varrho^2} \int_{-1}^{+\infty} \frac{J(\mu)}{(\varrho\mu+\tau-u')^{\frac{3}{2}}} d\mu.$$

Using the formula for $J(\mu)$ and Fubini, we infer

$$\begin{aligned}\Phi(u,\underline{u},u') &= \sqrt{2}\sqrt{\varrho}\sqrt{(\tau-u')^2-\varrho^2}\int_{-1}^{1}\frac{dx}{\sqrt{1-x^2}}\int_{-x}^{+\infty}\frac{d\mu}{\sqrt{\mu+x}(\varrho\mu+\tau-u')^{\frac{3}{2}}}\\ &= \sqrt{2}\sqrt{\varrho}\sqrt{(\tau-u')^2-\varrho^2}\int_{-1}^{1}\frac{dx}{\sqrt{1-x^2}}\int_{0}^{+\infty}\frac{d\mu'}{\sqrt{\mu'}(\varrho\mu'-\varrho x+\tau-u')^{\frac{3}{2}}}.\end{aligned}$$

We have

$$\begin{aligned}\int_{0}^{+\infty}\frac{d\mu'}{\sqrt{\mu'}(\varrho\mu'-\varrho x+\tau-u')^{\frac{3}{2}}} &= \frac{1}{\sqrt{\varrho}(-\varrho x+\tau-u')}\int_{0}^{+\infty}\frac{d\mu''}{\sqrt{\mu''}(\mu''+1)^{\frac{3}{2}}}\\ &\lesssim \frac{1}{\sqrt{\varrho}(-\varrho x+\tau-u')}.\end{aligned}$$

We deduce

$$\Phi(u,\underline{u},u') \lesssim \sqrt{(\tau-u')^2-\varrho^2}\int_{-1}^{1}\frac{dx}{\sqrt{1-x^2}(-\varrho x+\tau-u')}.$$

We have for $x\geq 0$

$$-\varrho x+\tau-u' \leq \varrho x+\tau-u'$$

and hence

$$\begin{aligned}\int_{-1}^{1}\frac{dx}{\sqrt{1-x^2}(-\varrho x+\tau-u')} &= \int_{-1}^{0}\frac{dx}{\sqrt{1-x^2}(-\varrho x+\tau-u')}+\int_{0}^{1}\frac{dx}{\sqrt{1-x^2}(-\varrho x+\tau-u')}\\ &= \int_{0}^{1}\frac{dx}{\sqrt{1-x^2}(\varrho x+\tau-u')}+\int_{0}^{1}\frac{dx}{\sqrt{1-x^2}(-\varrho x+\tau-u')}\\ &\leq 2\int_{0}^{1}\frac{dx}{\sqrt{1-x^2}(-\varrho x+\tau-u')}.\end{aligned}$$

This yields

$$\begin{aligned}\Phi(u,\underline{u},u') &\lesssim \sqrt{(\tau-u')^2-\varrho^2}\int_{0}^{1}\frac{dx}{\sqrt{1-x^2}(-\varrho x+\tau-u')}\\ &\lesssim \sqrt{(\tau-u')^2-\varrho^2}\int_{0}^{1}\frac{dx}{\sqrt{1-x}(-\varrho x+\tau-u')}.\end{aligned}$$

Changing variables, we obtain

$$\begin{aligned}\Phi(u,\underline{u},u') &\lesssim \sqrt{(\tau-u')^2-\varrho^2}\int_{0}^{1}\frac{dy}{\sqrt{y}(\varrho y-\varrho+\tau-u')}\\ &\lesssim \frac{\sqrt{(\tau-u')^2-\varrho^2}}{\sqrt{\rho}\sqrt{-\varrho+\tau-u'}}\int_{0}^{+\infty}\frac{dz}{\sqrt{z}(z+1)}\\ &\lesssim 1.\end{aligned}$$

Coming back to v, we obtain

$$|v(u,\underline{u})| \lesssim C_0+\varepsilon\beta(u)+C_0\int_{-2}^{u}\beta(u')du'.$$

Taking the supremum in $\underline{u}$ for $u\leq\underline{u}\leq 0$ yields

$$\beta(u) \lesssim C_0+\varepsilon\beta(u)+C_0\int_{-2}^{u}\beta(u')du'.$$

For ε small enough, this implies

$$\beta(u) \lesssim C_0 + C_0 \int_{-2}^{u} \beta(u')du'.$$

Using Gronwall's Lemma, we infer

$$\beta(u) \lesssim C_0,$$

and hence

$$|v| \lesssim C_0.$$

This concludes the proof of the proposition. □

8.4. **Consequences of Proposition 8.13.**

Lemma 8.14. *We have*

$$|\phi| \leq rC_0 \text{ and } |\partial_u\phi| + |\partial_{\underline{u}}\phi| \leq C_0$$

where the constant C_0 only depends on initial data.

Proof. We first derive a refined upper bound for ϕ. Recall that

$$v = \left(\partial_\varrho + \frac{1}{\varrho}\right)\phi.$$

This yields

$$\partial_\varrho(\varrho\phi) = \varrho\partial_\varrho\phi + \phi = \varrho v$$

and hence

$$\phi(\tau, \varrho) = \int_0^\varrho \lambda v(\tau, \lambda)d\lambda.$$

We infer

$$\begin{aligned} \varrho|\phi(\tau,\varrho)| &\lesssim \int_0^\varrho \lambda|v(\tau,\lambda)|d\lambda \\ &\lesssim C_0\int_0^\varrho \lambda d\lambda \\ &\lesssim C_0\varrho^2 \end{aligned}$$

and hence

$$|\phi| \lesssim C_0\varrho.$$

The above upper bound for ϕ together with the upper bound for v and the definition of v implies

$$|\partial_\varrho\phi| \leq |v| + \frac{1}{\varrho}|\phi| \lesssim C_0.$$

Next, we derive an upper bound for $\partial_u\phi$. Recall that $\Xi = r\partial_u\phi$ satisfies

$$\partial_{\underline{u}}\left(\frac{\Xi}{\sqrt{r}}\right) = -\frac{1}{2\sqrt{r}}\partial_u r\partial_{\underline{u}}\phi - \frac{\Omega^2 f(\phi)}{4r^{\frac{3}{2}}}.$$

We integrate between $(u, \underline{u})$ and (u, u). We deduce

$$\begin{aligned} \frac{\Xi(u,\underline{u})}{\sqrt{r(u,\underline{u})}} &= \frac{\Xi(u,u)}{\sqrt{r(u,u)}} - \int_u^{\underline{u}} \frac{1}{2\sqrt{r}}\partial_u r\partial_{\underline{u}}\phi(u,\sigma)d\sigma - \int_u^{\underline{u}} \frac{\Omega^2 f(\phi)}{4r^{\frac{3}{2}}}(u,\sigma)d\sigma \\ &= \frac{\Xi(u,u)}{\sqrt{r(u,u)}} - \left[\frac{1}{2\sqrt{r}}\partial_u r\phi(u,\sigma)\right]_u^{\underline{u}} + \int_u^{\underline{u}} \frac{1}{2\sqrt{r}}\partial_{\underline{u}}\partial_u r\phi(u,\sigma)d\sigma \\ &\quad - \int_u^{\underline{u}} \frac{1}{4r^{\frac{3}{2}}}\partial_{\underline{u}} r\partial_u r\phi(u,\sigma)d\sigma - \int_u^{\underline{u}} \frac{\Omega^2 f(\phi)}{4r^{\frac{3}{2}}}(u,\sigma)d\sigma. \end{aligned}$$

In view of the definition of Ξ and since (u,u) is on the axis of symmetry, we infer

$$\begin{aligned}\sqrt{r(u,\underline{u})}\partial_u\phi(u,\underline{u}) &= -\left[\frac{1}{2\sqrt{r}}\partial_u r\phi(u,\sigma)\right]_u^{\underline{u}} + \int_u^{\underline{u}} \frac{1}{2\sqrt{r}}\kappa\frac{f(\phi)^2}{r}\phi(u,\sigma)d\sigma\\ &\quad - \int_u^{\underline{u}}\frac{1}{4r^{\frac{3}{2}}}\partial_{\underline{u}}r\partial_u r\phi(u,\sigma)d\sigma - \int_u^{\underline{u}}\frac{\Omega^2 f(\phi)}{4r^{\frac{3}{2}}}(u,\sigma)d\sigma.\end{aligned}$$

Using again the fact that (u,u) is on the axis of symmetry, we deduce

$$\sqrt{r(u,\underline{u})}|\partial_u\phi(u,\underline{u})| \lesssim \frac{|\phi(u,\underline{u})|}{\sqrt{r}} + \int_u^{\underline{u}}\frac{|\phi(u,\sigma)|}{r^{\frac{3}{2}}}d\sigma.$$

Using the above upper bound for ϕ, we infer

$$\begin{aligned}\sqrt{r(u,\underline{u})}|\partial_u\phi(u,\underline{u})| &\lesssim C_0\sqrt{r(u,\underline{u})} + C_0\int_u^{\underline{u}}\frac{1}{r^{\frac{1}{2}}}d\sigma\\ &\lesssim C_0\sqrt{r(u,\underline{u})}.\end{aligned}$$

Thus, we obtain

$$|\partial_u\phi(u,\underline{u})| \lesssim C_0.$$

Together with the upper bound for $\partial_\varrho\phi$, we also obtain

$$|\partial_{\underline{u}}\phi| \leq |\partial_\varrho\phi| + |\partial_u\phi| \lesssim C_0.$$

This concludes the proof of the lemma. □

Lemma 8.15. *We have*

$$\left|\partial_u r + \frac{1}{2}\right| + \left|\partial_{\underline{u}} r - \frac{1}{2}\right| + |\Omega - 1| \leq C_0\varrho^2,$$

where the constant C_0 only depends on initial data.

Proof. Integrating

$$\partial_u\partial_{\underline{u}}r = r\kappa\frac{\Omega^2}{4}\frac{g(\phi)^2}{r^2}$$

from the axis along u of $\underline{u}$ together with the upper bound on ϕ of Lemma 8.14 yields

$$\left|\partial_u r + \frac{1}{2}\right| + \left|\partial_{\underline{u}} r - \frac{1}{2}\right| \leq C_0\varrho^2.$$

Together with

$$\partial_{\underline{u}}(\Omega^{-2}\partial_{\underline{u}}r) = -\Omega^{-2}r\kappa(\partial_{\underline{u}}\phi)^2$$

which we integrate from the axis along u yields in view of Lemma 8.14

$$\left|\Omega^{-2}\partial_{\underline{u}}r - \frac{1}{2}\right| \leq C_0\varrho^2$$

and hence

$$|\Omega - 1| \leq C_0\varrho^2.$$

This concludes the proof of the lemma. □

9. Small energy implies global existence

In this section, we conclude the proof of Theorem 6.2.

9.1. **A wave equation for ϕ/ϱ.** We first derive a wave equation for ϕ/ϱ.

Lemma 9.1. *We introduce*

$$w = \frac{\phi}{\varrho}.$$

Then, we have

$$-\partial_\tau^2 w + \partial_\varrho^2 w + \frac{3}{\varrho}\partial_\varrho w = \left(\Omega^2 - 1 + \partial_u r + \frac{1}{2} - \left(\partial_{\underline{u}} r - \frac{1}{2}\right)\right)\frac{w}{\varrho^2} + \Omega^2\frac{\varrho^2}{r^2}w^3\zeta(\varrho w).$$

Proof. We have

$$\begin{aligned}
\square_{\mathbf{g}} w &= \frac{\square_{\mathbf{g}}\phi}{\varrho} - 2\frac{\mathbf{D}^\alpha\phi\mathbf{D}_\alpha\varrho}{\varrho^2} + \phi\left(-\frac{\square_{\mathbf{g}}\varrho}{\varrho^2} + 2\frac{\mathbf{D}^\alpha\varrho\mathbf{D}_\alpha\varrho}{\varrho^3}\right)\\
&= \frac{f(\phi)}{r^2\varrho} - 2\frac{\mathbf{D}_\alpha\varrho}{\varrho}\left(\mathbf{D}^\alpha w + \frac{w}{\varrho}\mathbf{D}^\alpha\varrho\right) + w\left(-\frac{\square_{\mathbf{g}}\varrho}{\varrho} + 2\frac{\mathbf{D}^\alpha\varrho\mathbf{D}_\alpha\varrho}{\varrho^2}\right)\\
&= \frac{w}{r^2} + \frac{\varrho^2}{r^2}w^3\zeta(\varrho w) - 2\frac{\mathbf{D}_\alpha\varrho}{\varrho}\left(\mathbf{D}^\alpha w + \frac{w}{\varrho}\mathbf{D}^\alpha r\right) + w\left(-\frac{\square_{\mathbf{g}}\varrho}{\varrho} + 2\frac{\mathbf{D}^\alpha\varrho\mathbf{D}_\alpha\varrho}{\varrho^2}\right)\\
&= -\frac{2}{\varrho}\mathbf{D}_\alpha\varrho\mathbf{D}^\alpha w + (1 - \varrho\square_{\mathbf{g}}\varrho)\frac{w}{\varrho^2} + \frac{\varrho^2}{r^2}w^3\zeta(\varrho w)\\
&= \frac{2}{\varrho\Omega^2}(-\partial_{\underline{u}} w + \partial_u w) + \left(1 + \frac{1}{\Omega^2}(\partial_u r - \partial_{\underline{u}} r)\right)\frac{w}{\varrho^2} + \frac{\varrho^2}{r^2}w^3\zeta(\varrho w).
\end{aligned}$$

In view of Lemma 7.3, we infer

$$-4\partial_u\partial_{\underline{u}} w + \frac{3}{\varrho}(\partial_{\underline{u}} w - \partial_u w) = \left(\Omega^2 - 1 + \partial_u r + \frac{1}{2} - \left(\partial_{\underline{u}} r - \frac{1}{2}\right)\right)\frac{w}{\varrho^2} + \Omega^2\frac{\varrho^2}{r^2}w^3\zeta(\varrho w).$$

We deduce

$$-\partial_\tau^2 w + \partial_\varrho^2 w + \frac{3}{\varrho}\partial_\varrho w = \left(\Omega^2 - 1 + \partial_u r + \frac{1}{2} - \left(\partial_{\underline{u}} r - \frac{1}{2}\right)\right)\frac{w}{\varrho^2} + \Omega^2\frac{\varrho^2}{r^2}w^3\zeta(\varrho w).$$

This concludes the proof of the lemma. □

Remark 9.2. *In this paper, we first obtain improved uniform bounds for ϕ and then use the wave equation for w to obtain regularity, following the approach of* [30] *and* [9] *for the 2+1 wave map problem. Note that we can not use a more direct approach based on Strichartz estimates for the wave equation for w as in* [28] *for the 2+1 wave map problem. Indeed, this approach does not allow to deal with the following terms in the right-hand side of the equation for w*

$$\frac{\Omega^2 - 1}{\varrho^2}w,\ \ \frac{\partial_u r + \frac{1}{2}}{\varrho^2}w \text{ and } \frac{\partial_{\underline{u}} r - \frac{1}{2}}{\varrho^2}w.$$

9.2. **Embeddings for radial functions on $\mathbb{R}^4$.** Note that the right-hand side of the wave equation for w in Lemma 9.1 is the 4-dimensional wave radial wave operator. Therefore, we will control w in radial Sobolev spaces in 4 dimension. Furthermore, we only need to control w in the causal region

$$\{-1 \le \tau < 0,\ \ 0 \le \varrho \le |\tau|\}.$$

Thus, for any $-1 \le \tau < 0$, we introduce

$$L^2_{r,\tau} = \left\{\psi\ /\ \int_0^{|\tau|}(\psi(\varrho))^2\varrho^3 d\varrho < \infty\right\}$$

Recall the Hardy estimate

$$\int_0^{|\tau|}\frac{(\psi(\varrho))^2}{\varrho^2}\varrho^3 d\varrho \lesssim \|\psi'\|^2_{L^2_{r,\tau}}.$$

We will also use the following non sharp embedding.

Lemma 9.3. *We have*

$$\sup_{0\le\varrho\le|\tau|}\varrho^{\frac{5}{4}}|\psi| \lesssim \|\psi'\|_{L^2_{r,\tau}}.$$

Proof. We have

$$\begin{aligned}\varrho^{\frac{5}{4}}\psi(\varrho) &= \int_0^{\varrho}(\sigma^{\frac{5}{4}}\psi(\sigma))'d\sigma\\ &= \int_0^{\varrho}\sigma^{\frac{5}{4}}\psi'(\sigma)d\sigma+\frac{5}{4}\int_0^{\varrho}\sigma^{\frac{1}{4}}\psi(\sigma)d\sigma.\end{aligned}$$

Since $\varrho\le|\tau|\le 1$, we infer

$$\begin{aligned}\varrho^{\frac{3}{2}}|\psi(\varrho)| &\lesssim \int_0^{|\tau|}\varrho^{\frac{5}{4}}|\psi'(\varrho)|d\varrho+\int_0^{|\tau|}\varrho^{\frac{1}{4}}|\psi(\varrho)|d\varrho\\ &\lesssim |\tau|^{\frac{1}{4}}\left(\left(\int_0^{|\tau|}(\psi'(\varrho))^2\varrho^3 d\varrho\right)^{\frac{1}{2}}+\left(\int_0^{|\tau|}\frac{(\psi(\varrho))^2}{\varrho^2}\varrho^3 d\varrho\right)^{\frac{1}{2}}\right)\\ &\lesssim \|\psi'\|_{L^2_{r,\tau}}\end{aligned}$$

where we used in the last inequality the Hardy inequality and the fact that $|\tau|\le 1$. This concludes the proof of the lemma. □

9.3. **Proof of Theorem 6.2.** We are now ready to prove Theorem 6.2. Recall that

$$w=\frac{\phi}{\varrho}$$

satisfies the following wave equation

$$-\partial_\tau^2 w+\partial_\varrho^2 w+\frac{3}{\varrho}\partial_\varrho w = h, \tag{9.1}$$

where h is given by

$$h = \left(\Omega^2-1+\partial_u r+\frac{1}{2}-\left(\partial_{\underline{u}} r-\frac{1}{2}\right)\right)\frac{w}{\varrho^2}+\Omega^2\frac{\varrho^2}{r^2}w^3\zeta(\varrho w).$$

For $\ell\in\mathbb{N}$, we introduce the following notations

$$D_\ell(\tau):=\max_{|\alpha|=\ell}\sup_{0\le\varrho\le|\tau|}\left(\frac{1}{\varrho^{\frac{3}{4}}}\left(\left|\partial^\alpha\left(\partial_u r+\frac{1}{2}\right)\right|+\left|\partial^\alpha\left(\partial_{\underline{u}} r-\frac{1}{2}\right)\right|+|\partial^\alpha(\Omega-1)|\right)(\tau,\varrho)\right),$$

$$E_\ell(\tau):=\max_{|\alpha|=\ell}\|\partial^\alpha w\|_{L^2_{r,\tau}},$$

$$\begin{aligned}F_\ell(\tau):=\max_{|\alpha|=\ell}\Bigg(&\sup_{-1\le\underline{u}\le 0}\left(\int_{-2-\underline{u}}^{\min(\underline{u},2\tau-\underline{u})}|\partial^\alpha w|^2\varrho^3(u',\underline{u})du'\right)^{\frac{1}{2}}\\ &+\sup_{-2\le u\le 0}\left(\int_{\max(u,-2-u)}^{\min(0,2\tau-u)}|\partial^\alpha w|^2\varrho^3(u,\underline{u}')d\underline{u}'\right)^{\frac{1}{2}}\Bigg),\end{aligned}$$

and

$$D_{\le\ell}(\tau) := \max_{j\le\ell} D_j(\tau),\ E_{\le\ell}(\tau) := \max_{j\le\ell} E_j(\tau),\ F_{\le\ell}(\tau) := \max_{j\le\ell} F_j(\tau).$$

Lemma 9.4. *Let $-1 \le \tau < 0$ and $\ell \in \mathbb{N}^*$. Then, we have the following estimate for h*

$$\max_{|\alpha|=\ell} \|\partial^\alpha h(\tau,\cdot)\|_{L^2_{r,\tau}} \lesssim \vartheta\left(C_0 + D_{\le\ell-1}(\tau) + E_{\le\ell}(\tau)\right)\left(1 + D_\ell(\tau) + E_{\ell+1}(\tau)\right),$$

where the constant C_0 only depends on initial data and where the function ϑ can be chosen to be continuous and increasing.

We postpone the proof of Lemma 9.4 to section 9.4. Next, let

$$\widetilde{D}_\ell(\tau) := \sup_{-1\le\tau'\le\tau} D_\ell(\tau'),\ \widetilde{E}_\ell(\tau) := \sup_{-1\le\tau'\le\tau} E_\ell(\tau'),\ \widetilde{F}_\ell(\tau) := \sup_{-1\le\tau'\le\tau} F_\ell(\tau'),$$

and

$$\widetilde{D}_{\le\ell}(\tau) := \sup_{-1\le\tau'\le\tau} D_{\le\ell}(\tau'),\ \widetilde{E}_{\le\ell}(\tau) := \sup_{-1\le\tau'\le\tau} E_{\le\ell}(\tau'),\ \widetilde{F}_{\le\ell}(\tau) := \sup_{-1\le\tau'\le\tau} F_{\le\ell}(\tau').$$

Lemma 9.4 immediately implies the following corollary.

Corollary 9.5. *Let $-1 \le \tau < 0$ and $\ell \in \mathbb{N}^*$. Then, we have the following estimate for h*

$$\max_{|\alpha|=\ell} \sup_{-1\le\tau'\le\tau} \|\partial^\alpha h(\tau',\cdot)\|_{L^2_{r,\tau'}} \lesssim \vartheta\left(C_0 + \widetilde{D}_{\le\ell-1}(\tau) + \widetilde{E}_{\le\ell}(\tau)\right)\left(1 + \widetilde{D}_\ell(\tau) + \widetilde{E}_{\ell+1}(\tau)\right),$$

where the constant C_0 only depends on initial data and where the function ϑ can be chosen to be continuous and increasing.

Next, we derive an estimate for $\widetilde{D}_\ell$.

Lemma 9.6. *Let $-1 \le \tau < 0$ and let $\ell \in \mathbb{N}$. We have the following estimate*

$$\widetilde{D}_\ell(\tau) \lesssim \vartheta\left(C_0 + \widetilde{D}_{\le\ell-1}(\tau) + \widetilde{E}_{\le\ell}(\tau) + \widetilde{F}_{\le\ell}(\tau)\right)\left(1 + \widetilde{E}_{\ell+1}(\tau) + \widetilde{F}_{\ell+1}(\tau)\right),$$

where the constant C_0 only depends on initial data and where the function ϑ can be chosen to be continuous and increasing.

We postpone the proof of Lemma 9.6 to section 9.5. In view of Lemma 8.15, we have

$$\sup_{-1\le\tau<0} D_0(\tau) \le C_0.$$

By iteration, we infer from Lemma 9.6 that for all $\ell \in \mathbb{N}$, we have

$$\widetilde{D}_\ell(\tau) \lesssim \vartheta\left(C_0 + \widetilde{E}_{\le\ell}(\tau) + \widetilde{F}_{\le\ell}(\tau)\right)\left(1 + \widetilde{E}_{\ell+1}(\tau) + \widetilde{F}_{\ell+1}(\tau)\right).$$

Together with Corollary 9.5, we obtain

$$\max_{|\alpha|=\ell} \sup_{-1\le\tau'\le\tau} \|\partial^\alpha h(\tau',\cdot)\|_{L^2_{r,\tau'}} \lesssim \vartheta\left(C_0 + \widetilde{E}_{\le\ell}(\tau) + \widetilde{F}_{\le\ell}(\tau)\right)\left(1 + \widetilde{E}_{\ell+1}(\tau) + \widetilde{F}_{\ell+1}(\tau)\right).$$

Together with the energy estimate for the wave equation (9.1) and Gronwall Lemma, we deduce by iteration for all $\ell \in \mathbb{N}$

$$\sup_{-1\le\tau<0} \left(D_\ell(\tau) + E_{\ell+1}(\tau) + F_{\ell+1}(\tau)\right) \le C_\ell < +\infty.$$

We have thus obtained the regularity of $(M, \mathbf{g}, \phi)$ at the origin $(\tau, \varrho) = (0, 0)$. This concludes the proof of Theorem 6.2.

9.4. **Proof of Lemma 9.4.** We have

$$\begin{aligned}
\partial^\alpha h &= \left(\frac{\Omega^2-1+\partial_u r+\frac{1}{2}-\left(\partial_{\underline{u}} r-\frac{1}{2}\right)}{\varrho^2}\right)\partial^\alpha w \\
&\quad + \sum_{\beta+\gamma=\alpha,\beta\neq 0,\gamma\neq 0}\partial^\beta\left(\frac{\Omega^2-1+\partial_u r+\frac{1}{2}-\left(\partial_{\underline{u}} r-\frac{1}{2}\right)}{\varrho^2}\right)\partial^\gamma w \\
&\quad +\partial^\alpha\left(\frac{\Omega^2-1+\partial_u r+\frac{1}{2}-\left(\partial_{\underline{u}} r-\frac{1}{2}\right)}{\varrho^2}\right)w+\partial^\alpha(\Omega^2)\frac{\varrho^2}{r^2}w^3\zeta(\varrho w) \\
&\quad +\Omega^2\partial^\alpha\left(\frac{\varrho^2}{r^2}\right)w^3\zeta(\varrho w)+\sum_{\beta+\gamma+\mu=\alpha,\mu\neq 0}\partial^\beta(\Omega^2)\partial^\gamma\left(\frac{\varrho^2}{r^2}\right)\partial^\mu(w^3\zeta(\varrho w))
\end{aligned}$$

In view of Lemma 8.14, w satisfies the following a priori bound

$$|w|\leq C_0$$

where the constant C_0 only depends on initial data. Together with Lemma 8.15, we infer

$$\begin{aligned}
|\partial^\alpha h| &\lesssim C_0|\partial^\alpha w|+\sum_{\beta+\gamma=\alpha,\beta\neq 0,\gamma\neq 0}\varrho^{-\frac{9}{4}+|\gamma|}D_{\leq\ell-1}(\tau)|\partial^\gamma w|+C_0\varrho^{-\frac{5}{4}}D_\ell(\tau) \\
&\quad +C_0 D_\ell(\tau)+C_0(\vartheta(D_{\leq\ell-1}(\tau))+D_\ell(\tau)) \\
&\quad +\vartheta(C_0+D_{\leq\ell-1}(\tau)+E_{\leq\ell}(\tau))\left(1+\sum_{|\beta|=\ell}|\partial^\beta w|\right) \\
&\lesssim \sum_{\beta+\gamma=\alpha,\beta\neq 0,\gamma\neq 0}\varrho^{-\frac{9}{4}+|\gamma|}D_{\ell-1}(\tau)\varrho^{-\frac{5}{4}+\ell-|\gamma|}\sup_{0\leq\varrho\leq|\tau|}(\varrho^{\frac{5}{4}-\ell+|\gamma|}\partial^\gamma w) \\
&\quad +\varrho^{-\frac{5}{4}}\vartheta\left(C_0+D_{\leq\ell-1}(\tau)+E_{\leq\ell}(\tau)\right)\left(1+D_\ell(\tau)+E_{\ell+1}(\tau)\right) \\
&\lesssim \sum_{\beta+\gamma=\alpha,\beta\neq 0,\gamma\neq 0}\varrho^{-\frac{7}{2}+\ell}D_{\ell-1}(\tau)E_{\ell+1}(\tau) \\
&\quad +\varrho^{-\frac{3}{2}}\vartheta\left(C_0+D_{\leq\ell-1}(\tau)+E_{\leq\ell}(\tau)\right)\left(1+D_\ell(\tau)+E_{\ell+1}(\tau)\right).
\end{aligned}$$

Note that the sum over $\beta+\gamma=\alpha,\beta\neq 0,\gamma\neq 0$ is empty unless $\ell\geq 2$. We infer

$$|\partial^\alpha h| \lesssim \varrho^{-\frac{3}{2}}\vartheta\left(C_0+D_{\leq\ell-1}(\tau)+E_{\leq\ell}(\tau)\right)\left(1+D_\ell(\tau)+E_{\ell+1}(\tau)\right).$$

We infer

$$\|\partial^\alpha h(\tau,\cdot)\|_{L^2_{r,\tau}} \lesssim \vartheta\left(C_0+D_{\leq\ell-1}(\tau)+E_{\leq\ell}(\tau)\right)\left(1+D_\ell(\tau)+E_{\ell+1}(\tau)\right),$$

where we used the fact that

$$\int_0^{|\tau|}\left(\varrho^{-\frac{3}{2}}\right)^2\varrho^3 d\varrho\leq\int_0^1 d\varrho=1.$$

This yields

$$\max_{|\alpha|\leq\ell}\|\partial^\alpha h(\tau,\cdot)\|_{L^2_{r,\tau}} \lesssim \vartheta\left(C_0+D_{\leq\ell-1}(\tau)+E_{\leq\ell}(\tau)\right)\left(1+D_\ell(\tau)+E_{\ell+1}(\tau)\right),$$

which concludes the proof of Lemma 9.4.

9.5. **Proof of Lemma 9.6.** Recall that Ω satisfies

$$\Omega^{-2}(\partial_u\Omega\partial_{\underline{u}}\Omega - \Omega\partial_u\partial_{\underline{u}}\Omega) = \frac{1}{8}\Omega^2\kappa\left(\frac{4}{\Omega^2}\partial_u\phi\partial_{\underline{u}}\phi + \frac{g(\phi)^2}{r^2}\right)$$

and hence

$$\partial_u\partial_{\underline{u}}\log(\Omega) = -\frac{1}{2}\partial_u(\varrho w)\partial_{\underline{u}}(\varrho w) - \frac{\Omega^2}{8}\frac{g(\varrho w)^2}{r^2}.$$

Differentiating α times, we infer

$$|\partial_u\partial_{\underline{u}}(\partial^\alpha\log(\Omega))| \leq \vartheta\left(C_0 + \widetilde{D}_{\leq\ell-1}(\tau) + \widetilde{E}_{\leq\ell}(\tau)\right)\left(\frac{1}{\varrho^{\frac{5}{4}}} + |\partial^\alpha\Omega| + |\partial^\alpha w| + \varrho|\partial^{\alpha+1}w|\right).$$

We integrate once to obtain

$$\begin{aligned}
|\partial_{\underline{u}}\partial^\alpha\log(\Omega)| &\lesssim \vartheta\left(C_0 + \widetilde{D}_{\leq\ell-1}(\tau) + \widetilde{E}_{\leq\ell}(\tau)\right)\left(\int_{-2-\underline{u}}^{u}\left(\frac{1}{\varrho^{\frac{5}{4}}} + |\partial^\alpha\Omega| + |\partial^\alpha w| + \varrho|\partial^{\alpha+1}w|\right)du'\right)\\
&\lesssim \vartheta\left(C_0 + \widetilde{D}_{\leq\ell-1}(\tau) + \widetilde{E}_{\leq\ell}(\tau)\right)\\
&\quad\times\left(\int_{-2-\underline{u}}^{u}|\partial^\alpha\Omega|du' + (1 + \widetilde{E}_{\ell+1}(\tau) + \widetilde{F}_{\ell+1}(\tau))\int_{-2-\underline{u}}^{u}\frac{du'}{\varrho^{\frac{5}{4}}}\right)\\
&\lesssim \vartheta\left(C_0 + \widetilde{D}_{\leq\ell-1}(\tau) + \widetilde{E}_{\leq\ell}(\tau)\right)\left(\int_{-2-\underline{u}}^{u}|\partial^\alpha\Omega|du' + \frac{1}{\varrho^{\frac{1}{4}}}(1 + \widetilde{E}_{\ell+1}(\tau) + \widetilde{F}_{\ell+1}(\tau))\right).
\end{aligned}$$

We then integrate a second time, and obtain in view of Gronwall lemma and

$$\int_u^{\underline{u}}\frac{d\underline{u}}{\varrho^{\frac{1}{4}}} \lesssim \varrho^{\frac{3}{4}}$$

that

$$\frac{|\partial^\alpha\log(\Omega)|}{\varrho^{\frac{3}{4}}} \leq \vartheta\left(C_0 + \widetilde{D}_{\leq\ell-1}(\tau) + \widetilde{E}_{\leq\ell}(\tau)\right)(1 + \widetilde{E}_{\ell+1}(\tau) + \widetilde{F}_{\ell+1}(\tau)).$$

Hence, this yields in view of the L^∞ control for $\Omega - 1$ obtained in Lemma 8.15

$$\sup_{-1\leq\tau'\leq\tau,\ 0\leq\varrho\leq|\tau'|}\frac{|\partial^\alpha(\Omega-1)|}{\varrho^{\frac{3}{4}}}(\tau',\varrho) \leq \vartheta\left(C_0 + D_{\leq\ell-1}(\tau) + E_{\leq\ell}(\tau)\right)(1 + E_{\ell+1}(\tau) + F_{\ell+1}(\tau)).$$

Next, recall that r satisfies

$$\begin{aligned}
\partial_{\underline{u}}\partial_u r &= r\kappa\frac{\Omega^2}{4}\frac{g(\phi)^2}{r^2}\\
&= r\kappa\frac{\Omega^2}{4}\frac{g(\varrho w)^2}{r^2}.
\end{aligned}$$

Differentiating α times, we infer

$$\begin{aligned}
\left|\partial_{\underline{u}}\partial^\alpha\left(\partial_u r + \frac{1}{2}\right)\right| &\lesssim \vartheta\left(C_0 + \widetilde{D}_{\leq\ell-1}(\tau) + \widetilde{E}_{\leq\ell}(\tau)\right)\left(\frac{1}{\varrho^{\frac{1}{4}}} + |\partial^\alpha(\Omega-1)| + \varrho|\partial^\alpha\phi|\right)\\
&\lesssim \vartheta\left(C_0 + \widetilde{D}_{\leq\ell-1}(\tau) + \widetilde{E}_{\leq\ell}(\tau)\right)\frac{1}{\varrho^{\frac{1}{4}}}(1 + \widetilde{E}_{\ell+1}(\tau) + \widetilde{F}_{\ell+1}(\tau)),
\end{aligned}$$

where we used in particular the estimate obtained previously for Ω. Integrating, we get

$$\begin{aligned}\left|\partial^\alpha\left(\partial_u r+\frac{1}{2}\right)\right|(\tau',\varrho) &\lesssim \vartheta\left(C_0+\widetilde{D}_{\le\ell-1}(\tau)+\widetilde{E}_{\le\ell}(\tau)\right)(1+\widetilde{E}_{\ell+1}(\tau)+\widetilde{F}_{\ell+1}(\tau))\int_u^{\underline{u}}\frac{d\underline{u}'}{\varrho^{\frac{1}{4}}}\\ &\lesssim \varrho^{\frac{3}{4}}\vartheta\left(C_0+\widetilde{D}_{\le\ell-1}(\tau)+\widetilde{E}_{\le\ell}(\tau)\right)(1+\widetilde{E}_{\ell+1}(\tau)+\widetilde{F}_{\ell+1}(\tau))\end{aligned}$$

and hence

$$\sup_{-1\le\tau'\le\tau,\ 0\le\varrho\le|\tau'|}\frac{\left|\partial^\alpha\left(\partial_u r+\frac{1}{2}\right)\right|(\tau',\varrho)}{\varrho^{\frac{3}{4}}} \lesssim \vartheta\left(C_0+\widetilde{D}_{\le\ell-1}(\tau)+\widetilde{E}_{\le\ell}(\tau)\right)(1+\widetilde{E}_{\ell+1}(\tau)+\widetilde{F}_{\ell+1}(\tau)).$$

Finally, recall that r satisfies

$$\begin{aligned}\partial_{\underline{u}}(\Omega^{-2}\partial_{\underline{u}}r) &= -\Omega^{-2}r\kappa(\partial_{\underline{u}}\phi)^2\\ &= -\Omega^{-2}r\kappa(\partial_{\underline{u}}(rw))^2.\end{aligned}$$

Differentiating α times, we infer

$$\begin{aligned}&\left|\partial_{\underline{u}}\partial^\alpha\left(\Omega^{-2}\partial_{\underline{u}}r-\frac{1}{2}\right)\right|\\ &\lesssim \vartheta\left(C_0+\widetilde{D}_{\le\ell-1}(\tau)+\widetilde{E}_{\le\ell}(\tau)\right)\left(\frac{1}{\varrho^{\frac{1}{4}}}+|\partial^\alpha(\Omega-1)|+\varrho|\partial^\alpha\phi|+\varrho^2|\partial^{\alpha+1}\phi|\right)\\ &\lesssim \vartheta\left(C_0+\widetilde{D}_{\le\ell-1}(\tau)+\widetilde{E}_{\le\ell}(\tau)\right)\left(\frac{1}{\varrho^{\frac{1}{4}}}(1+\widetilde{E}_{\ell+1}(\tau)+\widetilde{F}_{\ell+1}(\tau))+\varrho^2|\partial^{\alpha+1}\phi|\right),\end{aligned}$$

where we used in particular the estimate obtained previously for Ω. Integrating, we get

$$\begin{aligned}\left|\partial^\alpha\left(\Omega^{-2}\partial_{\underline{u}}r-\frac{1}{2}\right)\right| &\lesssim \vartheta\left(C_0+\widetilde{D}_{\le\ell-1}(\tau)+\widetilde{E}_{\le\ell}(\tau)\right)\\ &\quad\times\left((1+\widetilde{E}_{\ell+1}(\tau)+\widetilde{F}_{\ell+1}(\tau))\int_u^{\underline{u}}\frac{d\underline{u}'}{\varrho^{\frac{1}{4}}}+\int_u^{\underline{u}}\varrho^2\partial^{\alpha+1}w\,d\underline{u}'\right)\\ &\lesssim \vartheta\left(C_0+\widetilde{D}_{\le\ell-1}(\tau)+\widetilde{E}_{\le\ell}(\tau)\right)\\ &\quad\times\left(\varrho^{\frac{3}{4}}(1+\widetilde{E}_{\ell+1}(\tau)+\widetilde{F}_{\ell+1}(\tau))+\widetilde{F}_{\ell+1}(\tau)\left(\int_u^{\underline{u}}\varrho\,d\underline{u}'\right)^{\frac{1}{2}}\right)\\ &\lesssim \varrho^{\frac{3}{4}}\vartheta\left(C_0+\widetilde{D}_{\le\ell-1}(\tau)+\widetilde{E}_{\le\ell}(\tau)\right)(1+\widetilde{E}_{\ell+1}(\tau)+\widetilde{F}_{\ell+1}(\tau))\end{aligned}$$

and hence

$$\begin{aligned}&\sup_{-1\le\tau'\le\tau,\ 0\le\varrho\le|\tau'|}\frac{\left|\partial^\alpha\left(\Omega^{-2}\partial_{\underline{u}}r-\frac{1}{2}\right)\right|(\tau',\varrho)}{\varrho^{\frac{3}{4}}}\\ &\lesssim \vartheta\left(C_0+\widetilde{D}_{\le\ell-1}(\tau)+\widetilde{E}_{\le\ell}(\tau)\right)(1+\widetilde{E}_{\ell+1}(\tau)+\widetilde{F}_{\ell+1}(\tau)).\end{aligned}$$

Together with the estimate for $\Omega-1$, we infer

$$\begin{aligned}&\sup_{-1\le\tau'\le\tau,\ 0\le\varrho\le|\tau'|}\frac{\left|\partial^\alpha\left(\partial_{\underline{u}}r-\frac{1}{2}\right)\right|(\tau',\varrho)}{\varrho^{\frac{3}{4}}}\\ &\lesssim \vartheta\left(C_0+\widetilde{D}_{\le\ell-1}(\tau)+\widetilde{E}_{\le\ell}(\tau)\right)(1+\widetilde{E}_{\ell+1}(\tau)+\widetilde{F}_{\ell+1}(\tau)).\end{aligned}$$

This concludes the proof of Lemma 9.6.

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