Lifting of Multicuts

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Abstract

For every simple, undirected graph $G = (V, E)$, a one-to-one relation exists between the decompositions and the multicuts of G. A decomposition of G is a partition Π of V such that, for every $U \in \Pi$, the subgraph of G induced by U is connected. A multicut of G is a subset $M \subseteq E$ of edges such that, for every (chordless) cycle $C \subseteq E$ of G, $|M \cap C| \neq 1$. The multicut induced by a decomposition is the set of edges that straddle distinct components. The characteristic function $x \in \{0,1\}^E$ of a multicut $M = x^{-1}(1)$ of G makes explicit, for every pair $\{v, w\} \in E$ of neighboring nodes, whether v and w are in distinct components. In order to make explicit also for non-neighboring nodes, specifically, for all $\{v, w\} \in E'$ with $E \subseteq E' \subseteq \binom{V}{2}$, whether v and w are in distinct components, we define a lifting of the multicuts of G to multicuts of $G' = (V, E')$. We show that, if G is connected, the convex hull of the characteristic functions of those multicuts of G' that are lifted from G is an $|E'|$-dimensional polytope in $\mathbb{R}^{E'}$. We establish some properties of some facets of this polytope.

1 Introduction

This note is concerned with the set of all decompositions of a graph (Fig. 1). A decomposition of a graph $G = (V, E)$ is a partition Π of the node set V such that, for every subset $U \in \Pi$ of nodes, the subgraph of G induced by U is connected. This definition of a decomposition of a graph generalizes the definition of a partition of a set, as partitions of sets can be identified with decompositions of complete graphs. Roughly analogous to the bijection between the partitions of a set and the equivalence relations on the set is a bijection between the decompositions of a graph and the multicuts of the graph (Fig. 1). A multicut of G is a subset $M \subseteq E$ of edges such that, for every (chordless) cycle $C \subseteq E$ of G, $|M \cap C| \neq 1$. The multicut induced by a decomposition is the set of edges that straddle distinct components.

This note is motivated by a trivial observation: For a partition of a set, the corresponding equivalence relation makes explicit, for every pair of elements, whether these elements are in the same set of the partition. For a decomposition of a graph, however, the corresponding multicut makes explicit only for neighboring nodes whether these nodes are in distinct components. In order to make explicit also for non-neighboring nodes whether these nodes are in distinct components, we identify every decomposition Π of G not with the multicut of G induced by Π but with the multicut of a larger graph $G' = (V, E')$, with $E \subseteq E'$, induced by Π. As each decomposition of G is a decomposition of G' but not necessarily the other way around, the multicuts of G' induced by the decompositions of G are, in general, a subset of the multicuts of G' (compare Fig. 2 and 3).

This note is about the polyhedral geometry of lifted multicuts. It builds on more comprehensive studies of the multicut polytope (Def. 4) by Grötschel and Wakabayashi [5], Chopra and Rao [2] and Deza and Laurent [4].

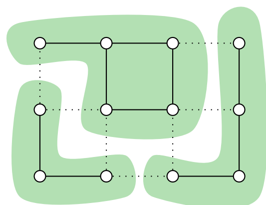

Figure 1 A decomposition of a graph (Def. 2) is a partition of the node set into connected subsets. Above, one decomposition is depicted in green. Every decomposition is characterized by the set of those edges (depicted as dotted lines) that straddle distinct components. Such edge sets are precisely the multicuts of the graph (Def. 3), by Lemma 2.

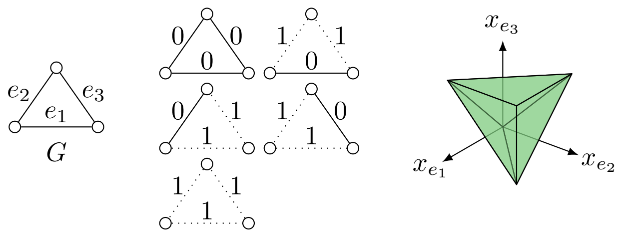

Figure 2 For every connected graph G (left), the characteristic functions of all multicuts of G (middle) span, as their convex hull in $\mathbb{R}^E$, the multicut polytope of G (right), a 01-polytope that is $|E|$-dimensional [2].

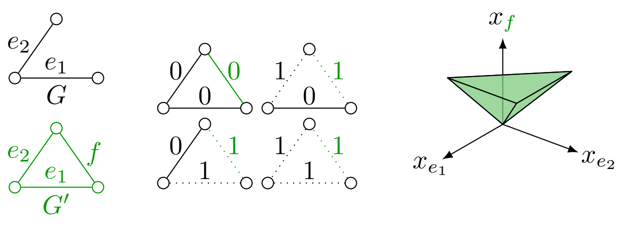

Figure 3 For every connected graph $G = (V, E)$ (top left) and every graph $G' = (V, E')$ with $E \subseteq E'$ (bottom left), those multicuts of G' that are lifted from G (middle) span, as their convex hull in $\mathbb{R}^E$, the lifted multicut polytope with respect to G and G' (right), a 01-polytope that is $|E'|$-dimensional (Thm. 1).

2 Decompositions and Multicuts

Definition 1 For every graph $G = (V, E)$, a subgraph $G' = (V', E')$ of G is called a *component* of G iff the following conditions hold:
(a) $V' \neq \emptyset$
(b) G' is induced by V', that is, $E' = E \cap \binom{V'}{2}$
(c) G' is connected.

Note that we do not require a component to be maximal w.r.t. the partial order that is the subgraph relation.

Definition 2 For every graph $G = (V, E)$, a partition Π of V is called a *decomposition* of G iff, for every $U \in \Pi$, the subgraph $(U, E \cap \binom{U}{2})$ of G induced by U is connected (and hence a component of G).

For every graph G, let $D_G \subset 2^{2^V}$ denote the set of all decompositions of G. Useful in the study of decompositions are the multicuts of a graph:

Definition 3 For every graph $G = (V, E)$, an $M \subseteq E$ is called a *multicut* of G iff, for every cycle $C \subseteq E$ of G, $|C \cap M| \neq 1$.

Lemma 1 [2] It is sufficient in Def. 3 to consider only the *chordless* cycles.

For every graph G, let $M_G \subseteq 2^E$ denote the set of all multicuts of G. One reason why multicuts are useful in the study of decompositions is that, for every graph G, a one-to-one relation exists between the decompositions and the multicuts of G, as depicted in Fig. 1:

Lemma 2 For every graph $G = (V, E)$, the map $\phi_G : D_G \to 2^E$ defined by (1) is a bijection between D_G and M_G.

$$\forall \Pi \in D_G \ \forall \{v, w\} \in E : \quad \{v, w\} \in \phi_G(\Pi) \ \Leftrightarrow \ \forall U \in \Pi (v \notin U \vee w \notin U) \tag{1}$$

A second reason why multicuts are useful in the study of decompositions is that, for every graph $G = (V, E)$ and every decomposition Π of G, the characteristic function of the multicut induced by Π is a 01-encoding of Π of fixed length $|E|$.

Lemma 3 [2] For every graph $G = (V, E)$ and every $x \in \{0, 1\}^E$, the set $x^{-1}(1)$ of those edges that are labeled 1 is a multicut of G iff (2) holds. It is sufficient in (2) to consider only *chordless* cycles.

$$\forall C \in \text{cycles}(G) \ \forall e \in C : \ x_e \leq \sum_{e' \in C \setminus \{e\}} x_{e'} \tag{2}$$

For every graph $G = (V, E)$, let X_G denote the set of all $x \in \{0, 1\}^E$ that satisfy (2).

Definition 4 [2, 4] For every graph $G = (V, E)$, the convex hull $\Xi_G := \operatorname{conv} X_G$ of X_G in $\mathbb{R}^E$ is called the *multicut polytope* of G.

An example is depicted in Fig. 2.

2.1 Complete Graphs

The decompositions of a complete graph $K_V := (V, \binom{V}{2})$ are precisely the partitions of the node set V (by Def. 2).

The multicuts of a complete graph K_V relate one-to-one to the equivalence relations on V:

Lemma 4 For every set V and the complete graph K_V, the map $\psi : M_{K_V} \to 2^{V \times V}$ defined by (3) is a bijection between M_{K_V} and the set of all equivalence relations on V.

$$\forall M \in M_{K_V} \ \forall v, w \in V : \quad (v, w) \in \psi(M) \ \Leftrightarrow \ \{v, w\} \notin M \tag{3}$$

The bijection between the decompositions of a graph and the multicuts of a graph (Lemma 2) specializes, for complete graphs, to the well-known bijection between the partitions of a set and the equivalence relations on the set (by Lemma 4). In this sense, decompositions and multicuts of graphs generalize partitions of sets and equivalence relations on sets.

3 Lifting of Multicuts

For every graph $G = (V, E)$, the characteristic function $x \in X_G \subseteq \{0, 1\}^E$ of a multicut $x^{-1}(1)$ of G makes explicit, for every pair $\{v, w\} \in E$ of neighboring nodes, whether v and w are in distinct components. In order to make explicit also for non-neighboring nodes, specifically, for all $\{v, w\} \in E'$ with $E \subseteq E' \subseteq \binom{V}{2}$, whether v and w are in distinct components, we define a lifting of the multicuts of G to multicuts of $G' = (V, E')$:

Definition 5 For any graphs $G = (V, E)$ and $G' = (V, E')$ with $E \subseteq E'$, the map $\lambda_{GG'} := \phi_{G'} \circ \phi_G^{-1}$ is called the *lifting* of multicuts from G to G'.

For any graphs $G = (V, E)$ and $G' = (V, E')$ with $E \subseteq E'$, let $F_{GG'} := E' \setminus E$ for brevity.

Lemma 5 For any connected graph $G = (V, E)$, any graph $G' = (V, E')$ with $E \subseteq E'$ and any $x \in \{0, 1\}^{E'}$, the set $x^{-1}(1)$ is a multicut of G' lifted from G iff

$$\forall C \in \text{cycles}(G) \forall e \in C : \ x_e \leq \sum_{e' \in C \setminus \{e\}} x_{e'} \tag{4}$$

$$\forall vw \in F_{GG'} \forall P \in vw\text{-paths}\,(G) : \ x_{vw} \leq \sum_{e \in P} x_e \tag{5}$$

$$\forall vw \in F_{GG'} \forall C \in vw\text{-cuts}\,(G) : \ 1 - x_{vw} \leq \sum_{e \in C} (1 - x_e) \tag{6}$$

For any graphs $G = (V, E)$ and $G' = (V, E')$ with $E \subseteq E'$ we denote by $X_{GG'}$ the set of all $x \in \{0, 1\}^{E'}$ that satisfy (4)–(6).

4 Lifted Multicut Polytope

Definition 6 For any connected graph $G = (V, E)$ and any graph $G' = (V, E')$ with $E \subseteq E'$, $\Xi_{GG'} := \operatorname{conv} X_{GG'}$ is called the *lifted multicut polytope* with respect to G and G'.

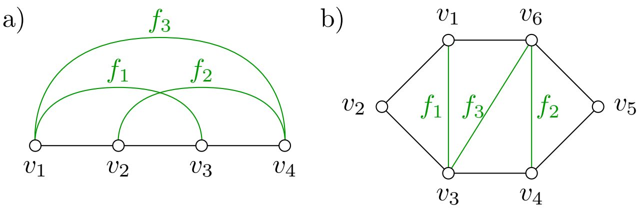

Figure 4 If two nodes $\{v, w\} = f \in F_{GG'}$ are in the same component, as indicated by $x_f = 0$, this can imply $x_{f'} = 0$ for one or more $f' \in F \setminus \{f\}$. In (a) $x_{f_3} = 0$ implies $x_{f_1} = 0$ and $x_{f_2} = 0$. In (b) $x_{f_3} = 0$ implies $x_{f_1} = 0$ or $x_{f_2} = 0$.

An example is depicted in Fig. 3. In this example, the lifted multicut polytope with respect to the graphs G and G' (Fig. 3) is a proper subset of the multicut polytope of the graph G' (Fig. 2), as not every multicut of G' is lifted from G.

4.1 Dimension

Theorem 1 *For each connected graph $G = (V, E)$ and each graph $G' = (V, E')$ with $E \subseteq E'$, $\dim \Xi_{GG'} = |E'|$.*

We prove Theorem 1 by constructing $|E'| + 1$ multicuts of G' lifted from G whose characteristic 01-vectors are affine independent. The strategy is to construct, for every $e \in E'$, an $x \in X_{GG'}$ with $x_e = 0$ and "as many ones as possible". The challenge is that edge labels are not independent. In particular, for $f \in F_{GG'}$, $x_f = 0$ can imply, for certain $f' \in F_{GG'} \setminus \{f\}$, that $x_{f'} = 0$, as illustrated in Fig. 4. This structure is made explicit below, in Def. 7 and 8 and Lemmata 6 and 7.

Definition 7 For each connected graph $G = (V, E)$ and every graph $G' = (V, E')$ such that $E \subseteq E'$, the sequence $(F_n)_{n \in \mathbb{N}}$ of subsets of $F_{GG'}$ defined below is called the *hierarchy* of $F_{GG'}$ with respect to G:
(a) $F_0 = \emptyset$
(b) For every $n \in \mathbb{N}$ and every $\{v, w\} = f \in F_{GG'}$, $\{v, w\} \in F_n$ iff there exists a vw-path in G such that, for any distinct nodes v' and w' in the path such that $\{v', w'\} \neq \{v, w\}$, either $\{v', w'\} \notin F_{GG'}$ or there exists a natural number $j < n$ such that $\{v', w'\} \in F_j$.

Lemma 6 For each connected graph $G = (V, E)$, every graph $G' = (V, E')$ with $E \subseteq E'$ and every $f \in F_{GG'}$, there exists an $n \in \mathbb{N}$ such that $f \in F_n$.

Definition 8 For each connected graph $G = (V, E)$ and every graph $G' = (V, E')$ with $E \subseteq E'$, the map $\ell : F_{GG'} \to \mathbb{N}$ such that $\forall f \in F_{GG'} \forall n \in \mathbb{N} : \ell(f) = n \Leftrightarrow f \in F_n \wedge f \notin F_{n-1}$ is called the *level function* of $F_{GG'}$.

Lemma 7 For each connected graph $G = (V, E)$, every graph $G' = (V, E')$ with $E \subseteq E'$ and for every $f \in F_{GG'}$, there exists an $x \in X_{GG'}$, called *f-feasible*, such that
(a) $x_f = 0$
(b) $x_{f'} = 1$ for all $f' \in F_{GG'} \setminus \{f\}$ with $\ell(f') \geq \ell(f)$.

4.2 Facets

4.2.1 Box

To begin with, we characterize those edges $e \in E'$ for which the inequality $x_e \leq 1$ defines a facet of the lifted multicut polytope $\Xi_{GG'}$.

Theorem 2 *For every connected graph $G = (V, E)$, every graph $G' = (V, E')$ with $E \subseteq E'$ and every $e \in E'$, the inequality $x_e \leq 1$ defines a facet of $\Xi_{GG'}$ iff there is no $vw = f \in F_{GG'}$ such that e connects a pair of v-w-cut-vertices*[1]*.*

Next, we give conditions that contribute to identifying those edges $e \in E'$ for which the inequality $0 \leq x_e$ defines a facet of the lifted multicut polytope $\Xi_{GG'}$.

Theorem 3 *For every connected graph $G = (V, E)$, every graph $G' = (V, E')$ with $E \subseteq E'$ and every $e \in E'$, the following assertions hold:*

In case $e \in E$, the inequality $0 \leq x_e$ defines a facet of $\Xi_{GG'}$ iff there is no triangle in G' containing e.

In case $uv = e \in F_{GG'}$, the inequality $0 \leq x_e$ defines a facet of $\Xi_{GG'}$ only if the following necessary conditions hold:
(a) There is no triangle in G' containing e.
(b) The distance of any pair of u-v-cut-vertices except $\{u, v\}$ is at least 3 in G and at least 2 in G'.
(c) There is no triangle of nodes s, s', t in G' such that $\{s, s'\}$ is a u-v-separating node set and t is a u-v-cut-vertex.

4.2.2 Cycles

Next, we characterize those inequalities of (4) and (5) that are facet-defining for $\Xi_{GG'}$. As stated in [2], an inequality of (2) defines a facet of the multicut polytope Ξ_G iff the cycle C is chordless. A similar statement (Theorem 4) holds true for (4) and (5) with respect to the lifted multicut polytope $\Xi_{GG'}$. For clarity, we introduce some notation:

For every cycle C of G and every $e \in C$, let

$$S_{GG'}(e, C) := \left\{ x \in X_{GG'} \middle| x_e = \sum_{e' \in C \setminus \{e\}} x_{e'} \right\} \tag{7}$$

$$\Sigma_{GG'}(e, C) := \operatorname{conv} S_{GG'}(e, C) \ . \tag{8}$$

Similarly, for each $vw = f \in F_{GG'}$ and each vw-path P in G, let

$$S_{GG'}(f, P) := \left\{ x \in X_{GG'} \middle| x_{vw} = \sum_{e \in P} x_e \right\} \tag{9}$$

$$\Sigma_{GG'}(f, P) := \operatorname{conv} S_{GG'}(f, P) \ . \tag{10}$$

Theorem 4 *For every connected graph $G = (V, E)$ and every graph $G' = (V, E')$ with $E \subseteq E'$, the following assertions hold:*

[1] For any graph $G = (V, E)$ and any $v, w \in V$, a *v-w-cut-vertex* is a node $u \in V$ that lies on every vw-path of G.

(a) For every cycle C in G and every $e \in C$, the polytope $\Sigma_{GG'}(e, C)$ is a facet of $\Xi_{GG'}$ iff C is chordless in G'.

(b) For every edge $vw = f \in F_{GG'}$ and every vw-path P in G, the polytope $\Sigma_{GG'}(f, P)$ is a facet of $\Xi_{GG'}$ iff $P \cup \{f\}$ is chordless in G'.

4.2.3 Cuts

Toward a characterization of those inequalities of (6) that define a facet of the lifted multicut polytope $\Xi_{GG'}$, we introduce additional notation:

For each connected graph $G = (V, E)$, any distinct nodes $v, w \in V$ and every $C \in vw$-cuts(G), denote by

$$G(v, C) = (V(v, C), E(v, C)) \quad (11)$$
$$G(w, C) = (V(w, C), E(w, C)) \quad (12)$$

the largest components of the graph $(V, E \setminus C)$ that contain v and w, respectively. Note that

$$V(v, C) \cap V(w, C) = \emptyset \quad (13)$$
$$\wedge \quad V(v, C) \cup V(w, C) = V \quad (14)$$

by connectedness of G and by definition of a vw-cut[2].

Let $F_{GG'}(vw, C)$ the set of those edges in $F_{GG'}$, except vw, that cross the vw-cut C of G, i.e.

$$F_{GG'}(vw, C) := \{f \in F_{GG'} \setminus \{vw\} \mid f \not\subseteq V(v, C) \wedge f \not\subseteq V(w, C)\} \ . \quad (15)$$

In addition, let

$$G'(vw, C) := (V, F_{GG'}(vw, C) \cup C) \quad (16)$$

denote the subgraph of G' that comprises all edges from $F_{GG'}(vw, C)$ and C.

Finally, let

$$S_{GG'}(vw, C) := \left\{ x \in X_{GG'} \middle| 1 - x_{vw} = \sum_{e \in C} (1 - x_e) \right\} \quad (17)$$

$$\Sigma_{GG'}(vw, C) := \operatorname{conv} S_{GG'}(vw, C) \ . \quad (18)$$

Definition 9 For each connected graph $G = (V, E)$, any distinct nodes $v, w \in V$ and every $C \in vw$-cuts(G), a component (V^*, E^*) of G is called *properly (vw, C)-connected* iff

$$v \in V^* \ \wedge \ w \in V^* \ \wedge \ |E^* \cap C| = 1 \ . \quad (19)$$

It is called *improperly (vw, C)-connected* iff

$$V^* \subseteq V(v, C) \ \vee \ V^* \subseteq V(w, C) \ . \quad (20)$$

It is called *(vw, C)-connected* iff it is properly or improperly (vw, C)-connected.

[2] For every graph $G = (V, E)$ and any distinct nodes $v, w \in V$, a vw-cut of G is a minimal (with respect to inclusion) set $C \subseteq E$ such that v and w are not connected in $(V, E \setminus C)$. Minimality means that no proper subset $C' \subset C$ is such that v and w are not connected in $(V, E \setminus C')$.

For any (vw, C)-connected component (V^*, E^*) of G, denote by

$$F_{V^*} := \{v'w' = f' \in F_{GG'}(vw, C) \mid v' \in V^* \wedge w' \in V^*\} \quad (21)$$

the set of those edges $v'w' = f' \in F_{GG'}(vw, C)$ such that (V^*, E^*) is also $(v'w', C)$-connected.

Theorem 5 *For each connected graph $G = (V, E)$, every graph $G' = (V, E')$ with $E \subseteq E'$, any $vw = f \in F_{GG'}$ and any $C \in vw$-cuts(G), $\Sigma_{GG'}(vw, C)$ is a facet of $\Xi_{GG'}$ only if the following necessary conditions hold:*

C1 For every $e \in C$, there exists a (vw, C)-connected component (V^*, E^*) of G such that $e \in E^*$.

C2 For every $\emptyset \neq F \subseteq F_{GG'}(vw, C)$, there exists an edge $e \in C$ and (vw, C)-connected components (V^*, E^*) and (V^{**}, E^{**}) of G such that $e \in E^*$ and $e \in E^{**}$ and $|F \cap F_{V^*}| \neq |F \cap F_{V^{**}}|$.

C3 For every $f' \in F_{GG'}(vw, C)$, every $\emptyset \neq F \subseteq F_{GG'}(vw, C) \setminus \{f'\}$ and every $k \in \mathbb{N}$ there exist (vw, C)-connected components (V^*, E^*) and (V^{**}, E^{**}) with $f' \in F_{V^*}$ and $f' \notin F_{V^{**}}$ such that

$$|F \cap F_{V^*}| \neq k \ \text{or} \ |F \cap F_{V^{**}}| \neq 0 \ . \quad (22)$$

C4 For each $v' \in V(v, C)$, each $w' \in V(w, C)$ and every $v'w'$-path $P = (V_P, E_P)$ in $G'(vw, C)$, there exists a properly (vw, C)-connected component (V^*, E^*) of G such that

$$(v' \notin V^* \vee \exists w'' \in V_P \cap V(w, C) : w'' \notin V^*)$$
$$\wedge \quad (w' \notin V^* \vee \exists v'' \in V_P \cap V(v, C) : v'' \notin V^*) \ . \quad (23)$$

C5 For every cycle $Y = (V_Y, E_Y)$ in $G'(vw, C)$, there exists a properly (vw, C)-connected component (V^*, E^*) of G such that

$$(\exists v' \in V_Y \cap V(v, C) : v' \notin V^*)$$
$$\wedge \quad (\exists w' \in V_Y \cap V(w, C) : w' \notin V^*) \ . \quad (24)$$

Examples in which specific conditions of Theorem 5 are violated are depicted in Fig. 8. In the special case of a cut $C = \{\bar{e}\}$ consisting of a single edge $\bar{e} \in E$ only, we show that Conditions (C1)-(C5) of Theorem 5 are also sufficient, i.e., they guarantee that $\Sigma_{GG'}(vw, C)$ is a facet of $\Xi_{GG'}$.

Theorem 6 *For each connected graph $G = (V, E)$, every graph $G' = (V, E')$ with $E \subseteq E'$, any $vw = f \in F_{GG'}$ and any 1-elementary vw-cut $C = \{\bar{e}\}$ of G, the polytope $\Sigma_{GG'}(vw, C)$ is a facet of $\Xi_{GG'}$ iff Conditions (C1)-(C5) of Theorem 5 are satisfied.*

Toward the proofs of Theorems 5 and 6, some aspects are discussed below. The proofs themselves are deferred to Appendix A. The proof of Theorem 5 exploits a relation between the elements of $S_{GG'}(vw, C)$ and the (vw, C)-connected components of G. This relation is established in Lemma 8:

Lemma 8 For every connected graph $G = (V, E)$, every graph $G' = (V, E')$ with $E \subseteq E'$, every $vw \in F_{GG'}$ and every $C \in vw\text{-cuts}(G)$, the following holds:

(a) Every $x \in S_{GG'}(vw, C)$ defines a decomposition of G into (vw, C)-connected components. That is, every maximal component of the graph $(V, \{e \in E | x_e = 0\})$ is (vw, C)-connected. At most one of these is properly (vw, C)-connected. It exists iff $x_{vw} = 0$.

(b) For every (vw, C)-connected component (V^*, E^*) of G, the $x \in \{0, 1\}^{E'}$ such that $\forall rs \in E' (x_{rs} = 0 \Leftrightarrow r \in V^* \wedge s \in V^*)$ is such that $x \in S_{GG'}(vw, C)$.

The proof of Theorem 6 makes use of the following more concrete formulation of Conditions (C1)-(C5) of Theorem 5 for the case of a 1-elementary cut $C = \{\bar{e}\}$. To this end, let $V_0(v, C) \subseteq V(v, C)$ denote the set of nodes $v' \in V(v, C)$ such that for every properly (vw, C)-connected component (V^*, E^*) it holds that $v' \in V^*$. Let $V_0(w, C)$ be defined analogously.

Lemma 9 For each connected graph $G = (V, E)$, every graph $G' = (V, E')$ with $E \subseteq E'$, any $vw = f \in F_{GG'}$ and any vw-cut $C = \{\bar{e}\}$ in G consisting of a single edge $\bar{e} \in E$, the polytope $\Sigma_{GG'}(vw, C)$ is a facet of $\Xi_{GG'}$ if (and only if) the following conditions hold:

(a) There is no edge $v'w' = f' \in F_{GG'}(vw, C)$ such that $v' \in V_0(v, C)$ and $w' \in V_0(w, C)$.

(b) There is no pair of edges $v'w' = f' \in F_{GG'}(vw, C)$ and $v''w'' = f'' \in F_{GG'}(vw, C)$ such that

$$\begin{aligned} &(v' = v'' \wedge w', w'' \in V_0(w, C)) \\ \vee \quad &(w' = w'' \wedge v', v'' \in V_0(v, C)) \ . \end{aligned} \tag{25}$$

Remark 1 In the case of a 1-elementary cut $C = \{\bar{e}\}$, the following assertions hold. Firstly, for a proof of Theorem 6, it suffices to show that Conditions (C1)-(C5) of Theorem 5 imply Conditions (a) and (b) of Lemma 9. Secondly, (C1) is trivially satisfied by connectedness of G. Thirdly, (a) $\Leftrightarrow$ (C2) $\Leftrightarrow$ (C4) and (b) $\Leftrightarrow$ (C3) $\Leftrightarrow$ (C5).

5 Minimum Cost Lifted Multicut Problem

Finally, we point to an application of lifted multicuts in the area of discrete optimization.

Definition 10 For each connected graph $G = (V, E)$, every graph $G' = (V, E')$ with $E \subseteq E'$ and every $c : E' \to \mathbb{Z}$, the instance of the *minimum cost lifted multicut problem* with respect to G, G' and c is the optimization problem

$$\min \left\{ \sum_{e \in E'} c_e x_e \;\middle|\; x \in X_{GG'} \right\} \ . \tag{26}$$

If $E' = E$, (26) specializes to the APX-hard *minimum cost multicut problem* with respect to G' and c [5, 2], a problem also known as *correlation clustering* [1, 3].

If $E' \supset E$, the minimum cost lifted multicut problem with respect to G, G' and c differs from the minimum cost multicut problem with respect to G' and c. It has a smaller feasible set $X_{GG'} \subset X_{G'}$, as we have shown in the previous sections and depicted for the smallest example in Fig. 2 and 3.

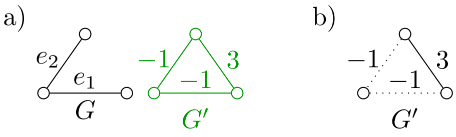

Figure 5 Depicted above in **a)** is an instance of the minimum cost lifted multicut problem (Def. 10) with respect to graphs G, G' and costs $c = (-1, -1, 3)$. Here, the cost 3 attributed to the additional edge in G' results in the edges e_1 and e_2 not being cut in the optimum $(0, 0, 0)$ which has cost 0. Depicted in **b)** is an instance of the minimum cost multicut problem with respect to the graph G' and the same cost function. Here, the cost 3 does not prevent the edges e_1 and e_2 from being cut in the optimum $(1, 1, 0)$ which has cost -2.

Unlike for the minimum cost multicut problem with respect to G' and c, for the minimum cost lifted multicut problem with respect to G, G' and c, it holds for every $vw \in E'$ that $x_{vw} = 0$ iff v and w are connected in G by a path of edges labeled 0. This property can be used to penalize by $c_{vw} > 0$ precisely those decompositions of G for which v and w are in distinct components. For nodes v and w that are not neighbors in G, such costs are sometimes referred to as *non-local* or *long-range attractive*. An example is depicted in Fig. 5.

6 Conclusion

For any graph $G = (V, E)$ and any graph $G' = (V, E')$ with $E \subseteq E'$, the decompositions of G relate one-to-one to those multicuts of G' that are lifted from G. A multicut of G' lifted from G makes explicit, for every $\{v, w\} \in E'$, whether v and w are in distinct components of G. If G is connected, the convex hull in $\mathbb{R}^{E'}$ of the characteristic functions of those multicuts of G' that are lifted from G is an $|E'|$-dimensional polytope.

Acknowledgements The examples depicted in Fig. 8h and 8j were proposed by Banafsheh Grochulla and Ashkan Mokarian, respectively.

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A Proofs

Proof of Lemma 2 First, we show that for any $\Pi \in D_G$, the image $\phi_G(\Pi)$ is a multicut of G. Assume the contrary, i.e. there exists a cycle C of G such that $|C \cap \phi_G(\Pi)| = 1$. Let $\{u, v\} = e \in C \cap \phi_G(\Pi)$, then for all $U \in \Pi$ it holds that $u \notin U$ or $v \notin U$. However, $C \setminus \{e\}$ is a sequence of edges $\{w_1, w_2\}, \dots, \{w_{k-1}, w_k\}$ such that $u = w_1, v = w_k$ and $\{w_i, w_{i+1}\} \notin \phi_G(\Pi)$ for all $1 \leq i \leq k-1$. Consequently, since Π is a partition of V, there exists some $U \in \Pi$ such that

$$w_1 \in U \wedge w_2 \in U \wedge \ldots \wedge w_{k-1} \in U \wedge w_k \in U.$$

This contradicts $w_1 = u \notin U$ or $w_k = v \notin U$.

To show injectivity of ϕ_G, let $\Pi = \{U_1, \dots, U_k\}$, $\Pi' = \{U'_1, \dots, U'_\ell\}$ be two decompositions of G. Suppose $\Pi \neq \Pi'$, then there exist some $u, v \in V, u \neq v$ and some $U_i \in \Pi$ such that $u, v \in U_i$ and for all $U'_j \in \Pi'$ it holds that $u \notin U'_j$ or $v \notin U'_j$. Thus, $\{u, v\} \in \phi_G(\Pi')$ but $\{u, v\} \notin \phi_G(\Pi)$, which means $\phi_G(\Pi) \neq \phi_G(\Pi')$.

For surjectivity, take some multicut $M \subseteq E$ of G. Let $\Pi = \{U_1, \dots, U_k\}$ collect the node sets of the connected components of the graph $(V, E \setminus M)$. Apparently, Π defines a decomposition of G. We have $\{u, v\} \in \phi_G(\Pi)$ if and only if for all $U \in \Pi$ it holds that $v \notin U$ or $u \notin U$. The latter holds true if and only if $\{u, v\}$ is not contained in any connected component of $(V, E \setminus M)$, which is equivalent to $\{v, w\} \in M$. Hence, $\phi_G(\Pi) = M$.

Proof of Lemma 4 First, we show that for any $M \in M_{K_V}$ the image $\psi(M)$ is an equivalence relation on V. Since K_V is simple, we trivially have $\{v, v\} \notin M$ for any $v \in V$. Therefore, $(v, v) \in \psi(M)$, which means $\psi(M)$ is reflexive. Symmetry of $\psi(M)$ follows from $\{u, v\} = \{v, u\}$ for all $u, v \in V$. Now, suppose $(u, v), (v, w) \in \psi(M)$. Then $\{u, v, \}, \{v, w\} \notin M$ and thus $\{u, w\} \notin M$ (otherwise $C = \{u, v, w\}$ would be a cycle contradicting the definition of a multicut). Hence, $(u, w) \in \psi(M)$, which gives transitivity of $\psi(M)$.

Let M, M' be two multicuts of K_V with $\psi(M) = \psi(M')$. Then

$$\begin{aligned} \{u, v\} \in M &\iff (u, v) \notin \psi(M) \\ &\iff (u, v) \notin \psi(M') \\ &\iff \{u, v\} \in M'. \end{aligned}$$

Hence $M = M'$, so ψ is injective.

Let R be an equivalence relation on V and define M by

$$\{u, v\} \in M \iff (u, v) \notin R.$$

Transitivity of R implies that M is a multicut of K_V. Moreover, by definition, it holds that $\psi(M) = R$. Hence, ψ is also surjective.

Proof of Lemma 5 Let $x \in \{0, 1\}^{E'}$ be such that $M' = x^{-1}(1)$ is a multicut of G' lifted from G. Every cycle in G is a cycle in G'. Moreover, for any path $vw = f \in F_{GG'}$ and any vw-path P in G, it holds that $P \cup \{f\}$ is a cycle in G'. Therefore, x satisfies all inequalities (4) and (5). Assume x violates some inequality of (6). Then there is an edge $vw \in F_{GG'}$ and some vw-cut C in G such that $x_{vw} = 0$ and for all $e \in C$ we have $x_e = 1$. Let Π be the partition of V corresponding to M' according to Lemma 2. There exists some $U \in \Pi$ with $v \in U$ and $w \in U$. However, for any $uu' = e \in C$ it holds that $u \notin U$ or $u' \notin U$. This means the subgraph $(U, E \cap \binom{U}{2})$ is not connected, because C is a vw-cut. Hence, Π is not a decomposition of G, which is a contradiction, because G is connected.

Now, suppose $x \in E'$ satisfies all inequalities (4)–(6). We show first that $M' = x^{-1}(1)$ is a multicut of G'. Assume the contrary, then there is a cycle C' in G' and some edge e' such that $C' \cap M' = \{e'\}$. For every $vw = f \in F_{GG'} \cap C' \setminus \{e'\}$ there exists a vw-path P in G such that $x_e = 0$ for all $e \in P$. Otherwise there would be some vw-cut in G violating (6), as G is connected. If we replace every such f with its associated path P in G, then the resulting cycle violates either (4) (if $e' \in E$) or (5) (if $e' \in F_{GG'}$). Thus, M' is a multicut of G'. By connectivity of G, the partition $\phi^{-1}_{G'}(M')$ is a decomposition of both G' and G. Therefore, $M = \lambda^{-1}_{GG'}(M') = \phi_G(\phi^{-1}_{G'}(M'))$ is a multicut of G and hence $M' = x^{-1}(1)$ is indeed lifted from G.

Proof of Lemma 6 Let $\{v, w\} = f \in F_{GG'}$ and let $d(v, w)$ the length of a shortest vw-path in G. Then, $d(v, w) > 1$ because $F_{GG'} \cap E = \emptyset$.

If $d(v, w) = 2$, there exists a $u \in V$ such that $\{v, u\} \in E$ and $\{u, w\} \in E$. Moreover, $\{v, u\} \notin F_{GG'}$ and $\{u, w\} \notin F_{GG'}$, as $F_{GG'} \cap E = \emptyset$. Thus, $f \in F_1$.

If $d(v, w) = m$ with $m > 2$, consider any shortest vw-path P in G. Moreover, let $F' \subseteq F_{GG'}$ such that, for any $\{v', w'\} = f' \in F_{GG'}$, $f' \in F'$ iff $v' \in P$ and $w' \in P$ and $f' \neq f$. If $F' = \emptyset$ then $f \in F_1$. Otherwise:

$$\forall \{v', w'\} \in F' : \quad d(v', w') < m$$

and thus:

$$\forall f' \in F' \; \exists n_{f'} \in \mathbb{N} : \quad f' \in F_{n_{f'}}$$

by induction (over m). Let

$$n = \max_{f' \in F'} n_{f'} \quad .$$

Then, $f \in F_{n+1}$.

Proof of Lemma 7 For any $\{v, w\} = f \in F_{GG'}$, let P be a shortest vw-path in G and let

$$F'_{GG'} := \{\{v', w'\} \in F_{GG'} \mid v' \in P \wedge w' \in P\} \tag{27}$$

$$F''_{GG'} := F_{GG'} \setminus F'_{GG'} \quad . \tag{28}$$

Moreover, let $x \in \{0, 1\}^{E'}$ with $x_P = 0$ and $x_{E \setminus P} = 1$ and $x_{F'_{GG'}} = 0$ and $x_{F''_{GG'}} = 1$. P has no chord in E, because it is a shortest path. Thus, $x \in X_{GG'}$.

Proof of Theorem 1 The all-one vector $\mathbb{1} \in \{0,1\}^{E'}$ is such that $\mathbb{1} \in X_{GG'}$.

For any $e \in E$, $x^e \in \{0,1\}^{E'}$ such that $x^e_e = 0$ and $x^e_{E\setminus\{e\}} = 1$ and $x^e_{F_{GG'}} = 1$ holds $x^e \in X_{GG'}$.

For any $f \in F_{GG'}$, any f-feasible $x^f \in \{0,1\}^{E'}$ is such that $x^f \in X_{GG'}$. Moreover, x^f can be chosen such that one shortest path connecting the two nodes in f is the only component containing more than one node.

For any $e \in E$, let $y^e \in \mathbb{R}^{E'}$ such that

$$y^e = \mathbb{1} - x^e \ . \tag{29}$$

For any $f \in F_1$, choose an f-feasible x^f and let $y^f \in \mathbb{R}^{E'}$ such that

$$y^f = \mathbb{1} - x^f - \sum_{\{e \in E \mid x^f_e = 0\}} y^e \ . \tag{30}$$

For any $n \in \mathbb{N}$ such that $n > 1$ and any $f \in F_n$, choose an f-feasible x^f and let $y^f \in \mathbb{R}^{E'}$ such that

$$y^f = \mathbb{1} - x^f \quad - \sum_{\{f' \in F_{GG'} \mid f' \neq f \wedge x^f_{f'} = 0\}} y^{f'} \quad - \sum_{\{e \in E \mid x^f_e = 0\}} y^e \ . \tag{31}$$

Here, $\ell(f') < \ell(f) \leq n$, by definition of f-feasibility. Thus, all $y^{f'}$ are well-defined by induction (over n).

Observe that $\{y^e \mid e \in E'\}$ is the unit basis in $\mathbb{R}^{E'}$. Moreover, each of its elements is a linear combination of $\{\mathbb{1} - x^e \mid e \in E'\}$ which is therefore linearly independent.

Thus, $\{\mathbb{1}\} \cup \{x^e \mid e \in E'\}$ is affine independent. It is also a subset of $X_{GG'}$ and, therefore, a subset of $\Xi_{GG'}$. Thus, $\dim \Xi_{GG'} = |E'|$.

Proof of Theorem 2 Let $S = \{x \in X_{GG'} \mid x_e = 1\}$ and put $\Sigma = \operatorname{conv} S$.

To show necessity, suppose there is some $vw = f \in F_{GG'}$ such that e connects a pair of v-w-cut-vertices. Then, for any vw-path P in G, either $e \in P$ or e is a chord of P. We claim that we have $x_f = 1$ for any $x \in S$. This gives $\dim \Sigma \leq |E'| - 2$, so the inequality $x_e \leq 1$ cannot define a facet of $\Xi_{GG'}$. If there are no vw-paths that have e as a chord, then $\{e\}$ is a vw-cut and the claim follows from the corresponding inequality of (6). Otherwise, every vw-path P that has e as a chord contains a subpath P' such that $P' \cup \{e\}$ is a cycle. Thus, for any $x \in S$, the inequalities (4) or (5) (for $e \in E$ or $e \in F_{GG'}$, respectively) imply the existence of some $e_{P'} \in P'$ such that $x_{e_{P'}} = 1$. Let $\mathcal{P}$ denote the set of all such paths P'. Apparently, the collection $\bigcup_{P' \in \mathcal{P}} \{e_{P'}\} \cup \{e\}$ is a v-w-separating set of edges. Therefore, it contains some subset C that is a vw-cut. This gives $x_f = 1$ via the inequality of (6) corresponding to C.

We turn to the proof of sufficiency. Assume there is no $vw = f \in F_{GG'}$ such that e connects a pair of v-w-cut-vertices in G. The construction of an affine independent $|E'|$-element-subset of $S \subset X_{GG'}$ is analogous to the proof of Theorem 1. The assumption guarantees for any $f \in F_{GG'}$ with $f \neq e$ the existence of an f-feasible $x \in S$ such that there is a vw-path P with $x_P = 0$. In particular, the hierarchy on $F_{GG'}$ defined by the level function ℓ remains unchanged (if $e \in F_{GG'}$, then $\ell(e) \geq \ell(f)$ for all $f \in F_{GG'}$). Hence, $\dim \Sigma = |E'| - 1$, which means Σ is a facet of $\Xi_{GG'}$.

Proof of Theorem 3 Let $S = \{x \in X_{GG'} \mid x_e = 0\}$ and put $\Sigma = \operatorname{conv} S$.

Consider the case that $e \in E$. Let $G_{[e]}$ and $G'_{[e]}$ be the graphs obtained from G and G', respectively, by contracting the edge e. The lifted multicuts $x^{-1}(1)$ for $x \in S$ correspond bijectively to the multicuts of $G'_{[e]}$ lifted from $G_{[e]}$. This implies $\dim \Sigma = \dim \Xi_{G_{[e]}G'_{[e]}}$. The claim follows from Theorem 1 and the fact that $G'_{[e]}$ has $|E'| - 1$ many edges if and only if e is not contained in any triangle in G'.

Now, suppose $uv = e \in F_{GG'}$. We show necessity of Conditions (a)-(c) by proving that if any of them is violated, then all $x \in S$ satisfy some additional, orthogonal equality and thus, $\dim \Sigma \leq |E'| - 2$.

First, assume that (a) is violated. Hence, there are edges $e', e'' \in E'$ such that $T = \{e, e', e''\}$ is a triangle in G'. Every $x \in S$ satisfies the cycle inequalities

$$x_{e'} \leq x_e + x_{e''} \tag{32}$$

$$x_{e''} \leq x_e + x_{e'} \tag{33}$$

by Lemma 3 applied to the multicut $x^{-1}(1)$ of G'. Every $x \in S$ satisfies $x_{e'} = x_{e''}$, by (32) and (33) and $x_e = 0$.

Next, assume that (b) is violated. Consider a violating pair $\{u', v'\} \neq \{u, v\}, u' \neq v'$ of u-v-cut-vertices. For every $x \in S$, there exists a uv-path P in G with $x_P = 0$, as $x_e = 0$. Any such path P has a sub-path P' from u' to v' because u' and v' are u-v-cut-vertices.

- If the distance of u' and v' in G or G' is 1, then $u'v' \in E'$. If $u'v' \in P$, then $x_{u'v'} = 0$ because $x_P = 0$. Otherwise, $x_{u'v'} = 0$ by $x_{P'} = 0$ and the cycle inequality

$$x_{u'v'} \leq \sum_{\hat{e} \in P'} x_{\hat{e}} \ . \tag{34}$$

  Thus $x_{u'v'} = 0$ for all $x \in S$.

- If the distance of u' and v' in G is 2, there is a $u'v'$-path P'' in G consisting of two distinct edges $e', e'' \in E$. We show that all $x \in S$ satisfy $x_{e'} = x_{e''}$:
  - If $e' \in P$ and $e'' \in P$ then $x_{e'} = x_{e''} = 0$ because $x_P = 0$.
  - If $e' \in P$ and $e'' \notin P$ then $x_{e'} = x_{e''} = 0$ by $x_P = 0$ and the cycle inequality

$$x_{e''} \leq \sum_{\hat{e} \in P' \setminus \{e'\}} x_{\hat{e}} \ . \tag{35}$$

  - If $e' \notin P$ and $e'' \notin P$ then $x_{e'} = x_{e''}$ by $x_P = 0$ and the cycle inequalities

$$x_{e''} \leq x_{e'} + \sum_{\hat{e} \in P'} x_{\hat{e}} \tag{36}$$

$$x_{e'} \leq x_{e''} + \sum_{\hat{e} \in P'} x_{\hat{e}} \ . \tag{37}$$

Now, assume that (c) is violated. Hence, there exists a u-v-cut-vertex t and a u-v-separating set of vertices $\{s, s'\}$ such that $\{ts, ts', ss'\}$ is a triangle in G'. We have that all $x \in S$ satisfy $x_{ss'} = x_{ts} + x_{ts'}$ as follows. At most one of x_{ts} and $x_{ts'}$ is 1, because t is a u-v-cut-vertex and $\{s, s'\}$ is u-v-separating as well. Moreover, $x_{ts} + x_{ts'} = 0$ if and only if $x_{ss'} = 0$.

Proof of Theorem 4 Note that both C and $P \cup \{f\}$ are cycles in G'. We show that, for any chordal cycle C' of G' and any $e \in C'$, the inequality

$$x_e \leq \sum_{e' \in C' \setminus \{e\}} x_{e'} \tag{38}$$

is not facet-defining for $\Xi_{G'}$. This implies that (38) cannot be facet-defining for $\Xi_{GG'}$ either, as $\Xi_{GG'} \subseteq \Xi_{G'}$ and $\dim \Xi_{GG'} = \dim \Xi_{G'}$. Hence, for facet-definingness of (4) and (5), it is necessary that C and $P \cup \{f\}$ be chordless in G'.

For this purpose, consider some cycle C' of G' with a chord $uv = e' \in E'$. We may write $C' = P_1 \cup P_2$ where P_1 and P_2 are edge-disjoint uv-paths such that $C_1 = P_1 \cup \{e'\}$ and $C_2 = P_2 \cup \{e'\}$ are cycles in G'. Let $e \in C'$, then either $e \in P_1$ or $e \in P_2$. W.l.o.g. we may assume $e \in P_1$. The inequalities

$$x_e \leq \sum_{e'' \in C_1 \setminus \{e\}} x_{e''}, \tag{39}$$

$$x_{e'} \leq \sum_{e'' \in C_2 \setminus \{e'\}} x_{e''} \tag{40}$$

are both valid for $\Xi_{G'}$. Moreover, since $e' \in C_1$, (39) and (40) imply (38) via

$$\begin{aligned} x_e &\leq \sum_{e'' \in C_1 \setminus \{e\}} x_{e''} = \sum_{e'' \in C_1 \setminus \{e, e'\}} x_{e''} + x_{e'} \\ &\leq \sum_{e'' \in C_1 \setminus \{e, e'\}} x_{e''} + \sum_{e'' \in C_2 \setminus \{e'\}} x_{e''} \\ &= \sum_{e'' \in C' \setminus \{e\}} x_{e''}. \end{aligned} \tag{41}$$

Thus, (38) is not facet-defining for $\Xi_{G'}$.

For the proof of sufficiency, suppose the cycle C of G is chordless in G' and let $e \in C$. Let Σ be a facet of $\Xi_{GG'}$ such that $\Sigma_{GG'}(e, C) \subseteq \Sigma$ and suppose it is induced by the inequality

$$\sum_{e' \in E'} a_{e'} x_{e'} \leq \alpha \tag{42}$$

with $a \in \mathbb{R}^{E'}$ and $\alpha \in \mathbb{R}$, i.e., $\Sigma = \operatorname{conv} S$, where

$$S := \left\{ x \in X_{GG'} \;\middle|\; \sum_{e' \in E'} a_{e'} x_{e'} = \alpha \right\}. \tag{43}$$

For convenience, we also define the linear space

$$L := \left\{ x \in \mathbb{R}^{E'} \;\middle|\; \sum_{e' \in E'} a_{e'} x_{e'} = \alpha \right\}. \tag{44}$$

As $0 \in S_{GG'}(e, C) \subseteq S$, we have $\alpha = 0$. We show that (42) is a scalar multiple of (4) and thus $\Sigma_{GG'}(e, C) = \Sigma$.

Let $y \in \{0, 1\}^{E'}$ be defined by

$$y_C = 0, \quad y_{E' \setminus C} = 1,$$

i.e. all edges except C are cut. Then $y \in S_{GG'}(e, C) \subseteq S$, since C is chordless.

For any $e' \in C \setminus \{e\}$, the vector $x \in \{0, 1\}^{E'}$ with

$$x_{C \setminus \{e, e'\}} = 0, \quad x_{E' \setminus C \cup \{e, e'\}} = 1$$

holds $x \in S_{GG'}(e, C) \subseteq S$. Therefore, $y - x \in L$. Thus,

$$\forall e' \in C \setminus \{e\} : \quad a_{e'} = -a_e \ . \tag{45}$$

It remains to show that $a_{e'} = 0$ for all edges $e' \in E' \setminus C$. We proceed by considering edges from E and $F_{GG'}$ separately. We consider the nodes $u, v \in V$ such that $uv = e'$. W.l.o.g., we assume that v does not belong to C. This is possible because C does not have a chord in G'.

Firstly, consider $e' \in E$ and distinguish the following cases:

(i) If e' connects two nodes not contained in C or it is the only edge connecting some node in C to v, then for $x \in \{0, 1\}^{E'}$, defined by

$$x_C = 0, \quad x_{e'} = 0, \quad x_{E' \setminus (C \cup \{e'\})} = 1,$$

it holds that $x \in S_{GG'}(e, C) \subseteq S$. Therefore, $y - x \in L$, which evaluates to $a_{e'} = 0$.

(ii) Otherwise, let $E'_{C,v} := \{\{u', v\} \in E' \mid u'$ belongs to $C\}$ denote the set of edges in E' that connect v to some node in C. By assumption, we have that $|E'_{C,v}| \geq 2$. Now, pick some direction on C and traverse C from one endpoint of e to the other endpoint of e. We may order the edges $E'_{C,v} = \{e_1, \ldots, e_k\}$ such that the endpoint of e_i appears before the endpoint of e_{i+1} in the traversal of C. We show that $a_{e_i} = 0$ for all $1 \leq i \leq k$:
For the vector $x \in \{0, 1\}^{E'}$ defined by

$$x_{e''} = \begin{cases} 0 & \text{if } e'' \in C \\ 0 & \text{if } e'' \in E'_{C,v} \\ 1 & \text{else,} \end{cases}$$

it holds that $x \in S_{GG'}(e, C) \subseteq S$. Therefore, $y - x \in L$. Thus:

$$\sum_{1 \leq i \leq k} a_{e_i} = 0. \tag{46}$$

Consider the $m \in \{1, \ldots, k\}$ such that $e' = e_m$. For any i with $1 \leq i \leq m - 1$, consider the following construction that is illustrated also in Fig. 6: Let $e'' \in C$ be some edge between the endpoints of e_i and e_{i+1}. If $e_i \in E$, define $x \in \{0, 1\}^{E'}$ via

$$\begin{gathered} x_e = x_{e''} = 1, \quad x_{C \setminus \{e, e''\}} = 0, \\ x_{e_j} = 0 \; \forall j \leq i, \quad x_{e_j} = 1 \; \forall j > i. \end{gathered}$$

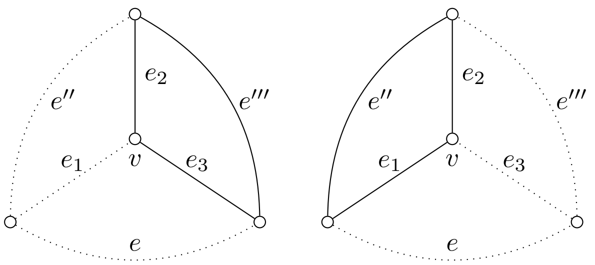

Figure 6 The figure illustrates the argument from case (ii) in the proof of Theorem 4 for the cycle $C = \{e, e'', e'''\}$. In this example, $e_3 = e'$, $e_1 \in F_{GG'}$ and $e_2 \in E$. The left multicut is chosen for $i = 1$ and the right one for $i = 2$.

If $e_i \in F_{GG'}$, define $x \in \{0,1\}^{E'}$ via

$$x_e = x_{e''} = 1, \quad x_{C \setminus \{e,e''\}} = 0,$$
$$x_{e_j} = 1 \; \forall j \le i, \quad x_{e_j} = 0 \; \forall j > i.$$

Either way, it holds that $x \in S_{GG'}(e, C) \subseteq S$ and thus, $y - x \in L$. If $e_i \in E$, this yields

$$0 = \sum_{1 \le j \le i} a_{e_j} - a_e - a_{e''} = \sum_{1 \le j \le i} a_{e_j}$$

by (45). If $e_i \in F_{GG'}$, we similarly obtain

$$0 = \sum_{i+1 \le j \le k} a_{e_j} - a_e - a_{e''} = \sum_{i+1 \le j \le k} a_{e_j}.$$

Together with (46), this yields $\sum_{1 \le j \le i} a_{e_j} = 0$ as well. Applying this argument repeatedly from $i = 1$ to $i = m-1$, we conclude that $a_{e_1} = \ldots = a_{e_{m-1}} = 0$. By reversing the order of the edges in $E'_{C,v}$, it can be shown analogously that $a_{e_k} = a_{e_{k-1}} = \ldots = a_{e_{m+1}} = 0$. Thus, by (46), $a_{e'} = a_{e_m} = 0$.

Next, consider $e' \in F_{GG'}$ and distinguish the following additional cases:

(iii) Suppose there is a uv-path P' in G that does not contain any node from C. Define $x \in \{0,1\}^{E'}$ via

$$x_{e''} = \begin{cases} 0 & \text{if } e'' \in C \\ 0 & \text{if } e'' = e' \\ 0 & \text{if } e'' \in P' \text{ or } e'' \text{ is a chord of } P' \\ 1 & \text{else.} \end{cases}$$

Then $x \in S_{GG'}(e, C) \subseteq S$ and thus $y - x \in L$. This gives

$$a_{e'} + \sum_{e'' \in P'} a_{e''} + \sum_{\substack{e'' \text{ chord} \\ \text{of } P'}} a_{e''} = 0.$$

We argue that all terms except $a_{e'}$ vanish by induction over the level function $\ell(e')$. If $\ell(e') = 1$, then P' does not have any chords from $F_{GG'}$, thus $a_{e'} = 0$, because $a_{e''} = 0$ for all $e'' \in E$ as shown previously in the cases (i) and (ii). If $\ell(e') > 1$, then for any chord $e'' \in F_{GG'}$ of P' it holds that $\ell(e'') < \ell(e')$. The induction hypothesis provides $a_{e''} = 0$ and hence we conclude $a_{e'} = 0$.

(iv) Suppose u is contained in C. Pick a shortest uv-path P' in G. We argue inductively over the length of P', which we denote by $d(P')$. If $d(P') = 1$, then P' consists of only one edge from E. This situation is in fact already covered by case (ii). If $d(P') > 1$, then we employ an argument similar to (ii) as follows. Let $F_{C,v} := \{\{u', v\} \in F_{GG'} \mid u' \text{ belongs to } C\} = \{f_1, \ldots, f_k\}$ be the set of edges $f_i \in F_{GG'}$ that connect v to some node in C. Again, assume they are ordered such that the endpoint of f_i appears before the endpoint of f_{i+1} on C in a traversal from e to itself. For the vector $x \in \{0,1\}^{E'}$ defined by

$$x_{e''} = \begin{cases} 0 & \text{if } e'' \in C \\ 0 & \text{if } e'' \in P' \text{ or } e'' \text{ is a chord of } P' \\ 0 & \text{if } e'' = u'v' \text{ where } u' \text{ belongs to } C, \\ & \qquad v' \neq v \text{ belongs to } P' \\ 0 & \text{if } e'' \in F_{C,v} \\ 1 & \text{else,} \end{cases}$$

it holds that $x \in S_{GG'}(e, C) \subseteq S$ and thus $y - x \in L$. This yields

$$\sum_{e'' \in P'} a_{e''} + \sum_{\substack{e'' \text{ chord} \\ \text{of } P'}} a_{e''} + \sum_{\substack{e''=u'v': \\ u' \text{ belongs to } C, \\ v' \neq v \text{ belongs to } P'}} a_{e''} + \sum_{e'' \in F_{C,v}} a_{e''} = 0$$

and thus

$$\sum_{1 \le i \le k} a_{f_i} = \sum_{e'' \in F_{C,v}} a_{e''} = 0, \tag{47}$$

as all other terms vanish (apply the induction hypothesis to all $u'v' \in F_{GG'}$ where u' belongs to C and $v' \neq v$ belongs to P').
Let m be the highest index such that the endpoint of f_m appears before the endpoint of P' on C. Now, for any i with $1 \le i \le m$, pick an edge $e'' \in C$ between the endpoint of f_i and the endpoint of f_{i+1} and before the endpoint of P' on C. Define $x \in \{0,1\}^{E'}$ by

$$x_g = \begin{cases} 0 & \text{if } g \in C \setminus \{e, e''\} \\ 0 & \text{if } g \in P' \text{ or } g \text{ is a chord of } P' \\ 0 & \text{if } g = u'v' \text{ where} \\ & u' \text{ appears before endpoint of } P' \text{ on } C, \\ & v' \neq v \text{ belongs to } P' \\ 0 & \text{if } g = f_j \; \forall j > i \\ 1 & \text{else.} \end{cases}$$

Then, it holds that $x \in S_{GG'}(e, C) \subseteq S$ and thus $y - x \in L$. This yields, after removing all zero terms (apply the induction hypothesis once more),

$$\sum_{i+1 \le j \le k} a_{f_j} = 0.$$

Together with (47), we obtain

$$\sum_{1 \le j \le i} a_{f_i} = 0.$$

Applying this argument repeatedly for $i = 1$ to $i = m$, we conclude $a_{f_1} = \ldots = a_{f_m} = 0$. Similarly, we obtain $a_{f_k} = a_{f_{k-1}} = \ldots = a_{f_m} = 0$, by reversing the direction of traversal of C and employing the same reasoning.

(v) Finally, suppose neither u nor v belong to the cycle C, but every uv-path in G shares at least one node with C. Let P' be such a uv-path. Define the vector $x \in \{0,1\}^{E'}$ by

$$x_{e''} = \begin{cases} 0 & \text{if } e'' \in C \\ 0 & \text{if } e'' = e' \\ 0 & \text{if } e'' \in P' \text{ or } e'' \text{ is a chord of } P' \\ 0 & \text{if } e'' = u'v' \text{ where } u' \text{ belongs to } C, \\ & \qquad v' \text{ belongs to } P' \\ 1 & \text{else.} \end{cases}$$

It holds that $x \in S_{GG'}(e, C) \subseteq S$ and thus $y - x \in L$. This gives

$$a_{e'} + \sum_{e'' \in P'} a_{e''} + \sum_{\substack{e'' \text{ chord} \\ \text{of } P'}} a_{e''} + \sum_{\substack{e''=u'v': \\ u' \text{ belongs to } C, \\ v' \text{ belongs to } P'}} a_{e''} = 0.$$

We argue inductively over the level function $\ell(e')$. If $\ell(e') = 1$, then P' does not have any chords and our consideration in cases (i)–(iv) yield that all terms except $a_{e'}$ vanish. If $\ell(e') > 1$, then we additionally employ the induction hypothesis to achieve the same result. Hence, it holds that $a_{e'} = 0$ as well.

The proof of sufficiency in the second assertion is completely analogous (replace C by $P \cup \{f\}$ and e by f). The chosen multicuts remain valid, because $e = f$ is the only edge in the cycle that is not contained in E.

Proof of Lemma 8 a) Let $x \in S_{GG'}(vw, C)$ arbitrary. Let $E_0 := \{e \in E | x_e = 0\}$ and let $G_0 := (V, E_0)$.

If $x_{vw} = 1$ then $\forall e \in C : x_e = 1$, by (17). Thus, every component of G_0 is improperly (vw, C)-connected.

If $x_{vw} = 0$ then

$$\exists e \in C (x_e = 0 \wedge \forall e' \in C \setminus \{e\} (x_{e'} = 1)) \tag{48}$$

by (17). Let (V^*, E^*) the maximal component of G_0 with

$$e \in E^* \ . \tag{49}$$

Clearly:

$$\forall e' \in C \setminus \{e\} : \ e' \notin E^* \tag{50}$$

by (48) and definition of G_0. There does not exist a $C' \in vw\text{-cuts}(G)$ with $x_{C'} = 1$, because this would imply $x_{vw} = 1$, by (6). Thus, there exists a $P \in vw\text{-paths}(G)$ with $x_P = 0$, as G is connected. Any such path P has $e \in P$, as $P \cap C \neq \emptyset$ and $C \cap E_0 = \{e\}$ and $P \subseteq E_0$. Thus:

$$v \in V^* \wedge w \in V^* \tag{51}$$

by (49). (V^*, E^*) is properly (vw, C)-connected, by (49), (50) and (51). Any other component of G_0 does not cross the cut, by (48), (49) and definition of G_0, and is thus improperly (vw, C)-connected.

b) We have

$$\forall st \in E : \ x_{st} = 0 \Leftrightarrow st \in E^* \tag{52}$$

by the following argument:

- If $st \in E^*$, then $s \in V^* \wedge t \in V^*$, as (V^*, E^*) is a graph. Thus, $x_{st} = 0$, by definition of x.
- If $st \notin E^*$ then $s \notin V^* \vee t \notin V^*$, as (V^*, E^*) is a component of G. Thus, $x_{st} = 1$, by definition of x.

Consider the decomposition of G into (V^*, E^*) and singleton components. $E_1 := \{e \in E | x_e = 1\}$ is the set of edges that straddle distinct components of this decomposition, by (52). Therefore, E_1 is a multicut of G, by Lemma 2. Thus, (4) holds, by Lemma 3.

For any $st \in F_{GG'}$ and any $P \in st\text{-paths}(G)$, distinguish two cases:

- If $P \subseteq E^*$, then $s \in V^* \wedge t \in V^*$, as (V^*, E^*) is a graph. Thus, $x_{st} = 0$, by definition of x. Moreover, $x_P = 0$, by (52). Hence, (5) evaluates to $0 = 0$.
- Otherwise, there exists an $e \in P$ such that $e \notin E^*$. Therefore, $x_e = 1$, by (52). Thus, (5) holds, as the r.h.s. is at least 1.

For any $st \in F_{GG'}$ and any $C' \in st\text{-cuts}(G)$, distinguish two cases:

- If $C' \cap E^* = \emptyset$ then $s \notin V^* \vee t \notin V^*$. Therefore, $x_{st} = 1$, by definition of x. Moreover, $x_{C'} = 1$, by (52). Thus, (6) evaluates to $0 = 0$.
- Otherwise, there exists an $e \in C'$ such that $e \in E^*$. Therefore, $x_e = 0$, by (52). Thus, (6) holds, as the r.h.s. is at least 1.

Proof of Theorem 5 Assume that C1 does not hold (as in Fig. 8a). Then, there exists an $e \in C$ such that no (vw, C)-connected component of G contains e. Thus, for all $x \in S_{GG'}(vw, C)$:

$$x_e = 1 \tag{53}$$

by Lemma 8. Now, $\dim \Sigma_{GG'}(vw, C) \le |E'| - 2$, by (17) and (53). Thus, $\Sigma_{GG'}(vw, C)$ is not a facet of $\Xi_{GG'}$, by Theorem 1.

Assume that C2 does not hold. Then, for any $e \in C$ there exists some number m such that for all (vw, C)-connected components (V^*, E^*) with $e \in E^*$ it holds that $|F \cap F_{V^*}| = m$. Thus, we can write

$$C = \bigcup_{m=0}^{|F|} C(F, m),$$

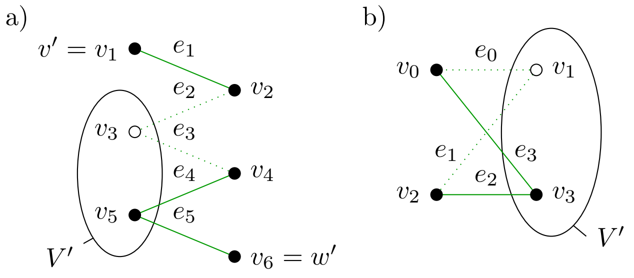

Figure 7 Depicted are the nodes (in black) and edges (in green) on a path **(a)** and on a cycle **(b)**, respectively. Nodes in the set V' are are either in V^* (filled circle) or not in V^* (open circle). Consequently, pairs of consecutive edges are either cut (dotted lines) or not cut (solid lines).

where $C(F,m) := \{e \in C \mid |F \cap F_{V^*}| = m \ \forall \ (vw,C)$-connected (V^*,E^*) with $e \in E^*\}$. It follows that for all $x \in S_{GG'}(vw,C)$ we have the equality

$$\sum_{m=0}^{|F|} m \sum_{e \in C(F,m)} (1 - x_e) = \sum_{f' \in F} (1 - x_{f'}) \tag{54}$$

by the following argument:

- If $x_e = 1$ for all $e \in C$, then $x_{f'} = 1$ for all $v'w' = f' \in F$, since C is also a $v'w'$-cut. Thus, (54) evaluates to $0 = 0$.
- Otherwise there exists precisely one edge $e \in C$ such that $x_e = 0$. Let m be such that $e \in C(F,m)$. By definition of $C(F,m)$, there are exactly m edges $f' \in F$ with $x_{f'} = 0$. Thus, (54) evaluates to $m = m$.

Assume that condition C3 does not hold. Then there exists an $f' \in F_{GG'}(vw,C)$, a set $\emptyset \neq F \subseteq F_{GG'}(vw,C)$ and some $k \in \mathbb{N}$ such that for all (vw,C) connected components (V^*,E^*) and (V^{**},E^{**}) with $f' \in F_{V^*}$ and $f' \notin F_{V^{**}}$ it holds that

$$|F \cap F_{V^*}| = k \text{ and } |F \cap F_{V^{**}}| = 0.$$

In other words, for all $x \in S_{GG'}(vw,C)$ it holds that $x_{f'} = 0$ iff there are exactly k edges $f'' \in F$ such that $x_{f''} = 0$. Similarly, it holds that $x_{f'} = 1$ iff for all $f'' \in F$ we have $x_{f''} = 1$. Therefore, all $x \in S_{GG'}(vw,C)$ satisfy the additional equality

$$k(1 - x_{f'}) = \sum_{f'' \in F} 1 - x_{f''}.$$

Assume that C4 does not hold. Then, there exist $v' \in V(v,C)$ and $w' \in V(w,C)$ and a $v'w'$-path $P = (V_P, E_P)$ in $G'(vw,C)$ such that every properly (vw,C)-connected component (V^*,E^*) of G holds:

$$(v' \in V^* \ \wedge \ V(w,C) \cap V_P \subseteq V^*) \tag{55}$$
$$\vee \quad (w' \in V^* \ \wedge \ V(v,C) \cap V_P \subseteq V^*) \ . \tag{56}$$

Let $v_1 < \cdots < v_{|V_P|}$ the linear order of the nodes V_P and let $e_1 < \cdots < e_{|E_P|}$ the linear order of the edges E_P in the $v'w'$-path P. Now, for all $x \in S_{GG'}(vw,C)$:

$$x_{vw} = \sum_{j=1}^{|E_P|} (-1)^{j+1} x_{e_j} \tag{57}$$

by the following argument: $|E_P|$ is odd, as the path P alternates between the set $V(v,C)$ where it begins and the set $V(w,C)$ where it ends. Thus,

$$\sum_{j=1}^{|E_P|} (-1)^{j+1} x_{e_j} = x_{e_1} - \sum_{j=1}^{(|E_P|-1)/2} (x_{e_{2j}} - x_{e_{2j+1}}) \ . \tag{58}$$

Distinguish two cases:

- If $x_{vw} = 1$, then $x_{E_P} = 1$, by (17) and (6). Thus, (57) evaluates to $1 = 1$, by (58).
- If $x_{vw} = 0$, the decomposition of G defined by x contains precisely one properly (vw,C)-connected component (V^*,E^*) of G, by Lemma 8. Without loss of generality, (55) holds. Otherwise, that is, if (56) holds, exchange v and w.

  Consider the nodes V_P as depicted in Fig. 7a: $v_1 = v' \in V^*$, by (55). For every even j, $v_j \in V(w,C)$, by definition of P. Thus:

  $$\forall j \in \{1, \ldots, (|E_P|+1)/2\} : \ v_{2j} \in V^* \tag{59}$$

  by (55).

  Consider the edges E_P as depicted in Fig. 7a: $e_1 = v_1v_2 \in E^*$, as $v_1 \in V^*$ and $v_2 \in V^*$ and as (V^*,E^*) is a component of G. Thus,

  $$x_{e_1} = 0 \tag{60}$$

  by Lemma 8. For every $j \in \{1, \ldots, (|E_P|-1)/2\}$, distinguish two cases:

  - If $v_{2j+1} \in V^*$, then $e_{2j} = v_{2j}v_{2j+1} \in E^*$ and $e_{2j+1} = v_{2j+1}v_{2j+2} \in E^*$, because $v_{2j} \in V^*$ and $v_{2j+2} \in V^*$, by (59), and because (V^*,E^*) is a component of G. Thus:

    $$x_{e_{2j}} = 0 \ \wedge \ x_{e_{2j+1}} = 0 \ . \tag{61}$$

  - If $v_{2j+1} \notin V^*$, then $e_{2j} = v_{2j}v_{2j+1}$ and $e_{2j+1} = v_{2j+1}v_{2j+2}$ straddle distinct components of the decomposition of G defined by x, because $v_{2j} \in V^*$ and $v_{2j+2} \in V^*$, by (59). Thus:

    $$x_{e_{2j}} = 1 \ \wedge \ x_{e_{2j+1}} = 1 \ . \tag{62}$$

  In any case:

  $$\forall j \in \{1, \ldots, (|E_P|-1)/2\} : \ x_{e_{2j}} - x_{e_{2j+1}} = 0 \ . \tag{63}$$

  Thus, (57) evaluates to $0 = 0$, by (58), (60), (63).

Assume that C5 does not hold. Then, there exists a cycle $Y = (V_Y, E_Y)$ in $G'(vw, C)$ such that every properly (vw, C)-connected component (V^*, E^*) of G holds:

$$V_Y \cap V(v, C) \subseteq V^* \tag{64}$$
$$\vee \quad V_Y \cap V(w, C) \subseteq V^* \ . \tag{65}$$

Let $v_0 < \dots < v_{|V_Y|-1}$ an order on V_Y such that $v_0 \in V(v, C)$ and, for all $j \in \{0, \dots, |E_Y| - 1\}$:

$$e_j := \{v_j, v_{j+1 \bmod |E_Y|}\} \in E_Y \ . \tag{66}$$

Now, for all $x \in S_{GG'}(vw, C)$:

$$0 = \sum_{j=0}^{|E_Y|-1} (-1)^j x_{e_j} \tag{67}$$

by the following argument: $|E_Y|$ is even, as the cycle Y alternates between the sets $V(v, C)$ and $V(w, C)$. Thus,

$$\sum_{j=0}^{|E_Y|-1} (-1)^j x_{e_j} = \sum_{j=0}^{(|E_Y|-2)/2} (x_{e_{2j}} - x_{e_{2j+1}}) \ . \tag{68}$$

Distinguish two cases:

- If $x_{vw} = 1$, then $x_{E_Y} = 1$, by (17) and (6). Thus, (67) evaluates to $0 = 0$, by (68).
- If $x_{vw} = 0$, the decomposition of G defined by x contains precisely one properly (vw, C)-connected component (V^*, E^*) of G, by Lemma 8. Without loss of generality, (64) holds. Otherwise, that is, if (65) holds, exchange v and w.

  Consider the nodes V_Y as depicted in Fig. 7b: For every even j, $v_j \in V(v, C)$, by definition of Y and the order. Thus:

  $$\forall j \in \{0, \dots, (|E_Y| - 2)/2\} : \ v_{2j} \in V^* \tag{69}$$

  by (64).

  Consider the edges E_Y as depicted in Fig. 7b: For every $j \in \{0, \dots, (|E_Y| - 2)/2\}$, distinguish two cases:

  - If $v_{2j+1} \in V^*$, then $e_{2j} = v_{2j}v_{2j+1} \in E^*$ and $e_{2j+1} = v_{2j+1}v_{2j+2 \bmod |E_Y|} \in E^*$, because $v_{2j} \in V^*$ and $v_{2j+2 \bmod |E_Y|} \in V^*$, by (69), and because (V^*, E^*) is a component of G. Thus:

    $$x_{e_{2j}} = 0 \ \wedge \ x_{e_{2j+1}} = 0 \ . \tag{70}$$

  - If $v_{2j+1} \notin V^*$, then $e_{2j} = v_{2j}v_{2j+1}$ and $e_{2j+1} = v_{2j+1}v_{2j+2 \bmod |E_Y|}$ straddle distinct components of the decomposition of G defined by x, because $v_{2j} \in V^*$ and $v_{2j+2 \bmod |E_Y|} \in V^*$, by (69). Thus:

    $$x_{e_{2j}} = 1 \ \wedge \ x_{e_{2j+1}} = 1 \ . \tag{71}$$

  In any case:

  $$\forall j \in \{0, \dots, (|E_Y| - 2)/2\} : \ x_{e_{2j}} - x_{e_{2j+1}} = 0 \ . \tag{72}$$

  Thus, (67) evaluates to $0 = 0$, by (68) and (72).

Proof of Lemma 9 We restrict ourselves to the proof of sufficiency, since necessity becomes apparent from the proof of Theorem 6.

Let $y^e \in \mathbb{R}^{E'}$ be the vector defined by $y^e_e = 1$ and $y^e_{E' \setminus \{e\}} = 0$. We show that $\Sigma_{GG'}(vw, C)$ has dimension at least $|E'| - 1$ by explicitly constructing the y^e for all $e \in E', e \neq f$ as linear combinations of elements in $S_{GG'}(vw, C)$, cf. the proof of Theorem 1. From $S_{GG'}(vw, C) = \{x \in X_{GG'} \mid x_f = x_{\bar{e}}\}$ we then conclude that $\dim \Sigma_{GG'}(vw, C) = |E'| - 1$.

The construction of all y^e for $e \in E \setminus C$ and all $y^{f'}$ for $f' \in F_{GG'} \setminus F_{GG'}(vw, C), f' \neq f$ is identical to the construction in the proof of Theorem 1. This is due to the fact that $x \in S_{GG'}(vw, C)$ for any $x \in X_{GG'}$ with $x_C = 1$ and $x_f = 1$. Therefore, we may assume that we have the vectors y^e for all $e \in E' \setminus \big(C \cup F_{GG'}(vw, C) \cup \{f\}\big)$ available for the remainder of the proof.

Now, let $v'w' = f' \in F_{GG'}(vw, C)$. Condition (a) implies that $v' \notin V_0(v, C)$ or $w' \notin V_0(w, C)$. We distinguish the following two cases.

(i) Suppose that $v' \notin V_0(v, C)$ and $w' \notin V_0(w, C)$. Since the cut $C = \{\bar{e}\}$ consists of a single edge $\bar{e} \in E$, there exists a vertex set $V^* \subseteq V$ with $v', w' \in V^*$ such that each of the sets $V^*, V^* \setminus \{v'\}, V^* \setminus \{w'\}$ and $V^* \setminus \{v', w'\}$ induces a properly (vw, C)-connected component of G. We write $x^{V^*} \in X_{GG'}$ for the vector that corresponds to the (vw, C)-connected component that is induced by V^* and put $y^{V^*} = \mathbb{1} - x^{V^*}$. The vector $y^{f'}$ may be constructed as follows. Let $z \in \mathbb{R}^{E'}$ be defined by

$$z = \big(y^{V^*} - y^{V^* \setminus \{v'\}}\big) - \big(y^{V^* \setminus \{w'\}} - y^{V^* \setminus \{v', w'\}}\big).$$

Observe that $z_{f'} = 1$ and $z_e = 0$ for all $e \in \{\bar{e}, f\} \cup F_{GG'}(vw, C) \setminus \{f'\}$. Thus, $y^{f'}$ is obtained as

$$y^{f'} = z - \sum_{e \in E' \setminus (C \cup F_{GG'}(vw, C) \cup \{f\})} z_e y^e.$$

(ii) Suppose (w.l.o.g.) that $v' \notin V_0(v, C)$ and $w' \in V_0(w, C)$. Otherwise exchange the roles of v' and w' accordingly. We may choose a vertex set $V^* \subseteq V$ with $v', w' \in V^*$ such that both V^* and $V^* \setminus \{v'\}$ induce some (vw, C)-connected component, respectively. Let $z \in \mathbb{R}^{E'}$ be defined by

$$z = y^{V^*} - y^{V^* \setminus \{v'\}}.$$

Consider any $v''w'' = f'' \in F_{GG'}(vw, C)$ with $v'' = v'$ and $w'' \neq w'$. It holds that $w'' \notin V_0(w, C)$, because otherwise f' and f'' would violate condition (b). Therefore, $y^{f''}$ can be constructed as shown in the previous case. Hence, we obtain $y^{f'}$ via

$$y^{f'} = z - \sum_{\substack{\{v''w'' = f'' \in F_{GG'}(vw, C) \mid \\ v'' = v', w'' \neq w'\}}} y^{f''} - \sum_{e \in E' \setminus (C \cup F_{GG'}(vw, C) \cup \{f\})} z_e y^e.$$

Finally, let $x \in S_{GG'}(vw, C)$ be the vector corresponding to some arbitrary properly (vw, C)-connected component of G. We obtain $y^{\bar{e}}$ via

$$y^{\bar{e}} = \mathbb{1} - x - \sum_{\substack{\{e \in E' | e \neq f, \\ e \neq \bar{e}, x_e = 0\}}} y^e .$$

Proof of Theorem 6 Theorem 5 gives necessity of the conditions C1-C5. To prove sufficiency, it suffices to show that conditions (a) and (b) presented in Lemma 9 are satisfied if C1-C5 hold. We proceed by contraposition.

Assume that condition (a) is violated. Then there exists an edge $v'w' = f' \in F_{GG'}(vw, C)$ such that $v' \in V_0(v, C)$ and $w' \in V_0(w, C)$. It follows that C2 is violated for $F = \{f'\}$, because $f' \in F_{V^*}$ for any properly (vw, C)-connected component (V^*, E^*) of G.

Assume that condition (b) is violated. Then there exist edges $v'w' = f' \in F_{GG'}(vw, C)$ and $v''w'' = f'' \in F_{GG'}(vw, C)$ such that (w.l.o.g.) $v' = v''$ and $w', w'' \in V_0(w, C)$. Otherwise exchange the roles of v', v'' and w', w'', respectively. It follows that C3 is violated for $F = \{f''\}$ and $k = 1$, because for any properly (vw, C)-connected component (V^*, E^*) of G it holds that

$$f' \in F_{V^*} \iff v' = v'' \in V^* \iff f'' \in F_{V^*} .$$

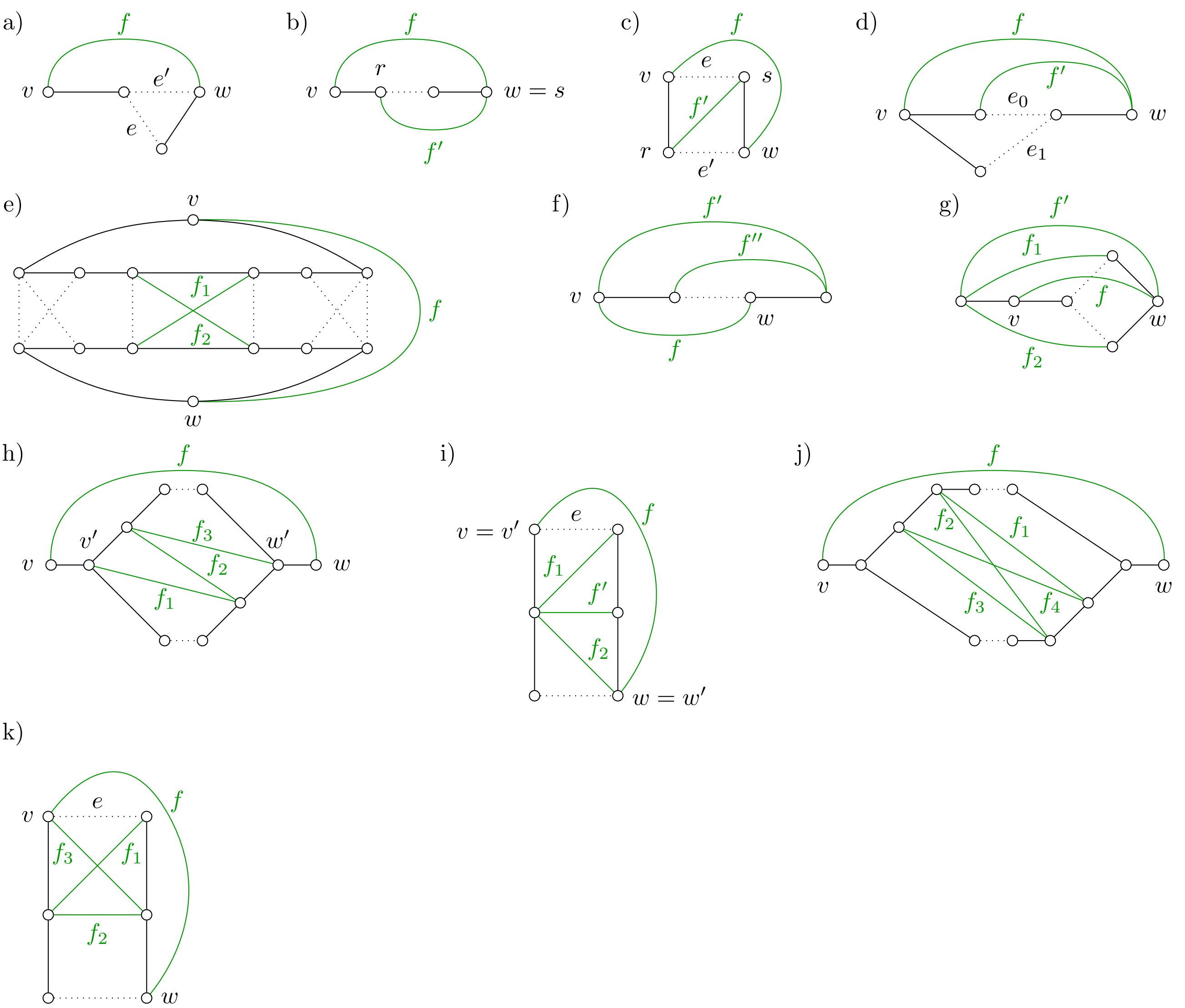

Figure 8 Depicted above are graphs $G = (V, E)$ (in black) and $G' = (V, E')$ with $E \subseteq E'$ (E' in green), distinct nodes $v, w \in V$ and a vw-cut C of G (as dotted lines). In any of the above examples, one condition of Theorem 5 is violated and thus, $\Sigma_{GG'}(vw, C)$ is not a facet of the lifted multicut polytope $\Xi_{GG'}$. **a)** Condition C1 is violated for e. **b)** Condition C2 is violated as r and s are connected in any (vw, C)-connected component. **c)** Condition C2 is violated as r and s are not connected in any (vw, C)-connected component. **d)** Condition C2 is violated. Specifically, $C(\{f'\}, 1) = \{e_0\}$ and $C(\{f'\}, 0) = \{e_1\}$ in the proof of Theorem 5. **e)** Condition C2 is violated for $F = \{f_1, f_2\}$. **f)** Condition C3 is violated. **g)** Condition C3 is violated for $F = \{f_1, f_2\}$ and $k = 1$. **h)** Condition C4 is violated for the $v'w'$-path $f_1 f_2 f_3$. **i)** Condition C4 is violated for the $v'w'$-path $e f_1 f_2$. **j)** Condition C5 is violated for the cycle $f_1 f_2 f_3 f_4$. **k)** Condition C5 is violated for the cycle $e f_1 f_2 f_3$.