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Phenomenology from the DSR-deformed relativistic symmetries of 3D quantum gravity via the relative-locality framework

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Abstract. During the last decade, there have been relevant advances in the study of quantum gravity coupled to point particles in 2+1 dimensions. In the emerging picture the momentum space of the particles is curved, the spacetime coordinates are noncommuting and the symmetries are DSR-deformed relativistic symmetries. In this article we study some phenomenological consequences of these features via the so-called "relative-locality" framework. We find that a "dual-gravity" lensing effect appears as a consequence of the relativity of spacetime locality associated to the deformed symmetries of the theory.

1. Introduction

The quantum properties of gravity are still elusive because of theoretical and experimental challenges.

From the theoretical point of view, the quantum-field-theory techniques that are very successful for phenomena described by electro-weak and strong interactions lead to severe conceptual and technical difficulties when applied to gravity. Also the brand new approaches that have been developed have reached some partial results, but they still face key open problems.

From the experimental side, the energy scale at which quantum gravity is expected to become relevant is given by the Planck mass $M_p = 10^{19} \text{ GeV}$, which is very distant from the energy scales that we can reach in current and near-future laboratory experiments. This means that in order to probe the quantum properties of gravity we need some amplification mechanisms that let the minute Planck-scale effects manifest.

The severe technical difficulties that one encounters when trying to formulate a full-comprehensive quantum theory of gravity have led to the extensive study of simpler systems that can provide theoretical and experimental insights into the quantum-gravity realm. In the last decade, there has been a significant progress in the quantum description of gravity coupled to point particles in 2+1 dimensions [1, 2, 3, 4, 5]. In particular, several approaches have led to the same class of results, which can now be considered as robust. Among these results there is the fact that in 2+1 dimensions and with vanishing cosmological constant the momentum space of a particle is described by the Lie group $SL(2, R)$, which has a curved (anti-deSitter) geometry. The departure from the flat (minkowskian) momentum space is controlled by the gravitational constant G , which provides an energy scale in 2+1 dimensions since its dimension is of the inverse of an energy. As a closely related feature one finds that the spacetime coordinates are characterized by a noncommutativity of the type $[x^\mu, x^\nu] = i\hbar\ell\epsilon^{\mu\nu\rho}x^\rho$, where by ℓ we denote the inverse



of the energy scale given by G . These features result in a deformation in ℓ of the relativistic symmetries, which are no more described by the Poincaré group, but by the quantum group $\mathcal{DSO}(2, 1)$.

It is interesting to note that the Planck-scale deformation of momentum space was suggested by an early argument by Born [6] and it is now supported by calculations in several quantum-gravity approaches in 3+1 dimensions, such as loop quantum gravity [7], group field theory [8], noncommutative geometry [9] and string theory [10].

It is important to remark that in the recent years there have been extensive studies on DSR-deformed special relativistic theories [11, 12, 13, 15, 16, 17]. These are theories in which the special relativistic symmetries are present but deformed by an energy scale which is expected to be the Planck scale, so that they are relativistic theories with two invariant constants: the speed of light c and the Planck scale. This class of theories have been shown to provide testable predictions for an important sort of quantum-gravity experiments, *i.e.* the observation of high-energy particles from astrophysical sources [18, 19, 20, 21] and, very recently, for cosmology [22, 23] (for more phenomenology coming from the Planck-scale deformation of the Poincaré symmetries we refer to [24]).

Recently a framework which enables to derive a DSR-deformed relativistic theory from a Planck-scale-curved momentum space has being developed and it goes under the name “relative-locality” framework [25, 26]. As we will see in more detail below, in the relative-locality framework momentum space is characterized by a metric and an affine connection which define respectively the (possibly-deformed) on-shell relation and the (possibly-deformed) composition law of momenta [25, 26]. So that thank to this framework one can study the kinematics of particles characterized by a curved momentum space.

This proceeding reports the results of Ref. [27], which considers the above mentioned results obtained in 3D quantum gravity with vanishing cosmological constant and extracts from them phenomenological predictions via the relative-locality framework. In particular we will make use of the classical limit of the structures appearing in 3D quantum gravity, as a first but very rich step of the analysis. In fact while in this limit we will not have spacetime noncommutativity, the curvature of momentum space and the deformation of the symmetries will be present, since, as we already noticed, in 3D the energy scale is given by the inverse of G . We will show that a “dual-gravity” lensing appears as a consequence of the relativity of spacetime locality, which, as we will explain in more detail below, is an important feature of DSR-theories.

We will adopt units such that $c = 1$ and, as already said, ℓ will denote the inverse of the Planck scale in 3D. The antisymmetric tensor $\varepsilon_{\mu\nu\rho}$ is such that $\varepsilon_{012} = -1$ and indices are raised and lowered with $\eta_{\mu\nu} = (-1, 1, 1)$.

2. Momentum space emerging from 3D quantum gravity

We here look in more detail at the momentum space and the spacetime noncommutativity that arise in 3D quantum gravity in the case of vanishing cosmological constant. As we have already said in the introduction the momentum space is described by the Lie group $SL(2, R)$. We can easily see that this momentum space is characterized by an anti-de-Sitter geometry. In fact we can write a generic element \mathbf{p} of $SL(2, R)$ as a combination of the identity matrix and of the elements of a basis of $sl(2, R)$

$$\mathbf{p} = u\mathbb{I} - 2\xi_{\mu}X^{\mu}, \quad (1)$$

where \mathbb{I} is the identity 2×2 matrix and the X^{μ} are

$$X^0 = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, X^1 = \frac{1}{2} \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, X^2 = \frac{1}{2} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad (2)$$

which is a basis of $sl(2, R)$. The requirement of having determinant equal to one ($\det \mathbf{p} = 1$) implies that the parameters (u, ξ_{μ}) satisfy

$$u^2 - \xi^{\mu}\xi_{\mu} = 1, \quad (3)$$

which is the definition of a 3 dimensional anti-de-Sitter hypersurface.

We notice that (2) implies that the X^μ satisfy (up to a dimensionful constant) the spacetime commutation relations that arise in 3D gravity

$$[X^\mu, X^\nu] = \epsilon^{\mu\nu}{}_\rho X^\rho . \tag{4}$$

For our purposes we will adopt coordinates p_μ on our momentum space¹ such that

$$\mathbf{p} = \sqrt{1 + \ell^2 p^\mu p_\mu} \mathbb{I} - 2\ell p_\mu X^\mu : \tag{5}$$

these p_μ will be the observable momenta. With this coordinatization of momentum space it has been shown [1, 2] that one finds a deformed onshell relation for the observable momenta of the type

$$\ell^{-2} \left(\arcsin \left(\sqrt{-\ell^2 p^\mu p_\mu} \right) \right)^2 = m^2 , \tag{6}$$

where m is the mass of the particle. Moreover the group structure of momentum space implies a deformation of the composition law of momenta. In fact if we multiply two group elements \mathbf{p} and \mathbf{q}

$$\begin{aligned} \mathbf{p} &= \sqrt{1 + \ell^2 p^\mu p_\mu} \mathbb{I} - 2\ell p_\mu X^\mu , \\ \mathbf{q} &= \sqrt{1 + \ell^2 q^\mu q_\mu} \mathbb{I} - 2\ell q_\mu X^\mu , \end{aligned} \tag{7}$$

we obtain a new element \mathbf{pq}

$$\mathbf{pq} = \left(\sqrt{1 + \ell^2 p^\mu p_\mu} \sqrt{1 + \ell^2 q^\nu q_\nu} + \ell^2 p^\mu q_\mu \right) \mathbb{I} - 2\ell \left(\sqrt{1 + \ell^2 q^\nu q_\nu} p_\mu + \sqrt{1 + \ell^2 p^\nu p_\nu} q_\mu - \ell \epsilon_\mu{}^{\nu\rho} p_\nu q_\rho \right) X^\mu , \tag{8}$$

where we made use of the identity

$$X^\mu X^\nu = \frac{1}{4} \eta^{\mu\nu} \mathbb{I} + \frac{1}{2} \epsilon^{\mu\nu}{}_\rho X^\rho . \tag{9}$$

From (8) we notice that the coordinates $(p \oplus q)_\mu$ of \mathbf{pq} are related to the coordinates of p_μ, q_μ of \mathbf{p}, \mathbf{q} in the following way

$$(p \oplus q)_\mu = \sqrt{1 + \ell^2 q^\nu q_\nu} p_\mu + \sqrt{1 + \ell^2 p^\nu p_\nu} q_\mu - \ell \epsilon_\mu{}^{\nu\rho} p_\nu q_\rho : \tag{10}$$

this is the deformed composition law for the momenta p_μ, q_μ .

3. Deformed relativistic kinematics

We here describe the DSR-relativistic symmetries, that arise of 3D quantum gravity, in the classical limit, which is the regime we will focus on in this article. These will be compatible with the spacetime coordinates with Poisson brackets given by

$$\{x^\mu, x^\nu\} = \ell \epsilon^{\mu\nu}{}_\rho x^\rho , \tag{11}$$

and by a momentum space with coordinates p_μ satisfying the onshell relation

$$\ell^{-2} \left(\arcsin \left(\sqrt{-\ell^2 p^\mu p_\mu} \right) \right)^2 = m^2 , \tag{12}$$

¹ An alternative coordinatization is reported in Ref. [1]). It consists in coordinates $P_\alpha, P_\beta, P_\gamma$ which are essentially Euler angles and are connected to the coordinates of our Eq. (5) by the relations $p_0 = \ell^{-1} \sin(P_\alpha \ell) \cosh(P_\beta \ell)$, $p_1 = \ell^{-1} \cos(P_\gamma \ell) \sinh(P_\beta \ell)$, $p_2 = \ell^{-1} \sin(P_\gamma \ell) \sinh(P_\beta \ell)$.

and obeying the composition law

$$(p \oplus q)_\mu = \sqrt{1 + \ell^2 q^\nu q_\nu} p_\mu + \sqrt{1 + \ell^2 p^\nu p_\nu} q_\mu - \ell \varepsilon_\mu^{\nu\rho} p_\nu q_\rho . \quad (13)$$

It is easy to see that the action of Lorentz-sector generators on momenta remains undeformed. In fact the mass shell (12) is invariant and the composition law (13) is covariant when adopting the standard

$$\{R, p_0\} = 0 , \quad \{N_1, p_0\} = p_1 , \quad \{N_2, p_0\} = p_2 , \quad (14)$$

$$\{R, p_1\} = -p_2 , \quad \{N_1, p_1\} = p_0 , \quad \{N_2, p_1\} = 0 , \quad (15)$$

$$\{R, p_2\} = p_1 , \quad \{N_1, p_2\} = 0 , \quad \{N_2, p_2\} = p_0 , \quad (16)$$

where R is the generator of rotations, while N_1 and N_2 are the generators of boosts.

Now we study the action of the symmetry generators on the spacetime coordinates. We will show that adopting undeformed Poisson brackets among translation generators

$$\{p_\mu, p_\nu\} = 0 \quad (17)$$

and undeformed action of rotations and boosts on the spacetime coordinates

$$\{R, x^0\} = 0 , \quad \{N_1, x^0\} = -x^1 , \quad \{N_2, x^0\} = -x^2 , \quad (18)$$

$$\{R, x^1\} = -x^2 , \quad \{N_1, x^1\} = -x^0 , \quad \{N_2, x^1\} = 0 , \quad (19)$$

$$\{R, x^2\} = x^1 , \quad \{N_1, x^2\} = 0 , \quad \{N_2, x^2\} = -x^0 , \quad (20)$$

the deformation will concern only the action of the translations generators on the spacetime coordinates in order to have the Jacobi identity satisfied. From now on our analysis will be mainly concentrated on results at leading order in ℓ , and we will start showing that there is a unique Poisson bracket of $\{p_\mu, x^\nu\}$, given by

$$\{p_\mu, x^\nu\} \simeq -\delta_\mu^\nu + \frac{\ell}{2} \varepsilon_\mu^{\nu\rho} p_\rho , \quad (21)$$

which satisfies all the Jacobi identities, taking into account (15)-(21). To do this we start considering the following general form for $\{p_\mu, x^\nu\}$ at leading order which gives the standard result in the limit $\ell \rightarrow 0$ limit

$$\{p_\mu, x^\nu\} = -\delta_\mu^\nu + \ell f_\mu^{\nu\rho} p_\rho . \quad (22)$$

We begin determining the parameters $f_\mu^{\nu\rho}$ by considering the Jacobi identity

$$\{p_\mu, \{x^\nu, x^\rho\}\} + \{x^\rho, \{p_\mu, x^\nu\}\} + \{x^\nu, \{x^\rho, p_\mu\}\} = 0 , \quad (23)$$

which turns out to imply

$$f_\mu^{\nu\rho} - f_\mu^{\rho\nu} = \varepsilon_\mu^{\nu\rho} . \quad (24)$$

This fixes the antisymmetric part of $f_\mu^{\nu\rho}$ in the indices $\nu\rho$ so that we can now write

$$f_\mu^{\nu\rho} = t_\mu^{\nu\rho} + \frac{1}{2} \varepsilon_\mu^{\nu\rho} , \quad (25)$$

where $t_\mu^{\nu\rho}$ must be symmetric in the indices $\nu\rho$. Then we consider the Jacobi identities for R, p_μ, x^ν , for N_1, p_μ, x^ν , and for N_2, p_μ, x^ν . These give the following equations

$$f_\mu^{\nu\rho} \varepsilon_{\gamma\rho}^\lambda + f_\rho^{\nu\lambda} \varepsilon_{\mu\gamma}^\rho + f_\mu^{\rho\lambda} \varepsilon_{\gamma\rho}^\nu = 0 . \quad (26)$$

We rewrite these equations using (25) and we find that $t_\mu^{\nu\rho}$ must satisfy

$$t_\mu^{\nu\rho} \varepsilon_{\gamma\rho}^\lambda + t_\rho^{\nu\lambda} \varepsilon_{\mu\gamma}^\rho + t_\mu^{\rho\lambda} \varepsilon_{\gamma\rho}^\nu = 0. \quad (27)$$

Now using that $t_\mu^{\nu\rho}$ is symmetric for the exchange of indices $\nu\rho$, one finds

$$t_\mu^{\nu\rho} = 0. \quad (28)$$

So looking back at (25) we realize that the choice $f_\mu^{\nu\rho} = \frac{1}{2} \varepsilon_\mu^{\nu\rho}$ is unique. As an aside we note that the following all-orders-in- ℓ formula

$$\{p_\mu, x^\nu\} = -\delta_\mu^\nu \sqrt{1 + \frac{\ell^2}{4} p^\rho p_\rho} + \frac{\ell}{2} \varepsilon_\mu^{\nu\rho} p_\rho \quad (29)$$

satisfies all the Jacobi identities exactly (no leading-order truncation), taking into account $\{x^\mu, x^\nu\} = \ell \varepsilon^{\mu\nu\rho} x^\rho$ and (15)-(21).

We note that there is another source of deformation of the translational symmetry, which appears in multiparticle systems. This deformation is due to the fact that the momentum charges are composed following the deformed addition law (13). In fact if we consider for example a system of two particles, with phase-space coordinates p_μ, x^ν and q_μ, y^ν , a translation with parameter b^ρ is generated by the total-momentum charge $(p \oplus q)_\rho$. For the particle with phase-space coordinates p_μ, x^ν this translation transformation results in

$$b^\rho \{(p \oplus q)_\rho, x^\nu\} \simeq b^\rho \{p_\rho, x^\nu\} - \ell b^\rho \varepsilon_\rho^{\sigma\gamma} q_\gamma \{p_\sigma, x^\nu\}, \quad (30)$$

which is a deformation of the usual translation due to the nonlinearity of the composition law and it can be described by saying that the symmetries have the structure of a quantum group with nontrivial coproduct for the translation generators.

We have thus characterized the DSR-symmetries that arise in 3D quantum gravity in the classical limit and we have seen that they are such that the action of Lorentz-sector generators is undeformed while the action of translation generators is deformed.

4. Kinematics within the relative-locality framework

In this section we will first show how the onshell relation and the composition law of momenta can be interpreted with the relative-locality framework in terms of the geometric properties of the momentum space. Then we will see how using this framework one is able to describe the classical relativistic kinematics of a system of particles characterized by the momentum space $SL(2, R)$ which emerges from 3D gravity.

4.1. On-shell relation from the relative-locality framework

In the relative-locality framework [25, 26], the onshell relation is encoded in the metric on momentum space. In particular it is defined by the geodesic distance $D(0, p_\mu)$ of the momentum p_μ from the origin:

$$D^2(0, p_\mu) = m^2, \quad (31)$$

where $D(0, p_\mu) = \int_0^1 g^{\mu\nu} \dot{k}_\mu(s) \dot{k}_\nu(s) ds$, $k_\mu(0) = 0$, $k_\mu(1) = p_\mu$ and the geodesic is defined by the Levi-Civita connection associated to the momentum space metric $g^{\mu\nu}$.

We now find a metric on our momentum space $SL(2, R)$ that encodes the onshell relation (42). To do this we embed $SL(2, R)$ in $\mathbb{R}^{2,2}$, which is characterized by the metric

$$ds^2 = -du^2 - (d\xi_0)^2 + (d\xi_1)^2 + (d\xi_2)^2. \quad (32)$$

So we can describe the metric on $SL(2, R)$ as the metric induced by the $\mathbb{R}^{2,2}$ metric. The embedding coordinates are $Y_I = (\sqrt{1 + \ell^2 p^\mu p_\mu}, \ell p_\mu)$ so that the pull-back of the metric (32) to our $SL(2, R)$ gives us²:

$$ds^2 = -(dp_0)^2 + (dp_1)^2 + (dp_2)^2 - \frac{\ell^2 p^\mu p^\nu dp_\mu dp_\nu}{1 + \ell^2 p^\mu p_\mu}, \tag{33}$$

which will be the metric we adopt on $SL(2, R)$.

Our next step is to compute $D(0, p_\mu)$ using the metric (33) and to show that we are able to reproduce the onshell relation (42). For our purpose it is convenient to describe the geodesics in the embedding space $\mathbb{R}^{2,2}$. And we start noticing that on the anti-deSitter hypersurface, which is the image of our embedding, a geodesic defined by the Levi-Civita connection associated to the metric (33) can be described by the Lagrangian

$$L = \dot{Y}^I \dot{Y}_I + \lambda(Y^I Y_I + 1), \tag{34}$$

where λ is a Lagrange multiplier, imposing that the motion should be on the anti-deSitter hypersurface. We note that the equations of motion coming from the Lagrangian (34) are very simple:

$$\begin{aligned} \ddot{Y}_I &= \lambda Y_I \\ Y^I Y_I + 1 &= 0. \end{aligned} \tag{35}$$

It is easy to see that when computing the geodesic distance we can distinguish three cases. In fact for the geodesic going out from the origin and arriving at a point $Y_I = (\sqrt{1 + \ell^2 p^\mu p_\mu}, \ell p_\mu)$ we have

$$\ell^2 p^\mu p_\mu = \sinh^2(D(0, Y_I)), \tag{36}$$

if $p^\mu p_\mu > 0$. If $p^\mu p_\mu = 0$ one gets $D(0, Y_I) = 0$ and finally, if $p^\mu p_\mu < 0$ one has

$$\ell^2 p^\mu p_\mu = -\sin^2(D(0, Y_I)). \tag{37}$$

Using that $D(0, Y_I) = \ell D(0, p_\mu) = \ell m$ we rewrite the previous equations as

$$\ell^2 p^\mu p_\mu = \sinh^2(\ell m) \qquad p^\mu p_\mu > 0, \tag{38}$$

$$p^\mu p_\mu = 0 \qquad p^\mu p_\mu = 0, \tag{39}$$

$$\ell^2 p^\mu p_\mu = -\sin^2(\ell m) \qquad p^\mu p_\mu < 0. \tag{40}$$

Since our mass-shell condition should be a perturbation of the special relativistic one, the relevant cases are the last two, that can be written together as

$$\ell^2 p^\mu p_\mu = -\sin^2(\ell m). \tag{41}$$

Rewriting (41) in the spirit of (31), we finally have

$$\ell^{-2} \left(\arcsin \left(\sqrt{-\ell^2 p^\mu p_\mu} \right) \right)^2 = m^2, \tag{42}$$

which reproduces the prediction (42) based on 3D-gravity results.

We note that the physical momentum space is defined by the condition

$$-\ell^{-2} \leq p^\mu p_\mu \leq 0, \tag{43}$$

where the first inequality comes from the anti-deSitter nature of our momentum space and the second one comes from the requirement for the mass-shell condition to have the right special relativistic limit.

² The metric (33) on $SL(2, R)$ can also be obtained by exploiting the fact that our momentum space is a Lie group and then defining the metric over this space using the Killing form of its Lie algebra.

4.2. Composition law of momenta from the relative-locality framework

In the relative-locality framework [25, 26], the law of composition of momenta is encoded in the affine connection on momentum space, which is in principle not related to the metric which gives the onshell relation. In the leading-order analysis we are interested in here, the momentum-space connection coefficients $\Gamma_\mu^{\nu\rho}(0)$ are the coefficients of the leading-order expansion of the composition law [25, 26]:

$$(p \oplus q)_\mu \simeq p_\mu + q_\mu - \ell \Gamma_\mu^{\nu\rho}(0) p_\nu q_\rho + \dots \quad (44)$$

Recalling that in our case we have

$$(p \oplus q)_\mu \simeq p_\mu + q_\mu - \ell \varepsilon_\mu^{\nu\rho} p_\nu q_\rho . \quad (45)$$

then the connection coefficients which encode the composition law in our case are

$$\Gamma_\mu^{\nu\rho}(0) = \varepsilon_\mu^{\nu\rho} . \quad (46)$$

Adopting this geometric description, the algebraic properties of the composition law become geometric properties, which can be analyzed through the computation of the torsion, curvature and nonmetricity tensors. Following the definitions given in Refs. [25, 26], the torsion describes the noncommutativity of the composition law and we find

$$T_\mu^{\nu\rho}(0) = -\frac{\partial}{\partial p_\nu} \frac{\partial}{\partial q_\beta} ((p \oplus q)_\mu - (q \oplus p)_\mu)_{p=q=0} = 2\ell \Gamma_\mu^{[\nu\rho]}(0) = \ell \Gamma_\mu^{\nu\rho}(0) - \Gamma_\mu^{\rho\nu}(0) = 2\ell \varepsilon_\mu^{\nu\rho} . \quad (47)$$

The curvature of the connection (evaluated in the origin) measures the nonassociativity of the composition law and in our case it vanishes

$$R_\mu^{\nu\rho\sigma}(0) = 2 \frac{\partial}{\partial p_{[\nu}} \frac{\partial}{\partial q_{\rho]}} \frac{\partial}{\partial k_\sigma} ((p \oplus q) \oplus k - p \oplus (q \oplus k))_\mu |_{p=q=k=0} = 0 , \quad (48)$$

meaning that the composition law is associative. We finally compute the value in the origin of the nonmetricity tensor, which turns out to be 0:

$$N^{\rho\mu\nu}(0) = \nabla^\rho g^{\mu\nu}(0) = g^{\mu\nu,\rho}(0) + \ell \Gamma_\sigma^{\mu\rho}(0) g^{\sigma\nu}(0) + \ell \Gamma_\sigma^{\nu\rho}(0) g^{\mu\sigma}(0) = 0.$$

4.3. Description of particles within the relative-locality framework

We here show with a simple example how one describes the kinematics of a system of interacting particles characterized by a curved momentum space in the relative-locality framework. As we will see, the key ingredients Refs. [25, 26] are the deformed onshell relation and boundary terms, enforcing momentum conservation, at endpoints of worldlines, when an interaction occurs. For our analysis we will adopt the $SL(2, R)$ momentum space and the associated deformed relativistic kinematics described previously, and we will be satisfied by considering formulas at leading order in ℓ . In particular our example is a single two-body-particle-decay process (see Fig. 1), and at leading order in ℓ with the prescription given above it is described by the action [25, 26]:

$$\begin{aligned} S = & \int_{-\infty}^{s_0} ((\delta_\nu^\mu - \frac{\ell}{2} \varepsilon^{\mu\sigma}{}_\nu k_\sigma) z^\nu \dot{k}_\mu + \mathcal{N}_k(k^\mu k_\mu - m^2)) ds + \int_{s_0}^{\infty} ((\delta_\nu^\mu - \frac{\ell}{2} \varepsilon^{\mu\sigma}{}_\nu p_\sigma) x^\nu \dot{p}_\mu + \mathcal{N}_p(p^\mu p_\mu - m'^2)) ds \\ & + \int_{s_0}^{\infty} ((\delta_\nu^\mu - \frac{\ell}{2} \varepsilon^{\mu\sigma}{}_\nu q_\sigma) y^\nu \dot{q}_\mu + \mathcal{N}_q(q^\mu q_\mu - m'^2)) ds - \xi_{[0]}^\mu \mathcal{K}_\mu^{[0]}(s_0) . \end{aligned} \quad (49)$$

Here the Lagrange multipliers $\mathcal{N}_k, \mathcal{N}_p, \mathcal{N}_q$ enforce the on-shell relation of particles, m represents the mass of the incoming particle and m' and m'' the masses of the outgoing particles. We note that the symplectic form is such that $\{x^\mu, x^\nu\} = \ell \varepsilon^{\mu\nu} p_\sigma x^\sigma$, so that (see (29)) one has

$$\{p_\mu, x^\nu\} = -\delta_\mu^\nu + \frac{\ell}{2} \varepsilon_\mu^{\nu\sigma} p_\sigma. \quad (50)$$

$\xi_{[0]}^\mu$ is a Lagrange multiplier enforcing the conservation law of momenta, and, on the basis of what we have established in the previous sections, we have

$$\mathcal{K}_\mu^{[0]}(s_0) = (k)_\mu - (p \oplus q)_\mu = k_\mu - p_\mu - q_\mu + \ell \varepsilon_\mu^{\nu\rho} p_\nu q_\rho. \quad (51)$$

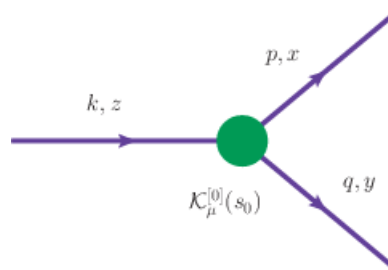


Figure 1. We here show the decay process described in Eq. (49), with one incoming and two outgoing particles.

It has been noticed [25, 26, 28] that the deformation of the relativistic symmetries produces relativity of spacetime locality³. This amounts to the fact that an interaction event which is local in the description of nearby observers, is not local in the description of distant observers. And this feature leads to the conclusion that the description of an event made by distant observers is misleading and that one should use the description of an event made by an observer which is near to it. We will see in the next section that the relativity of spacetime locality will require to use both equations of motion and symmetry transformations when deriving physical predictions from the theory.

5. Dual-gravity lensing

We here analyze a phenomenological consequence deriving from adopting the $SL(2, R)$ momentum space and associated deformed relativistic kinematics described in the previous sections. To do this we will consider the process depicted in Fig. 2, which is described within the relative-locality framework by the following action

$$\begin{aligned} \mathcal{S} = & \int_{-\infty}^{s_0} \left((\delta_\nu^\mu - \frac{\ell}{2} \varepsilon^{\mu\sigma} p_\sigma) x^\nu \dot{p}_\mu + \mathcal{N}_p (p^\mu p_\mu - m^2) \right) ds + \int_{s_0}^{+\infty} \left((\delta_\nu^\mu - \frac{\ell}{2} \varepsilon^{\mu\sigma} p_\sigma) z^\nu \dot{k}_\mu + \mathcal{N}_k (k^\mu k_\mu \right. \\ & \left. - m'^2) \right) ds + \int_{s_0}^{s_1} \left((\delta_\nu^\mu - \frac{\ell}{2} \varepsilon^{\mu\sigma} p'_\sigma) x'^\nu \dot{p}'_\mu + \mathcal{N}_{p'} (p'^\mu p'_\mu - m''^2) \right) ds + \int_{-\infty}^{s_1} \left((\delta_\nu^\mu - \frac{\ell}{2} \varepsilon^{\mu\sigma} q_\sigma) y^\nu \dot{q}_\mu \right. \\ & \left. + \mathcal{N}_q (q^\mu q_\mu - \mu^2) \right) ds + \int_{s_1}^{+\infty} \left((\delta_\nu^\mu - \frac{\ell}{2} \varepsilon^{\mu\sigma} p''_\sigma) x''^\nu \dot{p}''_\mu + \mathcal{N}_{p''} (p''^\mu p''_\mu - \mu'^2) \right) ds + \int_{s_1}^{+\infty} \left((\delta_\nu^\mu \right. \\ & \left. - \frac{\ell}{2} \varepsilon^{\mu\sigma} q'_\sigma) y'^\nu \dot{q}'_\mu + \mathcal{N}_{q'} (q'^\mu q'_\mu - \mu''^2) \right) ds - \xi_{(0)}^\mu \mathcal{K}_\mu^{[0]}(s_0) - \xi_{(1)}^\mu \mathcal{K}_\mu^{[1]}(s_1), \end{aligned} \quad (52)$$

³ The relativity of spacetime locality can also be seen as a consequence of the additional invariant (energy) scale that is present in DSR theories. In fact, in a similar way, in the passage from galilean to special relativity, the introduction of the invariant constant c , renders simultaneity an observer dependent feature.

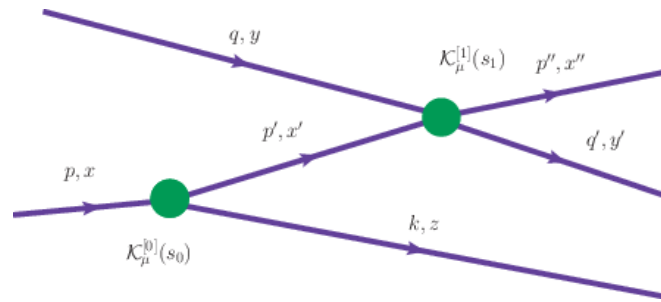


Figure 2. We here show the process described in Eq. (52), involving two causally-connected interactions.

where $(m, m', m'', \mu, \mu', \mu'')$ are the masses of the particles. The conservation laws are encoded in $\mathcal{K}_\mu^{[0]}(s_0)$ and $\mathcal{K}_\mu^{[1]}(s_1)$, for which we follow the prescription given in Ref. [26], for having causally-connected interactions preserving translational invariance, so that we have

$$\mathcal{K}_\mu^{[0]}(s_0) = (q \oplus p)_\mu - (q \oplus p' \oplus k)_\mu = p_\mu - p'_\mu - k_\mu - \ell \varepsilon_\mu^{\alpha\beta} (q_\alpha p_\beta - q_\alpha p'_\beta - q_\alpha k_\beta - p'_\alpha k_\beta) \quad (53)$$

and

$$\mathcal{K}_\mu^{[1]}(s_1) = (q \oplus p' \oplus k)_\mu - (p'' \oplus q' \oplus k)_\mu = q_\mu + p'_\mu - p''_\mu - q'_\mu - \ell \varepsilon_\mu^{\alpha\beta} (q_\alpha p'_\beta + q_\alpha k_\beta + p'_\alpha k_\beta - p''_\alpha q'_\beta - p''_\alpha k_\beta - q'_\alpha k_\beta) . \quad (54)$$

The equations of motion and the boundary conditions at endpoints of worldlines (which come by the presence of the boundary terms devoted to enforce the conservation laws for an interaction event) are derived by varying (52) keeping momenta fixed [25, 26] at $\pm\infty$. Then for the equations of motion one finds

$$\dot{p}_\mu = 0, \quad \dot{q}_\mu = 0, \quad \dot{q}'_\mu = 0, \quad \dot{k}_\mu = 0, \quad \dot{p}'_\mu = 0, \quad \dot{p}''_\mu = 0, \quad (55)$$

$$C_p = 0, \quad C_q = 0, \quad C_{q'} = 0, \quad C_k = 0, \quad C_{p'} = 0, \quad C_{p''} = 0, \quad (56)$$

$$\dot{x}^\mu = 2\mathcal{N}_p p^\mu, \quad \dot{y}^\mu = 2\mathcal{N}_q q^\mu, \quad \dot{y}'^\mu = 2\mathcal{N}_{q'} q'^\mu, \quad (57)$$

$$\dot{z}^\mu = 2\mathcal{N}_k k^\mu, \quad \dot{x}'^\mu = 2\mathcal{N}_{p'} p'^\mu, \quad \dot{x}''^\mu = 2\mathcal{N}_{p''} p''^\mu, \quad (58)$$

and the boundary conditions are

$$\begin{aligned} z^\mu(s_0) &= -\xi_{[0]}^v \frac{\delta \mathcal{K}_v^{[0]}}{\delta k_\sigma} (\delta_\sigma^\mu + \frac{\ell}{2} \varepsilon_\sigma^{\mu\rho} k_\rho) = \xi_{[0]}^\mu - \frac{\ell}{2} \varepsilon_v^{\mu\alpha} (k_\alpha - 2q_\alpha - 2p'_\alpha) \xi_{[0]}^v, \\ x^\mu(s_0) &= \xi_{(0)}^v \frac{\delta \mathcal{K}_v^{(0)}}{\delta p_\sigma} (\delta_\sigma^\mu + \frac{\ell}{2} \varepsilon_\sigma^{\mu\rho} p_\rho) = \xi_{[0]}^\mu - \frac{\ell}{2} \varepsilon_v^{\mu\alpha} (p_\alpha - 2q_\alpha) \xi_{[0]}^v, \\ x'^\mu(s_0) &= -\xi_{[0]}^v \frac{\delta \mathcal{K}_v^{[0]}}{\delta p'_\sigma} (\delta_\sigma^\mu + \frac{\ell}{2} \varepsilon_\sigma^{\mu\rho} p'_\rho) = \xi_{[0]}^\mu - \frac{\ell}{2} \varepsilon_v^{\mu\alpha} (p'_\alpha - 2k_\alpha + 2q_\alpha) \xi_{[0]}^v, \\ x'^\mu(s_1) &= \xi_{[1]}^v \frac{\delta \mathcal{K}_v^{[1]}}{\delta p'_\sigma} (\delta_\sigma^\mu + \frac{\ell}{2} \varepsilon_\sigma^{\mu\rho} p'_\rho) = \xi_{[1]}^\mu - \frac{\ell}{2} \varepsilon_v^{\mu\alpha} (p'_\alpha - 2k_\alpha + 2q_\alpha) \xi_{[1]}^v, \\ x''^\mu(s_1) &= -\xi_{[1]}^v \frac{\delta \mathcal{K}_v^{[1]}}{\delta p''_\sigma} (\delta_\sigma^\mu + \frac{\ell}{2} \varepsilon_\sigma^{\mu\rho} p''_\rho) = \xi_{[1]}^\mu - \frac{\ell}{2} \varepsilon_v^{\mu\alpha} (p''_\alpha - 2q'_\alpha - 2k_\alpha) \xi_{[1]}^v, \\ y^\mu(s_1) &= \xi_{[1]}^v \frac{\delta \mathcal{K}_v^{[1]}}{\delta q_\mu} (\delta_\sigma^\mu + \frac{\ell}{2} \varepsilon_\sigma^{\mu\rho} q_\rho) = \xi_{[1]}^\mu - \frac{\ell}{2} \varepsilon_v^{\mu\alpha} (q_\alpha - 2p'_\alpha - 2k_\alpha) \xi_{[1]}^v, \end{aligned} \quad (59)$$

$$\begin{aligned}
y^\mu(s_1) &= -\xi_{[1]}^v \frac{\delta \mathcal{K}_v^{[1]}}{\delta q'_\sigma} (\delta \sigma^\mu + \frac{\ell}{2} \varepsilon^{\mu\sigma\rho} q'_\rho) = \xi_{[1]}^\mu - \frac{\ell}{2} \varepsilon_v^{\mu\alpha} (q'_\alpha - 2k_\alpha + 2p''_\alpha) \xi_{[0]}^v, \\
\mathcal{K}_\mu^{[0]}(s_0) &= 0, \quad \mathcal{K}_\mu^{[1]}(s_1) = 0, \quad \frac{\delta \mathcal{K}_\mu^{[0]}}{\delta q_v} = 0, \quad \frac{\delta \mathcal{K}_\mu^{[1]}}{\delta k_v} = 0.
\end{aligned} \tag{60}$$

We note that when all the particles are "soft", *i.e.* they have energies small enough that the ℓ -deformation can be ignored, one finds the standard special-relativistic situation.

Here we will look in particular at what the theory predicts for the particle exchanged between the two interactions, assuming it is a massless particle. To do this we will consider an observer Alice who is placed where the interaction characterized by the conservation law $\mathcal{K}_\mu^{[0]} = 0$ occurs, and an observer Bob who is placed where the interaction with conservation law $\mathcal{K}_\mu^{[1]} = 0$ occurs. And for our purposes we will assume that Bob is at rest with respect to Alice and that Bob's and Alice's coordinatizations are related by a translation along the x^1 axis. We start analyzing what happens in the case in which all particles are soft, which gives the special-relativistic situation. In this case Alice's description of the exchanged massless particle is

$$x_{A(s)}'^1 = x_{A(s)}'^0, \quad x_{A(s)}'^2 = 0, \tag{61}$$

where we assumed that the first interaction happens in Alice's origin, and that the massless particle exchanged between the two interactions propagates along the x^1 axis in Alice's coordinatization. The index (s) introduced in (61) will be used to identify quantities related to a soft particle. Now we require that the massless particle exchanged between the two interactions is detected, through the second interaction, in Bob's origin. This implies that Bob is related to Alice by a translation of parameters $b^\mu = (b^0, b^1, 0)$ with $b^0 = b^1$, and it is easy to see that the worldline in Bob's description is

$$x_{B(s)}'^1 = x_{B(s)}'^0 + b^0 - b^1 = x_{B(s)}'^0, \quad x_{B(s)}'^2 = 0. \tag{62}$$

which has the same form of the worldline in Alice's description.

Now we analyze what happens if the interactions are not soft and then the ℓ -corrections become important. For our analysis we choose the momentum of the exchanged particle to be (in Alice's and Bob's frames):

$$\begin{aligned}
p'^1 &= p'^0 \cos \theta, \\
p'^2 &= -p'^0 \sin \theta,
\end{aligned} \tag{63}$$

where θ is a free parameter. Requiring that the particle is emitted from Alice's origin, one finds that the worldline in Alice's description is

$$\begin{aligned}
x_A'^1 &= x_A'^0 \cos \theta, \\
x_A'^2 &= -x_A'^0 \sin \theta.
\end{aligned} \tag{64}$$

Now we want to check for which values of θ the exchanged particle crosses Bob's origin. For this, due to the relativity of spacetime locality mentioned in the previous paragraph, we need to determine Bob's description of the worldline of the exchanged particle. We start noting that the translation transformations which are symmetry of the action (52) are [25, 26]:

$$\begin{aligned}
x_B'^\mu &= x_A'^\mu + b^\nu \{ (q \oplus p' \oplus k)_\nu, x_A'^\mu \} = x_A'^\mu - b^\mu + b^\nu \frac{\ell}{2} \varepsilon_\nu^{\mu\rho} (p'_\rho + 2k_\rho - 2q_\rho), \\
x_B^\mu &= x_A^\mu + b^\nu \{ (q \oplus p)_\nu, x_A^\mu \} = x_A^\mu - b^\mu + \frac{\ell}{2} \varepsilon_\nu^{\mu\rho} (p_\rho - 2q_\rho) b^\nu, \\
y_B^\mu &= y_A^\mu + b^\nu \{ (q \oplus p' \oplus k)_\nu, y_A^\mu \} = y_A^\mu - b^\mu + \frac{\ell}{2} \varepsilon_\nu^{\mu\rho} (q_\rho + 2p'_\rho + 2k_\rho) b^\nu, \\
z_B^\mu &= z_A^\mu + b^\nu \{ (q \oplus p' \oplus k)_\nu, z_A^\mu \} = z_A^\mu - b^\mu + \frac{\ell}{2} \varepsilon_\nu^{\mu\rho} (k_\rho - 2q_\rho - 2p'_\rho) b^\nu, \\
x_B''^\mu &= x_A''^\mu + b^\nu \{ (p'' \oplus q' \oplus k)_\nu, x_A''^\mu \} = x_A''^\mu - b^\mu + \frac{\ell}{2} \varepsilon_\nu^{\mu\rho} (p''_\rho + 2q'_\rho + 2k_\rho) b^\nu, \\
y_B''^\mu &= y_A''^\mu + b^\nu \{ (p'' \oplus q' \oplus k)_\nu, y_A''^\mu \} = y_A''^\mu - b^\mu + \frac{\ell}{2} \varepsilon_\nu^{\mu\rho} (q'_\rho + 2k_\rho - 2p''_\rho) b^\nu.
\end{aligned} \tag{65}$$

Then using the translation transformation for the coordinate of the exchanged particle, specializing it to the case of a translation from Alice to Bob for which the translation parameter is $b^\mu = (b, b, 0)$ and combining it with (64), one finds that Bob's description of the worldline is

$$x_B'^1 = \cos \theta x_B'^0 + \cos \theta \Delta^0 - \Delta^1, \quad (66)$$

$$x_B'^2 = -\sin \theta x_B'^0 - \sin \theta \Delta^0 - \Delta^2, \quad (67)$$

where $\Delta^\mu \equiv b^\mu - \frac{\ell}{2} \varepsilon_v^{\mu\rho} (p'_\rho + 2k_\rho - 2q_\rho) b^v$. Now we observe that the equation of motion can be easily rearranged as follows

$$\begin{aligned} x_B'^2 &= -\tan \theta x_B'^1 - \tan \theta \Delta^1 - \Delta^2, \\ x_B'^0 &= \frac{x_B'^1 - \cos \theta \Delta^0 + \Delta^1}{\cos \theta}. \end{aligned} \quad (68)$$

And finally enforcing that the particle goes through the space origin of Bob $x_B'^1 = x_B'^2 = 0$ we find that

$$\begin{aligned} \tan \theta &= -\frac{\Delta^2}{\Delta^1}, \\ x_B'^0 &= \frac{\Delta^1}{\cos \theta} - \Delta^0. \end{aligned} \quad (69)$$

From the first equation we have

$$\theta \simeq \tan \theta = -\frac{\Delta^2}{\Delta^1} = -\frac{b\ell(k_1 - q_1 + k_0 - q_0) + b\frac{\ell}{2}(p'_0 + p'_1)}{-b - b\ell(k_2 - q_2) - b\frac{\ell}{2}p'_2} \approx \frac{b\ell(k_1 - q_1 + k_0 - q_0)}{b + b\ell(k_2 - q_2)} \simeq \ell(k_1 - q_1 + k_0 - q_0), \quad (70)$$

which implies that the worldline of hard particle that reaches Bob from Alice is not parallel to the worldline of a soft particle that reaches Bob from Alice. This is the feature known as ‘‘dual-gravity lensing’’ in the relative-locality literature [29, 30]. Before commenting more on this effect, we now compute the time at which the hard particle crosses Bob's spatial origin. This can be done by using our result for θ in the second of Eqs. (69):

$$x_B'^0 = \frac{\Delta^1}{\cos \theta} - \Delta^0 \approx \Delta^1 - \Delta^0 \approx b + b(k_2 - q_2) - b - b(k_2 - q_2) = 0. \quad (71)$$

So we have that the hard particle crosses the spacetime origin of Bob.

We make two further comments on the dual-gravity lensing we have found here and that we represent pictorially in Fig. 3. First we notice that, as in the previous studies which have considered this effect, relative locality plays a key role [29, 30]. In fact if we look at the event where the soft (red) worldline crosses Bob and the event where the hard (blue) worldline crosses Bob, we notice that these two events are coincident in the coordinatization of the nearby observer Bob, while in the description of the distant observer Alice they are not coincident. So in order to characterize the dual-gravity lensing it has been crucial not to rely only on the description of one observer, but to consider the description of the observer local to the event under consideration. Finally we make a remark on the energy dependence of the dual-gravity lensing we have found here. We notice that the angle θ goes like ℓE_* , where E_* is the energy scale of the particles involved. Then considering a case with some E_* and a case with some E'_* bigger than E_* we have that the difference between the angles of the two cases is $\theta' - \theta \approx \ell E'_* - \ell E_*$. This contributes to the investigation of the energy dependence of the dual-gravity lensing. In fact, before our investigation, there have been the studies of [30] which predicts, as we do, $\theta' - \theta$ to be proportional to the difference of the energy scales involved, $E'_* - E_*$, while the study [29] predicts $\theta' - \theta$, to be proportional to the sum $E'_* + E_*$.

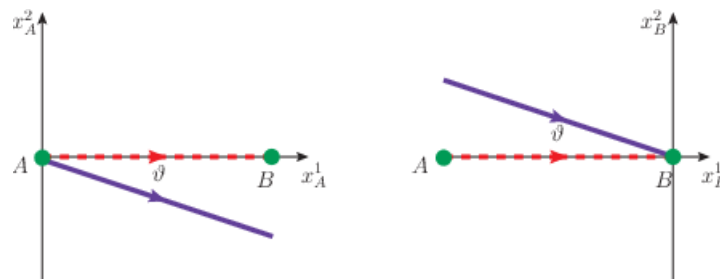


Figure 3. The worldline of a hard (high-energy) particle emitted at Alice and reaching Bob forms a non-zero angle with the worldline of a soft (low-energy) particle also emitted at Alice and reaching Bob. We note that the angle θ (whose value is determined by our Eq. (70)) is very small (for example if the energy of the particles are of the order of 1TeV , then the angle θ is of order 10^{-16}).

6. Outlook

We have seen how DSR-studies enhances the capacity of results from 3D quantum gravity, such as a curvature of momentum space, spacetime noncommutativity and deformed relativistic symmetries, to suggest interesting phenomenology. On the other hand 3D quantum gravity is helpful for DSR-research since DSR-models derived by 3D quantum gravity can be a guidance in the investigation of DSR-theories.

An interesting development of this study would be to investigate the phenomenology deriving by the quantum effects related to the deformation of the symmetries in 3D gravity. Similar studies have been performed only very recently in the context of κ -Minkowski noncommutativity [32, 33], and we feel that a good starting point to do this in the context of 3D gravity is constituted by the results in Ref. [31].

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