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A power counting theorem for a $p^{2a}\phi^4$ tensorial group field theory

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We introduce a tensorial group field theory endowed with weighted interaction terms of the form $p^{2a}\phi^4$. The model can be seen as a field theory over $d = 3, 4$ copies of $U(1)$ where formal powers of Laplacian operators, namely Δ^a, $a > 0$, act on tensorial ϕ^4-interactions producing, after Fourier transform, $p^{2a}\phi^4$ interactions. Using multi-scale analysis, we provide a power counting theorem for this type of models. A new quantity depending on the incidence matrix between vertices and faces of Feynman graphs is invoked in the degree of divergence of amplitudes. As a result, generally, the divergence degree is enhanced compared to the divergence degree of models without weighted vertices. The subleading terms in the partition function of the ϕ^4 tensorial models become, in some cases, the dominant ones in the $p^{2a}\phi^4$ models. Finally, we explore sufficient conditions on the parameter a yielding a list of potentially super-renormalizable $p^{2a}\phi^4$ models.

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For Vincent Rivasseau, in this year of his 60th birthday

"Je n'enseigne pas, je raconte." de Montaigne.

"Behandle die Menschen so, als wären sie, was sie sein sollten, und du hilfst ihnen zu werden, was sie sein können." Goethe.

I. INTRODUCTION

Tensorial Group Field Theory (TGFT) [1] is a field theoretical formulation of tensor models [2] which has also strong ties with Group Field Theory [3], placing at its heart the combinatorial duality between Feynman graphs and simplicial manifolds. It is also fair to mention that they all emanate from the study of matrix models [4], a well-known success addressing gravity in 2D, and all pertain to dedicated efforts for defining a discrete-to-continuum scenario for gravity.

Recently, the study of these tensor models has acknowledged a strong revival because of the discovery of a well behaved sub-class of these models, the colored ones [5]. Colored tensor models have been intensively studied because they possess an interesting 1/N expansion à la 't Hooft [6]. Such a 1/N expansion, in the 2D case, was a key aspect leading to the understanding of phase transitions in matrix models [4]. Concerning colored tensor models, the 1/N expansion also allows to analytically prove that there exists indeed phase transition in such models [7]. Nevertheless, one shows that the new phase does not describe geometries of the expected type but a singular branched polymeric one [8]. Because tensor models have a lot more structure, it has been suggested the existence of multiple-scaling limits to examinate other regimes of parameters which allow to incorporate a wider class of graphs (including subleading) such that the critical behavior of tensor models could be improved [9, 10]. This is clearly a greater challenge which is still under investigation. It might therefore be appropriate to ask "Is the branched polymer phase not simply the fate of tensor models?".[1] A question of this kind cannot find a short and authoritative answer, at least, within a short period of time. What is however certain is that, as a physically motivated mathematical framework, tensor models offer enough freedom to be enriched, hopefully towards reaching their initial goals. In the present work, going in that

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[1] This question was raised by Thibault Damour, during the conference "Quantum Gravity in Paris," IHES & LPT Orsay, March, 2014.

direction, we will discuss a class of tensor models which are susceptible to gain ground upon the above undesirable geometrical limit.

Supplementing the statistical point of view, colored tensor models find a field theory formulation that one calls TGFT.[2] The idea here is to understand in a field theory language the Renormalization Group analysis of tensor models and their ensuing flow of coupling constants. Several renormalizable models have been analyzed perturbatively and proved renormalizable and their Renormalization Group flow has been studied [11]-[23]. More recently, as a prominent way for addressing the issue of critical phenomena, the Functional Renormalization Group approach has been investigated for tensorial models [23]. The RG flow of the simplest rank 3 TGFT over the torus $U(1)^3$ is determined as a non-autonomous system of β-function. The explicit appearance of the IR cut-off in the RG equations results from the existence of a hidden scale which is the radius of the manifold and the nonlocal feature of the interactions. At large or small radius limit, one is able to set up a proper notion of dimensionless couplings and then infers the existence of fixed points in the flow. The presence of a non-Gaussian fixed point supports the existence of a phase transition again. As in the usual scalar field theory, one suggests that these phases must be related to a symmetric phase and broken (or condensate) phase. Such a program must be pursued to clarify this issue.

In order to make clear the objectives of this work, let us detail some aspects of these TGFTs. The expansion of the partition function of the colored tensor models admits a 1/N expansion in a new parameter called the degree of a colored tensor graph [6]. The degree is a combinatorial quantity replacing the genus, from 2D to arbitrary d dimensional colored simplicial complex. In that expansion, the leading terms are special class of graphs called "melons" [7], or "melonic graphs," which are dual to peculiar sphere triangulations in any dimension d. These graphs satisfy a kind of planarity condition (similar to planarity for ribbon graphs of matrix models) and are spanned by a recursive rule by inserting two-point graphs onto lines of two-point graphs. Among the ideas to escape from the branch polymer phase in tensor models, one proposal is to enhance the contribution of graphs of the non-melonic type. Such a program has been recently investigated in the statistical framework [24, 25]. Our following scheme inspires from these but, it is formulated in a radically different approach which is the field theory one.

Given a rank d complex tensor $\phi_{\mathbf{p}}$, where $\mathbf{p} = (p_i)$ is a multi-index. The known renormalizable TGFT actions are defined using [14]:

- a kinetic term of the form $\sum_{\mathbf{p}} \bar{\phi}_{\mathbf{p}}(|\mathbf{p}|^{2b} + \mu)\phi_{\mathbf{p}}$, where b is a positive parameter and μ a mass coupling, $|\mathbf{p}|^{2b} = \sum_i |p_i|^{2b}$; this choice is reminiscent of a sum of powers of eigenvalues of a Laplacian, if one looks at $\phi_{\mathbf{p}}$ as the Fourier components of a field;

- nonlocal interactions of the melonic type obtained by convoluting an even number of tensors that we write $\mathrm{Tr}_{2n}(\phi^{2n})$.

In this work, we restrict our attention to the ranks $d = 3, 4$, and in addition to the above parts, we introduce another vertex operator with a momentum weight obtained by convoluting $2n$ tensors, using the same pattern of contraction of melonic vertices but modified with an index dependent kernel. We write the new vertex in the suggestive form $\mathrm{Tr}_4(p^{2a}\phi^4)$. In direct space, for the value $a = 1$, such terms have clear-cut meaning: these are obtained by letting act Laplacians on $2n$ fields convoluted. For arbitrary a, these vertices can be certainly written in the momentum space. In a pictorial way, we propose to put a weight, p^{2a}, to the melonic vertex (attached to some strands of the vertex) in a way to generate, at the quantum and perturbative level, non-melonic graphs with enhanced power counting.

By multi-scale analysis [26], we identify a power counting theorem for the $p^{2a}\phi^4$-model. As expected, non-melonic contributions are enhanced and these can be even more divergent than melonic ones. To the degree of divergence of any graph obtained in previous analysis [14], we must now add a new quantity. This combinatorial object is obtained from the optimization of the integrations of internal momenta (equivalent to a momentum routine in ordinary QFT) which involves a new incidence matrix between vertices and faces or strands of the graph. We recall that a power counting theorem is essential in understanding a renormalization analysis. We do not perform this procedure here but simply undertake the first steps. Scrutinizing the degree of divergence of any graph in ranks $d = 3$ and 4, and at maximal valence of the vertex ϕ^4, a list of sufficient conditions on the parameter a strongly suggests the existence of super-renormalizable $p^{2a}\phi^4$ models. The formalism easily extends in any rank d with not much effort.

The paper is organized in the following way: the next section reviews the construction of tensorial models and presents the new vertices. Devoted to the perturbative analysis at all orders, Section III starts by the quantum model and its amplitudes, sets up the multi-scale analysis, leads to our main result, namely the power counting Theorem 1 and finally discusses potentially renormalizable models. We give a conclusion of this work in Section IV and the paper closes with an appendix providing a worked out example of the optimization procedure on which relies the multi-scale analysis.

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[2] We must stress that, in this work, we will interchangeably use TGFT and tensorial models.

II. MODELS

Consider a rank d complex tensor $\phi_{\mathbf{P}}$, with $\mathbf{P} = (p_1, p_2, \dots, p_d)$ a multi-index, and denote $\bar\phi_{\mathbf{P}}$ its complex conjugate. The indices p_k can be chosen of several types. For simplicity, in this work, we consider that these are integers: $p_k \in \mathbb{Z}$. This choice can be certainly motivated from a field theory point of view: introducing a complex function $\phi : U(1)^d \to \mathbb{C}$, $\phi_{\mathbf{P}}$ define nothing but the Fourier components of such a field. Hence, the development as found hereafter could be translated in a field theory language on a compact space like the d-torus (up to subtleties that we will give precisions on) and this is also the reason why, in most of our study, we regard $\mathbf{P}$ as a momentum index.

An action S of a tensorial model is built by convoluting several copies of $\phi_{\mathbf{P}}$ and $\bar\phi_{\mathbf{P}}$ using kernels. S is of the general form

$$
\begin{aligned}
&S[\bar\phi, \phi] = \mathrm{Tr}_2(\bar\phi \cdot K \cdot \phi) + \mu\, \mathrm{Tr}_2(\phi^2) + S^{\mathrm{int}}[\bar\phi, \phi]\,, \\
&\mathrm{Tr}_2(\bar\phi \cdot K \cdot \phi) = \sum_{\mathbf{P},\mathbf{P}'} \bar\phi_{\mathbf{P}}\, K(\mathbf{P};\mathbf{P}')\, \phi_{\mathbf{P}'}\,, \qquad \mathrm{Tr}_2(\phi^2) = \sum_{\mathbf{P}} \bar\phi_{\mathbf{P}} \phi_{\mathbf{P}}\,, \\
&S^{\mathrm{int}}[\bar\phi, \phi] = \sum_{n_b} \lambda_{n_b} \mathrm{Tr}_{n_b}(\bar\phi^{n_b} \cdot \mathcal{V}_{n_b} \cdot \phi^{n_b})\,,
\end{aligned}
\tag{1}
$$

where Tr_{n_b} are "generalized traces" over tensors, K and $\mathcal{V}_{n_b}$ kernels to be specified, μ (mass) and λ_{n_b} are coupling constants. Putting $\mathcal{V}_{n_b}$ to an identity kernel, it must be pointed out that Tr_{n_b} are convolutions of tensors which generate unitary invariants [27–29].

Let us be more specific at this stage and characterize the kinetic term involving K and the mass-like term $\mathrm{Tr}_2(\phi^2)$. We are interested in an action in rank d with a kinetic term determined by

$$
K(\{p_i\};\{p'_i\}) = \boldsymbol{\delta}_{p_i,p'_i}\Big(\sum_{i=1}^d p_i^2\Big)\,, \qquad \boldsymbol{\delta}_{p_i,p'_i} := \prod_{i=1}^d \delta_{p_i,p'_i}\,, \qquad \mathrm{Tr}_2(\phi^2) = \sum_{p_i\in\mathbb{Z}} |\phi_{12\dots d}|^2\,,
\tag{2}
$$

where we use a compact notation $\phi_{12\dots d} := \phi_{p_1,p_2,\dots,p_d}$. The kernel K is the sum of squared eigenvalues of d Laplacian operators over the d copies of $U(1)$. Mostly, we will restrict our attention to the rank $d = 3$ and $d = 4$ cases.

Focusing on the interaction part, given a parameter $a \in (0,\infty)$, the interaction terms are chosen such that

$$
\begin{aligned}
&S^{\mathrm{int}}[\bar\phi,\phi] = \frac{\lambda}{2}\,\mathrm{Tr}_4(\phi^4) + \frac{\eta}{2}\,\mathrm{Tr}_4(p^{2a}\,\phi^4)\,, \\
&\mathrm{Tr}_4(\phi^4) := \mathrm{Tr}_{4;1}(\phi^4) + \mathrm{Sym}(1 \to 2 \to \cdots \to d)\,, \\
&\mathrm{Tr}_4(p^{2a}\phi^4) := \mathrm{Tr}_{4;1}(p_1^{2a}\,\phi^4) + \mathrm{Sym}(1 \to 2 \to \cdots \to d)\,,
\end{aligned}
\tag{3}
$$

where λ and η are coupling constants and where the symbols $\mathrm{Tr}_{4;1}(\phi^4)$, $\mathrm{Tr}_{4;1}(p_1^{2a}\,\phi^4)$ and Sym must be now given a sense. In rank $d = 3$, the expression of the tensor traces in S^{int} have the explicit form

$$
\begin{aligned}
&\mathrm{Tr}_{4;1}(\phi^4) = \sum_{p_i,p'_i\in\mathbb{Z}} \phi_{123}\,\bar\phi_{1'23}\,\phi_{1'2'3'}\,\bar\phi_{12'3'}\,, \\
&\mathrm{Tr}_{4;1}(p_1^2\,\phi^4) = \sum_{p_i,p'_i\in\mathbb{Z}} \Big(p_1^{2a} + {p'}_1^{2a}\Big)\phi_{123}\,\bar\phi_{1'23}\,\phi_{1'2'3'}\,\bar\phi_{12'3'}\,,
\end{aligned}
\tag{4}
$$

and, in rank $d = 4$, these contractions easily generalize using the same cyclic pattern and by introducing another index (a graphical representation associated with these vertices will follow after introducing the quantum model and its Feynman rules, in the next section). The symbol Sym in (3) manifests the fact that we must add to the above terms colored symmetric ones. In the end, we shall write, in understandable and more compact notations,

$$
S^{\mathrm{int}}[\bar\phi,\phi] = \frac{1}{2}\,\mathrm{Tr}_4[(\lambda + \eta p^{2a})\,\phi^4]\,.
\tag{5}
$$

Note that the interaction kernels can be interpreted as well in the $U(1)$ formulation: we add formal operators $\Delta_s^a + \Delta_{s'}^a$ acting on the ϕ^4 terms. The formal character of these operators is in the sense that, for arbitrary a, we regard these only through the momentum space. Hence, the coupling η plays a role similar to a wave function renormalization now associated with the interaction. This means that, when performing a renormalization procedure, subleading contributions to the vertex operators must be investigated. As one might realize from this point, the present theory space becomes far richer than the usual unitary invariant potential ansatz where the vertices of the model do not have any momentum weight. A last thing must be noticed: in both class of models, with weighted vertices or not, the interaction terms are nonlocal and the new combinatorics they generate provide them with a genuinely different renormalization analysis than ordinary quantum field theories.

III. AMPLITUDES AND MULTISCALE ANALYSIS

The quantum model associated with the above action (1) with kinetic term (2) and interaction (3) is determined by the partition function

$$Z = \int d\nu_C(\bar{\phi}, \phi)\, e^{-S^{\text{int}}[\bar{\phi},\phi]}\,, \tag{6}$$

where $d\nu_C(\bar{\phi}, \phi)$ is a Gaussian field measure with covariance given by the inverse of the kinetic term and this is

$$C(\{p_i\};\{p'_i\}) = \tilde{C}(\{p_i\})\,\boldsymbol{\delta}_{p_i,p'_i}\,, \qquad \tilde{C}(\{p_s\}) = \frac{1}{\sum_s p_s^2 + \mu}\,. \tag{7}$$

At the graphical level the propagator is represented by a collection of d segments called strands (see Figure 1). Dealing with the interaction, we have the following vertex kernel amplitude associated with (4):

$$V_{4;s}(\{p_i\};\{p'_i\};\{p''_i\};\{p'''_i\}) = \frac{1}{2}\Big(\lambda + \eta(p_s^{2a} + {p'}_s^{2a})\Big)\delta_{4;s}(\{p_i\};\{p'_i\};\{p''_i\};\{p'''_i\})\,, \quad s = 1, 2, \dots, d\,, \tag{8}$$

where the operator $\delta_{4;s}(-)$ is a product of Kronecker deltas identifying the different momenta according to the pattern given by the vertex $\mathrm{Tr}_{4;s}(\phi^4)$. We remark that $V_{4;s}$ has a color index. Graphically, the interaction is represented by stranded vertex (see $V_{4;s=1}$ or equivalently $\frac{1}{2}\mathrm{Tr}_{4;1}[(\lambda + \eta p_1^{2a})\phi^4]$ in rank $d = 3$ and 4, in Figure 1). Strictly speaking, one should introduce two types of vertices, one for each coupling, λ and η. Such a requirement will be mandatory when the renormalization analysis will be carried out. Nevertheless, in the following, we are interested in a power counting theorem which can be achieved without any further distinction.

FIG. 1. Rank $d = 3$ and 4 propagator (stranded) lines and vertices $\frac{1}{2}\mathrm{Tr}_{4;1}[(\lambda + \eta p_1^{2a})\phi^4]$.

As emphasized before, the vertex operator has now a weight. A way to circumvent this feature (we may note that this is not necessarily a negative one) is to bring back to the propagator the total momentum dependence of the vertex. This amounts, in our present case, to redefine a propagator kernel of the form

$$C'_s(\{p_i\};\{p'_i\}) = \frac{\sqrt{\frac{1}{2}(\lambda + \eta(p_s^{2a} + {p'}_s^{2a}))}}{\sum_{s'} p_{s'}^2 + \mu}\boldsymbol{\delta}_{p_i,p'_i}\,, \qquad s = 1, 2, \dots, d\,, \tag{9}$$

which looks almost unusual and has now a manifest color index. By inspecting directly (9), we can infer that, *a priori*, the large momenta analysis of the amplitudes will be governed by the presence of η-terms. We will give a precise statement about this in the following. This being mentioned, we will use the direct approach, i.e. considering a symmetric propagator and colored vertices which altogether make still a tractable model in the present situation.

Amplitudes. The graph amplitudes of the model have the following structure: given a connected graph $\mathcal{G}$ with set $\mathcal{V}$ of vertices (with $V = |\mathcal{V}|$) and set $\mathcal{L}$ of propagator lines (with $L = |\mathcal{L}|$), we write (in loose notations)

$$A_{\mathcal{G}} = \sum_{p_{v;s}} \prod_{l\in\mathcal{L}} C_l(\{p_{v(l)}\};\{p'_{v'(l)}\}) \prod_{v\in\mathcal{V}} (-V_{4;v}(\{p_{v;s}\}))\,. \tag{10}$$

This expression is similar to the ordinary field theory amplitudes where propagators C_l have line indices l and momentum arguments $p_{v(l)}$ convoluted using vertex constraints $V_{4;v}$. The sum in (10) is performed over internal momenta $p_{v;s}$ appearing in the vertex operators $V_{4;v}$. Noting that the propagator and vertex operators are weighted discrete delta's, there is conservation of momenta along strands of the graph. In contrast with usual tensorial models where the amplitude factorizes along connected strands called also faces of the graph [11], the amplitude (10) here cannot be factorized but is a sum of strand-factorized terms. This new feature is described in the next paragraph.

Following strands in the tensor graphs, one defines one dimensional connected objects called "faces" (see Figure 2). One distinguishes two types of faces: open ones, homeomorphic to lines, and closed or internal ones, homeomorphic to circles which are sometimes called loops. The sets of external and internal faces are denoted by $\mathcal{F}_{\rm ext}$ and $\mathcal{F}_{\rm int}$, respectively. A face in a graph has a colored index $s = 1, \dots, d$, which refers to a color index in the tensor. A face f_s with color s has a colored conserved momentum p_{f_s} and passes through some vertices v_s, with vertex operator of the form $V_{4;s}$, and vertices $v_{s'}$, with vertex operator of the form $V_{4;s'}$, with $s' \neq s$. A face f can pass through a vertex v

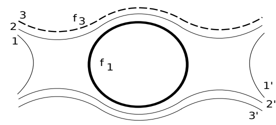

FIG. 2. A rank 3 4-point graph with an internal (closed) face f_1 with color 1 (in bold). Any face different from f_1 is open or external, for instance, see the face f_3 with color 3 (in dash).

a number of times denoted $\alpha = 0, 1, 2$. We symbolically write this as $v^\alpha \in f$ and then define a new type of incidence matrix by

$$\epsilon_{v_s f_{s'}} = \begin{cases} \alpha, & \text{if } s = s' \text{ and if } \; v_s^\alpha \in f_s, \\ 0, & \text{otherwise.} \end{cases} \tag{11}$$

For any color s, we write the contribution in the amplitude, at fixed vertex v_s, as $\lambda(1 + \tilde\eta \sum_{f_{s'}} \epsilon_{v_s, f_{s'}} p_{f_{s'}})$, with $\tilde\eta = \eta/\lambda$. A noteworthy fact must be reported: calculating bounds on amplitudes in usual/local quantum field theory and from the simple graph theory perspective, the incidence matrix $\epsilon^{(1)}_{vl}$ encoding the incidence between lines and vertices is of major importance. In particular tensorial models [15, 16], there is another matrix $\epsilon^{(2)}_{lf}$ which encodes the incidence between lines and faces and proves also to be useful. These two objects, in colored tensor models, are somehow natural because they are related with the existence of a homological structure on colored graphs [5]. In the present context, we discover a new matrix ϵ_{vf} which records the incidence vertices-faces. This is certainly a peculiarity of the present class of models which gives a weight to the vertices. On may naturally wonder if the product of the incidence matrices $\epsilon^{(1)}_{vl}$ by $\epsilon^{(2)}_{lf}$ does not generate ϵ_{vf}. The answer to this question is: "no, in general," $\sum_l \epsilon^{(1)}_{vl}\epsilon^{(2)}_{lf} \neq \epsilon_{vf}$.

Let us introduce the set $\mathcal{V}_s$ of vertices with vertex kernel $V_{4;s}$. Then, $\mathcal{V} = \sqcup_{s=1}^d \mathcal{V}_s$ (disjoint union). Using the Schwinger parametric form of the propagator kernel as

$$\tilde C(\{p_s\}) = \int_0^\infty d\alpha \; e^{-\alpha(\sum_s p_s^2 + \mu)} \,, \tag{12}$$

we put the amplitude (10) in the form

$$A_{\mathcal{G}} = \kappa(\lambda) \sum_{p_{f_s}} \int \Big[\prod_{l \in \mathcal{L}} d\alpha_l \, e^{-\alpha_l \mu} \Big] \prod_{f_s \in \mathcal{F}_{\rm ext}} e^{-(\sum_{l \in f_s} \alpha_l)(p^{\rm ext}_{f_s})^2} \prod_{f_s \in \mathcal{F}_{\rm int}} e^{-(\sum_{l\in f_s}\alpha_l) p^2_{f_s}} \prod_{s=1}^d \prod_{v_s \in \mathcal{V}_s} [1 + \tilde\eta(\epsilon\, \tilde p^{\,2a})_{v_s}] \,,$$

$$(\epsilon\, \tilde p^{\,2a})_{v_s} := \sum_{f_{s'}} \epsilon_{v_s, f_{s'}} (\tilde p_{f_{s'}})^{2a} \,, \tag{13}$$

where $\kappa(\lambda)$ includes symmetry factors and coupling constants, $p^{\rm ext}_{f_s}$ are external momenta which are not summed and, in the last line, $\tilde p_{f_s}$ refers to an internal or external momentum. Summing over arbitrary high momenta might produce divergent amplitudes (13), hence the need of a renormalization analysis. Our next goal will be not to perform this analysis but to initiate that program by providing a power counting theorem for any amplitude. The particular scheme that we use for working out such a theorem is the so-called multi-scale analysis [26]. So far, it has been proved enough powerful to address any tensorial model at the perturbative level. We will see that, in the present situation as well, the multi-scale analysis allows to reach a power counting theorem.

Multiscale analysis. We start the multiscale analysis by introducing a slice decomposition of the propagator in the parameter $M > 1$, and determine bounds on each sliced propagator:

$$\tilde C(\{p_s\}) = \int_0^\infty d\alpha \; e^{-\alpha(\sum_s p_s^2 + \mu)} = \sum_{i=0}^\infty C_i(\{p_s\}) \,,$$

$$C_i(\{p_s\}) = \int_{M^{-2(i+1)}}^{M^{-2i}} d\alpha\, e^{-\alpha(\sum_s p_s^2+\mu)} \leq K' M^{-2i} e^{-M^{-2i}(\sum_s p_s^2+\mu)} \leq K M^{-2i}\, e^{-\delta M^{-i}(\sum_s |p_s|+\mu)}\,,$$
$$C_0(\{p_s\}) = \int_1^{\infty} d\alpha\, e^{-\alpha(\sum_s p_s^2+\mu)} \leq K\,, \tag{14}$$

for some constants K, K' and δ. As in the standard field theory case, high values of i select high momenta of order M^i and this will be called UV. It is immediate that this regime also coincides with small distances on $U(1)$. Meanwhile, small momenta are selected by the slice $i = 0$, and corresponds to the IR. We introduce a cut-off Λ and the cut-offed propagator expresses as $C^\Lambda = \sum_{i=0}^{\Lambda} C_i$.

Beginning with the analysis, the next developments follow the same steps as detailed in [14] but extra features arise from the vertex weights and must be discussed. We slice all propagators in (10) and write, using the bounds (14),

$$A_{\mathcal{G}} = \sum_{\boldsymbol{\mu}} A_{\mathcal{G};\boldsymbol{\mu}}\,, \qquad A_{\mathcal{G};\boldsymbol{\mu}} = \sum_{p_{v;s}} \prod_{l\in\mathcal{L}} C_{i_l}(\{p_{v(l)}\};\{p'_{v'(l)}\}) \prod_{v\in\mathcal{V}} (-V_{4;v}(\{p_{v;s}\}))\,,$$
$$|A_{\mathcal{G};\boldsymbol{\mu}}| \leq \kappa(\lambda) K^L K_1^V\; K_2^{F_{\rm ext}} \prod_{l\in\mathcal{L}} M^{-2i_l} \sum_{p_{f_s}} \prod_{f_s\in\mathcal{F}_{\rm int}} e^{-\delta(\sum_{l\in f_s} M^{-i_l})|p_{f_s}|} \prod_{s=1}^{d} \prod_{v_s\in\mathcal{V}_s} [1+\tilde\eta(\epsilon\,\tilde p^{\,2a})_{v_s}]\,, \tag{15}$$

where $\boldsymbol{\mu} = \{i_l\}_{l\in\mathcal{L}}$ is a multi-index, called momentum assignment, which collects the propagator indices $i_l \in [0,\Lambda]$, $K_{1,2}$ are constants. The sum over the momentum assignments will be performed only after renormalization according to a standard procedure [26]. The object of interest is $A_{\mathcal{G};\boldsymbol{\mu}}$ and seeking an optimal bound for that amplitude is our next goal. Mainly, the sum over internal momenta must be performed in a way to bring the less possible divergences, i.e. positive powers of M^i. This can be done in a way compatible with the Gallavotti-Nicolò tree of quasi-local subgraphs [30].

Let us define the quasi-local or "dangerous" subgraphs which are, by essence, intimately related with a notion of locality of the theory. Consider a graph $\mathcal{G}$, with set $\mathcal{L}$ of lines and set $\mathcal{F}_{\rm int}$ of internal faces. Let i be a fixed slice index and define $\mathcal{G}^i$ the subgraph of $\mathcal{G}$ built with lines with indices such that $\forall \ell \in \mathcal{L}(\mathcal{G}^i)\cap\mathcal{L}$, $i_\ell \geq i$. In the case that $\mathcal{G}^i$ has several connected components, we note them G^i_k. We call $\{G^i_k\}_{(i,k)}$ the set of quasi-local subgraphs. Let g be a subgraph of $\mathcal{G}$ and call $\mathcal{L}(g)$ and $\mathcal{L}_{\rm ext}(g)$, the sets of internal and external lines of g, respectively. Given a momentum assignment $\boldsymbol{\mu}$ of $\mathcal{G}$, define $i_g(\boldsymbol{\mu}) = \inf_{\ell\in\mathcal{L}(g)} i_\ell$ and $e_g(\boldsymbol{\mu}) = \sup_{\ell\in\mathcal{L}_{\rm ext}(g)} i_\ell$, then g subgraph of $\mathcal{G}$ is quasi-local if and only if the following criterion is satisfied: $i_g(\boldsymbol{\mu}) > e_g(\boldsymbol{\mu})$.

There are well-known facts about the set of quasi-local subgraphs $\{G^i_k\}$: it is partially ordered under inclusion and forms an abstract tree called the Gallavotti-Nicolò (GN) tree (see Figure 12 in [14] for an example given in the tensorial setting). We want to perform the sums over internal momenta p_{f_s} in (15) in a way compatible with the GN tree, that is, such that the result can be uniquely expressed in terms of the graphs G^i_k.

Performing the sum over internal momenta p_{f_s}, we will need the particular index $i_f = \min_{l\in f} i_l$. This index corresponds to a line l_f, namely $i_{l_f} = i_f$. The following result which is also useful, can be simply obtained:

Proposition 1. *Given $n \in \mathbb{N}^*$ and a constant $B \in \mathbb{R}^*$,*

$$\sum_{p=1}^{\infty} p^n e^{-Bp} = c\, B^{-(n+1)}(1 + O(B^{(n+1)}))\,, \tag{16}$$

where c is a constant depending on n.

From the above, we can recognize that, at leading order, the discrete sum evaluates like an integral.

We are now in position to find an optimal bound for the amplitude. Because all external momenta are chosen such that $p^{\rm ext}_{f_s} \ll p_{f_{s'}}$, for all s, s', and there is no sum over these external momenta, one proves that the following bound holds

$$|A_{\mathcal{G};\boldsymbol{\mu}}| \leq \kappa(\lambda) K^L K_1^V\; K_2^{F_{\rm ext}} \prod_{l\in\mathcal{L}} M^{-2i_l} \sum_{p_{f_s}} \prod_{f_s\in\mathcal{F}_{\rm int}} e^{-\delta(\sum_{l\in f_s} M^{-i_l})|p_{f_s}|} \prod_{s=1}^{d} \prod_{v_s\in\mathcal{V}_s} [1+K_3\,\tilde\eta(\epsilon\, p^{\,2a})_{v_s}] \tag{17}$$

where K_3 is another constant, and the matrix ϵ (13) is now reduced to internal faces (we keep however the same notation ϵ).

The sum over internal momenta p_{f_s} must be performed in an optimal way, namely, in a way bringing the less possible divergences while the inequality (17) must remain a correct approximation. Each sum brings, according to Proposition 1, a "bad" factor of M^{+i}. The optimal way to evaluate the sum is made in two steps: given a f (forgetting a moment the subscript s), among the lines $l \in f$, we will simply use the line l_f with $i_{l_f} = \min_{l\in f} i_l = i_f$, which will generate the lowest factor M^{i_f}, such that the sum over $(p_f^{2a})^{\alpha_{p_f}} e^{-\delta M^{-i_f}|p_f|}$, with α_{p_f} an integer, is the lowest possible. Then, there is another difficulty which is to optimize the products of the vertex kernels. As we are searching an upper bound for the amplitude and the product generates a sum of positive terms, then we must target, in each factor of the product of the vertex kernels, the term p_f generating after summation a product of $M^{i_f(2a\alpha+1)}$ with the largest possible power. Thus, there is a monomial generated by the product over vertex kernels which yields an upper bound of an optimal kind for $A_{\mathcal{G};\boldsymbol{\mu}}$. To identify this monomial, we must investigate the combinatorics of the ϵ matrix.

Let us define the following optimization procedure using the matrix $\hat\epsilon_{v_s f_{s'}}$ obtained from $\epsilon_{v_s f_{s'}}$ (11) by simply putting all $\alpha = 1$. Thus $\hat\epsilon$ simply notes if a face is incident to a given vertex (the information about how many times this happens contribute to an overall constant factor in the bound amplitude (17)). First, for convenience, we organize the incidence matrix $\hat\epsilon$ by color blocks: we first list all vertices v_s of a given color as columns and list all faces f_s of the same color in row blocks. Next, organizing the line indices for fixed color s, we list the faces $f_{s;k}$ from the highest index $i_{f_{s;k}}$ to the lowest; if several faces have the same momentum index, we list them in an arbitrary way. Start with the face $f_{s;1}$, and count $\varrho_{f_{s;1}} = \sum_l \hat\epsilon_{v_{s,l} f_{s.1}}$, i.e. the number of vertices $v_{s;l}$ such that $\hat\epsilon_{v_{s;l} f_{s;1}} = 1$. Then, delete all these columns and the line $f_{s;1}$ and define a new reduced matrix that we denote again, for simplicity, $\hat\epsilon$. Pass to the next line $f_{s;2}$ and proceed in the same way with the reduced matrix. If there are no more vertices or the matrix trivializes, we define

$$\varrho(\mathcal{G}) = \sum_s \sum_{f_{s;k}} \varrho_{f_{s;k}} \,. \tag{18}$$

An illustration of this procedure is given in Appendix A. At an intermediate step labeled by f_s, $\varrho(\mathcal{G})$ increases its value by the number of times that this face f_s passes through remaining vertices v_s where it still possesses the dominant index i_{f_s}. It is immediate that $\varrho(\mathcal{G})$ is bounded from above by the number of vertices of $\mathcal{G}$, thus

$$\varrho(\mathcal{G}) \leq V(\mathcal{G}) \,. \tag{19}$$

In some case, $\varrho(\mathcal{G})$ coincides with the rank of $\hat\epsilon$ but, of course, it is not in general.

The above procedure leads us to an optimal bound and we can observe how this model has an interesting feature: it combines both an optimization which lowers the value of the indices of faces while, due to the vertex $p_s^{2a}\phi_s^4$, enhances the contribution by taking, in an independent way, the largest values among the i_f's. We write a new bound

$$|A_{\mathcal{G};\boldsymbol{\mu}}| \leq \kappa_1 K^L K_1^V \; K_2^{F_{\rm ext}} \prod_{l\in\mathcal{L}} M^{-2i_l} \sum_{p_{f_s}} \prod_{f_s\in\mathcal{F}_{\rm int}} e^{-\delta M^{-i_{f_s}}|p_{f_s}|} \prod_{s'=1}^{d} \prod_{f_{s'}} p_{f_{s'}}^{2a\varrho_{f_{s'}}} \,, \tag{20}$$

where κ_1 incorporates coupling constants λ and η. Performing the sum over internal momenta, one gets

$$|A_{\mathcal{G};\boldsymbol{\mu}}| \leq \kappa_2 \prod_{l\in\mathcal{L}} M^{-2i_l} \prod_{f_s\in\mathcal{F}_{\rm int}} M^{i_{f_s}(2a\varrho_{f_s}+1)} \,, \tag{21}$$

where κ_2 is another constant depending on the graph and including K, K_1, K_2 and κ_1 and new constants coming from the summation over internal momenta.

We now use the expansion in quasi-local subgraphs G_k^i to write the above bound in the form compatible with the Gallavotti-Nicolò tree:

$$\begin{aligned} |A_{\mathcal{G};\boldsymbol{\mu}}| &\leq \kappa_2 \Big[\prod_{l\in\mathcal{L}} \prod_{i=1}^{i_l} M^{-2}\Big] \prod_{f_s\in\mathcal{F}_{\rm int}} \Big[\Big(\prod_{i=1}^{i_{f_s}} M\Big)\Big(\prod_{i=1}^{i_{f_s}} M^{2a\varrho_{f_s}}\Big)\Big] \\ &\leq \kappa_2 \Big[\prod_{l\in\mathcal{L}} \prod_{(i,k)/\, l\in\mathcal{L}(G_k^i)} M^{-2}\Big] \prod_{f_s\in\mathcal{F}_{\rm int}} \Big[\Big(\prod_{(i,k)/\, l\in\mathcal{L}(G_k^{i_{f_s}})} M\Big)\Big(\prod_{(i,k)/\, l\in\mathcal{L}(G^{i_{f_s}})} M^{2a\varrho_{f_s}}\Big)\Big]. \end{aligned} \tag{22}$$

The product $\prod_{l\in\mathcal{L}} \prod_{(i,k)/\, l\in\mathcal{L}(G_k^i)} M^{-2}$ can be recast in a standard way [26], as $\prod_{(i,k)} M^{-2L(G_k^i)}$. The second product $\prod_{f_s\in\mathcal{F}_{\rm int}} \Big(\prod_{(i,k)/\, l\in\mathcal{L}(G_k^{i_{f_s}})} M\Big)$ has been studied in previous works [11, 14] as well. We re-express it as

$$\prod_{f_s\in\mathcal{F}_{\rm int}} \prod_{(i,k)/\, l\in\mathcal{L}(G_k^{i_{f_s}})} M = \prod_{f_s\in\mathcal{F}_{\rm int}} \prod_{(i,k)/\, l_{f_s}\in\mathcal{L}(G_k^i)} M = \prod_{(i,k)} \prod_{f_s\in\mathcal{F}_{\rm int}\cap G_k^i} M = \prod_{(i,k)} M^{F_{\rm int}(G_k^i)} \,, \tag{23}$$

where we use the fact that the face f becomes closed in the graph G_k^i if the line $l_f \in \mathcal{L}(G_k^i)$. The last product, namely, the one involving the new ingredient ϱ_{f_s}, introduces a new feature for the present model. We address it in the following way:

$$\prod_{f_s \in \mathcal{F}_{\rm int}} \prod_{(i,k)/\, l_{f_s} \in G_k^i} M^{2a\varrho_{f_s}} = \prod_{(i,k)} \prod_{f_s \in \mathcal{F}_{\rm int} \cap G_k^i} M^{2a\varrho_{f_s}} = \prod_{(i,k)} M^{2a\varrho(G_k^i)}\,, \tag{24}$$

where $\varrho(\cdot)$ has been defined in (18). We reach the following statement:

Theorem 1 (Power counting)**.** *Let $A_{\mathcal{G};\boldsymbol{\mu}}$ be the amplitude associated with the graph $\mathcal{G}$ of the $p^{2a}\phi_d^4$-model in the multi-scale index $\boldsymbol{\mu}$, then there exists a constant κ depending on the graph such that*

$$|A_{\mathcal{G};\boldsymbol{\mu}}| \leq \kappa \prod_{(i,k)\in\mathbb{N}^2} M^{\omega_{\rm d}(G_k^i)}\,, \tag{25}$$

where G_k^i are quasi-local subgraphs and

$$\omega_{\rm d}(G_k^i) = -2L(G_k^i) + F_{\rm int}\,(G_k^i) + 2a\varrho(G_k^i)\,. \tag{26}$$

Setting $a \to 0$ brings us back to the degree of divergence of usual tensorial models [11, 14]. The term $2a\varrho(G_k^i)$ enhances, as predicted and *a priori*, the divergence degree of any graph. In particular, it allows non-melonic graphs to diverge as well. Indeed, consider the non-melonic 4-point graph $\mathcal{G}_1$ of Figure 3. Such a graph has a (superficial) degree of divergence:

$$\omega_{\rm d}(\mathcal{G}_1) = -2\times 2 + 1 + 2a \times 2 = 4a - 3 \tag{27}$$

which is strictly positive, and so possesses a divergent amplitude, whenever $a > \frac{3}{4}$. Evaluating the 4-point melonic graph $\mathcal{G}_2$ in the same figure, one finds $\omega_{\rm d}(\mathcal{G}_2) = -2\times 2 + 2 = -2 < 0$ which implies a convergent amplitude. By

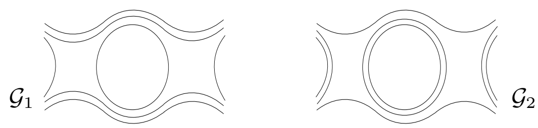

FIG. 3. Two rank 3 4-point graphs: $\mathcal{G}_1$ is a not a melon and $\mathcal{G}_2$ is.

this calculation, at large momenta, we simply realize that non-melonic diagrams can dominate melonic ones. This also shows that the type of graphs that need to be renormalized, in this framework, is radically different from any type known in previous tensorial models. In top of melonic graphs, we might have non-melonic graphs which might contribute to the flow of the coupling constants. However, the issue of renormalization is very subtle in general, and surely more, in the present context. For a given graph $\mathcal{G}$, the quantity $\varrho(\mathcal{G})$ must be fully analyzed before addressing the second stage for a renormalization procedure which should consist in identifying a locality principle for this class of models. This is left to a subsequent work. Nevertheless, even at this stage, from the above power counting theorem, we can extract additional information on the model which could enlighten its renormalizability property. For instance, we can discuss now how to fix the rank d and parameter a in a way which might lead to renormalizable models given a maximal valence of the interaction $\phi^{k_{\rm max}=4}$. The list the divergent graphs and their boundary data coming from these models will determine precisely the locality principle mentioned above. To proceed with this locality and renormalizability study, another important fact must be investigated: is there a hidden competition between $F_{\rm int}$ and ϱ, such that ϱ cannot be large if $F_{\rm int}$ is, and vice-versa? Our next analysis does not take any consideration of the kind and thus, it is possible that it might be improved (for e.g. turning sufficient conditions to necessary and sufficient ones).

Let $\mathcal{G}$ be a graph, then, as aforementioned $\varrho(\mathcal{G}) \leq V(\mathcal{G})$, and so we do have (from now on, we omit the dependency of the graph $\mathcal{G}$ in the basic quantities $V = V(\mathcal{G})$, $L = L(\mathcal{G})$, etc...)

$$\omega_{\rm d}(\mathcal{G}) \leq -2L + F_{\rm int} + 2aV\,. \tag{28}$$

and this bound can be saturated.

Let us note that the number of internal faces of a connected graph $\mathcal{G}$, in any rank $d \geq 3$ tensorial model, is given by [15]:

$$F_{\rm int} = -\frac{2}{(d-1)!}(\omega(\mathcal{G}_{\rm color}) - \omega(\partial\mathcal{G})) - (C_{\partial\mathcal{G}} - 1) - \frac{d-1}{2}N_{\rm ext} + d - 1 - \frac{d-1}{4}(4-2n)\cdot V\,, \tag{29}$$

where $\mathcal{G}_{\rm color}$ is called the colored extension of $\mathcal{G}$ (Definition 1i in [11]), $\partial\mathcal{G}$ defines the boundary of $\mathcal{G}$ (Definition 1iv in the same reference above), with number $C_{\partial\mathcal{G}}$ of connected components, V_k its number of vertices of coordination k, $V=\sum_k V_k$ its total number of vertices, $n\cdot V=\sum_k kV_k$ its number of half lines exiting from vertices, $N_{\rm ext}$ its number of external legs. The number $\omega(\mathcal{G}_{\rm color})=\sum_J g_{\widetilde{J}}$ is called the degree of $\mathcal{G}_{\rm color}$, $\widetilde{J}$ is the "pinched" jacket associated with J, a jacket of $\mathcal{G}_{\rm color}$ (Definition 1iii), $\omega(\partial\mathcal{G})=\sum_{J_\partial} g_{J_\partial}$ is the degree of $\partial\mathcal{G}$ (Definition 1v). The interested reader can have a proof of the formula (29) after Proposition 3.7 in [15]. Using now the combinatorial formula

$$-2L=-(n\cdot V-N_{\rm ext}), \tag{30}$$

we obtain, setting $d^-=d-1$,

$$\begin{aligned}\omega_{\rm d}(\mathcal{G}) &= -\frac{2}{(d^-)!}(\omega(\mathcal{G}_{\rm color})-\omega(\partial\mathcal{G}))-(C_{\partial\mathcal{G}}-1)-\frac{d^-}{2}N_{\rm ext}+d^--\frac{d^-}{4}(4-2n)\cdot V-(n\cdot V-N_{\rm ext})+2a\varrho \\ &= -\frac{2}{(d^-)!}(\omega(\mathcal{G}_{\rm color})-\omega(\partial\mathcal{G}))-(C_{\partial\mathcal{G}}-1)-\frac{1}{2}\Big((d^--2)N_{\rm ext}-2d^-\Big)-\frac{1}{2}\Big[\big(2d^-+(2-d^-)n\big)\cdot V-4a\varrho\Big]. \end{aligned}\tag{31}$$

It has been proved in [12], that either $\omega(\mathcal{G}_{\rm color})-\omega(\partial\mathcal{G})=0$ or it satisfies the bound

$$-\frac{2}{(d^-)!}(\omega(\mathcal{G}_{\rm color})-\omega(\partial\mathcal{G}))\leq-(d^--1). \tag{32}$$

Thus, using (32), we have the following bounds

• $\omega(\mathcal{G}_{\rm color})-\omega(\partial\mathcal{G})=0$ and

$$\omega_{\rm d}(\mathcal{G})\leq-(C_{\partial\mathcal{G}}-1)-\frac{1}{2}\Big((d^--2)N_{\rm ext}-2d^-\Big)-\frac{1}{2}\Big[\big(2d^-+(2-d^-)n\big)\cdot V-4a\varrho\Big]; \tag{33}$$

• $\omega(\mathcal{G}_{\rm color})-\omega(\partial\mathcal{G})>0$ and

$$\omega_{\rm d}(\mathcal{G})\leq-(d^--1)-(C_{\partial\mathcal{G}}-1)-\frac{1}{2}\Big((d^--2)N_{\rm ext}-2d^-\Big)-\frac{1}{2}\Big[\big(2d^-+(2-d^-)n\big)\cdot V-4a\varrho\Big]. \tag{34}$$

On potentially renormalizable models. We investigate the particular cases of ranks $d=3$ and 4 and restrict to $k_{\rm max}=4$ as the maximal valence of the theory vertices. Hence, $V=V_2+V_4$ and $n\cdot V=2V_2+4V_4$. We now aim at (1) finding sufficient conditions such that only graphs with at most four external fields might diverge and at (2) showing that, for some values of a, there are non-melonic graphs which might be involved in the renormalization analysis. The number V_2 of mass vertices does not add much to the discussion in the following, we will simply neglect it.

(i) In rank 3 (**$\phi^4_{d=3}$-model**), $d^-=2$, the relation (31) exhibits a crucial fact: the divergence degree does not depend on the number of external legs of the graph. The bounds (33) and (34) take the form

• $\omega(\mathcal{G}_{\rm color})-\omega(\partial\mathcal{G})=0$ and

$$\omega_{\rm d}(\mathcal{G})\leq-(C_{\partial\mathcal{G}}-1)-2(V_4-1-a\varrho); \tag{35}$$

• $\omega(\mathcal{G}_{\rm color})-\omega(\partial\mathcal{G})>0$ and

$$\omega_{\rm d}(\mathcal{G})\leq-(C_{\partial\mathcal{G}}-1)-[2(V_4-a\varrho)-1]. \tag{36}$$

We start by the second inequality (36) which is quite coercive. Whenever $2(V_4-a\varrho)\leq 1$, we might have divergent graphs. As we can build graphs with number of external fields higher than 4 which diverge and satisfy that condition (see, for e.g., the 6-point graph in Figure 4 which is linearly divergent with $C_{\partial\mathcal{G}}=1$, $\varrho=V_4$ and fixing $a=1$, the external data of which do not correspond to a term initially present in our action (2) and (3)[3]), thus, without any further assumptions, this is a signal of a nonrenormalizable ϕ^4-model. We must therefore find conditions which can lead us to a better control on the number of external legs.

---

[3] Note that one can easily generalize this example to an arbitrary number of legs $N_{\rm ext}=2n\geq 2$, $V_4=n$, the divergence occurs when $1+2an>2\times n$, i.e. when $a>1-\frac{1}{2n}$. The case $n=1=V_4$ is discussed in the following as the non-melonic tadpole, Figure 5; the case $n=2$ was already discussed in Figure 3.

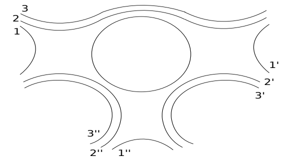

FIG. 4. A rank 3 ϕ^6-graph.

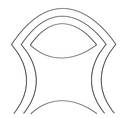

FIG. 5. Rank 3 non-melonic tadpole.

$\sim$ (i1) The case $V_4 = 1$ is very particular: these are tadpole graphs with one propagator (to avoid triviality), $C_{\partial\mathcal{G}} = 1$, $N_{\rm ext} = 2$ and $\omega_{\rm d}(\mathcal{G}) \leq -1 + 2a\varrho$. Since $\varrho \leq V_4 = 1$, ϱ can only be 0 or 1. The case leading to divergence is given by $\omega_{\rm d}(\mathcal{G}) = -1+2a$. This is a non-melonic tadpole (see Figure 5) with $\omega(\mathcal{G}_{\rm ext}) - \omega(\partial\mathcal{G}) = 1$. This contribution is divergent if $a \geq 1/2$, and converges otherwise. Any divergence must be re-absorbed by the mass or wave function renormalization.

$\sim$ (i2) Now, consider $V_4 = 2$ and a number of external legs $2 \leq N_{\rm ext} \leq 4$ (noting that $N_{\rm ext} = 6$ defines a graph without loops or disconnected). We get $\omega_{\rm d}(\mathcal{G}) \leq -3+2a\varrho \leq -3+4a$ which can be positive ($a \geq \frac{3}{4}$) or negative. Any divergences must be tackled by mass and wave function, if $N_{\rm ext} = 2$, and ϕ^4-coupling, if $N_{\rm ext} = 4$.

$\sim$ (i3) We finally consider $V_4 \geq 3$ and $N_{\rm ext} \geq 2$. Then a solution of the renormalizability problem is to enforce a sufficient condition making all these amplitudes convergent. And such a condition is given by $2(V_4 - a\varrho) > 1$, $\forall V_4 \geq 3$. In that situation, one finds $0 < a < \frac{5}{6} \leq 1 - \frac{1}{2V_4}$.

The first relation (35) dealing with only melonic graphs is now discussed. From the above analysis, we restrict the investigation to the interval $a \in (0, \frac{5}{6})$. Before undertaking any case-by-case analysis depending on the number of vertices, one must observe that if $\varrho = 0$, $\omega_{\rm d}(\mathcal{G})$ can be at most 0, and this occurs if $C_{\partial\mathcal{G}} = 1$, since $V_4 \geq 1$. The possible divergent terms will renormalize the mass (these are log-divergent terms and the expansion of the amplitude generates subleading terms which are convergent). As noticed, we do not have any constraint on a. Now concentrating $\varrho > 0$,

$\sim$ (ii1) $V_4 = 1$, there is a single possibility to construct a melonic tadpole and for this case $\varrho = 0$.

$\sim$ (ii2) $V_4 = 2$, $N_{\rm ext} = 2$, there is a finite number of ways to construct melonic graphs such that $\varrho > 0$, by mixing $V_{4;s}$ and $V_{4;s'}$, $s \neq s'$. For such graphs, $\varrho = 1$, $C_{\partial\mathcal{G}} = 1$, $\omega_{\rm d}(\mathcal{G}) \leq -2(2-1-a) = -2(1-a)$ which cannot diverge if $a < 1$. For $N_{\rm ext} = 4$, there is no such occurrence.

$\sim$ (ii3) $V_4 \geq 3$, a condition making all amplitudes convergent is $V_4 - 1 - a\varrho > 0$. This is satisfied if $a < 1 - \frac{1}{V_4}$. Thus $0 < a < \frac{2}{3}$ guarantees the convergence of all such amplitudes, and this hints at super-renormalizability. In the case $\frac{2}{3} \leq a < \frac{5}{6}$, one can show that, for $V_4 \leq 6$, the amplitudes might diverge and whenever $V_4 > 6$, we necessarily have a convergent amplitude. As a result, the number of divergent configurations is finite and this is a strong signal of super-renormalizability. The sole problem is that, in this case, one must check if there is no $(N > 4)$-graph which diverge. If yes then, it means that ϕ^4-model is not renormalizable in this truncation.

As a result, the above analysis suggests the following table (below tadpole graphs are with $V_4 = 1, 2$)

$$
\begin{aligned}
0 < a < \frac{1}{2}\,, &\quad p^{2a}\phi^4_{d=3} \ \text{ is super-renormalizable with melonic divergent tadpole graphs;} \\
\frac{1}{2} \leq a < \frac{2}{3}\,, &\quad p^{2a}\phi^4_{d=3} \ \text{ is super-renormalizable with melonic and non-melonic divergent tadpole graphs;} \\
\frac{2}{3} \leq a < \frac{5}{6}\,, &\quad \text{Inconclusive: melonic and non-melonic tadpole diverge; non-melonic 4pt-graphs diverge} \\
&\quad\ \text{if } a \geq \frac{3}{4}\text{; melonic graphs with } \varrho > 0 \text{ and } V_4 \leq 6 \text{ might diverge;} \\
a \geq \frac{5}{6}\,, &\quad p^{2a}\phi^4_{d=3} \ \text{ is non-renormalizable.}
\end{aligned} \tag{37}
$$

(ii) In rank 4 ($\phi^4_{d=4}$**-model**), $d^- = 3$, the situation is a little more involved but, still, we can address it in a similar way as above. We have:

• $\omega(\mathcal{G}_{\rm color}) - \omega(\partial\mathcal{G}) = 0$ and

$$\omega_{\rm d}(\mathcal{G}) \leq -(C_{\partial\mathcal{G}} - 1) - \frac{1}{2}\big(N_{\rm ext} - 6\big) - (V_4 - 2a\varrho)\,; \tag{38}$$

• $\omega(\mathcal{G}_{\rm color}) - \omega(\partial\mathcal{G}) > 0$ and

$$\omega_{\rm d}(\mathcal{G}) \leq -(C_{\partial\mathcal{G}} - 1) - \frac{1}{2}\big(N_{\rm ext} - 6\big) - \big(V_4 + 2 - 2a\varrho\big)\,. \tag{39}$$

The case $N_{\rm ext} = 6$ is enough particular to be stressed right away: it seems that a $p^{2a}\phi^6$-model would have, in this rank, interesting renormalizability properties. Indeed such a model would already have features similar to the model ϕ^6 of [11] (divergent melonic contributions with $\varrho = 0$ from (38)). Nevertheless, it will require a greater challenge to work out in details, because the model will have with two types of trace invariants ${\rm Tr}_{6;1;i}(\phi^6)$ and ${\rm Tr}_{6;2;ij}(\phi^6)$ invariants (with i and j parametrizing color indices).

To start with, let us consider a graph with $N_{\rm ext} \geq 6$, then we obtain from the previous bounds

• $\omega(\mathcal{G}_{\rm color}) - \omega(\partial\mathcal{G}) = 0$ and

$$\omega_{\rm d}(\mathcal{G}) \leq -(C_{\partial\mathcal{G}} - 1) - \big(V_4 - 2a\varrho\big)\,; \tag{40}$$

• $\omega(\mathcal{G}_{\rm color}) - \omega(\partial\mathcal{G}) > 0$ and

$$\omega_{\rm d}(\mathcal{G}) \leq -(C_{\partial\mathcal{G}} - 1) - \big(V_4 + 2 - 2a\varrho\big)\,. \tag{41}$$

Noting that $V_4 - 2a\varrho \leq V_4 + 2 - 2a\varrho$, for having all graphs convergent, it is sufficient to require from (40) that $V_4 - 2a\varrho \geq V_4(1-2a) > 0$ implying $a < \frac{1}{2}$. A rapid checking shows that, indeed, for $a \geq \frac{1}{2}$, there are graphs with higher valency which diverge and so the model is non-renormalizable. A specific example is given in Figure 6. Fixing $a = \frac{3}{4}$, we have a quadratic divergent amplitude which is of the ϕ^6-form.

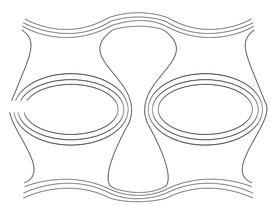

FIG. 6. Rank 4 divergent 6-point graph.

Now, we treat the case of a graph with $N_{\rm ext} = 4$ and $V_4 \geq 2$ ($V_4 < 2$ is trivial) with the bounds

• $\omega(\mathcal{G}_{\rm color}) - \omega(\partial\mathcal{G}) = 0$ and

$$\omega_{\rm d}(\mathcal{G}) \leq -(C_{\partial\mathcal{G}} - 1) - (V_4 - 1 - 2a\varrho)\,; \tag{42}$$

• $\omega(\mathcal{G}_{\rm color}) - \omega(\partial\mathcal{G}) > 0$ and

$$\omega_{\rm d}(\mathcal{G}) \leq -(C_{\partial\mathcal{G}} - 1) - \big(V_4 + 1 - 2a\varrho\big)\,. \tag{43}$$

Let us set $\varrho = 0$, then it is immediate that all amplitudes under this condition are convergent: $1 < V_4 - 1 < V_4 + 1$ and $\omega_{\rm d}(\mathcal{G}) < 0$. We now focus on $\varrho > 0$. From the study of $N_{\rm ext} \geq 6$, we must fix $0 < a < \frac{1}{2}$. Because $\varrho \leq V_4$, $1 < V_4 + 1 - 2a\varrho$, and then we infer from the relation (43) that all non-melonic graph amplitudes with $\varrho > 0$ are convergent. The relation (42) is now scrutinized at $\varrho > 0$ and $0 < a < \frac{1}{2}$. One must observe that, if $V_4 - \varrho = x \geq 1$ then $V_4 - 1 - 2a\varrho = V_4 - 1 - 2a(V_4 - x) = (V_4 - 1)(1 - 2a) + 2a(x - 1) > 0$. Therefore, the only case which might bring divergence is given by $\varrho = V_4$. In the interval $-1 < V_4(1-2a) - 1 \leq 0$, we could generate divergences and this is choosing a such that $\frac{1}{2}(1 - \frac{1}{V_4}) \leq a < \frac{1}{2}$, for all V_4. If there exists an infinite family of graphs with arbitrary number of vertices V_4 such that a satisfies this bound, this means that $\frac{1}{2} \leq a < \frac{1}{2}$ which is absurd. So the only way to have divergent graphs is that these belong to a finite family with a maximal finite number V_4. This suggests super-renormalizability. At this stage, we did not have any graph example such that $\varrho = V_4 \geq 2$ which is melonic and divergent. What is obviously true instead is that restricting $a < \frac{1}{4}$ leads directly to the convergence of any amplitude. Thus, at this stage, for $\frac{1}{4} \leq a < \frac{1}{2}$, the model renormalizability behavior is not determined.

Next, the case $N_{\rm ext} = 2$, $V_4 \geq 1$, is now clarified for $0 < a < \frac{1}{4}$.

• $\omega(\mathcal{G}_{\rm color}) - \omega(\partial\mathcal{G}) = 0$ and

$$\omega_{\rm d}(\mathcal{G}) \leq -(C_{\partial\mathcal{G}} - 1) - (V_4 - 2 - 2a\varrho)\,; \tag{44}$$

• $\omega(\mathcal{G}_{\rm color}) - \omega(\partial\mathcal{G}) > 0$ and

$$\omega_{\rm d}(\mathcal{G}) \leq -(C_{\partial\mathcal{G}} - 1) - \big(V_4 - 2a\varrho\big)\,. \tag{45}$$

Let us inspect the case $\varrho = 0$, (45) and $V_4 \geq 1$ mean convergence for amplitudes of non-melonic graphs, and (44) shows that 2-point melonic graphs can be at most linearly divergent if $V_4 \leq 2$ and they are convergent otherwise. They could renormalize the mass and possibly the wave function.

We focus now on $\varrho > 0$. From (45), $V_4 - 2a\varrho \geq V_4(1-2a) > 0$, thus all non-melonic graphs of this kind are finite. Starting by (44), we have $\frac{V_4}{2} - 2 \leq V_4 - 2 - \frac{\varrho}{2} < V_4 - 2 - 2a\varrho < V_4 - 2$. As a result, all graphs with $V_4 > 4$ are convergent. There is a finite number of configurations and the maximal degree of divergence is 2. These graph amplitudes must renormalize the mass and wave function, at subleading order.

The study of the rank $d = 4$ suggests that

$$\begin{aligned}
0 < a < \frac{1}{4}\,, &\quad p^{2a}\phi^4_{d=4} \;\text{ is super-renormalizable with divergent melonic 2-point} \\
&\quad \text{graphs made with } V_4 \leq 4 \text{ vertices;} \\
\frac{1}{4} \leq a < \frac{1}{2}\,, &\quad \text{Inconclusive: melonic 4pt-graphs might diverge;} \\
a \geq \frac{1}{2}\,, &\quad p^{2a}\phi^4_{d=4} \;\text{ is non-renormalizable.}
\end{aligned} \tag{46}$$

IV. CONCLUSION

We have introduced a tensorial field theory of the type $p^{2a}\phi^4$ and studied its power counting theorem using multi-scale analysis. We find a new degree of divergence of the amplitudes generalizing that of [11] and [14] by adding a new quantity related to the incidence matrix of vertices and faces. The main motivation for introducing such a $p^{2a}\phi^4$ model is to inquire other types of continuum limits and phases different from the branched polymer one, namely, the continuum limit of the simplest tensorial models. The present model shows that previous suppressed contributions (called non-melonic graphs) in ordinary tensorial models become enhanced in the present context. At the field theory level, our present analysis reveals that, indeed, non-melonic contributions can be of the same degree of divergence than the melonic ones and even, in some case, more relevant. This is encouraging for the above program on the continuum limit. Concerning renormalizable models, we have strong indications that, for a range of value of a, there are several ϕ^4-models which might be super-renormalizable although did not have yet any hint for just-renormalizable models of the ϕ^4-type. In general, the role played by non-melonic contributions is not yet really sensible in the ϕ^4-truncation. The $\phi^6_{d=4}$ model seems to possess relevant properties within the above scheme.

Finally, let us mention that other exotic choices for models are possible to complete the $p^{2a}\phi^4$-model. For instance, we can introduce a different power of the momenta in the kinetic term $\boldsymbol{\delta}_{p_i,p'_i}(\sum_{i=1}^d p_i^{2b})$ (as highlighted in [14]) which will allow to explore more models in two parameters (a,b) and might be a better approach to find just-renormalizable models. We might also change the dimension of the group from $U(1)$ to $U(1)^D$ and perhaps consider the group $SU(2)$ in the way of [14]. This must be fully addressed elsewhere.

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APPENDIX

Appendix A: Optimization of the momentum sums: a worked out example

Consider the graph $\mathcal{G}$ of Figure 7. It is defined by a set $\mathcal{V}$ of vertices decomposed in two disjoint subsets: $\{V_1^{(1)}, V_2^{(1)}\}$ which includes vertices with vertex kernel $V_{4;1}$ and $\{V_1^{(2)}\}$ identified by a kernel like $V_{4;2}$. The set $\mathcal{L}$ of lines includes $l_1, \dots, l_5$. We associate these lines with a scale index $i_{l_1} = 15$, $i_{l_2} = 12$, $i_{l_3} = 10$, $i_{l_4} = 9$, $i_{l_5} = 3$. The set of closed faces of $\mathcal{G}$ decomposes into faces of color 1, namely $\{f_{(1),1}, f_{(1),2}, f_{(1),3}\}$ (in red), and a face of color 3 denoted simply $\{f_3\}$ (in blue).

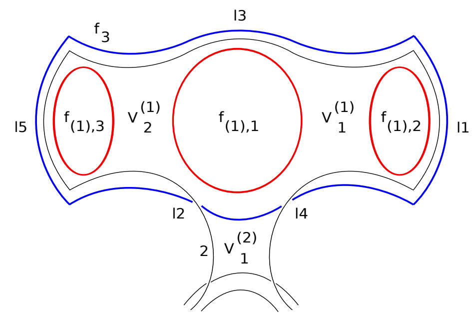

FIG. 7. A rank 3 graph with set $\{V_1^{(1)}, V_2^{(1)}\} \sqcup \{V_1^{(2)}\}$ of vertices, set $\{l_1, \dots, l_5\}$ of lines, sets $\{f_{(1),1}, f_{(1),2}, f_{(1),3}\} \sqcup \{f_3\}$ of internal faces.

The optimization of the sums over internal momenta first requires to specify the index i_f of each face such that $i_f = \min_{l \in f} i_l$. This yields

$$i_{f_{(1),1}} = 9\,, \qquad i_{f_{(1),2}} = 15\,, \qquad i_{f_{(1),3}} = 3\,, \qquad i_{f_3} = 3\,. \tag{A.1}$$

Each face f has an index i_f and momentum p_f which is of order M^{i_f}. According to our prescription, we can now arrange the $\hat{\epsilon}$ matrix as follow, from the highest i_f to the lowest and by color blocks:

	$V_1^{(1)}$	$V_2^{(1)}$	$V_1^{(2)}$
$f_{(1),2}$	1	0	0
$f_{(1),1}$	1	1	0
$f_{(1),3}$	0	1	0
f_3	0	0	0

(A.2)

We start by $f_{(1),2}$ and count

$$\varrho_{f_{(1),2}} = \sum_{V_k} \hat{\epsilon}_{V_k, f_{(1),2}} = \hat{\epsilon}_{V_1^{(1)}, f_{(1),2}} = 1\,. \tag{A.3}$$

then we erase the column $V_1^{(1)}$ and $f_{(1);2}$, and get the reduced matrix

	$V_2^{(1)}$	$V_1^{(2)}$
$f_{(1),1}$	1	0
$f_{(1),3}$	1	0
f_3	0	0

(A.4)

For $f_{(1),1}$, we count, using the above reduced matrix that we call again $\hat{\epsilon}$,

$$\varrho_{f_{(1),1}} = \sum_{V_k} \hat{\epsilon}_{V_k, f_{(1),1}} = \hat{\epsilon}_{V_2^{(1)}, f_{(1),1}} = 1\,. \tag{A.5}$$

Erasing the column $V_2^{(1)}$ leads to a trivial matrix and the procedure stops. Here, forgetting any reference to scales, one concludes that $\varrho(\mathcal{G}) = 1 + 1 = 2$. Note that in the above example $\varrho(\mathcal{G})$ is the rank of the $\hat{\epsilon}$ matrix but, more generally, it might be not the case. We get the following optimal amplitude bound, for a constant κ,

$$A_{\mathcal{G};\boldsymbol{\mu}} \leq \kappa \big[\prod_{l \in \mathcal{L}} M^{-2i_l} \big] M^{9(1+2a)+15(1+2a)+3+3} \,. \tag{A.6}$$

It can be checked that, since $9(1 + 2a) + 15(1 + 2a) + 3 + 3 = 30 + 24 \times 2a$,

$$\sum_{(i,k)} F_{\rm int}\,(G_k^i) = 30\,, \qquad \sum_{(i,k)} \varrho(G_k^i) = 24\,, \tag{A.7}$$

with $F_{\rm int}\,(G_k^{i\in[10,15]}) = 1$, $F_{\rm int}\,(G_k^{i\in[4,9]}) = 2$, $F_{\rm int}\,(G_k^{i\in[1,3]}) = 4$, $\varrho(G_1^{i\in[10,15]}) = 1$, $\varrho(G_2^{i\in[11,12]}) = 0$, and $\varrho(G_2^{i\in[1,9]}) = 2$.

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