On the Equivalence Principle and Electrodynamics of Moving Bodies

Maciej Trzetrzelewski *

Abstract

Consider an observer surrounded by a charged, conducting elevator (assume that the charge is isolated from the observer). In the presence of the external electric field the elevator will accelerate however, due to the screening effect, the observer will not be able to detect any electromagnetic field. According to the equivalence principle, the observer may identify the cause of the acceleration with the external gravitational field. However the elevator's motion is given by Lorentz-force equation. Therefore there should exist a metric, depending on electromagnetic potential, for which the geodesics coincide with the trajectories of the charged body in the electromagnetic field.

We give a solution to this problem by finding such metric. In doing so one must impose a constraint on the electromagnetic field in a certain way. That constraint turns out to be achievable by marginal gauge transformations whose phase is closely related to the Hamilton-Jacobi function.

Finally we show that for weak fields the Einstein-Hilbert action for the proposed metric results in the Stueckelberg massive electrodynamics. For strong fields (e.g. at small scales) the correspondence is broken by a term that at the same time makes the theory non-renormalizable. We conjecture the existence of a quantum theory whose effective action reproduces the non-renormalizable term and hence the Einstein-Hilbert action.

*e-mail: maciej.trzetrzelewski@gmail.com

1 Introduction

In Physics literature one can find many attempts to formulate so called unified field theories i.e. field theories that unify electromagnetic and gravitational phenomena at classical level. Most important examples (in chronological order) are Weyl's conformal gravity [1], Kaluza-Klein approach [2], Eddington's affine gravity [3] and later development by Einstein and Straus [4] and by Schrödinger [5] - see [6] for a comprehensive review. Nowadays, these attempts are not considered, by most physicists, as realistic directions towards the unification of known interactions since they do not involve weak and strong interactions as well as they do not use the formalism of quantum theory.

On the other hand, these two missing ingredients are subject to small scales and hence it is not entirely impossible that some, incomplete, unification can be made only for long ranged forces i.e. gravity and electromagnetism. However, considering the lack of success of the previous unifying approaches, it seems unlikely that this can be achieved by simply guessing the mathematical formalism, guided mostly by aesthetic reasoning. Some guiding, *physical* principle is needed. Because the problem involves general relativity and electrodynamics, it is desirable to take advantage of the equivalence principle applied in a concrete situation involving electromagnetic fields.

In this paper we choose this route by considering a particular modification of Einstein's elevator thought experiment. We will assume that the elevator is charged and freely falling with an observer inside it. It follows that there is no electromagnetic field inside the elevator, moreover the observer cannot even tell if the elevator is indeed charged. At this stage the relative distance between the observer and the elevators walls are constant in time.

We then assume that the elevator falls into the region with electric field and so it will start accelerating toward the observer - i.e. the relative distance between the observer and one on the elevator's walls will start decreasing. Entering the electromagnetic field is the real cause of that acceleration. However, the observer cannot conclude that unambiguously as he cannot detect the electromagnetic field locally. The observer may suspect that the elevator is charged and that it has entered the region with the electric field but he may as well conclude that apparently the elevator hit another body (e.g. it has landed) which would explain why the observer now accelerates towards one side of the elevator. We have therefore arrived at a certain modifica-

tion of equivalence principle i.e under above assumptions, locally, the observer cannot distinguish between gravitational and electromagnetic field.

2 Equivalence of equations of motion

From the charged elevator thought experiment, it follows that the trajectory given by the Lorentz-force equation should coincide with the geodesic in a appropriately chosen metric depending on the electromagnetic potential. Therefore, we will be looking for a metric $g_{\mu\nu}$ which depends on the electromagnetic field A_μ in such a way that the equations for the geodesics imply equations of motion for a charged particle in the electromagnetic field. In doing so we are allowed to use the dimensional parameters of the problem i.e. the charge q and the mass m of the body. Because the metric should be symmetric in indices a natural guess is

$$g_{\mu\nu} = \eta_{\mu\nu} + k\frac{q^2}{m^2}A_\mu A_\nu \tag{1}$$

where the coefficient q^2/m^2 is chosen such that the metric is dimensionless (we work with $c = 1$ units), k is some dimensionless parameter, the signature of the metric is $(+, -, -, -)$.

We will now derive condition under which the equations of motion for the action

$$S = -m \int \sqrt{g_{\mu\nu}u^\mu u^\nu}d\tau \tag{2}$$

where $u^\mu = \dot{x}^\mu$ is the four-velocity, are equivalent to the Lorenz-force equation. Performing the variation of S w.r.t. x_μ we find

$$\left(\frac{\eta_{\mu\nu}u^\nu}{\sqrt{gu^2}}\right)^{\cdot} = k\frac{\kappa}{\sqrt{gu^2}}\frac{q}{m}F_{\mu\nu}u^\nu - k\frac{q}{m}\left(\frac{\kappa}{\sqrt{gu^2}}\right)^{\cdot}A_\mu, \tag{3}$$

$$gu^2 := g_{\mu\nu}u^\mu u^\nu, \qquad \kappa := \frac{q}{m}A_\mu u^\mu$$

where we introduced a key dimensionless quantity κ. This result should be compared with the corresponding action for a charged particle and the resulting equations of motion

$$S_A = -m \int \sqrt{\eta_{\mu\nu}u^\mu u^\nu}d\tau - q \int A_\mu u^\mu d\tau, \tag{4}$$

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$$\delta_x S_A = 0 \quad \Longrightarrow \quad \left(\frac{\eta_{\mu\nu}u^\nu}{\sqrt{u^2}}\right)^{\cdot} = \frac{q}{m}F_{\mu\nu}u^\nu. \tag{5}$$

Note that, since we are in curved space-time, the lowering or rising of the induces should done by the metric $g_{\mu\nu}$ or $g^{\mu\nu}$, which is why we kept $\eta_{\mu\nu}$ explicitly in (3). On the other hand we want the resulting equations of motion to be equivalent to Lorentz-force equations written in Minkowski space. In practice the expressions like $g^{\mu\nu}A_\mu$ will appear rarely while the ones like $\eta^{\mu\nu}A_\mu$ quite often. Therefore in this paper we will adopt a non-standard convention and use $\eta_{\mu\nu}/\ \eta^{\mu\nu}$ to lower/rise indices of A_μ, p_μ, x^μ and $F_{\mu\nu}$. The indices of $g_{\mu\nu}$ will never be risen or lowered - $g_{\mu\nu}$ will always be written explicitly while $g^{\mu\nu}$ is defined as a reciprocal of $g_{\mu\nu}$.

In order to have (3) equivalent to the Lorentz-force equation we should at least get rid of the gauge dependent term on the r.h.s. of equation (3). Therefore we set

$$\frac{\kappa}{\sqrt{gu^2}} = C \tag{6}$$

where C is a constant. It is tempting to set $C = 1/k$ so that the r.h.s. of (3) is exactly the Lorentz force but in fact we should not do that because on the l.h.s. we still have the incorrect factor $\sqrt{gu^2}$ instead of $\sqrt{u^2}$. Condition (6) implies that $\kappa^2 = C^2 u^2/(1 - kC^2)$ which in turn gives

$$\sqrt{gu^2} = \frac{1}{\sqrt{1 - kC^2}}\sqrt{u^2} \tag{7}$$

and so the equations of motion are now

$$\dot{p}_\mu = \frac{kC}{\sqrt{1 - kC^2}}qF_{\mu\nu}u^\nu, \quad p_\mu = mu_\mu/\sqrt{u^2} \tag{8}$$

where we introduced momentum p_μ. Now, to recover the Lorentz force law exactly, all we need to do is to set $kC = \sqrt{1 - kC^2}$ in which case the condition (6) becomes

$$qA_\mu p^\mu = m^2 \frac{1}{k}. \tag{9}$$

Therefore we have shown the desired equivalence. Note that if we now replace $\eta_{\mu\nu}$ in (1) with arbitrary metric $h_{\mu\nu}$ we will arrive at the Lorenz force law in curved space corresponding to $h_{\mu\nu}$.

Equations $\kappa^2 = C^2 u^2/(1-kC^2)$ together with (9) imply that $C^2 = 1/(k+k^2)$ and so $k \in (-\infty, -1) \cup (0, \infty)$. However the case $k = -1$ is interesting since (9) can also be written as

$$m\sqrt{u^2} - qkA_\mu u^\mu = 0 \tag{10}$$

and so, for $k = -1$, (10) is equivalent to saying that the Lagrangian of the charged particle vanishes on real trajectories. This value is also distinguished from the point of view of gauge transformations as we now show. We will come back to this value many times.

3   A gauge choice

The constraint (9) is a necessary condition for the potential A_μ and needs to be satisfied on the trajectory of the particle, if geodesics in the metric (1) are supposed to coincide with the trajectories of the charged particle in the electromagnetic field in the Minkowski space. This constraint is also a necessary one to make equation (3) gauge invariant. Therefore we can say that in order to save gauge invariance of equation (3) one needs to, nevertheless, fix the field A_μ in a marginal way, i.e. on the world-line of the particle. From this point of view gauge invariance and equivalence principle are not independent concepts.

One can look at (9) as some sort of gauge fixing - if A'_μ is an arbitrary potential then we can always make a gauge transformation

$$A_\mu = A'_\mu + \partial_\mu \chi \tag{11}$$

in such a way that (9) will be satisfied. This is obtained by choosing such χ that

$$qA'_\mu u^\mu + q\dot\chi = m\sqrt{u^2}/k$$

and so

$$\chi = \frac{1}{q} \int_\gamma \left(\frac{m}{k}\sqrt{u^2} - qA'_\mu u^\mu \right) d\tau \tag{12}$$

where γ is the world-line of the particle. Therefore for $k = -1$ we find that the phase $q\chi$ is in fact given by the action of the charged particle for the A'_μ field, evaluated at the classical trajectory i.e. by the Hamilton-Jacobi function. We are not allowed to set $k = -1$ however we can set k arbitrary close to -1 and so our phase $q\chi$ can

be arbitrary close to the Hamilton-Jacobi function. Let us keep k arbitrary and consider a k dependent Hamilton-Jacobi function

$$S_{HJ}^{(k)} := q\chi.$$

For $k = -1$ we obtain the usual Hamilton-Jacobi function which we will denote by $S_{HJ} := S_{HJ}^{(-1)}$. The relation between $S_{HJ}^{(k)}$ and S_{HJ} is simply

$$S_{HJ}^{(k)} = S_{HJ} + \left(1 + \frac{1}{k}\right) m \int_\gamma \sqrt{u^2} d\tau. \qquad (13)$$

We shall use this relation later on. Now, in view of (12) the derivative of $S_{HJ}^{(k)}$ and the k dependent Hamilton-Jacobi equation are

$$\partial_\mu S_{HJ}^{(k)} = \frac{1}{k} p_\mu - qA'_\mu, \quad (\partial S_{HJ}^{(k)} + qA')^2 = m^2/k^2 \qquad (14)$$

and so we are able to recover the usual Hamilton-Jacobi equation for $k = 1$ or $k = -1$. Returning to the gauge transformation (11) we see that (14) results in

$$q^2 A_\mu A^\mu = m^2/k^2. \qquad (15)$$

Important consequences of this equation will be discussed in Sections 6 and 7. Here let us only note that in view of (15) the determinate of the metric (1) is

$$\det g = -1 - k\frac{q^2}{m^2}A^2 = -1 - \frac{1}{k}$$

and so $g_{\mu\nu}$ would change the signature if we had $k \in (-1, 0)$. Now it is clear that the requirement $k \in (-\infty, -1) \cup (0, \infty)$ which we derived earlier is in fact equivalent to saying that the signature does not change.

Using the formula for p_μ in (14) and (11) we also obtain

$$p_\mu = kqA_\mu \qquad (16)$$

which has to interpreted as follows: the particle's direction is indicated by the field A_μ. At this point it is appropriate to make the following comment. We have arrived at equation (16) using equivalence principle for charged bodies. That equation is a consequence and must be regarded as the necessary condition. However the l.h.s. of (16) is an observable while the r.h.s. is proportional to the electromagnetic potential. Therefore we must conclude that A_μ is as physical as the

momentum. This conclusion is striking since we have arrived at it not referring to quantum theory where one obtains a similar conclusion using the Aharonov-Bohm effect.

There are several important consequences of equation (16).

- First, returning to (13), we obtain another formula for the derivative of $S_{HJ}^{(k)}$, we have

$$\partial_\mu S_{HJ}^{(k)} = \partial_\mu S_{HJ} + \left(1 + \frac{1}{k}\right) p_\mu. \qquad (17)$$

  However $\partial_\mu S_{HJ}$ should satisfy the usual Hamilton-Jacobi equation $(\partial S_{HJ} + qA')^2 = m^2$. Indeed, using (17) we have

$$\partial_\mu S_{HJ} + qA'_\mu = \partial_\mu S_{HJ}^{(k)} + qA'_\mu - \left(1 + \frac{1}{k}\right) p_\mu = -p_\mu \qquad (18)$$

  where in the last step we used the definition of A'_μ via gauge transformation (11) and substituted (16). Now, taking the square of (18) we arrive at the Hamilton-Jacobi equation for S_{HJ}.

- Second, equation (18) together with (16) imply that if A'_μ is not pure gauge then the value of k is unique. To see that assume that there are two possible values k and $\bar{k}$. Substituting (16) to (18) and expressing A_μ in terms of A'_μ via (11), we find that

$$\partial_\mu S_{HJ} + qA'_\mu = -kqA'_\mu - k\partial_\mu S^{(k)} = -\bar{k}qA'_\mu - \bar{k}\partial_\mu S^{(\bar{k})} \qquad (19)$$

  where in the last step we used the fact that S_{HJ} and A'_μ are independent of k and so the whole l.h.s. of (19) is k independent. This however implies that

$$A'_\mu = \partial_\mu \left(\frac{kS^{(k)} - \bar{k}S^{(\bar{k})}}{q(\bar{k} - k)}\right).$$

  We have therefore arrived at the statement that potential A'_μ is pure gauge. This implies that our assumption about existence of k and $\bar{k}$ is incorrect hence k is unique.

- Third, introducing the generalised momentum $\pi^\mu = p^\mu + qA^\mu$ we see that

$$\pi^\mu = p^\mu \left(1 + \frac{1}{k}\right) \implies \pi^2 = \left(1 + \frac{1}{k}\right)^2 m^2$$

  and so for $k = -1$ we would have $\pi^\mu = 0$.

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- Forth, substituting (16) to the Lorenz-force law (5) we obtain a consistency condition for A_μ, we have

$$q k \dot{A}_\mu = q F_{\mu\nu} u^\nu \implies (1+k)\dot{A}_\mu = \partial_\mu A_\nu u^\nu.$$

We see again that $k = -1$ plays a special role. Contracting the above equations with u^μ we see that for arbitrary k we have

$$u^\mu \dot{A}_\mu = u^\mu u^\nu \partial_\nu A_\mu = 0.$$

However for $k = -1$ the condition is even stronger

$$\partial_\mu A_\nu p^\nu = 0 \implies A^\nu \partial_\mu A_\nu = 0 \tag{20}$$

where we used (16) in the last step.

4   A normalisation choice

So far we performed the calculations not imposing normalisation constraints on u^μ. In curved space-time we can always set $g u^2 = C_1$ while in Minkowski space we may set $u^2 = C_2$ where C_1 and C_2 are constants. In our problem we should be able to set these conditions simultaneously since u^μ is the same 4-velocity from curve space and from Minkowski space perspective. That this is possible follows from equation (7) which implies that C_1 and C_2 are not independent but satisfy $C_1 = C_2/\sqrt{1 - kC^2}$. In fact, we can turn this argument around and say that: the possibility that C_1 and C_2 can be set constant simultaneously *implies* that $\kappa/\sqrt{g u^2} = const. = C_3$. Therefore the condition (6) may have been deduced already at the level of the action (2) while consistency with the Lorentz force merely implies that $C_3 = C$ (cp. (6)).

It is clearly most convenient to set $u^2 = 1$ and we shall use this convention in the next section. We showed that the choice of k is very subtle - in fact we have no choice since k is unique. Whatever that value is we are not able to determine it at classical level.

5   Consistency check

The fact that condition (9) involves both u^μ and A_μ is not a surprise since the geodesic equation is quadratic in four-velocities while the

Lorentz-force equation is linear in u^μ. Here we investigate this remark directly looking at the geodesic equation

$$\dot{u}^\mu + \Gamma^\mu_{\alpha\beta} u^\alpha u^\beta = 0. \qquad (21)$$

Moreover, considering remarks from previous section, we will assume that $gu^2 = const$. A priori it is not obvious (or at least not *that* obvious) that (21) is already equivalent to (8) since the Christoffel symbols in (21) involve the inverse metric $g^{\mu\nu}$ which was never used in the previous derivation. The inverse of (1) is

$$g^{\mu\nu} = \eta^{\mu\nu} - \frac{a^\mu a^\nu}{k + a^2}, \qquad a^\mu := k\frac{q}{m} A^\mu \qquad (22)$$

and so the Christoffel symbols are

$$\Gamma^\mu_{\alpha\beta} = \frac{1}{2k} \left(g^{\mu\nu} a_{(\alpha} f_{\beta)\nu} + g^{\mu\nu} a_\nu s_{\alpha\beta} \right), \qquad (23)$$

$$f_{\mu\nu} := \partial_{[\mu} a_{\nu]}, \qquad s_{\mu\nu} := \partial_{(\mu} a_{\nu)}.$$

Note that we could at this point take advantage of the condition $a^2 = 1$ however it turns out not to be necessary (this is expected since we did not use this condition in Section 2 when deriving the Lorentz-force law). Substituting (23) and (22) to (21) we obtain

$$\dot{u}^\mu + \frac{1}{k} f_\beta{}^\mu u^\beta - \frac{a^\mu}{k + a^2} \left(-\frac{1}{k} a^\nu f_{\beta\nu} u^\beta + \frac{1}{2} s_{\alpha\beta} u^\alpha u^\beta \right) = 0 \qquad (24)$$

where we used $a \cdot u = 1$. Now, we observe that

$$a \cdot u = 1 \quad \implies \quad \frac{1}{2} s_{\alpha\beta} u^\alpha u^\beta = -a^\nu \dot{u}_\nu$$

hence, returning to the A_μ variables, we finally obtain

$$\dot{u}^\mu + \Gamma^\mu_{\alpha\beta} u^\alpha u^\beta = g^{\mu\nu} \left(\dot{u}_\nu - \frac{q}{m} F_{\nu\alpha} u^\alpha \right).$$

The above calculation clarifies at the same time the following problem. We used the equivalence principle to argue that the Lorentz force should be derivable from the geodesic equation. However the inverse assertion should also be true i.e. it should be possible to use the Lorentz force equations and some relation involving $g_{\mu\nu}$ and A_μ so that the resulting equation looks like the geodesic one. Clearly, such relation should be as in (1). Then we can use the above calculation in

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reversed order to arrive at geodesic equation. (The only caveat in this argument is related to the question if every metric can be represented as in (1). Globally, this assertion is not true however locally - which is enough here - it is fairly justified since locally one can make even stronger choice by introducing the Fermi normal co-ordinates. Note however that we need to maintain $k \notin [-1, 0]$.)

6   Dirac's new electrons

Constraints (9), (15) and (16) can be considered as kinematic requirements for charged particles - an extension of the relativistic mass-shell constraint to the case of interaction with electromagnetic field. These constraints can be written in a compact form as

$$
\boxed{
\begin{array}{ll}
p^2 = m^2, & p_\mu = kqA_\mu \\
qA \cdot p = m^2/k, & k^2 q^2 A^2 = m^2
\end{array}
}
\tag{25}
$$

Clearly they are not independent - the first pair, of the above equations, in fact implies the second pair. Moreover, the equations in the right column imply the ones in the left column. The same conclusion holds for the pair of equations on the anti-diagonal of the above square.

All of them are supposed to hold only on the particle's real trajectory. Therefore they do not fix A_μ at all if we are considering regions of space-time away from particle's world line. However, due to continuity of A_μ equation $k^2 q^2 A^2 = m^2$ should approximately hold for regions very close to the trajectory. One can imagine a narrow tube, with radius ϵ, extended along the trajectory, inside which $k^2 q^2 A^2 = m^2 + O(\epsilon)$. Then one could use the remaining gauge degrees of freedom to actually make $k^2 q^2 A^2 = m^2$ inside the tube. Moreover the k-dependent Hamilton-Jacobi function $S_{HJ}^{(k)}$ can be now defined inside the tube, not just on the world-line of the particle. We may assume that $S_{HJ}^{(k)}$ is constant in the direction perpendicular to the world-line.

If condition $k^2 q^2 A^2 = m^2$ holds in a region of space-time (not just on the world-line) then we are allowed to differentiate it w.r.t. x^μ which results in $A^\mu \partial_\nu A_\mu = 0$. Then, using $p_\mu = kqA_\mu$, we conclude that $p^\mu \partial_\nu A_\mu = 0$ and therefore we have arrived at the third pair of equations which are equivalent to (20) where we needed to use $k = -1$.

Apparently the additional pair (20) is achievable using the remaining gauge degrees of freedom inside the tube - not requiring that $k = -1$.

It is interesting to note that one of the constraints we have derived, $k^2 q^2 A^2 = m^2$, was considered by Dirac in early 50' during his attempts to formulate quantum electrodynamics without divergences [7]. Dirac considered the gauge $A^2 = k^2$ for some constant k (Dirac's k^2 is our $m^2/q^2 k^2$) disregarding the commonly accepted choice $\partial A = 0$. He then concluded that any other potential A_μ^* was related with A_μ by a gauge transformation $A_\mu = A_\mu^* + \partial_\mu S$ where S is the Hamilton-Jacobi function of the electron, *provided* that $k = m/e$ where m and e are mass and charge of the electron. Dirac's conclusions are (almost) equivalent to ours although his motivation and reasoning are different. In particular we have arrived at similar conclusions from the constraint (6) which results from the equivalence principle - nothing is assumed about the structure of the electron while in Dirac's approach, the electron is not longer a point like object but a stream of electricity. In later publications Dirac re-examined the existence of ether - a substance that was supped to drive charged particles. All these additional interpretations made by Dirac are not necessary in our approach. One could hope however that Dirac's attempts to construct QED without infinities, can nevertheless be finalised using the ideas presented in this paper.

7 Einstein's equations

Let us now investigate what are the vacuum Einstein equations if the metric (1) is used with a constraint $k^2 q^2 A^2 = m^2$. The resulting equations will describe a gravitational field from the point of view of a charged particle when the map (1) is used. To simplify calculations we will be working with the dimensionless field a_μ as in (22) so that the metric and its inverse are

$$g_{\mu\nu} = \eta_{\mu\nu} + \frac{1}{k} a_\mu a_\nu, \quad g^{\mu\nu} = \eta^{\mu\nu} - \frac{1}{k+1} a^\mu a^\nu \qquad (26)$$

where we used $a^2 = 1$. The convention where rising/lowering of indices is done by $\eta^{\mu\nu}/\eta_{\mu\nu}$ will be very useful in this section. The condition $a^2 = 1$ results in several very useful identities. We have

$$a^\mu \partial_\nu a_\mu = 0, \qquad (27)$$

$$- a_\mu f^{\mu\nu} f_{\nu\rho} a^\rho = a_\mu f^{\mu\nu} s_{\nu\rho} a^\rho = a_\mu s^{\mu\nu} s_{\nu\rho} a^\rho = (a\partial a)^2 \qquad (28)$$

where we use a shorthand notation $(a\partial a)^2 = (a^\mu \partial_\mu a^\rho)(a^\nu \partial_\nu a_\rho)$. Let us now express the Christoffel symbols in a convenient way as

$$\Gamma^\mu_{\nu\rho} = \frac{1}{2k} a_{(\nu} f_{\rho)}{}^\mu + \frac{1}{2(k+1)} a^\mu s_{\nu\rho} - \frac{1}{2k(k+1)} a^\mu a^\sigma a_{(\nu} f_{\rho)\sigma} \qquad (29)$$

where $f_{\mu\nu}$ and $s_{\mu\nu}$ are as in (23). Using (27) we observe that $\Gamma^\mu_{\mu\nu} = 0$ and so only two terms contribute to the Ricci scalar, they are

$$R = g^{\mu\nu} \partial_\rho \Gamma^\rho_{\mu\nu} - g^{\mu\nu} \Gamma^\rho_{\mu\sigma} \Gamma^\sigma_{\nu\rho}.$$

Calculating these terms is straightforward using (26), (29) and identities (27), (28), although it is a bit lengthy. We find that

$$\eta^{\mu\nu} \Gamma^\rho_{\mu\sigma} \Gamma^\sigma_{\nu\rho} = a^\mu a^\nu \Gamma^\rho_{\mu\sigma} \Gamma^\sigma_{\nu\rho} = -\frac{1}{4k^2} f_{\mu\nu} f^{\mu\nu} + \frac{1}{2k^2}(a\partial a)^2,$$

$$a^\mu a^\nu \partial_\rho \Gamma^\rho_{\mu\nu} = \frac{1}{2k} f_{\mu\nu} f^{\mu\nu} - \frac{1}{k}(a\partial a)^2 + \frac{1}{k} \partial_\mu(a^\nu f_\nu{}^\mu)$$

and so, up to the irrelevant total derivative (note that $\sqrt{-g} = \sqrt{1 + 1/k}$), the curvature is

$$R = -\frac{1}{4k(k+1)} f_{\mu\nu} f^{\mu\nu} + \frac{1}{2k(k+1)}(a\partial a)^2. \qquad (30)$$

Returning now to the A_μ variables, we should augment the final Lagrangian by a quadratic term in A_μ so that the constraint $k^2 q^2 A^2 = m^2$ is incorporated at the action level. In the end we arrive at the following action principle

$$S = \frac{1}{16\pi G} \int d^4 x \sqrt{-\det g}\, R \quad \rightarrow$$

$$\sqrt{\frac{k}{k+1}} \frac{q^2}{16\pi G m^2} \int d^4 x \left[-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{q^2 k^2}{2m^2}(A\partial A)^2 + \lambda m^2 \left(k^2 q^2 A^2 - m^2 \right) \right] \qquad (31)$$

where λ is dimensionless Lagrange multiplier. Returning back to the general field $A'_\mu = A_\mu - \frac{1}{q} \partial S$ we obtain (up to a constant) the following Lagrangian

$$\mathcal{L} = \sqrt{\frac{k}{k+1}} \frac{q^2}{16\pi G m^2}(\mathcal{L}_S + \mathcal{L}_Q), \qquad (32)$$

$$\mathcal{L}_S = -\frac{1}{4} F'_{\mu\nu} F'^{\mu\nu} + \lambda m^2 k^2 (qA' - \partial S)^2, \qquad (33)$$

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$$\mathcal{L}_Q = -\frac{k^2}{2m^2}(qA'_\mu - \partial_\mu S)F'^{\mu\nu}F'_{\nu\rho}(qA'^\rho - \partial^\rho S)$$

where we used one of the identities in (28). The $\mathcal{L}_S$ part in (33) is recognised as the Stueckelberg Lagrangian. Two important observations are now in order.

Weak field unification

It is important to interpret result (33) properly. This Lagrangian describes the gravitational filed as seen locally by the charged particle with mass m. The $g_{\mu\nu}$ field is replaced by the A_μ using the map (1). This is still a theory of gravitational field but rewritten in terms of A_μ in a very particular situation. However, since Einstein-Hilbert action is non-renormalizable [8] the action (31) should also have the same issues. We observe that non-renormalizability of the action (31) is guaranteed by the $(A\partial A)^2 = -AFFA$ term. The existence of this term is therefore expected. At this point we can also turn this argument around and say that gravity cannot be renormalizable because of the $(A\partial A)^2$ term in (30).

Moreover, this term is very small compared to the F^2 term and becomes significant when $A_\mu \sim m/qk$ i.e. when the filed is strong (e.g. if we consider the Coulomb potential then the $(A\partial A)^2$ term becomes non-negligible at distances of order kq^2/m i.e. Compton wavelength times $kq^2/2\pi$). Therefore for weak fields we conclude that:

Gravitation $\approx$ Stueckelberg massive electrodynamics.

A prototype of quantum gravity

The term $(A\partial A)^2$ that causes renormalization issues is at the same time the only term that makes the Lagrangian (33) different from the renormalizable Stueckelberg part. Therefore it is natural to seek for the renormalizable theory that contains $\mathcal{L}_S$ and whose quantum corrections produce $\mathcal{L}_Q$. If such theory exists then we would be able to derive the Einstein-Hilbert action from first principles. The non-renormalizable part of the Einstein-Hilbert action would be interpreted as a quantum correction of the underlying theory. A natural candidate for such theory would include fermionic field which, when integrated out along the lines similar to the Euler-Heisenberg procedure, would produce the desired term. That field would be a fundamental

ingredient of the quantum theory of gravity and hence it is most likely not given by the ordinary Dirac field. One should be looking for a theory with a very special fermionic field. A good candidate along these lines would be the spin $3/2$ field since, among all possible fields with spin ≤ 2, it is the only half-integer spin field that was not discovered yet.

8 Summary

It is commonly argued that the Lorentz-force equation cannot be considered as a geodesic one due to the fact that it depends on the mass of the particle. There is however a caveat in this argument namely, we can make the metric depending on the mass too. This, in fact, is a necessity considering that such metric should depend on the dimensional field A_μ. If one wishes to obtain the equivalence between the geodesic and Lorentz-force trajectories it is almost a logical necessity to take advantage of some characteristic feature of the electromagnetic field. That feature in our opinion is the Gauss law - the consequence of which is the screening of the filed inside a charged conducting body. This allows one to consider a modification of Einstein's elevator thought experiment in which the elevator is charged. The observer inside the elevator cannot detect the electromagnetic field and so the equivalence between the geodesics and the Lorentz-force trajectories follows.

We have derived consistency condition (9) from the requirement that the geodesic equation in the metric (1) coincides with the Lorentz force equation in Minkowski space. Such condition can be achieved by choosing a certain gauge. Therefore we must conclude that the equivalence principle, exploited in they way, fixes the gauge of the electromagnetic potential (albeit marginally) even though the equations of motion are gauge invariant.

A striking conclusion that follows is that the potential A_μ is physical (cp. (16)). This one can deduce already from the Aharonov-Bohm experiment and so one perhaps should not be surprised that its value can be determined/constrained by some physical considerations. What is surprising however is that, unlike in the Aharonov-Bohm experiment, we are not using the rules of quantum mechanics. Contrary to common belief it follows that A_μ is physical, relying on entirely classical considerations.

Using the A_μ dependent metric in the Einstein-Hilbert action re-

sults in a theory that for weak fields (in particular regions far away from the sources) coincides with massive electrodynamics with a scalar field. We have therefore arrived at a certain unification scheme already at classical level i.e. for weak fields equations of electromagnetic field follow from Einstein's equations. For strong fields (in particular for small scales) the correspondence is broken by a term that makes the theory non-renormalizable - as expected. We conjecture the existence of a renormalizable theory involving the gauge potential, the scalar field and the spinor field, whose effective action reproduces the non-renormalizable term and hence the complete Einstein-Hilbert action.

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