Probability distributions of turbulent energy

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Probability density functions (PDFs) of scale-dependent energy fluctuations, $P[\delta E(\ell)]$, are studied in high-resolution direct numerical simulations of Navier-Stokes and incompressible magnetohydrodynamic (MHD) turbulence. MHD flows with and without a strong mean magnetic field are considered. For all three systems it is found that the PDFs of inertial range energy fluctuations exhibit self-similarity and monoscaling in agreement with recent solar-wind measurements [B. Hnat et al., Geophys. Res. Lett. 29(10), 86-1 (2002)]. Furthermore, the energy PDFs exhibit similarity over all scales of the turbulent system showing no substantial qualitative change of shape as the scale of the fluctuations varies. This is in contrast to the well-known behavior of PDFs of turbulent velocity fluctuations. In all three cases under consideration the $P[\delta E(\ell)]$ resemble Lévy-type gamma distributions. In contrast the PDFs of velocity coincide in all three investigated systems and a simple reaction-rate model.

The observed gamma distributions exhibit a scale-dependent exponent, say $\gamma(t)$, in the direction of a fixed unit vector $\hat{e}_\ell$, in the direction of a fixed unit vector $\hat{e}_\ell$, since they yield a comprehensive and scale-dependent characterization of the statistical properties of turbulent fluctuations via the associated probability density function (PDF) [3].

PDFs of temporal fluctuations [16] in the solar wind, e.g. of total (magnetic + kinetic) energy density, as measured by the WIND spacecraft are self-similar over all observed scales, exhibit monoscaling, and closely resemble gamma distributions. In contrast the PDFs of velocity and magnetic field are found to display well-known multifractal characteristics, i.e. the associated PDFs change from Gaussian at large scales to leptocuritic (fat-tailed) at small scales [4–6]. The solar wind plasma is a complex and inhomogeneous mixture of mutually interacting regions with different physical characteristics and dynamically important kinetic processes [7, 8].

The appearance of Lévy-type gamma distributions apparently results from nonlinear turbulent transfer as suggested by similar findings in all three investigated systems and a simple reaction-rate model.

The dimensionless equations of incompressible MHD, formulated in Elsässer variables $z^{\pm} = v \pm b$ with the fluid velocity $v$ and the magnetic field $b$ which is given in Alfvén-speed units [9], read

$$\nabla \cdot z^{\pm} = 0 \quad (1)$$

$$\partial_t z^{\pm} - \nabla z^{\pm} \cdot \nabla P + \eta \Delta z^{\pm} + \eta \Delta z^{\mp} \quad (2)$$

with the total pressure $P = p + \frac{1}{2} b^2$. The dimensionless kinematic viscosity $\mu$ and magnetic diffusivity $\eta$ appear in $\eta = 1/2(\mu \pm \eta)$.

The data used in this work stems from pseudospectral high-resolution direct numerical simulations [10] based on a set of equations equivalent to Eqs. (1) and (2). It describes homogeneous fully-developed turbulent MHD and Navier-Stokes ($b \equiv 0$) flows in a cubic box of linear size $2\pi$ with periodic boundary conditions. The initial conditions for the decaying simulation run consist of random fluctuations with total energy equal to unity. In the MHD cases total kinetic and magnetic energy are approximately equal. The initial spectral energy distri-
bution is peaked at small wave numbers around $k = 4$ and decreases like a Gaussian towards small scales. In the MHD setups magnetic and cross helicity are small implying $\mathbf{z}^+ \simeq \mathbf{z}^-$. The driven turbulence simulations were run towards quasi-stationary states whose energetic and helicity characteristics as mentioned above are roughly equal to the decaying run. The MHD magnetic Prandtl number $Pr_m = \mu/\eta$ is unity. The Reynolds numbers of all configurations are of order $10^5$.

Three cases are considered. Setup (a) represents decaying macroscopically isotropic 3D MHD turbulence. The dataset contains 9 states of fully developed turbulence each comprising $1024^3$ Fourier modes. The samples are taken equidistantly in time over a period of about 3 large eddy turnover times. The angle-integrated energy spectrum of this system exhibits a Kolmogorov-like scaling law [11] in the inertial range, i.e. $E \sim k^{-5/3}$. The second dataset (b) contains simulation data of a driven quasi-stationary macroscopically anisotropic MHD flow with a strong constant mean magnetic field. The driving method as in case (b) and exhibits Kolmogorov-like scaling observed, e.g., for two-point increments of a turbulent velocity field.

To test if the abovementioned observations in the solar wind are a phenomenon related to inherent properties of turbulence time- and space-averaged increment series $\delta z^+ (\ell)$ and $\delta E (\ell)$ for different $\ell$, ranging between $\pi/512$ up to $\pi$ are computed. In system (a) the increments are normalized using $(E^T)^{1/2}$ with $E^T = 1/4 \int_V dV \langle (z^+)^2 \rangle$ to compensate for the decaying amplitude of the turbulent fluctuations. The PDFs are generated as normalized histograms of the respective increments taken over all positions in the $2\pi$-periodic box which contains the real space fields, $\mathbf{v}(\mathbf{r})$ and $\mathbf{b}(\mathbf{r})$, computed from the available Fourier-coefficients. Fig. 1 shows $P[\delta E (\ell)]$ for various $\ell$ in the isotropic case (a). The non-Gaussian nature of the PDFs over all scales is evident. Similar behavior is found in the anisotropic case (b) where the increments are taken perpendicularly to the direction of the mean field as well as in the Navier-Stokes simulation (c). The PDFs are highly symmetric and become increasingly broader with growing $\ell$ reflecting the increase of turbulent energy towards larger scales. Interestingly, the PDFs at all scales have the same leptokurtic shape resembling Lévy laws. In particular, away from the center, $\delta E = 0$, the PDFs are close to gamma distributions $\sim \exp(-\delta E/\Delta)\delta E^{-\gamma}$ of different widths $\Delta$. The exponent $\gamma$ of the best fits is constant in the inertial range and amounts approximately to 3.4 (a), 4.2 (b), and 3.1 (c). In the solar wind a similar finding however with $\gamma \approx 2.5$ was reported [4].

The similarity of the $P[\delta E (\ell)]$ on different scales $\ell$ suggests the possibility of monoscaling. The monoscaling
exponent is expected to be scale-independent in the inertial range only since the energy increments are not Galilei invariant. Therefore, small-scale $\delta E$ also comprise contributions by larger eddies which advect the small-scale fluctuations. A linearization of $\delta E$ with respect to the largest-scale contribution $(z_0^+)^2 \gg (\delta z^+)^2$ yields to lowest order $\delta E \approx (z_0^+ + \delta z^+)^2 \sim z_0^+ \delta z^+$. As a consequence, the energy increments reflect the inertial-range scaling of the turbulent Elsässer fields, i.e. $\delta E \sim \delta z^+ \sim \ell^\alpha$. To apply the rescaling procedure given by Eq. (3) (cf. also \cite{[4]}) the exponent $\alpha$ is extracted from the PDFs by two independent techniques.

Firstly, the standard deviation is considered which is defined as $\sigma(\ell) = \langle (\delta E(\ell))^2 \rangle^{1/2}$. In the inertial range $\sigma$ exhibits power-law behavior with respect to the increment distance, $\sigma(\ell) \sim \ell^\alpha$, Fig. 2 shows the standard deviation of total energy fluctuations in the inertial range for the isotropic case (a) in double logarithmic presentation. A linear least-squares fit is carried out to obtain $\alpha$. The characteristic exponents deduced in this way are $\alpha = 0.29 \pm 0.025$ for the isotropic case (a), $\alpha = 0.23 \pm 0.025$ for the anisotropic case (b), and $\alpha = 0.28 \pm 0.03$ for the Navier-Stokes flow (c). As expected these values are close to the non-intermittent scaling exponents observed for the turbulent field fluctuations, i.e. $\alpha_{K41} = 1/3$ for cases (a) and (c) while $\alpha_{K41} = 1/4$ for case (b).

Secondly, in the inertial range the characteristic exponents can be obtained via the amplitude of $P(0, \ell) \sim \ell^{-\alpha}$ profiting from the fact that the peaks of the PDFs are statistically the least noisy part of the distributions. The scaling exponent obtained by using this method is in good agreement with the value of $\alpha$ obtained via the PDF variance. Fig. 3 shows the rescaled PDFs according to Eq. (3) for MHD case (a) (similar for (b), not shown) while Fig. 4 displays the rescaled PDFs obtained from the Navier-Stokes simulation (c). The corresponding increment distances $\ell$ are all lying in the respective inertial range. Evidently the PDFs are self-similar and collapse up to $20\sigma$ with weak scattering on the master PDF, $P_x$, when using the characteristic exponents given above. The dashed lines in both figure display the best fitting gamma laws.

The PDFs of the Elsässer field fluctuations, $P[\delta z^+ (\ell)]$, in system (a) (systems (b) and (c) likewise) display a different and well-known behavior as $\ell$ increases. Due to the lacking correlation of distant turbulent fluctuations the associated distributions become approximately Gaussian at large scales. Because of the resulting multifractal scaling of

\[ \log_{10}(\sigma_l) \]

\[ \text{Slope: } 0.29 \ (2.5E-02) \]

\[ \text{FIG. 2: Standard deviation of total energy increments within the inertial range in case (a) (triangles) with linear least-squares fit (solid line).} \]

\[ \frac{\delta E_s}{\langle \delta E_s^2 \rangle^{1/2}} \]

\[ \frac{P_s(\delta E_s, \ell)}{20 \sigma} \]

\[ \text{FIG. 3: Rescaled PDFs of total energy fluctuations in the inertial range of the isotropic case (a). The gamma law } 10^{-3} \exp(-|\delta E|/0.35) |\delta E|^{-3.1} \text{ is represented by the dashed curve.} \]

\[ \frac{\delta E_s}{\langle \delta E_s^2 \rangle^{1/2}} \]

\[ \frac{P_s(\delta E_s, \ell)}{20 \sigma} \]

\[ \text{FIG. 4: Rescaled PDFs of total energy fluctuations in the inertial range of the Navier-Stokes case (c). The gamma law } 10^{-3} \exp(-|\delta E|/0.4) |\delta E|^{-3.4} \text{ is represented by the dashed curve.} \]
fluctuations with energy \( e \) as a result of turbulent transfer from fluctuations with energy \( e' \) while \( \tau_{-}(e) \) is the respective characteristic decay time. Normalization of \( n(e) \) by \( \int_{0}^{\infty} \text{d}e' n(e') \) yields the corresponding PDF \( P(e) \). In a statistically stationary state Eq. (4) then gives

\[
P(e) = C_1 \int_{e}^{\infty} \text{d}e' P(e') \frac{\tau_{-}(e)}{\tau_{+}(e', e)}
\]

where \( C_1 \) is a normalization constant. For \( \tau_{-}(e)/\tau_{+}(e', e) \sim (e'/e)^{5/2} \) this integral equation has the solution \( P(e) = C_2 e^{-\gamma} \exp(-e/\Delta) \). Thus, the model (4) which mimics in combination with the aforementioned assumptions a direct spectral transfer process yields the observed gamma distributions. Note that the lower bound of the integral in Eq. (5) implies that energy flows from higher to lower levels where for technical simplicity very large differences between \( e \) and \( e' \) are allowed. A finite upper bound of the integral in Eq. (5) does, however, not change the result fundamentally. This suggests that the observed gamma distributions are an indication of turbulent spectral transfer.

In summary it has been shown by high-resolution direct numerical simulations of incompressible turbulent magnetohydrodynamic and Navier-Stokes flows that the monoscaling of energy fluctuation PDFs observed in the solar wind is the consequence of lacking Galilei invariance of energy increments in combination with self-similar scaling of the underlying turbulent fields. The closeness of the PDFs to Lévy-type gamma distributions is made plausible by a simple model mimicking nonlinear spectral transfer.

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[16] In the specific configuration time scales can be linearly related to spatial scales (Taylor’s hypothesis).