arXiv:1503.07855v1 [gr-qc] 26 Mar 2015

Loop quantum cosmology with self-dual variables

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Using the complex-valued self-dual connection variables, the loop quantum cosmology of a closed Friedmann universe coupled to a massless scalar field is studied. It is shown how the reality conditions can be imposed in the quantum theory by choosing a particular measure for the inner product in the kinematical Hilbert space. While holonomies of the self-dual Ashtekar connection are not well-defined in the kinematical Hilbert space, it is possible to introduce a family of generalized holonomy-like operators, some of which are well-defined; these operators in turn are used in the definition of a Hamiltonian constraint operator where the scalar field can be used as a relational clock. The resulting quantum dynamics are similar, although not identical, to standard loop quantum cosmology constructed from the Ashtekar-Barbero variables with a real Immirzi parameter. Effective Friedmann equations are derived, which provide a good approximation to the full quantum dynamics for sharply-peaked states whose volume remains much larger than the Planck volume, and they show that for these states quantum gravity effects resolve the big-bang and big-crunch singularities and replace them by a non-singular bounce. Finally, the loop quantization in self-dual variables of a flat Friedmann space-time is recovered in the limit of zero spatial curvature and is identical to the standard loop quantization in terms of the real-valued Ashtekar-Barbero variables.

I. INTRODUCTION

One of the key developments in loop quantum gravity (LQG) was the introduction of the complex-valued self-dual Ashtekar connection variables [1, 2]. These variables played an important role as they not only made it possible to rewrite general relativity as a (constrained) gauge theory and thus suggested the use of holonomies in the quantum theory [3, 4], but also because the form of the constraints is greatly simplified when expressed in terms of the self-dual variables.

However, since the self-dual variables are complex-valued, it is necessary to impose reality conditions in order to recover general relativity. The imposition of these reality conditions for generic space-times in the quantum theory has turned out to be very difficult, and remains an open problem. In fact, it is precisely this difficulty that has motivated the current use of the real-valued Ashtekar-Barbero connection variables [5] in loop quantum gravity.

While the use of real-valued variables obviates the need of any reality conditions, there is a price to pay. First, it is necessary to introduce a new parameter in the theory, the Barbero-Immirzi parameter γ [6] (with $\gamma = \pm i$ for the original self-dual variables). Second, due to the appearance of a new term, the form of the scalar constraint becomes significantly more complicated. While this can be handled in the quantum theory [7], a number of supplementary quantization ambiguities associated to the presence of this additional term arise. Finally, unlike the self-dual connection, the real-valued connection does not admit an interpretation as a space-time gauge field: it only tranforms as a connection under diffeomorphisms that are tangential to the spatial surface [8, 9]. (An alternative way to render four-dimensional Lorentz covariance explicit, rather than using a space-time connection, is via projected spin networks where the SU(2) spin networks are viewed as projected SL(2,$\mathbb{C}$) spin networks [10]. A number of the recent proposals for spin foam models can be expressed in this framework [11, 12].)

Furthermore, there are a number of recent papers studying black holes in loop quantum gravity that point out that a black hole entropy of $S = A/4\ell_{\rm Pl}^2$ is naturally obtained after performing an analytic continuation sending $\gamma \to i$ [13–16]. See also [17–20] for further discussions on this topic as well as some additional results supporting the potential importance of setting the Barbero-Immirzi parameter to $\gamma = i$.

The recent results in the studies of black holes, together with the drawbacks outlined above that are associated with the use of the real-valued connection, motivate a reexamination of the possibility of using the self-dual variables and especially of the problem of imposing the reality conditions in the quantum theory. This last question has previously been studied in some detail for the case of vacuum spherically symmetric and asymptotically flat space-times [21], but has not been addressed in any of the other highly symmetric space-times where the problem becomes simpler, like the homogeneous cosmological space-times. One of the main goals of this paper is to explicitly show how the reality conditions can be imposed for the Friedmann-Lemaître-Robertson-Walker (FLRW) space-time. While the presence of the symmetries of the FLRW space-times will simplify this task, the hope is that this result may give some insight into how the reality conditions can be applied in a more general context.

Indeed, the imposition of the reality conditions will be one of the key steps in the study of the loop quantum cosmology (LQC) of the spatially closed FLRW space-time

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in terms of the self-dual variables that is presented in this paper. Since there exists a vast literature in the field of loop quantum cosmology (for recent reviews, see e.g. [22–24]), many of the details of the quantization procedure —when following steps analogous to those already well known in loop quantum cosmology— will be only briefly described in this paper in order to avoid unnecessary repetition and the reader is referred to the original papers studying the 'improved dynamics' LQC of spatially closed FLRW space-times in terms of the real-valued Ashtekar-Barbero connection [25, 26], which I will refer to as 'standard LQC' from now on. Rather, the focus of this paper will be on the differences that arise when one works with self-dual variables.

The outline of the paper is as follows: there is a brief introduction to the self-dual Ashtekar variables in Sec. II, which is followed by a description in Sec. III of the symmetry reduction necessary in order to study the spatially closed FLRW space-time in terms of these variables. Then in Sec. IV the quantum theory is presented in two steps, the first being a definition of the kinematical Hilbert space where following the procedure suggested in [27] the reality conditions are imposed via the choice of the measure in the inner product, and the second being the construction of the Hamiltonian constraint operator, with the important result that the classical big-bang and big-crunch singularities are resolved. The effective equations for the quantum theory are studied in Sec. V, and there is a discussion in Sec. VI which includes a comparison of the results obtained in this paper with those of [28] where an analytic continuation is used in order to define a different version of LQC with a Barbero-Immirzi parameter of $\gamma = i$.

II. SELF-DUAL VARIABLES

This section contains a very brief review of the complex-valued self-dual variables that will be used in this paper. The review is restricted to the definitions and results that will be needed in the following sections and is thus necessarily incomplete. For further details, see e.g. [27] and the many references therein.

The self-dual variables are the Ashtekar connection A_a^k and the densitized triad E_k^a, which are related to the triads e_k^a, the determinant of the spatial metric q, the spin-connection Γ_a^k and the extrinsic curvature $K_a^k = K_{ab} e^{bk}$ as follows:

$$A_a^k = \Gamma_a^k - i K_a^k, \qquad E_k^a = \sqrt{q}\, e_k^a. \tag{1}$$

These variables are canonically conjugate,

$$\{A_a^j(x), E_k^b(y)\} = i \cdot 8\pi G\, \delta_a^b\, \delta_k^j\, \delta^{(3)}(x-y), \tag{2}$$

and one of their main advantages is that the scalar constraint takes a simple form,

$$\mathcal{H} = \frac{E_i^a E_j^b}{16\pi G \sqrt{q}} \epsilon^{ij}{}_k F_{ab}{}^k + \frac{p_\phi^2}{2\sqrt{q}} \approx 0, \tag{3}$$

with the field strength of the self-dual connection being

$$F_{ab}{}^k = 2\, \partial_{[a} A_{b]}^k + \epsilon_{ij}{}^k A_a^i A_b^j, \tag{4}$$

and a massless scalar field is taken as the matter content. The Gauss and diffeomorphism constraints also have a similarly simple expression [27].

Since the variables A_a^k and E_k^a are complex-valued (and the spatial metric and extrinsic curvature appearing in the ADM formulation of general relativity are not), it is necessary to impose reality conditions which take the form

$$A_a^k + \left(A_a^k\right)^\star = 2\, \Gamma_a^k, \qquad E_k^a E^{bk} > 0. \tag{5}$$

As mentioned in the Introduction, it is currently not known how to correctly implement these reality conditions in the full quantum theory and this is one of the main reasons the real-valued Ashtekar-Barbero connection is now typically used in loop quantum gravity.

The fundamental variables in the quantum theory will correspond to (i) what I will call 'generalized holonomies' along paths

$$h = \mathcal{P} \exp\left[\int_{\text{path}} A_a\right], \tag{6}$$

with $A_a := A_a^i\, \alpha \sigma_i$, where the σ_i are the Pauli matrices and α is a complex-valued parameter, and (ii) areas of surfaces. For standard holonomies $\alpha = 1/2i$, but as shall be shown below these holonomies are not well-defined in the kinematical Hilbert space of self-dual loop quantum cosmology given in Sec. IV A. Instead, it is necessary that $\alpha \in \mathbb{R}$ for the operators corresponding to generalized holonomies to be well-defined.

Here an important remark is in order. Defining

$$F_{ab} = 2\partial_{[a} A_{b]} + [A_a, A_b], \tag{7}$$

and

$$F_{ab}{}^k = 2\partial_{[a} A_{b]}^k + \epsilon_{ij}{}^k A_a^i A_b^j, \tag{8}$$

the relation

$$F_{ab} = F_{ab}{}^k\, \tilde{\alpha} \sigma_k \tag{9}$$

holds if and only if $\alpha = 1/2i$ (in which case $\tilde{\alpha} = \alpha$). For this reason, if α is left free, it is important to remember that while generalized holonomies of A_a will be related in a simple fashion to A_a^i,

$$A_a^i \cdot t^a = \lim_{z \to 0} \frac{\text{Tr}[(\mathcal{P} e^{\int_0^z A_a t^a}) \sigma^i]}{2\alpha z}, \tag{10}$$

(where t^a is the tangent vector to the path and z is the length of the path), the relation between the field strength F_{ab} obtained from the generalized holonomy of A_a around a closed loop and the quantity $F_{ab}{}^k$ which appears in the scalar constraint is considerably more complicated. Because of this, it will be more convenient to express the scalar constraint operator in self-dual loop quantum cosmology in terms of a non-local connection operator (constructed from generalized holonomies) rather than a non-local field strength operator as is commonly done for standard isotropic loop quantum cosmology.

III. COSMOLOGY

The metric of a closed FLRW space-time is

$$ds^2 = -dt^2 + a(t)^2 \mathring{\omega}_a^i \mathring{\omega}_b^j \delta_{ij} dx^a dx^b, \tag{11}$$

where the fiducial co-triads $\mathring{\omega}_a^i$ satisfy the relation

$$\mathrm{d}\mathring{\omega}^i + \frac{1}{2}\mathring{\epsilon}^i{}_{jk}\mathring{\omega}^j \wedge \mathring{\omega}^k = 0, \tag{12}$$

where $\mathring{\epsilon}_{ijk}$ is totally anti-symmetric and $\mathring{\epsilon}_{123} = 1$. The fiducial triads $\mathring{e}_i^a$ (the inverse of the co-triads $\mathring{\omega}_a^i$) provide a basis to a three-sphere of radius 2 and volume $V_o = 16\pi^2$.

The spatial part of the metric (11) can be rewritten in terms of co-triads as $\omega_a^k = a(t)\,\mathring{\omega}_a^k$ and from this and (12), it follows that the spin-connection has the simple form

$$\Gamma_a^k = -\epsilon^{ijk} e_j^b \left(\partial_{[a}\omega_{b]k} + \frac{1}{2} e_k^c \omega_a^l \partial_{[c}\omega_{b]l} \right) = \frac{1}{2}\,\mathring{\omega}_a^k. \tag{13}$$

A convenient parametrization of the self-dual Ashtekar connection and the densitized triads is

$$A_a^k = \frac{c}{\ell_o}\,\mathring{\omega}_a^k, \qquad E_k^a = \frac{p}{\ell_o^2}\,\sqrt{\mathring{q}}\,\mathring{e}_k^a, \tag{14}$$

where $\mathring{q}$ is the determinant of the fiducial metric $\mathring{q}_{ab} = \mathring{\omega}_a^i \mathring{\omega}_{bi}$ and $\ell_o := V_o^{1/3}$. In classical general relativity, it is easy to show that $|p| = a^2 \ell_o^2$ and $c = \ell_o[1/2 - i\dot{a}]$, where the dot indicates a derivative with respect to the proper time t. Also, note that p can be negative as it encodes the orientation of the triads: a positive p corrresponds to right-handed triads and a negative p to left-handed triads. Since the handedness of the triads does not affect the dynamics of the space-time, neither does the sign of p. For further details on this parametrization of a closed FLRW space-time, see [25].

This parametrization induces from the full theory the symplectic structure

$$\{c, p\} = i \cdot \frac{8\pi G}{3}, \tag{15}$$

and the reality conditions reduce to

$$c + c^\star = \ell_o, \qquad p^\star = p. \tag{16}$$

Finally, it is easy to check that the parametrization (14) automatically satisfies the diffeomorphism and Gauss constraints, while the scalar constraint has the form

$$\mathcal{H} = \frac{3\sqrt{|p|}}{8\pi G}\mathrm{sgn}(p)^2 \Big[c^2 - \ell_o c \Big] + \frac{p_\phi^2}{2|p|^{3/2}} \approx 0. \tag{17}$$

Here $p_\phi = \dot{\phi}/V_o$ and $\{\phi, p_\phi\} = 1$. In the classical theory, $\mathrm{sgn}(p)^2$ can simply be set equal to 1, but the presence of two $\mathrm{sgn}(p)$ terms will be useful in order to simplify the action of the Hamiltonian constraint operator in the quantum theory.

Note that for the flat FLRW space-time, (12) becomes simply $\mathrm{d}\mathring{\omega}^i = 0$, which leads to a vanishing spin-connection. Thus, a convenient shortcut in order to obtain the reality conditions and the Hamiltonian constraint for the flat FLRW cosmology is to set $\ell_o = 0$ in (16) and (17). This is what can be done at the end of this paper in order to extend the results obtained here for the case of vanishing spatial curvature.

The classical Friedmann equation is obtained by squaring the relation

$$\dot{p} = \{p, \mathcal{H}\} = -i \cdot 3\sqrt{|p|}(2c - \ell_o), \tag{18}$$

and using the constraint $\mathcal{H} = 0$ in order to obtain

$$H^2 = \frac{\dot{p}^2}{4p^2} = \frac{8\pi G}{3}\rho - \frac{\ell_o^2}{4|p|}. \tag{19}$$

Here $H = \dot{a}/a$ is the Hubble rate and $\rho = p_\phi^2/2|p|^3$ is the energy density of the massless scalar field. Note that the factor of 4 in the denominator of the second term appears due to the conventions in (12) where the radius of the three-sphere is taken to be 2.

Together with the continuity equation

$$\dot{\rho} + 3H(\rho + P) = 0, \tag{20}$$

where P is the pressure of the matter field, the Friedmann equation determines the dynamics of the space-time. For a massless scalar field, $P = \rho$ and the continuity equation simplifies to $\dot{p}_\phi = 0$.

An important point here is that there is no need to dynamically impose reality conditions if one is working with the equations (19) and (20): if the scale factor and energy density are initially taken to be real, their reality will be preserved by the dynamics generated by (17).

Finally, as explained above in Sec. II, generalized holonomies of the Ashtekar connection will play a key role in the quantum theory. An important result following from (6) is that the generalized holonomy taken along a path tangential to the fiducial triad $\mathring{e}_k$ and of a length $\mathring{\mu}\ell_o$ with respect to the fiducial metric is

$$\begin{aligned} h_k(\mathring{\mu}) &= \exp\left(\alpha\mathring{\mu}c\sigma_k\right) \\ &= \cosh\left(\alpha\mathring{\mu}c\right)\mathbb{I} + \sinh\left(\alpha\mathring{\mu}c\right)\sigma_k. \end{aligned} \tag{21}$$

IV. QUANTUM COSMOLOGY

This section is divided in two parts: the first contains the definition of the kinematical Hilbert space of the theory —including the implementation of the reality conditions by making a specific choice of the measure in the inner product— while the second defines the Hamiltonian constraint operator for the theory and makes explicit its action on generic states.

A. The Kinematical Hilbert Space

The kinematical Hilbert space of the theory has two sectors, namely the gravitational and matter sectors, and the total Hilbert space is simply given by the tensor product between them,

$$H_{\rm tot} = H_g \otimes H_m, \tag{22}$$

and therefore the wave functions have the form

$$\Psi(c,\phi) = \psi(c) \otimes \chi(\phi). \tag{23}$$

Starting with the gravitational sector, the fundamental operators in the quantum theory (following loop quantum gravity) are taken to be generalized holonomies of A_a^k and areas of surfaces. Due to the homogeneity of the spacetime, it is enough to consider generalized holonomies that are parallel to the fiducial triads $\mathring{e}_k^a$ [in which case the c-dependence only appears in sums and differences of terms of the form $\exp(\alpha\mathring{\mu}c)$ as can be seen in (21)], and simply define an operator corresponding to the classical variable p for an area operator.

Since the self-dual variables are complex, the wave functions $\psi(c)$ are required to be holomorphic functions of c. Then, following the standard procedure of LQC, a natural choice for the definition of the fundamental operators is

$$\hat{p}\,\psi(c) = -\frac{8\pi G\hbar}{3}\frac{d}{dc}\,\psi(c), \qquad \widehat{e^{\mu c}}\,\psi(c) = e^{\mu c}\,\psi(c), \tag{24}$$

where $\mu = \alpha\mathring{\mu}$. As explained above, since linear combinations of $\widehat{e^{\mu c}}$ encode the non-trivial information about a generalized holonomy of length $\mathring{\mu}\ell_o$ (with respect to the fiducial metric $\mathring{q}_{ab}$) along an edge parallel to any one of the $\mathring{e}_k^a$, and $\hat{p}$ is proportional to the equatorial area of the 3-sphere, these operators are indeed the LQC equivalent of the fundamental LQG operators, namely (generalized) holonomies and areas.

It terms of these basic operators, the reality conditions are

$$p^\dagger = p, \qquad \left(\widehat{e^{\mu c}}\right)^\dagger = e^{\bar{\mu}\ell_o}\,\widehat{e^{-\bar{\mu}c}}, \tag{25}$$

which shall be implemented by making a specific choice for the measure in the inner product as suggested in [27]. Note that since ℓ_o is a real number for the closed Friedmann universe, the $\exp(\bar{\mu}\ell_o)$ term simply provides a numerical pre-factor and there is no operator ordering ambiguity.

The eigenstates of $\hat{p}$ (with eigenvalue $p \in \mathbb{C}$) are given by

$$|p\rangle = e^{-3pc/8\pi G\hbar}, \tag{26}$$

and it is easy to check that

$$\widehat{e^{\mu c}}\,|p\rangle = |p - 8\pi G\hbar\mu/3\rangle \tag{27}$$

acts as a shift operator on $|p\rangle$.

Already from this result it can be seen that it will be necessary to require $\alpha \in \mathbb{R}$ for the following reason: the reality conditions impose that p be real. Therefore, assuming that p is initially real, for the shifted state to also represent a real-valued area, $\mu = \alpha\mathring{\mu}$ must be real. Since $\mathring{\mu}$ is by definition real, this condition requires that α be real. A further reason to require α to be real, as shall be shown below, is to ensure that the fundamental operators (24) are well defined.

For the inner product, the use of the Bohr compactification of the real line in standard LQC suggests the ansatz

$$\langle\psi_1|\psi_2\rangle = \lim_{L\to\infty}\frac{1}{2L}\int_{-L}^{L} dc_R \int_{-L}^{L} dc_I\,\mathfrak{u}(c,\bar{c})\,\bar{\psi}_1\,\psi_2, \tag{28}$$

where $c = c_R + ic_I$, and the measure $\mathfrak{u}(c,\bar{c})$ must be chosen so that the reality conditions (25) are implemented.

The first reality condition, after integrating by parts and using the holomorphicity of the wave function $\psi(c)$, gives the result that $\mathfrak{u}(c,\bar{c}) = \mathfrak{u}(c+\bar{c})$. The second condition implies that

$$\langle e^{\mu c}\psi_1|\psi_2\rangle = \int_c \mathfrak{u}(c,\bar{c})\,e^{\bar{\mu}\bar{c}}\bar{\psi}_1\,\psi_2, \tag{29}$$

[where $\int_c$ is shorthand for the integrals over c_R and c_I given in (28), including the prefactor of $1/2L$ and the limit of $L\to\infty$] and

$$\langle\psi_1|e^{\bar{\mu}(\ell_o - c)}\psi_2\rangle = \int_c \mathfrak{u}(c,\bar{c})\,\bar{\psi}_1\,e^{\bar{\mu}(\ell_o - c)}\psi_2, \tag{30}$$

must be equal. This requires that the measure $\mathfrak{u}(c,\bar{c})$ be distributional and have the form

$$\mathfrak{u}(c,\bar{c}) = \delta(c + \bar{c} - \ell_o), \tag{31}$$

where the overall numerical factor is set to 1 for simplicity. Note that this result also satisfies the requirement of the first reality condition that $\mathfrak{u}(c,\bar{c}) = \mathfrak{u}(c+\bar{c})$.

With this measure, it is easy to check that the eigenstates $|p\rangle$ are normalizable only for real p, and that the inner product between two eigenstates $|p_1\rangle$ and $|p_2\rangle$ (assuming p_1 and p_2 are real) is proportional to the Kronecker delta,

$$\langle p_1|p_2\rangle = e^{-3\ell_o p_1/8\pi G\hbar}\,\delta_{p_1,p_2}. \tag{32}$$

Thus, requiring the normalizability of the wave function automatically ensures the reality of p.

From this last result, it follows immediately that any normalized wave function in H_g can be written as

$$|\psi\rangle = \sum_{p\in\mathbb{R}} C_p\, |p\rangle, \tag{33}$$

with $\sum_p |C_p|^2\, e^{-3\ell_o p/8\pi G\hbar} = 1$.

Also, it is easy to check that if the imaginary part of α does not vanish, then $\langle\psi_1|\exp(\mu c)|\psi_2\rangle$ diverges for all normalized $|\psi_1\rangle, |\psi_2\rangle$ and for all values of $\mathring{\mu} \neq 0$ (recall that $\mathring{\mu} \in \mathbb{R}$ by definition). For this reason, it is necessary to impose that $\alpha \in \mathbb{R}$. While at present there is no reason to prefer a specific numerical value for α, it seems reasonable to assume that it is of the order of unity, with $1/2$ appearing to be a somewhat natural choice since for standard holonomies $\alpha = 1/2i$.

Note that it is possible to define a family of normalized eigenfunctions of $\hat{p}$,

$$|p\rangle_n = e^{3\ell_o p/16\pi G\hbar}\, |p\rangle, \quad \Rightarrow \quad {}_n\langle p_1|p_2\rangle_n = \delta_{p_1,p_2}, \tag{34}$$

in which case it follows that

$$\hat{p}|p\rangle_n = p|p\rangle_n, \quad \widehat{e^{\mu c}}|p\rangle_n = e^{\ell_o\mu/2}|p - 8\pi G\hbar\mu/3\rangle_n. \tag{35}$$

While the basis $|p\rangle_n$ can be convenient for some calculations, e.g. showing that the reality conditions (25) are satisfied, the basis $|p\rangle$ behaves in a simpler fashion under the action of the Hamiltonian constraint operator and therefore $\psi(p)$ shall be taken to be of the form (33) in Sec. IV B.

Finally, since the sign of p classically only determines the orientation of the triads and does not affect the metric, the wave function $\psi(p)$ should be invariant under the parity transformation $p \to -p$. Equivalently, $\psi(c)$ should be invariant under the parity transformation $c \to -c$ as both Γ_a^k and K_a^k change sign under a parity transformation (see the discussion on parity transformations in [29] for further details on this point). Since the parity transformation sends $c \to -c$, from the definition of the eigenstates $|p\rangle$ in (26) it follows that

$$\Pi\, |p\rangle = |-p\rangle, \tag{36}$$

and then the requirement that the wave function be invariant under a parity transformation is simply

$$\Pi\, \psi(p) = \psi(p). \tag{37}$$

This completes the definition of the kinematical Hilbert space of the gravitational sector of the theory. For the matter sector, the kinematical Hilbert space H_m corresponds to the space of square-integrable functions $\chi(\phi)$ over the real line, and the fundamental operators correspond to the scalar field ϕ and its momentum p_ϕ,

$$\hat{\phi}\chi(\phi) = \phi\chi(\phi), \qquad \hat{p}_\phi\chi(\phi) = -i\hbar\frac{d\chi(\phi)}{d\phi}, \tag{38}$$

which act by multiplication and differentiation respectively.

It is clear that in order to obtain the kinematical Hilbert space for a flat FLRW cosmology, it is sufficient to set $\ell_o = 0$ in all of the results in this section.

B. The Hamiltonian Constraint Operator

In order to determine the physical Hilbert space, it is necessary to construct the Hamiltonian constraint operator and find the states that it annihilates. In the classical Hamiltonian constraint, the quantities p, c and p_ϕ appear. While operators corresponding to $\hat{p}$ and $\hat{p}_\phi$ exist, there is no operator corresponding to the connection. Instead, it is necessary to use operators of the type (24).

This can be done by employing the relation (10) and, instead of taking the limit of the path length going to zero (in which case the result would not be well-defined as an operator on the kinematical Hilbert space), the path length (with respect to the physical metric) is set to be a minimal length ℓ_m, assumed to be of the order of the Planck length. This choice is of course motivated by the Planck-scale discreteness of (self-dual) LQG [30]. Note that a similar procedure has also been used in standard LQC for Bianchi space-times with a non-vanishing spatial curvature where it was also impossible to properly define a non-local field strength operator as is done for isotropic or spatially flat space-times [29, 31].

The task is then to choose an appropriate value for the length $\mathring{\mu}$ appearing in the definition of the generalized holonomy. Recall from (21) that the length of the generalized holonomy with respect to the fiducial metric $\mathring{q}_{ab}$ is $\mathring{\mu}\ell_o$ and that therefore the physical length is $\mathring{\mu}\ell_o a(t) = \mathring{\mu}\sqrt{|p|}$. Since the physical length of the generalized holonomy has been chosen to be set to ℓ_m, this gives $\mathring{\mu} = \ell_m/\sqrt{|p|}$, or equivalently

$$\mu = \frac{\lambda_m}{\sqrt{|p|}}, \tag{39}$$

where $\lambda_m = \alpha\ell_m$. This choice for μ corresponds to the 'improved dynamics' of standard loop quantum cosmology, as first introduced in [32]. The choice of ℓ_m —which gives the scale of the underlying discreteness of the quantum theory— in standard LQC is set to be the square root of the minimal non-zero eigenvalue of the area operator of loop quantum gravity. A similar procedure can be followed here using known results about the spectrum of the area operator for self-dual variables which would set $\ell_m^2 = 4\sqrt{3}\pi G\hbar$ [30], but in principle ℓ_m should be determined by a derivation of LQC from full LQG.

From this, it follows that the non-local connection operator has the form (ignoring factor-ordering ambiguities for now)

$$\begin{aligned}\hat{c} &= \frac{\sqrt{|p|}}{\lambda_m}\sinh\frac{\lambda_m c}{\sqrt{|p|}} \\ &= \frac{\sqrt{|p|}}{2\lambda_m}\left(e^{\lambda_m c/\sqrt{|p|}} - e^{-\lambda_m c/\sqrt{|p|}}\right).\end{aligned} \tag{40}$$

The final step, before choosing a factor-ordering and calculating the action of the Hamiltonian constraint, is to define how the operator $\exp(\lambda_m c/\sqrt{|p|})$ acts. Since the prefactor to c depends on p, the simple shift operation defined in (24) for the case of a constant prefactor cannot be applied here.

A solution is offered by the fact that the classical variable conjugate to $\beta = c/\sqrt{|p|}$ is $V = \mathrm{sgn}(p)|p|^{3/2}$,

$$\{\beta, V\} = i \cdot 4\pi G, \tag{41}$$

and therefore this operator should act as a simple shift operator on the eigenkets $|V\rangle$ corresponding to the $\hat{V}$ operator,

$$e^{\lambda_m c/\sqrt{|p|}}|V\rangle = |V - 4\pi G\hbar\lambda_m\rangle, \tag{42}$$

where the eigenvectors

$$|V\rangle = \exp[-3\,\mathrm{sgn}(V)|V|^{2/3}c/8\pi G\hbar]$$

are simply a relabeling of the eigenvectors $|p\rangle$ defined in (26) with $p = \mathrm{sgn}(V)|V|^{2/3}$. Note that $\mathrm{sgn}(p) = \mathrm{sgn}(V)$ and therefore the sign of V encodes the handedness of the triads. Finally, in order to use a variable as similar to that of standard LQC as possible,

$$\nu = \frac{V}{2\pi G\hbar\lambda_m} \tag{43}$$

is defined and then

$$e^{\lambda_m c/\sqrt{|p|}}|\nu\rangle = |\nu - 2\rangle. \tag{44}$$

As an aside, the action of $\exp(\lambda_m c/\sqrt{|p|})$ acts on the normalized eigenvectors $|\nu\rangle_n$ as

$$e^{\lambda_m c/\sqrt{|p|}}|\nu\rangle_n = e^{-3\ell_o\lambda_m^{2/3}\mathrm{sgn}(\nu-2)|\nu-2|^{2/3}/8(2\pi G\hbar)^{1/3}} \times e^{3\ell_o\lambda_m^{2/3}\mathrm{sgn}(\nu)|\nu|^{2/3}/8(2\pi G\hbar)^{1/3}}|\nu - 2\rangle_n.$$

Since the action of this shift operator is very complicated when expressed in terms of the normalized eigenvectors, it is more convenient to use the non-normalized basis with the inner product (32) and this is what shall be done for the remainder of the paper.

The final step is to choose a lapse and a factor-ordering for the Hamiltonian constraint operator corresponding to $\mathcal{C}_H = N\mathcal{H}$. While the factor-ordering choices should ultimately be determined by the full theory of quantum gravity, at this time no particular choice is preferred, apart perhaps from aesthetical considerations. There are obviously many possibilities, and a particularly convenient choice that simplifies the action of the constraint is to take the lapse to be $N = |p|^{3/2}$ following [33] and a factor-ordering similar to [34],

$$\begin{aligned}\widehat{\mathcal{C}_H} = &\frac{3\pi G\hbar^2}{8}\sqrt{|\nu|}(\mathcal{N}_+ - \mathcal{N}_-)|\nu|(\mathcal{N}_+ - \mathcal{N}_-)\sqrt{|\nu|} \\ &+ \frac{3\ell_o}{8}(2\pi G\hbar\lambda_m)^{2/3}|\nu|^{5/6}(\mathcal{N}_+ - \mathcal{N}_-)|\nu|^{5/6}\mathrm{sgn}(\nu) \\ &- \frac{\hbar^2}{2}\partial_\phi^2,\end{aligned} \tag{45}$$

where the hats on operators have been dropped in order to simplify the notation, and

$$\begin{aligned}\mathcal{N}_\pm|\nu\rangle :=& \frac{1}{2}\left[e^{\mp\lambda_m c/\sqrt{p}}\mathrm{sgn}(\nu) + \mathrm{sgn}(\nu)e^{\mp\lambda_m c/\sqrt{p}}\right]|\nu\rangle \\ =& \frac{1}{2}\big[\mathrm{sgn}(\nu) + \mathrm{sgn}(\nu \pm 2)\big]|\nu \pm 2\rangle.\end{aligned} \tag{46}$$

The factor-ordering choices made in (45) and (46) significantly simplify the action of the Hamiltonian constraint operator, primarily because the operator $\mathcal{N}_\pm$ annihilates any eigenket $|\nu\rangle$ that would otherwise change sign. Note also that because of this effect, $\mathrm{sgn}(\nu)$ commutes with $\mathcal{N}_\pm$.

Demanding that $\widehat{\mathcal{C}_H}$ annihilate wave functions $\Psi(\nu, \phi) = \psi(\nu) \otimes \chi(\phi)$ in the physical Hilbert space gives

$$-\hbar^2\partial_\phi^2\Psi(\nu, \phi) = \Theta\,\Psi(\nu, \phi), \tag{47}$$

where Θ, being -2 times the first two lines of (45), acts only on $\psi(v)$. It is then clear that ϕ can play the role of a relational clock, with $\sqrt{\Theta}$ being a true Hamiltonian (assuming that Θ is essentially self-adjoint and has a positive spectrum), and the physical Hilbert space is given by the positive frequency solutions to (47), namely

$$-i\hbar\,\partial_\phi\Psi(\nu, \phi) = \sqrt{\Theta}\,\Psi(\nu, \phi). \tag{48}$$

Then, if one is given an initial state with respect to the relational clock ϕ, $\psi_o(\nu) = \Psi(\nu, \phi_o)$, its evolution (again with respect to the relational clock ϕ) is

$$\Psi(\nu, \phi) = e^{i\sqrt{\Theta}(\phi-\phi_o)/\hbar}\,\psi_o(\nu). \tag{49}$$

Due to the form of $\mathcal{N}_\pm$, the action of Θ on $\psi(\nu)$ splits nicely into three parts, one part for $\nu > 0$, another for $\nu < 0$, and a final part for $\nu = 0$. It is immediately obvious that the state $|\nu = 0\rangle$ is annihilated by Θ,

$$\Theta|\nu = 0\rangle = 0, \tag{50}$$

and that it is therefore a stationary state with respect to the relational clock ϕ. Furthermore, it is also clear that there is no eigenket $|\nu\rangle$ with $\nu \neq 0$ that under the action of the Hamiltonian constraint operator is mapped to $|\nu = 0\rangle$. While the states $|\nu = \pm 4\rangle$ and $|\nu = \pm 2\rangle$ are shifted (in part) to the zero volume state, as can be seen in (45) the new zero volume states will be annihilated by a prefactor of ν (to some positive power) whose eigenvalue is of course 0 for these newly-shifted states.

It is in this sense that the cosmological singularity is resolved in this model: if an initial state $\psi_o(\nu)$ is non-singular (i.e., it has no support on the 'singular' state $|\nu = 0\rangle$ which corresponds to the points in phase space of $a(t) = 0$ where the cosmological singularities occur in the classical theory), then under the evolution of the Hamiltonian constraint operator with respect to the relational clock ϕ (49), the wave function will always remain non-singular.

Also, as already mentioned, due to the factor-ordering choice made in the definition of the shift operators $\mathcal{N}_\pm$, any superposition of eigenkets $|\nu\rangle$ with $\nu > 0$ will, under the action of Θ, continue to have only support on $\nu > 0$ since the $\mathrm{sgn}(\nu)$ operators will annihilate any state that would be shifted to a negative value of ν, as seen in (46). For the same reason, any superposition of eigenkets $|\nu\rangle$ with $\nu < 0$ will continue to have support only on $\nu < 0$ when acted upon by Θ. Thus it follows that the positive and negative sectors of ν also decouple under the action of the Hamiltonian constraint operator.

It is easy to verify that under the parity transformation (37), $\Pi\,\psi(\nu) = \psi(-\nu)$ and the shift operators $\mathcal{N}_\pm$ transform as

$$\Pi\,\mathcal{N}_\pm\,\Pi = \mathcal{N}_\mp, \tag{51}$$

and then it immediately follows that Θ is invariant under a parity transformation,

$$\Pi\,\Theta\,\Pi = \Theta, \tag{52}$$

and that any state that initially satisfies (37) will continue to do so under the evolution (with respect to ϕ) generated by the Hamiltonian constraint operator. Due to these properties of Θ, it is enough to calculate the action of this operator on the restriction of $\psi(\nu)$ to strictly positive ν, given by

$$\begin{aligned}\Theta\,\psi(\nu) = \frac{3\pi G\hbar^2}{4}\sqrt{\nu}\Big[&(\nu+2)\sqrt{\nu+4}\,\psi(\nu+4) \\ &-\sqrt{\nu}\Big(\nu+2+\theta_{\nu-2}(\nu-2)\Big)\psi(\nu) \\ &+\theta_{\nu-4}(\nu-2)\sqrt{\nu-4}\,\psi(\nu-4)\Big] \\ +\frac{3\ell_o}{4}(2\pi G\hbar\lambda_m)^{2/3}\Big[&\Big(\nu(\nu+2)\Big)^{5/6}\psi(\nu+2) \\ &-\theta_{\nu-2}\Big(\nu(\nu-2)\Big)^{5/6}\psi(\nu-2)\Big],\end{aligned} \tag{53}$$

where θ_x is the Heaviside function equal to 1 for positive x and zero otherwise. Then, from this it is possible to determine the action of Θ on strictly negative ν via the relation (37). (Recall that $\Theta|\nu = 0\rangle = 0$.) With this explicit definition of the action of the Hamiltonian constraint operator, it is now possible to numerically solve for the quantum dynamics (with respect to ϕ) of any initial states of interest.

As in standard LQC, a superselection in ν occurs here, where the Hamiltonian constraint operator only couples values of ν that are separated by an integer multiple of 2. Therefore, it is sufficient to restrict our attention to a superselection lattice denoted by the parameter $\epsilon \in (0, 2]$ and given by $L_\epsilon = \{\epsilon + 2n, n \in \mathbb{N}\}$. From the parity condition (37), it follows that once ϵ is chosen for positive ν, the superselection lattice for negative ν must be $L_{-\epsilon} = \{-\epsilon - 2n, n \in \mathbb{N}\}$.

Due to the presence of quantization ambiguities already in standard LQC, there exist a handful of loop quantizations of the spatially closed FLRW space-time [25, 26, 35, 36]. It has been shown numerically that the differences in these various quantum theories, while extant, are very small and do not significantly change the qualitative behaviour of the space-time, at least for states where the bounce occurs at a large volume compared to the Planck volume [35, 37]. For this reason, the qualitative results obtained in previous studies of the loop quantization of spatially closed FLRW space-times are expected to also hold for the self-dual loop quantization given here.

The most important result established for standard LQC is the resolution of the big-bang singularity and its replacement by a bounce, which are expected to occur in self-dual LQC as well following the arguments above. The effective equations, presented in the following section, provide further evidence for the presence of a bounce resolving the singularity in self-dual LQC.

Most tellingly, however, is the fact that the Hamiltonian constraint operator for a flat FLRW space-time [obtained from (45) by setting $\ell_o = 0$] is identical[1] to the Hamiltonian constraint operator for the standard loop quantum cosmology of a flat FLRW space-time, for the factor-ordering prescription called 'sMMO' in [37]. (This equivalence arises due to the choice of the lapse and the factor-ordering choices made in the definition of (45), motivated by choices made in standard LQC in [33] and [34] respectively.) This specific model has been studied in some detail, and it has been shown that the big-bang singularity is resolved and is generically replaced by a bounce —even for states that have no semi-classical interpretation [34, 37]. Thus, in the limit of vanishing spatial curvature, it is clear that a bounce occurs in self-dual LQC, and that this bounce has all of the same properties as the 'sMMO' prescription of standard LQC.

Since spatial curvature is typically negligible in the high-curvature regime of isotropic cosmology, it is reasonable to expect that a bounce will occur for generic solutions to (45), including for $\ell_o \neq 0$. Nonetheless, it is necessary to study the full quantum dynamics in further detail in order to confirm this expectation. In addition, it would be interesting to determine how the small differences between self-dual LQC and standard LQC in the definition of the Hamiltonian constraint operator affect the quantum dynamics.

V. THE EFFECTIVE THEORY

It is by now well known that in standard LQC, for physical states that are sharply peaked at some initial 'time' ϕ_o, there exist effective equations that provide an excellent approximation to the full quantum dynamics at all times [25, 38].

[1] The form is identical, and the numerical factors agree if λ_m is set equal to the square root of the area gap of standard LQG.

These effective equations are extremely reliable because quantum fluctuations (if initially small) do not grow significantly as the high-curvature regime is approached so long as the volume of the space-time remains large compared to the Planck volume. The reason for this is that in minisuperspace models, it is global quantities such as the total volume that are of interest and if $V \gg \ell_{\rm Pl}^3$ these correspond to heavy degrees of freedom where quantum fluctuations are negligible [39].

These results, originally obtained for standard LQC, can easily be extended for the case of self-dual LQC, and therefore the effective equations will provide an excellent approximation to the full dynamics of sharply-peaked states in this setting also, so long as the volume of the space-time remains much larger than $\ell_{\rm Pl}^3$. Thus, in this section any terms of the order of $\ell_{\rm Pl}^3/V$ can be dropped as these terms are negligible in the regime where the effective equations are reliable. If one wishes to study the dynamics of a space-time that reaches a volume comparable to the Planck volume, then a full quantum evolution using the Hamiltonian constraint operator is necessary.

The effective equations are obtained by taking the Hamiltonian constraint operator (45), treating it as a classical constraint,

$$C_H^{eff} = \frac{3p^{3/2}}{8\pi G\lambda_m^2}\Big[\sinh^2\frac{\lambda_m c}{\sqrt{p}} - \frac{\lambda_m \ell_o}{\sqrt{p}}\sinh\frac{\lambda_m c}{\sqrt{p}}\Big] + p^{3/2}\rho \approx 0, \qquad (54)$$

and using the standard Poisson relations $\dot{\mathcal{O}} = \{\mathcal{O}, C_H^{eff}\}$ in order to determine the dynamics. In the classical and effective theories it is possible to consider only positive p and that is what shall be done here. Note that it is easy to allow for any perfect fluid simply by replacing $p_\phi^2/2p^{3/2}$ by $p^{3/2}\rho$, with ρ being the energy density of the perfect fluid.

Recalling the fundamental Poisson brackets given in (15), one of the effective equations of motion is

$$\dot{p} = -\frac{2ip}{\lambda_m}\Big[\sinh\frac{\lambda_m c}{\sqrt{p}}\cosh\frac{\lambda_m c}{\sqrt{p}} - \frac{\lambda_m \ell_o}{\sqrt{p}}\cosh\frac{\lambda_m c}{\sqrt{p}}\Big], \qquad (55)$$

which, after using the effective scalar constraint (54) twice, gives the modified Friedmann equation

$$H^2 = \left(\frac{\dot{p}}{2p}\right)^2 = \left(\frac{8\pi G}{3}\rho - \frac{\ell_o^2}{4p}\right)\times\left(1 - \frac{\rho}{\rho_c}\right), \qquad (56)$$

where $\rho_c = 3/8\pi G\lambda_m^2$, and terms of the order $\lambda_m/\sqrt{p}$ have been dropped since the derivation of these equations makes the assumption that the volume of the space-time always remains much larger than the Planck volume, and it is also assumed that the discreteness scale λ_m is of the order of the Planck length.

Note that this is not the same effective Friedmann equation as the one derived in [25] (although the qualitative aspects of the effective dynamics are unchanged): it actually has a simpler form, which provides additional evidence that working with self-dual variables can simplify calculations.

Since there are only corrections to the gravitational sector of the Hamiltonian constraint in the effective theory, the continuity equation is unchanged,

$$\dot{\rho} + 3H(\rho + P) = 0. \qquad (57)$$

Finally, taking the time derivative of (56) and using the continuity equation gives

$$\frac{\ddot{a}}{a} = \frac{8\pi G}{3}\rho\left(1 - \frac{\rho}{\rho_c}\right) - 4\pi G(\rho + P)\left(1 - \frac{2\rho}{\rho_c}\right) - \frac{3\ell_o^2}{8p}\left(\frac{\rho + P}{\rho_c}\right). \qquad (58)$$

Clearly, the classical Friedmann equations are recovered in the limit $\rho_c \to \infty$.

The effective Friedmann equation (56) clearly shows that a bounce occurs when the energy density of the matter field equals the critical energy density ρ_c, and therefore, for sharply-peaked states, the big-bang and big-crunch singularities that generically arise in general relativity are automatically resolved in self-dual loop quantum cosmology and are replaced by a bounce, just as in standard loop quantum cosmology.

Finally, setting $\ell_o = 0$ gives the effective equations for the spatially flat FLRW space-time, which are identical to the effective equations in standard LQC for that space-time [32].

VI. DISCUSSION

In this paper, it was shown how a loop quantum cosmology for spatially closed FLRW space-times can be constructed starting from self-dual variables. The resulting quantum theory is very similar to standard loop quantum cosmology in that: (i) the dynamics are given by a difference equation under which the big-bang and big-crunch singularities are resolved since the non-singular states decouple from the singular states under the action of the Hamiltonian constraint operator, and (ii) for sharply peaked states (and where $\langle |V| \rangle$ is always much larger than the Planck volume) the full quantum dynamics are well-approximated by effective equations which show that the classical singularity is replaced by a non-singular bounce due to quantum gravity effects. Note however that there are some differences between self-dual LQC and standard LQC in the explicit expressions of both the Hamiltonian constraint operator and the effective equations that, while they do not affect the qualitative behaviour, would be interesting to study in greater detail. Finally, in the limit of a spatially flat space-time (obtained by setting $\ell_o = 0$ in the relevant equations), one recovers exactly the results of standard LQC (with the same factor-ordering ambiguities) that have been studied in considerable depth in the literature.

It is interesting to compare the results obtained here with those of [28] where a different procedure is followed in order to determine the form LQC takes when the Barbero-Immirzi parameter is set to $\gamma = i$. This other procedure is to take the results of standard LQC and to simultaneously (i) perform an analytic continuation $\gamma \to i$ and (ii) change from calculating the holonomy in the $j = 1/2$ representation of SU(2) to the lowest non-trivial continuous representation of SU(1,1). It is then found that the resulting theory is different from what is obtained here. Most strikingly, while the effective equations show that the big-bang singularity is also replaced by a bounce, the bounce is found to be necessarily strongly asymmetric in [28]. The differences between these two approaches probably arise due to the fact that in this paper the generalized holonomies were taken to be in the $j = 1/2$ representation of SU(2), just as in standard LQC. Thus, this suggests that while using self-dual variables does not strongly affect the qualitative features of LQC, representing the holonomies in terms of the continuous representation of SU(1,1) does lead to significant changes in the resulting theory.

One of the two key new steps that are necessary in order to define self-dual LQC as presented here is the implementation of the reality conditions in the kinematical Hilbert space by choosing an appropriate measure for the inner product. However, due to the isotropy of the space-time, the spin-connection is independent of the phase space variables and this simplification made the task of imposing the reality conditions relatively easy. Nonetheless, the fact that this could be done in this setting raises the hope that it may be possible for more complicated space-times as well, and a natural next step in this direction would be to consider the Bianchi models with non-vanishing spatial curvature. If the reality conditions can also be handled in more complex minisuperspaces like the Bianchi type IX space-time, it may be worth revisiting the possibility of using self-dual variables in full loop quantum gravity.

Finally, the other key new step is also one of the more interesting results of this paper: operators corresponding to standard holonomies are not well-defined in the kinematical Hilbert space. Instead, it was necessary to introduce a family of generalized holonomies parametrized by α (with the standard holonomies recovered for $\alpha = 1/2i$), and it was only for $\alpha \in \mathbb{R}$ that the generalized holonomy operators were well-defined. This can be understood to come from the fact that it is the extrinsic curvature part of the Ashtekar connection that is conjugate to the densitized triad, and thus —because the kinematical Hilbert space is, loosely speaking, essentially the Bohr compactification of the extrinsic curvature since the spin-connection is a constant— it is operators corresponding to complex exponentials of the extrinsic curvature that are well-defined in the quantum theory. This would explain why only generalized holonomies with $\alpha \in \mathbb{R}$ are well-defined for self-dual LQC where $\gamma = i$, while normal holonomies (with $\alpha = 1/2i$) are well-defined in standard LQC where $\gamma \in \mathbb{R}$.

While it is necessary to go beyond the study of the simplest FLRW space-times in quantum cosmology before drawing any major conclusions, this result does raise the possibility that if one chooses to work with the self-dual variables, then standard holonomies —which as pointed out above are perfectly well-defined in LQC for the real-valued Ashtekar-Barbero variables— perhaps should not be chosen to be one of the basic variables to be promoted unambiguously to operators in the quantum theory. Instead, when working with the self-dual variables it may be necessary to consider a different type of operator such as the generalized holonomies with $\alpha \in \mathbb{R}$ used here.

ACKNOWLEDGMENTS

I would like to thank Karim Noui, Daniele Oriti, Alejandro Perez, Jorge Pullin and Hanno Sahlmann for helpful discussions. This work is supported by a grant from the John Templeton Foundation.

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