

# A Comment on Continuous Spin Representations of the Poincaré Group and Perturbative String Theory

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ABSTRACT: We make a simple observation that the massless continuous spin representations of the Poincaré group are not present in perturbative string theory constructions. This represents one of the very few model-independent low-energy consequences of these models.

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## 1. Introduction and Conclusions

String theory, just like standard relativistic quantum field theories, has very few model independent consequences at low energies. In quantum field theory we can name the existence of anti-particles, the CPT theorem, the running of couplings in terms of the renormalisation group and the identity of all particles of the same type. String theory, for vacua with non-compact dimensions, ‘predicts’ gravity and at least one neutral scalar, the dilaton, antisymmetric tensors of different ranks and usually also charged matter, and supersymmetry (see for instance [1]).

By the same nature of a theory (string theory or otherwise) with a naturally large energy scale to address the issue of quantum gravity, it is very difficult to identify model independent low-energy implications subject to experimental verification which can put to test the theory and not just particular models or scenarios.

The purpose of this note is to make a simple but general remark. We point out a low-energy consequence of all string constructions, that is the absence of massless continuous spin representations (CSR) of the Poincaré group [2]. This fact has no straightforward explanation within standard particle physics field theoretical analysis and is consistent with all experiments so far since particles fitting into those representations have not been found in nature; see however [3] or a recent discussion of their phenomenology.

One of the most elegant theoretical developments in particle physics, pioneered by Wigner, is the description of one-particle states in terms of unitary representations of the four-dimensional Poincaré group [2] (see also [4]).

One-particle states are classified according to the quantum numbers of the invariant Casimir operators  $C_1 = P^\mu P_\mu$  and  $C_2 = W^\mu W_\mu$  with  $P^\mu$  and  $W^\mu = \epsilon^{\mu\nu\rho\sigma} P_\nu M_{\rho\sigma}$  the momentum and Pauli-Ljubansky vectors respectively and  $M_{\rho\sigma}$  the Lorentz generators.  $C_1$  and  $C_2$  label the representation in terms of their eigenvalues that essentially correspond to mass  $m$  and spin  $J$  in a representation with fixed momentum  $p_\mu$ .

The representations differ according to whether  $C_1$  is positive, zero or negative. For massive particles ( $C_1 > 0$ ) the remaining space-time quantum numbers come from the fact that the stabilising or Little Group in four-dimensions is  $SO(3)$ , the subgroup of the Poincaré group leaving invariant a state in its rest mass frame described by a four-momentum  $p = (m, 0, 0, 0)$ . The corresponding states are the different spin states of the multiplet. This defines a particle in terms of quantum numbers  $|m, J; p_\mu, s\rangle$  with  $s = -J, -J + 1, \dots, J$  and  $p^2 = m^2$ .

For massless particles ( $C_1 = 0$ )<sup>1</sup> the momentum can be written  $p = (E, 0, 0, E)$  and the corresponding Little Group is not only the naively guessed  $SO(2)$  but actually the whole Euclidean group in two dimensions  $E_2$  or  $ISO(2)$ . This complicates matters since this group has infinite dimensional unitary representations, known as continuous spin representations (CSR), that would correspond to a continuous spin-like label on the elementary particles, something that has not been observed in nature.

A standard way to proceed is to simply restrict to the finite dimensional representations that correspond to those of  $SO(2)$ . This defines helicity  $\lambda$ , as the good quantum number which is quantised in half integers. Since the two Casimirs vanish in the reference frame defined by  $p = (E, 0, 0, E)$  all observed massless particles<sup>2</sup> are then labelled only by  $p_\mu$  and  $\lambda$ :  $|p_\mu, \lambda\rangle$  with  $\lambda = 0, \pm 1/2, \pm 1, \dots$ . But there is no satisfactory explanation why to restrict only to representations of  $SO(2)$  instead of the full  $E_2$ . Contrary to the massive case for which matter fields fit into generic representations of  $SO(3)$ , there are massless representations (infinite dimensional) that are allowed by the basic principles of special relativity and quantum mechanics but do not seem to be realised in nature. A theoretical understanding of this fact is needed.

Over the years the continuous spin representations have been discussed in several different contexts (see for instance [6] and references therein) and attempts have been made to describe them in terms of quantum field theoretical interactions, but without much success. The question of their relevance becomes even stronger in the description of higher dimensional theories, such as ten and eleven dimensional supergravities, for which the argument that they have not been observed in nature does not directly apply. Therefore we may wonder if either these representations exist and may have an important role to play in a fundamental theory or if the structure of the fundamental theory may provide a first-principles explanation of why these particles are not observed in nature.

In this note we would like to address the relevance of string theory for the existence or not of continuous spin representations. One may ask if these states could be present in string theory. In perturbative string theory we can observe an obstruction since in the standard quantisation, particles of different masses and spin are in the same multiplet in the sense that upon application of the creation and annihilation operators one relates particles of different masses. It is then clear that if the massive representations do not carry a continuous label the massless states should not carry it either. Then the continuous spin representations are not present in perturbative string constructions. This argument can be turned into a model-independent prediction of string constructions at the same level as the other two general ‘predictions’ of the theory, the presence of gravity and supersymmetry. It can be said that if these states are detected experimentally, all string theory constructions known so far would be ruled out. From the perspective of the CSR’s string theory provides a straightforward explanation of why the relevant part of the Little group for massless states is  $O(D - 2)$  for a  $D$  dimensional theory instead of the full  $ISO(D - 2)$ .

The fact that these representations are not realised in perturbative string theory does not pre-

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<sup>1</sup>The case  $C_1 < 0$  corresponds to tachyonic states that are usually a signature of instability. In supersymmetric string theories this particle is projected out of the spectrum, although being the ground state of the quantisation it has played important roles in the understanding of branes and with potential cosmological implications [5].

<sup>2</sup>Recall that in the standard model all known particles, except the Higgs particle itself, are described by massless states and the massive ones acquire their mass via the Higgs effect.

clude a potential explanation in terms of field theory itself (see for instance [7]). In principle it may be the that interacting field theories for these states may prove inconsistent <sup>3</sup>. But in perturbative string theory the statement is much cleaner since these representations simply do not appear in the standard way of quantising the theory. It may also be argued that in terms of the standard gravity/gauge theory correspondence, if these states are not present on the gravity side they should not be present in any field theory with a string dual. Furthermore if there exist interacting field theories for these states, they should belong to the swampland of the string landscape [8].

Finally it has been suggested that CSR's could be realised in the zero tension limit of string theory in which the infinite tower of massive states collapses to zero mass (see for instance [9]). This could be very interesting, but even if true it does not correspond to the standard string constructions that upon compactifications lead to realistic low energy effective field theories. In the remainder of this note we will make the argument for the absence of CSR explicit by actually computing the action of the extra generators on the string states and show that they vanish.

## 2. Review of CSRs in $d$ dimensions

The generators of the Poincaré group satisfy the well known algebra

$$[M^{\mu\nu}, M^{\alpha\beta}] = i(\eta^{\mu\alpha}M^{\nu\beta} - \eta^{\nu\alpha}M^{\mu\beta} + \eta^{\nu\beta}M^{\mu\alpha} - \eta^{\mu\beta}M^{\nu\alpha}) \quad (2.1)$$

$$[M^{\mu\nu}, P^\alpha] = i(\eta^{\mu\alpha}P^\nu - \eta^{\nu\alpha}P^\mu) \quad ; \quad [P^\alpha, P^\beta] = 0 \quad (2.2)$$

where the Greek indices run from 0 to  $d-1$ . In light cone frame the momenta are the  $P^i$  together with

$$P^\pm = \frac{1}{\sqrt{2}}(P^0 \pm P^{d-1}) \quad (2.3)$$

while the Lorentz generators split into the  $M^{ij}$  and

$$M^{+-} = M^{d-10} \quad ; \quad M^{\pm i} = \frac{1}{\sqrt{2}}(M^{0i} \pm M^{d-1i}) \quad (2.4)$$

where  $i = 1, \dots, d-1$ .

For massless particles the representative momentum is  $p^\mu = (E, 0, \dots, E)$ , so that  $p^+ = \sqrt{2}E$  while  $p^- = 0$  and  $p^i = 0$ . Therefore, the generators of the little group are  $M^{ij}$  and  $\Pi^i \equiv M^{-i}$  which leave the representative momentum invariant [10]. These generators satisfy the  $ISO(d-2)$  algebra

$$[M^{ij}, M^{kl}] = i(\delta^{ik}M^{jl} - \delta^{jk}M^{il} + \delta^{jl}M^{ik} - \delta^{il}M^{jk}) \quad (2.5)$$

$$[M^{ij}, \Pi^k] = i(\delta^{ik}\Pi^j - \delta^{jk}\Pi^i) \quad ; \quad [\Pi^i, \Pi^j] = 0 \quad (2.6)$$

The helicity representation is obtained when  $\Pi^i \equiv 0$  on the states and the algebra reduces to that of  $SO(d-2)$ . The CSR representations are obtained for  $\sum_i (\Pi^i)^2 \neq 0$ .

To see what this condition implies, consider for simplicity  $d = 4$ , i.e.  $i = 1, 2$ . With the definitions  $M^{12} = J_3$  and  $\Pi^\pm = \Pi^1 \pm i\Pi^2$ , the algebra (2.6) becomes  $[\Pi^+, \Pi^-] = 0$  and  $[J_3, \Pi^\pm] = \pm\Pi^\pm$ . Given an eigenstate of  $J_3$  with integer or half-integer eigenvalue  $\sigma$ , i.e.  $J_3|\sigma\rangle = \sigma|\sigma\rangle$ , we

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<sup>3</sup>However a series of recent articles indicate the opposite, making these states much more interesting and physical than usually appreciated [3].

can create states  $|\sigma \pm n\rangle = (\Pi^\pm)^n |\sigma\rangle$  with eigenvalues  $(\sigma \pm n)$  for any positive integer  $n$ . Going to Fourier space we can construct states  $|\theta\rangle = \sum_\sigma e^{-i\sigma\theta} |\sigma\rangle$  which simultaneously diagonalize  $\Pi^\pm$ , i.e.  $\Pi^\pm |\theta\rangle = e^{\pm i\theta} |\theta\rangle$ , whereas a rotation acts as  $e^{-i\alpha J_3} |\theta\rangle = |\theta + \alpha\rangle$ . It is this continuous label  $\theta \in [0, 2\pi)$  which gives rise to the name ‘continuous spin representation’.

To prepare for the discussion of the string spectrum, we first review the point particle, mainly following ref. [6, 11] which are, in parts, based on [12].

In systems with reparametrization invariance, such as the point particle, it is often convenient to work in the light-cone gauge in which  $x^+$  is fixed. In this case  $p^-$  is no longer a conjugate momentum. Instead  $p^-$  is determined to be

$$p^- = \frac{p^i p^i + m^2}{2p^+} \quad (2.7)$$

which follows from the on-shell constraint for a particle of mass  $m$ . The conjugate pairs satisfy  $[x^i, p^j] = i\delta^{ij}$ , and  $[x^-, p^+] = -i$ . The translation operators are  $P^+ = p^+$  and  $P^i = p^i$ , while  $P^- = p^-$  is the light-cone Hamiltonian.

The Lorentz generators can be decomposed into orbital and spin parts. Concretely,

$$M^{ij} = x^i p^j - x^j p^i + S^{ij} \quad (2.8)$$

where the  $S^{ij}$  are  $SO(d-2)$  generators. Moreover,

$$M^{+i} = -x^i p^+ \quad ; \quad M^{+-} = -\frac{1}{2} \{x^-, p^+\} \quad (2.9)$$

In  $d = 4$  it can be shown explicitly that  $S^{+i} = 0$  and  $S^{+-} = 0$  [11]. In general one can show that the above  $M^{+-}$  and  $M^{+j}$ , together with

$$M^{-i} = x^- p^i - \frac{1}{2} \{x^i, p^-\} + \frac{1}{p^+} (T^i - p^j S^{ij}) \quad (2.10)$$

satisfy the Lorentz algebra. The  $T^i$  are  $SO(d-2)$  vectors, i.e.  $[S^{ij}, T^k] = i(\delta^{ik} T^j - \delta^{jk} T^i)$ , and further satisfy

$$[T^i, T^j] = im^2 S^{ij} \quad (2.11)$$

In  $d = 4$  the  $T^i$  are constructed explicitly and shown to verify these properties [11]. In general they follow imposing that the  $M^{-i}$  in (2.10) fulfill the Lorentz algebra. In particular, (2.11) guarantees that  $[M^{-i}, M^{-j}] = 0$ .

When  $m = 0$  the  $T^i$  commute among themselves and become the translation operators of  $ISO(d-2)$ . Thus, in the helicity representation  $T^i = 0$  whereas in the CSR  $T^i \neq 0$ .

In [6] the authors generalize the above discussion to continuous spin representations of the supersymmetry algebra. The question whether these CSRs can be incorporated into an 11-dimensional supergravity theory is possibly of interest in relation to M-theory and the continuous excitation spectrum of the membrane [13].<sup>4</sup>

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<sup>4</sup>We thank M. Green for mentioning this possibility.

### 3. CSRs in String Theory

In the (open) bosonic string, in the critical dimension  $d = 26$ , the translation operators  $\Pi^i \equiv M^{-i}$  in the light cone gauge are given by [14, 15]<sup>5</sup>

$$\Pi^i = x_0^- p^i - \frac{1}{2} \{x_0^i, p^-\} - i \sum_{n=1}^{\infty} \frac{1}{n} (\alpha_{-n}^- \alpha_n^i - \alpha_{-n}^i \alpha_n^-) \quad (3.1)$$

where  $[x_0^i, p^j] = i\delta^{ij}$ ,  $[x_0^-, p^+] = -i$ ,  $[\alpha_m^i, \alpha_{-n}^j] = m\delta^{ij}\delta_{m,n}$ , and

$$p^- = \frac{1}{2\alpha' p^+} \left( \alpha' p^i p^i + \sum_{n=1}^{\infty} n \alpha_{-n}^i \alpha_n^i - 1 \right) \quad (3.2)$$

$$\alpha_n^- = \frac{1}{\sqrt{2\alpha' p^+}} \sum_{p=-\infty}^{\infty} \alpha_{n-p}^i \alpha_p^i \quad ; \quad \alpha_{-n}^- = (\alpha_n^-)^\dagger \quad (3.3)$$

Normal ordering is understood. Notice that  $[x_0^i, p^-] = ip^i/p^+$ .

The  $\alpha_n^i$ ,  $n \geq 1$ , annihilate the vacuum  $|p^+, \vec{p}\rangle$ . The massless states are ( $\vec{p} = \{p^i\}$ )

$$|j\rangle \equiv \alpha_{-1}^j |p^+, \vec{p}\rangle \quad (3.4)$$

which transform as a vector of  $SO(d-2)$ . This indicates that the little group is  $SO(d-2)$ . By consistency we then expect the  $\Pi^i$  to be zero acting on these states. Using  $\alpha_0^i = \sqrt{2\alpha'} p^i$  we find

$$p^- |j\rangle = \frac{\vec{p}^2}{2p^+} |j\rangle \quad ; \quad \alpha_{-1}^i \alpha_1^- |j\rangle = \frac{p^j}{p^+} |i\rangle \quad ; \quad \alpha_{-1}^- \alpha_1^i |j\rangle = \delta^{ij} \frac{p^k}{p^+} |k\rangle \quad (3.5)$$

Thus, it follows that  $\Pi^i |j\rangle = 0$  when the transverse momentum of the massless states verifies  $p^i = 0$ .

In light-cone gauge it is actually more convenient to employ the decomposition of the Lorentz generators into orbital and spin parts. In the bosonic string the  $M^{ij}$  are in fact written as in (2.8) with the spin piece given by

$$S^{ij} = -i \sum_{n=1}^{\infty} \frac{1}{n} (\alpha_{-n}^i \alpha_n^j - \alpha_{-n}^j \alpha_n^i) \quad (3.6)$$

Furthermore, comparing (2.10) and (3.1), and using the above expression for  $S^{ij}$ , we obtain

$$T^i = i \sum_{n=1}^{\infty} \frac{1}{n} \left[ \alpha_{-n}^i (p^+ \alpha_n^- - p^j \alpha_n^j) - (p^+ \alpha_{-n}^- - \alpha_{-n}^j) \alpha_n^i \right] \quad (3.7)$$

It can be shown that these  $T^i$ 's satisfy (2.11) with  $m^2$  replaced by the mass operator of the open bosonic string .

Since the massless states  $|j\rangle$  belong in the helicity representation we again expect that the  $T^i$  are zero acting on these states. Indeed, using the last two results in (3.5), we find  $T^i |j\rangle = 0$  and this holds for all  $p^i$ . This is a consistency check that the CSRs do not appear in the perturbative string spectrum.

Here we have only considered the contribution to the Lorentz generators from the oscillators which arise in the quantization of the non-compact space-time dimensions. In the critical dimension

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<sup>5</sup>We mostly use the notation of [16].

this is all there is. In the compactified theory there are also contributions from the internal CFT but they will not change the argument and result. The same is true for the extension to the fermionic string.

We have observed that using the standard rules for the spectrum of perturbative string models, continuous spin representations do not appear. It would be interesting to study if these representations could be present in the full-fledged string theory.

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