Uniformizing higher-spin equations

2015 J. Phys. A: Math. Theor. 48 015401

(http://iopscience.iop.org/1751-8121/48/1/015401)

IOP Publishing Journal of Physics A: Mathematical and Theoretical

J. Phys. A: Math. Theor. **48** (2015) 015401 (27pp) doi:10.1088/1751-8113/48/1/015401

Uniformizing higher-spin equations

K B Alkalaev[1,2], **M A Grigoriev**[1,2] **and E D Skvortsov**[1,3]

[1] I.E. Tamm Department of Theoretical Physics, P.N. Lebedev Physical Institute, Leninsky ave. 53, 119991 Moscow, Russia
[2] Moscow Institute of Physics and Technology, Dolgoprudnyi, 141700 Moscow region, Russia
[3] Albert Einstein Institute, Golm, D-14476, Am Mühlenberg 1, Germany

E-mail: grig@lpi.ru

Received 6 October 2014, revised 16 November 2014
Accepted for publication 17 November 2014
Published 8 December 2014

Abstract

Vasiliev's higher-spin (HS) theories in various dimensions are uniformly represented as a simple system of equations. These equations and their gauge invariances are based on two superalgebras and have a transparent algebraic meaning. For a given HS theory these algebras can be inferred from the vacuum HS symmetries. The proposed system of equations admits a concise AKSZ formulation. We also discuss novel HS systems including partially-massless and massive fields in AdS, as well as conformal and massless off-shell fields.

Keywords: higher spin theory, AdS/CFT, gauge field theories

1. Introduction

In this paper we study algebraic structures underlying Vasiliev's higher-spin (HS) theories in various dimensions [1–5] (see [6–8] for reviews). These HS theories are defined by classical equations of motion whose underlying geometrical principles are not entirely settled. More precisely, the equations of interacting HS fields are encoded [9] in the flatness condition imposed on the connection of a sufficiently large algebra, called the *embedding algebra* in this work. The connection is further constrained through its coupling to a set of 0-form fields which in turn are subject to certain algebraic constraints.

Another algebraic ingredient of the HS theory is an algebra of *dynamical symmetries* which is somewhat hidden in the usual formulations. This is a finite-dimensional Lie superalgebra that remains undeformed at the interacting level. On the other hand the infinite-dimensional *HS algebra* is a subalgebra of the *embedding algebra* preserving the most symmetric vacuum, which corresponds to empty AdS space. We argue that in HS theories the embedding algebra can be constructed as a twisted tensor product of the HS algebra and the algebra of dynamical symmetries. In particular, both factors turn out to be subalgebras of the embedding algebra.

In this paper we propose a formulation of HS equations, where these algebraic structures are realized in a manifest way. More precisely, in addition to the 1-form connection of the embedding algebra, there is a set of 0-form fields which are associated to the basis elements of the dynamical symmetry algebra. The constraints on these fields are simply the (anti)commutation relations of the algebra.

In this approach all the ingredients needed to formulate HS equations are the algebra of dynamical symmetries and the embedding algebra. More generally, this gives a new class of gauge invariant equations defined by a pair of superalgebras so that known HS theories form a particular subclass. Furthermore, we demonstrate that the system admits a concise AKSZ formulation in terms of the Chevalley–Eilenberg cohomology differentials of the two superalgebras.

Depending on the realization of the HS algebra the equations may describe off-shell theory in which case the system can be put on-shell through a version of the factorization procedure from [5]. We cover the variety of models including Vasiliev's HS theories in various dimensions as well as variations of the d-dimensional HS theory involving partially-massless or massive fields. The latter two are AdS/CFT dual to non-unitary singletons and generalized free fields respectively. Finally, we also discuss conformal and massless off-shell fields.

2. Uniform representation

2.1. Equations of motion and gauge symmetries

The equations of motion are built out of the following data: a Lie (super)algebra $\mathfrak{g}$, which is the algebra of *dynamical symmetries* and the associative (super)algebra $\mathfrak{A}$, which is the *embedding algebra*. In the known examples $\mathfrak{g}$ is $u(1)$, $sp(2)$ and $osp(1|2)$ or its extension. It is convenient to pick basis elements e_a in $\mathfrak{g}$ so that the graded commutation relations are $[e_a,\ e_b] = C^c_{ab}\, e_c$ and the (anti)symmetry relations are $[e_a,\ e_b] = -(-1)^{|e_a||e_b|}[e_b,\ e_a]$, where $|e_a|$ is the Grassmann degree of e_a. $\mathfrak{A}$ is an embedding (usually star-product) algebra for a HS algebra under consideration and the product in $\mathfrak{A}$ is denoted by $\star$. In particular, $\mathfrak{A}$ typically contains a Lie subalgebra isomorphic to $so(d,\ 2)$, i.e. AdS$_{d+1}$ spacetime isometries.

Given a space-time manifold with coordinates $x^{\underline{m}}$, where $\underline{m} = 0,\ \ldots,\ d$, the fields are 1-form $W = W_{\underline{m}}(x)\mathrm{d}x^{\underline{m}}$ and 0-forms $T_a = T_a(x)$, all taking values in $\mathfrak{A}$. The full set of equations is

$$\begin{aligned} \mathrm{d}W + W\star W &= 0,\\ \mathrm{d}T_a + \left[W,\ T_a\right]_\star &= 0,\\ \left[T_a,\ T_b\right]^\pm_\star - C^c_{ab}\, T_c &= 0, \end{aligned} \tag{2.1}$$

where $[T_a,\ T_b]^\pm_\star = T_a\star T_b - (-1)^{|e_a||e_b|}T_b\star T_a$. The above system is invariant under the gauge transformations of the form

$$\delta W = \mathrm{d}\xi + [W,\ \xi]_\star, \qquad \delta T_a = \left[T_a,\ \xi\right]_\star, \tag{2.2}$$

where $\xi = \xi(x)$ also takes values in $\mathfrak{A}$.

Disregarding x-dependence, fields T_a can be seen as components of a map τ: $\mathfrak{g} \to \mathfrak{A}$ with respect to the basis e_a so that $T_a = \tau(e_a)$. Then the last equation in (2.1) simply implies that τ is compatible with the algebra, i.e., τ is a homomorphism. The Grassmann degree in $\mathfrak{g}$ and $\mathfrak{A}$

determines the parity of the component fields. More precisely, if E_A denote basis in $\mathfrak{A}$ and component fields are introduced through $T_a = T_a^A E_A$ and $W = \mathrm{d}x^{\underline{m}} W_{\underline{m}}^A E_A$ then $|T_a^A(x)| = |e_a| + |E_A|$ and $|W_{\underline{m}}^A| = |E_A|$. Note that one can consistently put to zero all the fermionic component fields. This bosonic truncation corresponds to τ being a homomorphism of superalgebras.

System (2.1) determines a background independent field theory in the sense that the equations do not involve any background fields. Moreover, the space-time manifold plays a passive role in the formulation and can be, at least formally, taken arbitrary. However, background fields enter through the choice of the vacuum solution whose existence in general restricts the space-time geometry. In what follows we assume that a fixed vacuum solution $W = W^0$, $T_a = T_a^0$ is given and a system is interpreted as a perturbative expansion around W^0, T_a^0. In HS theories T_a^0 are space-time independent, i.e. $\mathrm{d}T_a^0 = 0$, while W^0 is a flat connection of the anti-de Sitter algebra $so(d, 2) \subset \mathfrak{A}$.

Given a vacuum solution global symmetries are by definition gauge transformations that leave the vacuum solution intact,

$$\mathrm{d}\xi_0 + \left[W^0, \xi_0 \right]_\star = 0, \qquad\qquad \left[T_a^0, \xi_0 \right]_\star = 0. \tag{2.3}$$

The first equation uniquely fixes the space-time dependence of global symmetry parameters ξ_0, while the second one implies that ξ_0 is $\mathfrak{g}$-invariant in $\mathfrak{A}$. At least locally in space-time this means that global symmetries are one-to-one with subalgebra $\mathfrak{hs} \subset \mathfrak{A}$ of $\mathfrak{g}$-invariants. As we are going to see, in the specific HS theories considered later $\mathfrak{hs}$ is the respective HS algebra.

It is worth mentioning that the choice of the vacuum is closely related to the precise definition of $\mathfrak{A}$. Indeed, two distinct vacua can either be equivalent being related by a gauge transformation or nonequivalent depending on particular functional class used to define a star-product in $\mathfrak{A}$. Not to mention that the entire physical content of the HS theory critically depends on the choice of the functional class.

Although system (2.1) looks very natural and similar systems are known in lower dimensions, in the HS theory context this was first considered in [10], where it was shown to describe off-shell constraints and gauge symmetries for HS fields on AdS for $\mathfrak{g} = sp(2)$ and suitable choice of algebra $\mathfrak{A}$ and the vacuum. Also, the simplest version with $\mathfrak{g} = u(1)$ has been proposed in the earlier work [11] to describe HS fields at the off-shell level. It is important to note, however, that now we are mainly concerned with Vasiliev equations in various dimensions, where the representation (2.1) is implicit in the literature, and where $\mathfrak{A}$ and $\mathfrak{g}$ originate from the conventional formulation of HS theory [1–5].

Depending on the realization of $\mathfrak{A}$, the equations (2.1), (2.2) may describe off-shell or reducible system. In this case, in addition to the equations of motion one needs to define a consistent factorization needed to eliminate unphysical components. The factorization is determined by an ideal $\mathfrak{h} \subset \mathfrak{g}$ whos associated fields $\{T_\alpha\} \subset \{T_a\}$ are generators of extra gauge symmetry $\delta T_a = \lambda_a^\alpha \star T_\alpha$ such that the system (2.1) is well defined on equivalence classes. Details of this procedure are explained in section 3.3.2, where we discuss d-dimensional HS equations.

2.2. Nontriviality and cohomology

It is instructive to show the way system (2.1) can yield propagating degrees of freedom. To this end, let us consider its linearization. Denoting the perturbations by w and t_a, the linearized system reads

$$
\begin{aligned}
D_0 w &= 0, & \delta w &= D_0 \xi, \\
D_0 t_a + \left[w, T_a^0 \right]_\star &= 0, & & \\
\left[t_a, T_b^0 \right]_\star^\pm + \left[T_a^0, t_b \right]_\star^\pm - C_{ab}^c\, t_c &= 0, & \delta t_a &= \left[T_a^0, \xi \right]_\star,
\end{aligned}
\tag{2.4}
$$

where $D_0 = d + [W^0, \cdot]_\star^\pm$ is the background covariant derivative, $D_0^2 = 0$.

The last line represents a standard cohomology problem. Indeed, t_a are maps $\mathfrak{g} \to \mathfrak{A}$ or, equivalently, elements of $\Lambda^1(\mathfrak{g}^*) \otimes \mathfrak{A}$, which by definition is the space of 1-cochains. From this perspective the last line contains the cocycle and the coboundary conditions for t_a. If, for instance, $\mathfrak{g}$ is such that $\mathbb{H}^1(\mathfrak{g}, \mathfrak{A})$ is empty, then one can use the gauge symmetry to set $t_a = 0$ so that we are left with

$$
D_0 w = 0, \qquad \left[w, T_a^0 \right]_\star = 0. \tag{2.5}
$$

This system does not contain 0-forms and hence does not describe local degrees of freedom. More precisely, taking into account residual gauge symmetry and making use of natural technical assumptions, the space of inequivalent solutions is empty. Indeed, w is just a linearized flat connection.

We now turn to the case where $\mathbb{H}^1(\mathfrak{g}, \mathfrak{A})$ is not empty and show that the linearized system directly leads to the familiar unfolded representation of the linearized HS equations [12, 13] (the form is also known as 'on-mass-shell theorem' in the literature [13, 14]). Let r_a parameterize representatives of $\mathbb{H}^1(\mathfrak{g}, \mathfrak{A})$. This means that using gauge symmetry we can replace t_a by r_a so that the system reads

$$
D_0 w = 0, \qquad D_0 r_a = ad_a w, \tag{2.6}
$$

where $ad_a = [T_a^0, \bullet]_\star$. The second equation can be solved for w modulo elements of $\mathbb{H}^0(\mathfrak{g}, \mathfrak{A})$: $w = \omega + ad_a^{-1} D_0 r_a$ and $ad_a\, \omega = 0$. Note that ω can be seen as taking values in $\mathbb{H}^0(\mathfrak{g}, \mathfrak{A})$ which in turn is the HS algebra $\mathfrak{hs}$ (cf (2.3)). Plugging this into the first equation of the system (2.4) we find

$$
D_0 \omega = -D_0\, ad_a^{-1} D_0 r_a, \tag{2.7}
$$

which has the structure of unfolded HS equations if one identifies ω as HS connection 1-form and 0-forms r_a as parameterizing curvatures. This equation merely expresses the linearized curvature of ω in terms of generalized Weyl tensors (or Riemann tensors at the off-shell level) ensuring that the system may describe local degrees of freedom. In other words, it specifies a deviation from being a flat connection. However, to see what the dynamical content exactly is the general considerations are not enough and one is to study concrete choice for $\mathfrak{g}$, $\mathfrak{A}$, and the vacuum solution.

The embedding algebra $\mathfrak{A}$ is a key element of the whole construction. It should be large enough to contain the HS algebra $\mathfrak{hs}$ and the algebra $\mathfrak{g}$ of dynamical symmetries, actually the full image of $U(\mathfrak{g})$. That the dynamics should be nontrivial is a crucial restriction, which is manifested by nonzero right-hand side (rhs) of (2.7). In particular, it follows that the algebra $\mathfrak{A}$ cannot be just a tensor product $\mathfrak{hs} \otimes U(\mathfrak{g})$, for which the two factors commute. Indeed, in this case $D_0 ad_a^{-1} = ad_a^{-1} D_0$ since operators associated to the commuting subalgebras also commute. Then, using $D_0^2 = 0$ one finds out that the rhs of the relation (2.7) vanishes identically. A reasonable way out is provided by the twisted product of associative algebras [15]. It seems to be the most general construction that allows one to build an algebra that contains two given algebras as subalgebras but their images do not necessarily commute to each other. In practice, the twisted product is realized through a specific star-product (see section 4.3).

The following comments are in order.

(i) The linear equations of the form (2.4) naturally appear in describing various HS fields at the free level in the so-called parent approach. In particular, for $\mathfrak{g} = u(1)$ and $\mathfrak{g} = sp(2)$ and suitable choice of $\mathfrak{A}$ the system (2.4) describes free HS fields on respectively flat [16] and AdS space-time [17] (see also [10, 11, 18]). Furthermore, equations of motion for nearly generic (including mixed-symmetry, partially-massless, etc) HS fields can naturally be formulated in the form (2.4) [19–21] or (2.7) [22–26], which for certain class of fields requires an extension of (2.1) with higher degree forms, the modification that we do not discuss in detail.

(ii) The system (2.4) as well as its equivalent reduction to (2.7) have a simple homological interpretation. Namely, if one combines both w and t_a into a homological complex with the differential being $Q = D_0 + \Delta$, where Δ is the Lie algebra differential of $\mathfrak{g}$, the system (2.4) takes the form $Q\Psi = 0$, $\delta\Psi = Q\Xi$. The differential of the form $D_0 + D_0\ ad_a^{-1} D_0$ in (2.7) as defined on $\mathbb{H}(\mathfrak{g}, \mathfrak{A})$-valued fields is just the differential induced by Q in the cohomology of Δ (In this form the homological technique for elimination of auxiliary and Stueckelberg fields was developed in [16, 17]. Note also a related σ_--cohomology approach of [27]).

3. Specialization to the Vasiliev equations

At present, there exist several HS theories in various dimensions, namely, $2d$ HS theory of matter fields interacting via topological HS fields [28]; the minimal $3d$ HS theory with massless matter fields [3]; Prokushkin–Vasiliev $3d$ theory that admits a one-parameter family of vacua with massive excitations [29]; $4d$ bosonic HS theory [1, 30] and its supersymmetric extensions with any $\mathcal{N}$ [31–33]; d-dimensional bosonic system [5]. All of them can be further extended by adding internal (Yang–Mills) symmetries, while certain truncations of the spectra are also possible (see [6, 33] for review). There are also topological HS systems in three [34] and two dimensions [35, 36] which are HS extensions of Chern–Simons and Jackiw–Teitelboim gravity models. A brief review of the Vasiliev equations in $d = 2, 3, 4$ and any d is given in the appendix.

In this section we show that the system (2.1) reduces to the Vasiliev equations provided the appropriate choice of the Lie superalgebra $\mathfrak{g}$. We observe that for all $d \geqslant 3$ HS systems a Lie superalgebra $\mathfrak{g}$ contains $osp(1|2)$ subalgebra. More precisely, in the Vasiliev system $\mathfrak{g}$ -relations are encoded in terms of the polynomial Serre type relations imposed on a subset of $\mathfrak{g}$-generators.

3.1. Serre type relations for osp(1, 2)

Let us first show how the Serre type realization works in the case of $sp(2)$ algebra and then consider its supersymmetric extension. Indeed, the conventional definition of $sp(2)$ algebra relies on the three commutation relations among three basis elements,

$$[H, E] = +2E, \qquad [H, F] = -2F, \qquad [E, F] = H, \tag{3.1}$$

where $[\cdot\,,\,\cdot]$ is the Lie product. Treating the last relation as a definition we reduce them to two cubic Serre type relations,

$$[[E, F], E] = +2E, \qquad [[E, F], F] = -2F. \tag{3.2}$$

Note that there are two independent equations imposed on two basis elements of the Lie algebra.

In the $osp(1|2)$ superalgebra case three even basis elements E, F, H are now supplemented with two odd ones e, f. Their non-vanishing graded commutation relations take the form

$$\{e, e\} = -2E, \qquad \{f, f\} = 2F, \qquad \{e, f\} = H, \tag{3.3a}$$

$$[H, e] = e, \qquad [H, f] = -f, \qquad [E, f] = e, \qquad [F, e] = f, \tag{3.3b}$$

$$[H, E] = +2E, \qquad [H, F] = -2F, \qquad [E, F] = H, \tag{3.3c}$$

where $[\,\cdot\,,\,\cdot\,]$ and $\{\,\cdot\,,\,\cdot\,\}$ define the graded Lie product. Relations between odd basis elements in the first line can be considered as definitions of even basis elements, while the second line contains nontrivial cubic relations between odd elements. Relations in the third line are algebraic consequences of the first two lines. In this way, we arrive at four cubic relations between two odd elements

$$[\{e, f\}, e] = e, \qquad [\{e, f\}, f] = -f, \qquad [\{e, e\}, f] = -2e, \qquad [\{f, f\}, e] = 2f, \tag{3.4}$$

which define the $osp(1|2)$ superalgebra. However, the above Serre type relations can be reduced even further. Assuming that $\mathfrak{g}$ is embedded into its universal enveloping algebra $U(\mathfrak{g})$, and expanding the (anti)commutators we see that the first two relations are equivalent to the last two relations.

While everything above is true for any Lie (super)algebra with appropriate modifications, the following property is special for $osp(1|2)$. We define an even element $\Upsilon \in U(osp(1|2))$,

$$\Upsilon = [e, f] + \frac{1}{2}, \tag{3.5}$$

which has a rather special property to (anti)commute with (odd)even elements

$$\{\Upsilon, e\} = 0, \qquad \{\Upsilon, f\} = 0, \tag{3.6a}$$

$$[\Upsilon, E] = 0, \qquad [\Upsilon, F] = 0, \qquad [\Upsilon, H] = 0. \tag{3.6b}$$

Obviously, Υ squared is the quadratic Casimir of $osp(1|2)$ superalgebra, i.e., $\mathbb{C}_2 = \Upsilon^2$. Using Υ one can show that defining relations of $osp(1|2)$ superalgebra, namely the first two of (3.4) follow from

$$\{\Upsilon, e\} = 0, \qquad \{\Upsilon, f\} = 0, \tag{3.7}$$

by simply plugging the definitions (3.3a) into (3.3b). Two equations (3.7) can be rewritten as

$$\left[e^2, f\right] = -e, \qquad \left[f^2, e\right] = f. \tag{3.8}$$

Again, just as in the $sp(2)$ case, there are two cubic equations for two basis elements—five basis elements of $osp(1|2)$ can be reduced to only two that obey (3.7). The associative algebra generated by e, f subjected to the above relations is isomorphic to $U(osp(1|2))$ [37].

3.2. HS equations in three dimensions

In the $3d$ HS theory [3, 29], the Lie superalgebra of dynamical symmetries is chosen to be

$$\mathfrak{g} = osp(1|2). \tag{3.9}$$

Recalling that fields T_a determine a map τ: $\mathfrak{g} \to \mathfrak{A}$, we unify the images of odd elements $\tau(e)$ and $\tau(f)$ into a doublet S_α, where $\alpha = 1, 2$ and the images $\tau(E), \tau(F), \tau(H)$ into a symmetric tensor $T_{\alpha\beta} = T_{\beta\alpha}$. The $osp(1|2)$ graded relations (3.3*a*)–(3.3*c*) now read as

$$\left[T_{\alpha\beta},\, T_{\gamma\rho}\right] = \epsilon_{\alpha\gamma} T_{\beta\rho} + 3 \text{ terms}, \quad \left[T_{\alpha\beta},\, S_\gamma\right] = \epsilon_{\alpha\gamma} S_\beta + \epsilon_{\beta\gamma} S_\alpha, \quad \frac{\mathrm{i}}{4}\left\{S_\alpha,\, S_\beta\right\} = T_{\alpha\beta}, \tag{3.10}$$

where $\epsilon^{\alpha\beta} = -\epsilon^{\beta\alpha}$ is $sp(2)$ invariant form. The $sp(2)$ indices are raised and lowered as $S_\alpha = S^\beta \epsilon_{\beta\alpha}$ and $S^\alpha = \epsilon^{\alpha\beta} S_\beta$. In particular, $\epsilon^{\alpha\beta}\epsilon_{\alpha\gamma} = \delta^\beta_\gamma$. The factor $\frac{\mathrm{i}}{4}$ is introduced anticipating the star-product realization of $\mathfrak{A}$.

The Serre type relations (3.8) uniquely determine τ and hence all T_a so that the system (2.1) can equivalently be rewritten in terms of 1-form fields W and 0-form fields S_α as follows

$$\mathrm{d}W + W \star W = 0, \tag{3.11a}$$

$$\mathrm{d}S_\alpha + \left[W,\, S_\alpha\right]_\star = 0, \tag{3.11b}$$

$$\frac{\mathrm{i}}{4}\, S_\beta \star S_\alpha \star S^\beta = S_\alpha. \tag{3.11c}$$

Here, a component form of the last equation (3.11*c*) reproduces relations (3.8) upon rescaling.

In the Vasiliev theory, the image $\tau(\Upsilon)$ of the element (3.3*a*) also denoted as Υ has been introduced as a new but not independent field. Taking into account that (3.8) can be equivalently represented as (3.5) and (3.7) the constraint (3.11*c*) can be split into the following two equations

$$\left[S_\alpha,\, S_\beta\right]_\star = -2\mathrm{i}\, \epsilon_{\alpha\beta}(1 + \Upsilon), \qquad \left\{S_\alpha,\, \Upsilon\right\}_\star = 0, \tag{3.12}$$

which have a more familiar form of the deformed oscillator algebra [38]. In order to match Vasiliev equations the field Υ needs to be redefined as $\Upsilon = B \star \varkappa$, where B is an arbitrary field, while $\varkappa$ satisfying $\varkappa^2 = 1$, called Klein operator, is a specific element of the embedding algebra $\mathfrak{A}$. In our formulation, the Klein operator is introduced for convenience in the process of solving equation (3.11) over a specific vacuum, see section 4.5. For a suitable choice of $\mathfrak{A}$ the system (3.11) is exactly the 3d Vasiliev system [3, 29] (see also the appendix). To be precise, one usually adds the covariant constancy $\mathrm{d}\Upsilon + [W,\, \Upsilon]_\star = 0$, which is a consequence of (3.11*b*) and (3.12).

For systems in four and any dimensions establishing a relation with the known Vasiliev systems requires extra steps, which can be traced back to specific realizations of the HS algebras. Furthermore, these Vasiliev systems involve in addition factorization/extra constraints needed to describe irreducible systems. All these subtleties are analyzed in some more details in sections 3.3 and 3.4.

3.3. HS equations in any dimensions

3.3.1. Off-shell system. The Vasiliev system in $(d + 1)$-dimensions is formulated in two steps: first one formulates the off-shell system and then performs the consistent factorization which puts the system on-shell [5] (see also review [7]). The off-shell system can be reformulated in the form (2.1). As an algebra $\mathfrak{g}$ one takes a semi-direct sum

$$\mathfrak{g} = sp(2) \Subset osp(1|2). \tag{3.13}$$

If S_α, $T_{\alpha\beta}$ denote fields associated to $osp(1|2)$ basis elements and $F_{\alpha\beta}$ to $sp(2)$ ones the last equation in (2.1) reads as

$$\left[F_{\alpha\beta},\, F_{\gamma\delta}\right]_\star = \epsilon_{\beta\gamma}F_{\alpha\delta} + \cdots,\ \left[F_{\alpha\beta},\, T_{\gamma\delta}\right]_\star = \epsilon_{\beta\gamma}T_{\alpha\delta} + \cdots,\ \left[F_{\alpha\beta},\, S_\gamma\right]_\star = \epsilon_{\alpha\gamma}S_\beta + \cdots,$$
$$\frac{\mathrm{i}}{4}\left\{S_\alpha,\ S_\beta\right\}_\star = T_{\alpha\beta},\ \left[T_{\alpha\beta},\, S_\gamma\right]_\star = \epsilon_{\alpha\gamma}S_\beta + \cdots,\ \left[T_{\alpha\beta},\, T_{\gamma\delta}\right]_\star = \epsilon_{\beta\gamma}T_{\alpha\delta} + \cdots, \tag{3.14}$$

where the ellipsis denote proper symmetrizations. The first line contains $sp(2)$-relations and $sp(2)$-module structure of $osp(1|2)$ while the second line contains $osp(1|2)$ relations.

Let $F^0_{\alpha\beta}$ denote specific vacuum values of $F_{\alpha\beta}$. This implies that $F^0_{\alpha\beta}$ form $sp(2)$ which makes $\mathfrak{A}$ into an $sp(2)$-module. If in the above system one puts $F_{\alpha\beta}$ to its vacuum value $F^0_{\alpha\beta}$ by hand (and hence $F_{\alpha\beta}$ are not anymore on equal footing with S, T, W fields) it is not difficult to see that the system coincides with the Vasiliev system [5] (see the appendix) provided one redefines $\Upsilon = B\star\varkappa$ and reformulates the $osp(1|2)$ relations solely in terms of S_α and $B\star\varkappa$ just like in the $d = 3$ case presented in section 3.1.

It turns out, however, that there is no need to put $F_{\alpha\beta} = F^0_{\alpha\beta}$ by hand. Indeed, consider the first equation in (3.14) around the vacuum solution $F_{\alpha\beta} = F^0_{\alpha\beta}$. Then the cohomological argument given in section 2.2 applies because $sp(2)$ is simple and the corresponding cohomology is empty. It follows that $F_{\alpha\beta}$ can be set to its vacuum value $F^0_{\alpha\beta}$ (at least perturbatively). In so doing one also restricts gauge parameters to preserve this on-shell gauge choice: $[F^0_{\alpha\beta},\, \xi]_\star = 0$. At the same time equations $\mathrm{d}F_{\alpha\beta} + [W,\, F_{\alpha\beta}]_\star = 0$ in this gauge imply $[F^0_{\alpha\beta},\, W]_\star = 0$ so that both the gauge parameter and the connection belong to the off-shell HS algebra. To conclude, the system (2.1) for the constraint algebra given by (3.13) yields a more general theory than the original Vasiliev equations, but they are perturbatively equivalent over the specific vacuum $F_{\alpha\beta} = F^0_{\alpha\beta}$.

The above argument is based on the Whitehead lemma which, strictly speaking, only applies to cohomology with coefficients in finite-dimensional modules. This is enough if, for instance, the $sp(2)$-cohomology differential preserves the decomposition of $\mathfrak{A}$ into a direct sum of finite-dimensional subspaces. It turns out that this is indeed the case if one takes a standard vacuum solution $F^0_{\alpha\beta}$ for $F_{\alpha\beta}$ because $[F^0_{\alpha\beta},\, \cdot]_\star$ as well as the cohomology differential is of vanishing homogeneity in all the oscillators and hence preserve the subspaces of definite homogeneity. This is shown in section 4.4, where explicit definitions for $\mathfrak{A}$ and $F^0_{\alpha\beta}$ are also given. An example where such a decomposition does not exist and $sp(2)$ relations do have nontrivial solutions can be found in section 4.6.

3.3.2. Factorization. Now we elaborate on the consistent factorization needed to put the off-shell system on-shell. We show that factorization can be performed at the level of the system determined by (3.14). Moreover, it can be seen as a certain gauge symmetry at the price of introducing extra fields. This gives a better understanding of the factorization procedure even in the conventional formulation of the Vasiliev system.

In this context it is often convenient to use the following set of fields[4] S_α, $T_{\alpha\beta}$, $\bar{F}_{\alpha\beta}$, where $\bar{F}_{\alpha\beta} = F_{\alpha\beta} - T_{\alpha\beta}$. In terms of the new fields, relations (3.14) involving $\bar{F}_{\alpha\beta}$ take the form

$$\left[\bar{F}_{\alpha\beta},\, \bar{F}_{\gamma\delta}\right]_\star = \epsilon_{\beta\gamma}\bar{F}_{\alpha\delta} + \cdots, \qquad \left[\bar{F}_{\alpha\beta},\, S_\gamma\right]_\star = 0, \qquad \left[\bar{F}_{\alpha\beta},\, T_{\gamma\delta}\right]_\star = 0, \tag{3.15}$$

while the relations between S_α, $T_{\alpha\beta}$ stay unchanged.

In terms of the off-shell system the factorization means to eliminate the ideal generated by $\bar{F}_{\alpha\beta}$. It turns out that system (2.1) is well-defined on equivalence classes of fields with respect to the following equivalence relation

[4] The similar trick is used within the Vasiliev equations, when enforcing $sp(2)$ invariance at the nonlinear level [7].

$$W \sim W + \lambda^i \star \bar{F}_i, \qquad T_a \sim T_a + \lambda_a^{\,j} \star \bar{F}_j, \tag{3.16}$$

which we present in the infinitesimal form and where $T_a = \{S_\alpha, T_{\alpha\beta}, F_{\alpha\beta}\}$ and $\bar{F}_l = \{\bar{F}_{\alpha\beta}\}$. Equations (2.1) understood as those on equivalence classes can be explicitly written as

$$\begin{aligned} \mathrm{d}W + W \star W &= u^l \star \bar{F}_l, \\ \mathrm{d}T_a + \left[W, T_a\right]_\star &= u_a^{\,l} \star \bar{F}_l, \\ \left[T_a, T_b\right]_\star^\pm - C_{ab}^c\, T_c &= u_{ab}^{\,l} \star \bar{F}_l, \end{aligned} \tag{3.17}$$

where u-fields are not treated as dynamical (in other words the equations only imply that such u do exist). Then the equivalence relations (3.16) can be seen as the gauge transformations of W, T_a. In these terms the consistency of the factorization is just the invariance of the above equations under (3.16) supplemented by appropriate transformations of u-fields. What one actually checks is that variation of the equations under (3.16) is proportional to $\bar{F}_i$.

The above construction applies to a generic system (2.1) provided the factorization is performed with respect to the generators of the ideal $\mathfrak{h} \subset \mathfrak{g}$. In the case at hand, ideal $\mathfrak{h}$ is the diagonal $sp(2)$ in $sp(2) \Subset osp(1|2)$, i.e. we have the following coset

$$\frac{sp(2) \Subset osp(1|2)}{sp(2)}. \tag{3.18}$$

More generally, one can consider consistent factorizations based on ideals in the enveloping algebra $U(\mathfrak{g})$ (see below).

Consider now the linearized system by taking $W = W^0 + w$, $S_\alpha = S_\alpha^0 + s_\alpha$, $T_{\alpha\beta} = T_{\alpha\beta}^0 + t_{\alpha\beta}$, $F_{\alpha\beta} = F_{\alpha\beta}^0 + f_{\alpha\beta}$. Linearization of the gauge symmetry (3.16) allows to gauge away components of w, s_α, $t_{\alpha\beta}$, $f_{\alpha\beta}$ proportional to $\bar{F}_{\alpha\beta}^0$. Upon the elimination of $f_{\alpha\beta}$, $t_{\alpha\beta}$ (using gauge invariance and equations of motion) the system becomes equivalent to the linearized on-shell Vasiliev system describing massless fields of all integer spins.

One can go even further and consider u-fields entering (3.17) at the equal footing with W, T_a. In this interpretation in addition to gauge transformations of u induced by (3.16) extra gauge symmetry may be needed to ensure that u do not bring new degrees of freedom.

The core of the above system is the last equation in (3.17). Its gauge symmetry is given by $\delta T_a = \lambda_a^{\,i} \star \bar{F}_i + [T_a, \xi]_\star$, where the standard gauge symmetry with parameter ξ has been reinstated and the corresponding gauge transformations for u are assumed. These can be written as follows

$$\begin{aligned} \left[T_a, T_a\right]_\star^\pm = U_{ab}^c \star T_c, \qquad \delta T_a &= \lambda_a^{\,b} \star T_b + \left[T_a, \xi\right]_\star, \\ \delta U_{ab}^c = \lambda_a^{\,d} \star U_{db}^c + \left[T_a, \lambda_b^c\right]_\star &- (-1)^{|a||b|}\left(\lambda_b^{\,d} \star U_{da}^c + \left[T_b, \lambda_a^c\right]_\star\right) \\ &+ U_{ab}^d \star \lambda_d^{\,c} + \left[U_{ab}^c, \xi\right]_\star, \end{aligned} \tag{3.19}$$

where U_{ab}^c is to be identified as $C_{ab}^c + u_{ab}^{\,c}$ and some of the components in $\lambda_b^{\,a}$ and u_{ab}^c vanish identically. More precisely, only those corresponding to the ideal $\mathfrak{h} \subset \mathfrak{g}$ are nonvanishing. Equation (3.19) and gauge symmetries are precisely those defining the constrained Hamiltonian system with constraints T_a and structure functions U_{ab}^c . In particular, gauge transformation with parameter $\lambda_b^{\,a}$ is nothing but the infinitesimal redefinition of the constraints which is a natural equivalence of constrained systems. Further details on the field-theoretical interpretation of constrained Hamiltonian systems can be found in, e.g. [18, 39]. Note, however, that in contrast to the constrained system where all $\lambda_b^{\,a}$ can be nonvanishing in our case $\lambda_b^a = 0$ if e_a is not in $\mathfrak{h}$. In other words, the equations for a usual constrained system correspond to $\mathfrak{h} = \mathfrak{g}$.

To complete the discussion of totally-symmetric HS fields in $(d + 1)$-dimensions let us mention that the off-shell system based on $\mathfrak{g}$ (3.13) may describe other on-shell systems if one allows for more general factorizations. For instance, consider an ideal generated by $\mathcal{F}_0 = C_2 - (\lambda^2 - 1)$, where $C_2 = -\frac{1}{2}\bar{F}_{\alpha\beta}\star\bar{F}^{\alpha\beta}$ is the $sp(2)$ Casimir element. When $\lambda = l$ is an integer there appears an additional ideal, which can be seen to be generated by

$$\mathcal{F}_{\alpha_1\ldots\alpha_{2\ell}} = \bar{F}_{(\alpha_1\alpha_2}\star\ldots\star\bar{F}_{\alpha_{2\ell-1}\alpha_{2\ell})}, \tag{3.20}$$

for $l \geqslant 1$ and where all the $sp(2)$-indices in the second expression are symmetrized. Replacing $\bar{F}_l$ with $\mathcal{F}_0$ and $\mathcal{F}_{\alpha_1\ldots\alpha_{2\ell}}$ in (3.17) (and hence replacing $u_\cdot^{\,l}$ with $u_\cdot^{\,\alpha_1\ldots\alpha_{2\ell}}$ and $u_\cdot^{\,0}$ as well) one ends up with a consistent system. It is clear that for $\ell = 1$ one recovers (3.17) describing massless fields. The resulting HS system describes an interacting system of (partially)-massless fields of depth $t = 1, 3, \ldots, 2\ell - 1$ on AdS_{d+1}. The generalization of the Vasiliev theory to partially-massless fields was suggested in [36, 39, 40].

In order to see partially-massless fields in the spectrum let us consider 1-form gauge fields subjected to $sp(2)$ singlet condition $[\bar{F}_{\alpha\beta}, W]_\star = 0$. These can be decomposed in $\bar{F}_{\alpha\beta}$ so that expansion coefficients are identified with (partially)-massless fields of odd depth [41, 42]. At the same time the factorization eliminates the coefficients associated to the ideal in the algebra of $sp(2)$-singlets generated by $C_2 - (\lambda^2 - 1)$ and $\mathcal{F}_{\alpha_1\ldots\alpha_{2\ell}}$ [36] so that only fields of depth $1, 3, \ldots, 2\ell - 1$ remain.

Let us say a few words about the AdS/CFT interpretation of various HS theories [43, 44]. The bosonic Vasiliev theory should be generically dual to a free boson theory in d dimensions [45]. For $\lambda = l$ the dual theory is $\Box^\ell\phi = 0$ or multi-critical points of vector models [39]. At generic λ we find massive HS fields in the spectrum, the CFT dual should be a mean field theory—a generalized free field of dimension $\frac{d}{2} \pm \lambda$, depending on the boundary conditions imposed.

One way to see that massive or partially-massless fields appear in the spectrum is to make use of the linear relation between quadratic Casimir elements of the Howe dual algebras $o(d, 2)$ and $sp(2)$. Indeed, in our setting $c_2 = -\frac{1}{4}(d^2 - 4) + C_2$, where c_2 and C_2 are respectively orthogonal and symplectic Casimir operators [46]. For c_2 one then finds $c_2 = -\Delta_\lambda(\Delta_\lambda - d)$, where $\Delta_\lambda = \frac{d}{2} - \lambda$. The irreducible conformal module with this value is $D(\frac{d}{2} - \lambda, 0)$. The spectrum of the respective HS theory in the bulk is $D(\frac{d}{2} - \ell, 0) \otimes D(\frac{d}{2} - \ell, 0)$ which decomposes into irreducible modules of particular AdS fields. Note that possible energy values of these fields are determined by λ. For $\lambda = \ell$ the module is the (higher order) singleton $D(\frac{d}{2} - \ell, 0)$ whose square decomposes into (partially)-massless fields of depth $1, 3, \ldots, 2\ell - 1$ (for $\ell = 1$ this is the well-known Flato–Fronsdal theorem, the case of $\ell > 1$ was in [39]). For λ generic, modules with the non-special values of energy (these are associated to massive fields) appear in the tensor square. Note also that in this case $D(\frac{d}{2} - \lambda, 0)$ is a Verma module and hence the conformal scalar it describes is off-shell (from the Verma module perspective equations are associated to singular vectors which are not present as $D(\frac{d}{2} - \lambda, 0)$ is irreducible, see e.g. [47]).

There is a group theoretical explanation for the ideals $\mathcal{F}_0 = C_2 - (\lambda^2 - 1)$ and (3.20). We have an image of $U(\mathfrak{g})$ in $\mathfrak{A}$ generated by T_a and can consider a more general quotient

$$\frac{U(sp(2) \Subset osp(1|2))}{C_2 - \left(\lambda^2 - 1\right)} = gl_\lambda \otimes U(osp(1|2)). \tag{3.21}$$

The first factor is the Feigin's gl_λ [48], which is defined as a quotient of $U(sp(2))$ by a two-sided ideal generated by $\mathcal{F}_0$. At generic λ algebra gl_λ is infinite-dimensional and simple. When $\lambda = l$ is an integer the value of the Casimir is that of the l-dimensional irreducible representation of $sp(2)$. In this case gl_l contains a two-sided ideal generated by (3.20) with the quotient being $gl(l)$.

3.4. HS equations in four dimensions

In the $4d$ case the Lie superalgebra of dynamical symmetries is given by a direct sum

$$\mathfrak{g} = osp(1|2) \oplus osp(1|2) \tag{3.22}$$

where each factor can be defined along the lines of the previous section using $sp(2)$ vectors S_α and $\bar{S}_{\dot\alpha}$. Namely, using (3.12) one finds

$$\left[S_\alpha, S_\beta\right]_\star = -2\mathrm{i}\,\epsilon_{\alpha\beta}(1 + \Upsilon), \qquad \left[\bar{S}_{\dot\alpha}, \bar{S}_{\dot\beta}\right]_\star = +2\mathrm{i}\,\epsilon_{\dot\alpha\dot\beta}\left(1 + \bar\Upsilon\right), \tag{3.23a}$$

$$\left\{S_\alpha, \Upsilon\right\}_\star = 0, \qquad \left\{\bar{S}_{\dot\alpha}, \bar\Upsilon\right\}_\star = 0, \tag{3.23b}$$

$$\left\{S_\alpha, \bar{S}_{\dot\alpha}\right\}_\star = 0, \tag{3.23c}$$

where Υ and $\bar\Upsilon$ are elements (3.5) associated to two copies of $osp(1|2)$. The last condition accounts for a direct sum of Lie superalgebras $osp(1|2)$.

In the standard realization of $\mathfrak{A}$ (see section 4.4), this system does not describe an irreducible HS theory. The problem is that Υ and $\bar\Upsilon$ are related to the selfdual and anti-selfdual components of the Weyl tensor and its HS generalizations and hence cannot be independent. It follows that some further constraints that belong to $U(\mathfrak{g})$ are needed.

In order to get an irreducible system we need to take the elementary extension of the Lie superalgebra $\mathfrak{g}$ (3.22). It means adding an odd element K that (anti)commutes with $osp(1|2) \oplus osp(1|2)$ basis elements. In particular, we have

$$\left\{K, S_\alpha\right\}_\star = 0, \quad \left\{K, \bar{S}_{\dot\alpha}\right\}_\star = 0, \quad [K, \Upsilon]_\star = 0, \quad \left[K, \bar\Upsilon\right]_\star = 0. \tag{3.24}$$

The additional restriction that makes the system irreducible reads

$$\Upsilon = \bar\Upsilon \star K. \tag{3.25}$$

We can think of $U(osp(1|2))$ as of a noncommutative 2-sphere $\mathbb{S}_R$, whose radius squared is given by the Casimir operator $R^2 = \mathbb{C}_2 = \Upsilon^2$. The additional constraint implies $\Upsilon^2 = \bar\Upsilon^2$, i.e. it is a square root of $R^2 = \bar{R}^2$, so that we have $\mathbb{S}_R \times \mathbb{S}_R$.

To get a system that is explicitly equivalent to the original Vasiliev equations one redefines $\bar{S}_{\dot\alpha} \to \bar{S}_{\dot\alpha} \star K$ provided that K is invertible. It follows that the system (3.23*a*) remains mainly intact while the only changes are that $\{S_\alpha, \bar{S}_{\dot\alpha}\}_\star = 0$ goes into $[S_\alpha, \bar{S}_{\dot\alpha}]_\star = 0$, and $[\bar{S}_{\dot\alpha}, \bar{S}_{\dot\beta}]_\star$ is now opposite in sign. In the Vasiliev system element K is identified with the so-called total Klein operator, see the appendix. The supersymmetric extensions of the $4d$ equations follow the same logic, but we do not discuss them here.

The $4d$ theory must be isomorphic to the d-dimensional theory, discussed in the previous section, when $d = 4$. However, they are realized differently. This difference can be attributed to the fact that there are two realizations of the same HS algebra available in $4d$, which we discuss in section 4.

4. HS dynamics and star-product

Here we describe a standard oscillator realization of the embedding algebra $\mathfrak{A}$ and respective HS algebra $\mathfrak{hs}$ in various spacetime dimensions. Also, we review relevant types of the star-products with particular emphasis to the so-called twisted star-product. It will be shown that the choice of the star-product and/or specific vacuum solution essentially determines whether respective HS theory has local degrees of freedom or not.

4.1. HS algebras

In section 2 the HS algebra $\mathfrak{hs}$ was introduced as the symmetry algebra of the vacuum. Indeed, according to the defining relation (2.3) HS algebra is spanned by $\mathfrak{g}$-invariants. Equivalently, its elements are given by the cohomology $\mathbb{H}^0(\mathfrak{g}, \mathfrak{A})$. Obviously, being a global symmetry algebra, $\mathfrak{hs}$ plays a fundamental role in HS theory in contrast to the embedding algebra $\mathfrak{A}$ which is basically a convenient tool for generating interaction vertices.

Apart from the above definition of $\mathfrak{hs}$, there are more physical, but equivalent, definitions that naturally reflect various aspects of HS field dynamic. Basically, this happens because of the AdS/CFT correspondence that identifies gauge symmetries of the bulk theory with the global symmetries of its boundary dual. Below we briefly consider a few relevant realizations.

- From the CFT perspective, algebra $\mathfrak{hs}$ is the associative algebra of global symmetries of a conformally invariant field equation. In the simplest case, one considers the massless Klein–Gordon equation in $\mathbb{R}^{d-1,1}$ spacetime, i.e., $\Box\phi = 0$. It is known to be conformally invariant, enjoying the symmetry generated by $so(d, 2)$ conformal Killing vectors, which are first order differential operators. Since the equation is linear one can multiply symmetries to generate an infinite-dimensional symmetry algebra formed by the differential operators associated to conformal Killing tensors. It turns out that this algebra exhaust all global symmetries of $\Box\phi = 0$ and hence is $\mathfrak{hs}$ itself [49]. The corresponding Noether currents are generated by totally-symmetric conserved tensors
$$j_{\mu_1\ldots\mu_s} = \partial_{(\mu_1}\cdots\partial_{\mu_k}\phi^*\,\partial_{\mu_{k+1}}\cdots\partial_{\mu_s)}\phi - \text{traces}, \qquad \partial^{\mu_1}j_{\mu_1\ldots\mu_s} = 0, \tag{4.1}$$
where $s = 1, 2, \ldots, \infty$ [50].
- Algebra $\mathfrak{hs}$ can also be thought of as a unique algebra generated by the stress tensor $T_{\mu\nu}$ and at least one HS conserved tensor $j_{\mu_1\ldots\mu_s}$, $s > 2$ of some CFT [51–55]. The uniqueness theorem holds under further assumptions that the only conserved tensors are totally symmetric ones and there is a special case of $4d$ where one finds a one-parameter family [53, 56–58], provided the locality is relaxed.
- From the AdS perspective, algebra $\mathfrak{hs}$ is the global symmetry algebra of a given HS theory at the linearized level, or, equivalently, a symmetry algebra of the most symmetric vacuum. It can also be seen as a gauge algebra of HS theory in the sense that the 1-form HS connections take values in $\mathfrak{hs}$ [14, 59–61]. One should stress, however, that the algebra structure of $\mathfrak{hs}$ is deformed at the nonlinear level.
- One can formalize above realizations by taking the universal enveloping algebra $U(so(d, 2))$ and factor out a two-sided ideal $\mathrm{Ann}(S)$ that annihilates the irreducible $so(d, 2)$-module S [49, 62],
$$\mathfrak{hs} = U(so(d, 2))/\mathrm{Ann}(S). \tag{4.2}$$
The module S is a spin-0 Dirac singleton representation spanned by solutions of $\Box\phi = 0$.

The other way around, the symmetry algebra of S is the endomorphism algebra of S, i.e., $S \otimes S^*$, which coincides with $U(so(d, 2))/\mathrm{Ann}(S)$.

4.2. Oscillator realization

For applications, an efficient realization of HS algebras is needed. It turns out that in all known cases bosonic HS algebra can be realized as the Weyl algebra (or its quotient) which is the associative algebra generated by operators with canonical commutation relations.

In particular, it implies that AdS algebra $so(d, 2) \subset \mathfrak{hs}$ enjoys an oscillator realization. Indeed, a simple but crucial fact is that given a set of n variables $\zeta^{\mathcal{A}}$ satisfying

$$\left[\zeta_{\mathcal{A}}, \zeta_{\mathcal{B}}\right]_\star = 2\mathrm{i}C_{\mathcal{AB}}, \tag{4.3}$$

their quadratic combinations form $sp(2n)$ algebra with $C^{\mathcal{AB}}$ being the invariant tensor,

$$\left[T_{\mathcal{AB}}, T_{\mathcal{CD}}\right]_\star = C_{\mathcal{AC}}T_{\mathcal{BD}} + 3\text{ terms}, \qquad T_{\mathcal{AB}} = \frac{1}{4\mathrm{i}}\left\{\zeta_{\mathcal{A}}, \zeta_{\mathcal{B}}\right\}_\star. \tag{4.4}$$

The symplectic algebra is either isomorphic to AdS algebra in lower dimensions or contains AdS algebra as a subalgebra in higher dimensions as listed in the table below.

Dim	AdS-algebra	Isomorphism	Lorentz-algebra	Isomorphism
2	$so(1,2)$	$sp(2, \mathbb{R})$	$so(1,1)$	u(1)
3	$so(2,2)$	$sp(2, \mathbb{R}) \oplus sp(2, \mathbb{R})$	$so(2,1)$	$sp(2, \mathbb{R})$
4	$so(3,2)$	$sp(4, \mathbb{R})$	$so(3,1)$	$sp(2, \mathbb{C})_{\mathbb{R}}$
d+1	$so(d,2) \subset sp(2d+4)$		$so(d,1) \subset sp(2d+2)$	

The oscillator realizations of the AdS algebra together with its splitting into Lorentz subalgebra and translations are summarized in the table below (see, e.g., reviews [6, 7])[5].

Dim	Generators	AdS	Lorentz	Translations
2	$[y_\alpha, y_\beta]_\star = 2\mathrm{i}\epsilon_{\alpha\beta}$	$T_{\alpha\beta} = \frac{1}{4\mathrm{i}}\{y_\alpha, y_\beta\}_\star$	$L = \frac{1}{4\mathrm{i}}\{y_1, y_2\}_\star$	$P_1 = \frac{1}{2\mathrm{i}}y_1 y_1$ $P_2 = \frac{1}{2\mathrm{i}}y_2 y_2$
3	$[y_\alpha, y_\beta]_\star = 2\mathrm{i}\epsilon_{\alpha\beta}$ $\psi^2 = 1$	$L_{\alpha\beta}, P_{\alpha\beta}$	$L_{\alpha\beta} = \frac{1}{4\mathrm{i}}\{y_\alpha, y_\beta\}_\star$	$P_{\alpha\beta} = \psi L_{\alpha\beta}$
3	$[y_\alpha, y_\beta]_\star = 2\mathrm{i}\epsilon_{\alpha\beta}(1+\nu k)$ $\{y_\alpha, k\}_\star = 0$ $\psi^2 = 1,\ k^2 = 1$	$L_{\alpha\beta},\ P_{\alpha\beta}$	$L_{\alpha\beta} = \frac{1}{4\mathrm{i}}\{y_\alpha, y_\beta\}_\star$	$P_{\alpha\beta} = \psi L_{\alpha\beta}$
4	$[y_{\mathsf{A}}, y_{\mathsf{B}}]_\star = 2\mathrm{i}C_{\mathsf{AB}}$ $y_{\mathsf{A}} = (y_\alpha, \bar{y}_{\dot\alpha})$	$T_{\mathsf{AB}} = \frac{1}{4\mathrm{i}}\{y_{\mathsf{A}}, y_{\mathsf{B}}\}_\star$	$L_{\alpha\beta} = \frac{1}{4\mathrm{i}}\{y_\alpha, y_\beta\}_\star$ $L_{\dot\alpha\dot\beta} = \frac{1}{4\mathrm{i}}\{\bar{y}_{\dot\alpha}, \bar{y}_{\dot\beta}\}_\star$	$P_{\alpha\dot\alpha} = \frac{1}{2\mathrm{i}}y_\alpha \bar{y}_{\dot\alpha}$
d + 1	$[Y^A_\alpha, Y^B_\beta]_\star = 2\mathrm{i}\eta^{AB}\epsilon_{\alpha\beta}$ $Y^A_\alpha = (y^a_\alpha, y^a_\alpha)$	$T^{AB} = \frac{1}{4\mathrm{i}}\{Y^A_\alpha, Y^{B\alpha}\}_\star$	$L^{ab} = \frac{1}{4\mathrm{i}}\{y^a_\alpha, y^{b\alpha}\}_\star$	$P^a = \frac{1}{2\mathrm{i}}y^a_\alpha y^\alpha$

Note that the $3d$ AdS algebra is a direct sum of two $sp(2)$. This doubling is achieved using an additional element ψ, $\psi^2 = 1$, which makes Clifford algebra in one dimension. Also, both two and three dimensional AdS algebra generators can be built via the so-called deformed

[5] The $sp(2)$ indices are $\alpha, \beta, \ldots = 1, 2$ and $\dot\alpha, \dot\beta, \ldots = 1, 2$; the $sp(4)$ indices $\mathsf{A}, \mathsf{B}, \ldots = 1, \ldots, 4$ can be split into a pair of $sp(2)$ ones, $\mathsf{A} = (\alpha, \dot\alpha)$; the AdS $so(d, 2)$ indices are $A, B, \ldots = 0, \ldots, d+1$, the Lorentz $o(d, 1)$ indices are $a, b, \ldots = 0, \ldots, d$. Tensors η^{AB}, C_{AB} and $\epsilon^{\alpha\beta}$ are the invariant metrics of $so(d, 2)$, $sp(4)$ and $sp(2)$, respectively.

oscillators with commutation relations parameterized by a continuous ν [38, 63, 64], see the third line in the table.

According to (4.2), the HS algebra is defined as a quotient. In lower dimensions, using $sp(2n)$ oscillators allows to resolve the ideal (see, e.g., [65]). It follows that in $d = 2, 3, 4$ dimensions HS algebras are identified with the enveloping algebra of the relations in the table above, i.e. an element of the HS algebra is a function $f(y)$ or $f(y, k)$ in two dimensions, $f(y, \psi)$ or $f(y, \psi, k)$ in three dimensions, and $f(y, \bar{y})$ in four dimensions, see, e.g. [6] for review.

It is worth noting that without the ψ element in $3d$ case, the enveloping algebra of the deformed commutation relations is isomorphic to $U(osp(1|2))/I$, where I is the two-sided ideal generated by the shifted Casimir element $(C_2 - \frac{1}{4}(\nu^2 - 1))$ [37]. The subalgebra of even in y elements decomposes into a direct sum of two gl_λ (which was defined after (3.21)) for $C_2 = \frac{1}{4}(\nu^2 \pm 3\nu - 3)$.

In $d + 1$ dimension, the AdS algebra is only a subalgebra of $sp(2d + 4)$, so that the ideal is only partially resolved and certain further constraints are needed. The oscillator approach developed in [5] makes the Weyl algebra generated by oscillators Y^A_α a bimodule of two algebras $so(d, 2) - sp(2)$, where $\tau_{\alpha\beta} = -\frac{i}{4}\{Y^A_\alpha, Y^B_\beta\}\eta_{AB}$ are the $sp(2)$ generators, which form a Howe dual pair, i.e. $[T^{AB}, \tau_{\alpha\beta}] = 0$. Then, $\mathfrak{hs}$ can be realized as an $sp(2)$-invariant subspace of the Weyl algebra further quotiented by a two-sided ideal spanned by the elements proportional to $sp(2)$ generators

$$\mathfrak{hs} \ni f\left(Y^A_\alpha\right): \quad \left[f(Y), \tau_{\alpha\beta}\right]_\star = 0, \qquad f(Y) \sim f(Y) + g^{\alpha\beta}(Y)\star\tau_{\alpha\beta}. \quad (4.5)$$

The Taylor coefficients of the quotient representatives $f(Y)$ carry $so(d, 2)$ indices described by traceless two-row rectangular Young diagrams of arbitrary length

$$f\left(Y^A_\alpha\right) = \sum_k f_{A_1\ldots A_k, B_1\ldots B_k} T^{A_1B_1}\ldots T^{A_kB_k}, \qquad f_{A_1\ldots A_k, B_1\ldots B_k} = \begin{array}{|c|c|c|}\hline & K & \\ \hline & & \\ \hline\end{array}\,. \quad (4.6)$$

Naturally, these tensor expansion coefficients are in one-to-one correspondence with both connections of the HS algebra $\mathfrak{hs}$, and conformal Killing tensors discussed in section 4.1.

Let us mention again that considering more general ideals one can get HS algebras for partially-massless and massive fields following the same procedure as in section 3.3.2 with $\bar{F}_{\alpha\beta}$ replaced by $\tau_{\alpha\beta}$. The CFT interpretation is that these are the algebras of higher symmetries of the polywave equation $\square^\ell\phi = 0$ [39], when $\lambda = l$ is an integer and the algebra of symmetries of generalized free field of dimension $\frac{d}{2} \pm \lambda$.

We stress that the $2d$, $3d$ (at $\nu = 0$) and $4d$ algebras are isomorphic to the d-dimensional algebra for $d = 2, 3, 4$ provided that the functions of respectively y_α, y_α, and y_{A}, are restricted to be even, i.e., half-integer spins are projected out[6]. Each of the oscillator realizations given above has its own features that do not bear any invariant meaning in contrast to the HS algebra itself. However, these features affect the choice of $\mathfrak{g}$ and, hence, the realization of the embedding algebra $\mathfrak{A}$.

4.3. Twisted star-product

As we argued in the introduction the embedding algebra $\mathfrak{A}$ has the structure of a twisted product, where it is the twist that is responsible for nontriviality of the theory. The factors are

[6] A relation between vector and spinor realizations of $d = 2, 3, 4$ HS algebras is explicitly discussed in [46].

the HS algebra $\mathfrak{hs}$ and the full algebra of dynamical symmetries, i.e. the image of $U(\mathfrak{g})$ in $\mathfrak{A}$. This is the structure present in all HS theories constructed so far. Because HS algebras admit oscillator realizations it is possible to give a concrete realization of the twisted product as the star-product [9]. Below we collect some basic definitions and properties of star-products.

The star-product algebra is the algebra of functions in commuting variables $\zeta^{\mathcal{A}}$ that is equipped with a non-commutative product, called star-product,

$$(f\star g)(\zeta) = f(\zeta)\exp \mathrm{i}\left(\frac{\overleftarrow{\partial}}{\partial\zeta^{\mathcal{A}}}\,\Omega^{\mathcal{AB}}\frac{\overrightarrow{\partial}}{\partial\zeta^{\mathcal{B}}}\right)g(\zeta). \tag{4.7}$$

The anti-symmetric component of $\Omega^{\mathcal{AB}}$, $C = \frac{1}{2}(\Omega - \Omega^{T})$ is the symplectic metric, which specifies the commutation relations $[\zeta^{\mathcal{A}}, \zeta^{\mathcal{B}}]_\star = 2\mathrm{i}C^{\mathcal{AB}}$. The symmetric part of $\Omega^{\mathcal{AB}}$ is responsible for ordering prescription for the operators that $\zeta^{\mathcal{A}}$ are symbols of. For example, the matrix $\Omega^{\mathcal{AB}}$ that corresponds to the totally-symmetric ordering is just $C^{\mathcal{AB}}$:

$$\text{symmetric:}\qquad (f\star g)(\zeta) = f(\zeta)\exp \mathrm{i}\left(\frac{\overleftarrow{\partial}}{\partial\zeta^{\mathcal{A}}}C^{\mathcal{AB}}\frac{\overrightarrow{\partial}}{\partial\zeta^{\mathcal{B}}}\right)g(\zeta). \tag{4.8}$$

Another commonly used prescription is the normal ordering, which corresponds to particular splitting of $\zeta^{\mathcal{A}}$ into q^m and p_n, i.e. $\zeta^{\mathcal{A}} = (q^m, p_n)$. Then, the product implementing the qp -ordering is

$$\text{normal:}\quad (f\star g)(q, p) = f(q, p)\exp \mathrm{i}\left(\frac{\overleftarrow{\partial}}{\partial p_n}\frac{\overrightarrow{\partial}}{\partial q^n}\right)g(q, p),\qquad \Omega = \begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix}. \tag{4.9}$$

It is remarkable that the HS theory uses a specific mixture of normal and symmetric orderings, which was introduced by Vasiliev [9]. We will refer to it as the twisted star-product. Suppose $\zeta^{\mathcal{A}}$ with $\mathcal{A} = 1, \ldots, 4$ splits into a pair of variables $\zeta^{\mathcal{A}} = (y^\alpha, z^\beta)$, where $\alpha, \beta = 1, 2$. The twisted star-product corresponds to the symmetric ordering among y^α and among z^β with the symplectic structure given by the epsilon-symbol $\epsilon^{\alpha\beta}$ in both the sectors, while it is normal ordered with respect to $q^\alpha = y^\alpha + z^\alpha$ and $p^\alpha = y^\alpha - z^\alpha$. Namely, let $\mathfrak{A}_0$ be an algebra of functions in y^α and z^α endowed with the following product:

$$(f\star g)(y, z) = f(y, z)\exp \mathrm{i}\left(\frac{\overleftarrow{\partial}}{\partial y^\alpha} + \frac{\overleftarrow{\partial}}{\partial z^\alpha}\right)\epsilon^{\alpha\beta}\left(\frac{\overrightarrow{\partial}}{\partial y^\beta} - \frac{\overrightarrow{\partial}}{\partial z^\beta}\right)g(y, z). \tag{4.10}$$

or, more generally, with a one-parameter family of star-products:

$$\text{twisted}_\vartheta\text{:}\quad (f\star g)(y, z) = f(y, z)\exp\left(\frac{\overleftarrow{\partial}}{\partial y^\alpha}, \frac{\overleftarrow{\partial}}{\partial z^\alpha}\right)\begin{pmatrix} \epsilon^{\alpha\beta} & -\vartheta\epsilon^{\alpha\beta} \\ \vartheta\epsilon^{\alpha\beta} & \epsilon^{\alpha\beta} \end{pmatrix} \times \begin{pmatrix} \dfrac{\overrightarrow{\partial}}{\partial y^\beta} \\ \dfrac{\overrightarrow{\partial}}{\partial z^\beta} \end{pmatrix} g(y, z), \tag{4.11}$$

interpolating between (4.10) at $\vartheta = 1$ and the symmetric product (4.8) at $\vartheta = 0$. Note, that at $\vartheta = 0$ there is no mixing among y^α and z^β. In section 4.5 we show that for $\vartheta \neq 0$ the HS theory is nontrivial, while taking the limit $\vartheta = 0$ yields a topological theory.

It is important to stress that all star-products are equivalent when restricted to polynomials. However, in the HS theory certain non-polynomial elements appear in perturbative solution of (2.1). With this being said, we have to consider $\vartheta \neq 0$ twisted and $\vartheta = 0$ symmetric star-products as non-equivalent. Then, the twisted star-product can be thought of as a particular example of the general concept of twisted tensor product of associative algebras.

Indeed, let A and B be two associative algebras. According to [15], algebra C is a twisted tensor product of A and B iff there exists injective algebra homomorphisms $i_A: A \to C$ and $i_B: B \to C$ such that the linear map $i_A \otimes i_B: A \otimes B \to C$ defined by $(i_A \otimes i_B)(a \otimes b) = i_A(a) \star i_B(b)$ is a linear isomorphism. Here $a \in A$, $b \in B$ and $\star$ is a product in C.

In the case of interest we have the associative HS algebra $\mathfrak{hs}$ and the algebra of dynamical symmetries. In all known cases the nontrivial part of the embedding algebra $\mathfrak{A}$ has the same realization as the twisted star-product (4.10), where A and B are the star-product algebras of functions $a(y^\alpha)$ and $b(z^\alpha)$, respectively, with the products μ_A and μ_B given by

$$\mu_A = \exp + \mathrm{i}\frac{\overleftarrow{\partial}}{\partial y^\alpha}\epsilon^{\alpha\beta}\frac{\overrightarrow{\partial}}{\partial y^\beta}, \qquad \mu_B = \exp - \mathrm{i}\frac{\overleftarrow{\partial}}{\partial z^\alpha}\epsilon^{\alpha\beta}\frac{\overrightarrow{\partial}}{\partial z^\beta}. \tag{4.12}$$

The algebra C is the algebra of functions $c(y, z)$ equipped with the twisted product (4.11). The map $\mu_A \otimes \mu_B$ determined by

$$a \otimes b \mapsto i_A(a) \star i_B(b) = a(y)\tau(\vartheta)b(z), \qquad \tau(\vartheta) = \exp - \mathrm{i}\vartheta\frac{\overleftarrow{\partial}}{\partial y^\alpha}\epsilon^{\alpha\beta}\frac{\overrightarrow{\partial}}{\partial z^\beta}, \tag{4.13}$$

is an isomorphism for a suitable class of functions because τ is formally invertible: $\tau^{-1}(\vartheta) = \tau(-\vartheta)$. At $\vartheta = 0$ we get back to the usual product of associative algebras. It is crucial for nontriviality of the theory that $[i_A(a(y)), i_B(b(z))] \neq 0$ while both $\mathfrak{hs} = A$ and $U(\mathfrak{g}) = B$ are subalgebras of C.

4.4. Embedding algebra and vacuum

The section is aimed at defining the embedding algebra $\mathfrak{A}$ and the vacuum solution for T_a and W. The general structure of the algebra $\mathfrak{A}$ is a twisted product of the HS algebra $\mathfrak{hs}$ and the algebra of dynamical symmetries $U(\mathfrak{g})$. Therefore, $\mathfrak{A}$ always includes the generators/relations we used to define $\mathfrak{hs}$. The twist enters through one or more factors of the algebra $\mathfrak{A}_0$, which is the twisted star-product algebra generated by y_α, z_α, where y_α's belong to the realization of $\mathfrak{hs}$. A number of discrete elements, which can be combined into Clifford algebras, can also appear. It is always possible to take the tensor product with matrix algebras Mat_n, which allow HS fields to carry Yang–Mills indices. The algebras so defined can be truncated by some reality conditions and other (anti)-automorphisms. For example, the Yang–Mills factor can be truncated to compact forms su, so, usp [31, 32].

In the table below we list some of the cases where $\mathfrak{A}$ is known. Below, A_{d+1} denotes the Weyl algebra formed by y^a_α in the $(d+1)$-dimensional HS algebra and the relations determining the star-product in the sector of (y_α, z_α) and $(y_{\dot\alpha}, z_{\dot\alpha})$ variables are those of $\mathfrak{A}_0$ and are omitted.

Dim	Generators, relations	$\mathfrak{A}$	Vacuum
3	$y_\alpha, z_\alpha, \{\psi_i, \psi_j\} = 2\delta_{ij}$	$\mathfrak{A}_0 \otimes Cl_{2,0}$	$S^0_\alpha = z_\alpha$
4	$y_\alpha, z_\alpha, \quad \bar{y}_{\dot\alpha}, \bar{z}_{\dot\alpha}$	$\mathfrak{A}_0 \otimes \mathfrak{A}_0$	$S^0_\alpha = z_\alpha, \ S^0_{\dot\alpha} = \bar{z}_{\dot\alpha}$

(Continued).

Dim	Generators, relations	$\mathfrak{A}$	Vacuum
d+1	y_α, z_α $[y^a_\alpha, y^b_\beta]_\star = 2i\eta^{ab}\epsilon_{\alpha\beta}$	$\mathfrak{A}_0 \otimes A_{d+1}$	$S^0_\alpha = z_\alpha \quad F^0_{\alpha\beta} = \frac{1}{4i}\left(\{Y^a_\alpha, Y^b_\beta\}_\star\eta_{ab} + \{y_\alpha, y_\beta\}_\star - \{z_\alpha, z_\beta\}_\star\right)$

An essential ingredient of the theory is the vacuum solution W^0, T^0_a. In HS theories 1-form W^0 is a flat connection of the anti-de Sitter algebra $so(d, 2) \subset \mathfrak{hs}$. By definition, the background W^0 has vielbein h^a and spin-connection $\varpi^{a,b}$ as its components along Lorentz and translation generators. For instance, in the d-dimensional notation we have $W^0 = \frac{1}{2}L_{ab}\varpi^{a,b} + P_a h^a$. Then, any non-degenerate solution of the flatness condition $\mathrm{d}W^0 + W^0\star W^0 = 0$ (2.1) describes empty anti-de Sitter space.

According to (2.3), the HS algebra $\mathfrak{hs}$ is the centralizer of the vacuum. Since the generators T_a of the $osp(1|2)$ part can be reduced to a two-component field S_α, it is sufficient to specify a vacuum value only for S_α. It is always $S^0_\alpha = z_\alpha$. It is obviously consistent with W_0 since $[z_\alpha, f]_\star = -2i\partial_\alpha f$, where $\partial_\alpha = \frac{\partial}{\partial z^\alpha}$ and the generators of the HS algebra are z-independent. Therefore, $\mathrm{d}T^0_a + [W^0, T^0_a]_\star = 0$ is satisfied. For the same reason, the global symmetries of the vacuum, which are solved from (2.3), belong to $\mathbb{H}^0(\mathfrak{g}, \mathfrak{A})$ and form the HS algebra.

In section 3.3.1 we proved that fluctuations of $F^0_{\alpha\beta}$ are trivial. The proof crucially relies on the assumption that $\mathrm{ad}(F^0_{\alpha\beta}) = [F^0_{\alpha\beta}, \cdot]_\star$ are of vanishing homogeneity degree in Y^a_α, y_α, z_α. While it is obviously true for the y^a_α part of the generators, its validity for y_α, z_α relies on an important property of the twisted star-product. Because of the twisting that take place in (4.11) the action of each of the two $sp(2)$ subalgebras associated with y_α and z_α is deformed, for example,

$$\frac{1}{2}\xi^{\alpha\beta}\left[L^y_{\alpha\beta}, f(y, z)\right]_\star = \xi^{\alpha\beta}\left(y_\alpha - i\vartheta\frac{\partial}{\partial z^\alpha}\right)\frac{\partial}{\partial y^\beta}f(y, z), \qquad L^y_{\alpha\beta} = -\frac{i}{4}\left\{y_\alpha, y_\beta\right\}_\star, \tag{4.14}$$

but the diagonal $sp(2)$ algebra that contributes to $F^0_{\alpha\beta}$ still acts canonically

$$\frac{1}{2}\xi^{\alpha\beta}\left[L^y_{\alpha\beta} + L^z_{\alpha\beta}, f(y, z)\right]_\star = \xi^{\alpha\beta}\left(y_\alpha\frac{\partial}{\partial y^\beta} + z_\alpha\frac{\partial}{\partial z^\beta}\right)f(y, z), \quad L^z_{\alpha\beta} = \frac{i}{4}\{z_\alpha, z_\beta\}_\star. \tag{4.15}$$

Therefore the Whitehead lemma used in section 3.3.1 can be applied.

4.5. Linearized HS dynamics and star-product

In what follows, it will be shown that the nontriviality of a given HS theory depends essentially on the choice of the star-product. With an appropriate choice of the star-product we show that the cohomology groups $\mathbb{H}^1(\mathfrak{g}, \mathfrak{A})$ that parameterize the deviation from the flat connection are not empty and (2.7) describes free fields of all spins.

Let us consider first-order perturbations (2.4) of the system (2.1). We expand $W = W^0 + w$ and $S_\alpha = z_\alpha + s_\alpha$. The linearized equations together with the gauge transformations have the form[7]

[7] For the 4d system there is a doubling s_α, $s_{\dot\alpha}$, which we do not consider in detail.

$$D_0 w = 0, \qquad \delta w = D_0 \xi, \tag{4.16a}$$

$$D_0 s_\alpha = \partial_\alpha w, \tag{4.16b}$$

$$\left\{z_\alpha, \partial_\gamma s^\gamma\right\}_\star = 0, \qquad \delta s_\alpha = \partial_\alpha \xi, \tag{4.16c}$$

where D_0 is the background covariant derivative $D_0 = d + [W^0, \cdot]_\star \equiv d + ad_{W^0}$ and $\partial_\gamma s^\gamma$ is the linearization of Υ, see e.g. (3.12). To get these equations we used $[z_\alpha, \cdot]_\star = -2\mathrm{i}\partial_\alpha$, where $\partial_\alpha = \frac{\partial}{\partial z^\alpha}$.

Equation (4.16c) is convenient to solve assuming that $\mathfrak{A}$ contains an element called Klein operator (see, e.g., [6]) that implements the $\mathbb{Z}_2$ automorphism,

$$\rho\left(\zeta^{\mathcal{A}}\right) = \varkappa \star \zeta^{\mathcal{A}} \star \varkappa^{-1} = -\zeta^{\mathcal{A}}. \tag{4.17}$$

The choice of ordering prescription affects the functional form of the Klein operator. For instance, see also appendix B in [66] for discussion of different orderings,

$$\begin{aligned} &\text{symmetric:} \quad \varkappa = \delta(\zeta) && \text{normal:} \quad \varkappa = \exp\left(\mathrm{i}q^m p_m\right), \\ &\text{twisted:} \quad \varkappa = \exp\left(\mathrm{i}z_\alpha y_\beta \epsilon^{\alpha\beta}\right) && \text{twisted}_\vartheta\text{:} \quad \varkappa = \exp\left(\mathrm{i}\vartheta^{-1} z_\alpha y_\beta \epsilon^{\alpha\beta}\right). \end{aligned} \tag{4.18}$$

Ignoring the issue of functional class we can map anti-commutators to commutators using the Klein operator. Indeed, according to (4.17), the vacuum satisfies $\varkappa \star z_\alpha \star \varkappa^{-1} = -z_\alpha$, so that

$$\partial_\gamma s^\gamma \equiv \Upsilon: \qquad \left\{z_\alpha, \Upsilon\right\}_\star = \left[z_\alpha, \Upsilon \star \varkappa\right]_\star \star \varkappa^{-1} = 0 \qquad \Longleftrightarrow \qquad \partial_\alpha(\Upsilon \star \varkappa) = 0, \tag{4.19}$$

where the last equation is true thanks to invertibility of $\varkappa$. It implies that $\Upsilon \star \varkappa$ is z-independent,

$$\Upsilon \star \varkappa = C(y|x), \tag{4.20}$$

where $C(y|x)$ is an arbitrary function of all variables but z_α, i.e. of $x^{\underline{m}}$, y_α and possibly of y_α^a or ψ_i, depending on the theory we consider. In order to reconstruct the gauge potential s_α from Υ we represent equation $\partial_\gamma s^\gamma = \Upsilon$ in the dualized form as follows

$$\partial_\alpha s_\beta - \partial_\beta s_\alpha = \epsilon_{\alpha\beta} \Upsilon = \epsilon_{\alpha\beta} C(y|x) \star \varkappa. \tag{4.21}$$

At this point it is useful to translate everything into the language of differential forms, by contracting all indices with anticommuting differentials $\mathrm{d}z^\alpha$. Then the last equation is simply

$$\partial s = \Upsilon\, \mathrm{d}z^\alpha \wedge \mathrm{d}z^\beta \epsilon_{\alpha\beta}, \tag{4.22}$$

where $\partial = \mathrm{d}z^\alpha \partial_\alpha$ is the $2d$ de Rham differential. Then, the general solution is $s = \partial^{-1}(\Upsilon) + \partial\xi$, where ∂^{-1} is any representative of anti-derivative, and the last term represents exact forms (4.16c). For example, one can use the standard contracting homotopy for the de Rham complex to obtain

$$s = \partial^{-1}(\Upsilon) \equiv \partial^{-1}\left(\Upsilon\, \epsilon_{\alpha\beta} \mathrm{d}z^\alpha \wedge \mathrm{d}z^\beta\right) = z^\alpha \int_0^1 t\, \mathrm{d}t\, \epsilon_{\alpha\beta} \Upsilon(zt) \mathrm{d}z^\beta, \tag{4.23}$$

where we worked in the Schwinger–Fock gauge $z^\alpha s_\alpha = 0$. We would like to stress that Υ does depend on z_α because of the Klein operator $\varkappa$, cf (4.20).

Now we lift expression for s_α (4.23) to equation (4.16b), which again has a form of $\partial w = D_0 s$ and can be solved as before, $w(y, z|x) = \omega(y|x) + \partial^{-1} D_0 s(y, z|x)$, where homogeneous part satisfies $\partial\omega(y|x) = 0$. According to the general discussion of section 2.2, ω represents cohomology $\mathbb{H}^0(\mathfrak{g}, \mathfrak{A})$. It follows that the HS field ω is identified with $\mathbb{H}^0(\mathfrak{g}, \mathfrak{A})$ connection. Using identities $\{d, \partial^{-1}\} \equiv 0$ and $\partial^{-1}\partial^{-1} \equiv 0$ the solution can be simplified to

$w = \omega + \partial^{-1}(ad_{W_0} s)$. On substituting this to the first equation (4.16*a*) we can restrict to $z = 0$ surface to get

$$D_0\omega = \ -ad_{W^0}\, \partial^{-1} ad_{W^0}\, \partial^{-1}\Upsilon\Big|_{z=0}. \tag{4.24}$$

Thus, we arrive at the particular realization of (2.7) used in the Vasiliev theory. The dynamical content of the equation (4.24) relies on the particular choice of the star-product. E. g., using the twisted star-product at $\vartheta \neq 0$ in d dimensions yields the standard unfolded equations

$$D_0\omega(y|x) = h^a \wedge h^b \epsilon_{\alpha\beta} \frac{\partial^2}{\partial y_\alpha^a \partial y_\beta^b} C\left(y_\alpha^a,\, y_\alpha = 0\, |x\right), \tag{4.25}$$

where function $C(y\,|x)$ on the rhs parameterizes all spin-s Weyl tensors (see [7] for more details).

Let us consider again equation (4.16*c*). Taking the limit $\vartheta = 0$ one finds out that the star-product corresponds to the symmetric ordering and the Klein operator is realized as the δ-function, (4.18). Even without Klein operator it is obvious that equation $\{z_\alpha, \Upsilon\}_\star = 2z_\alpha\Upsilon = 0$ has only a trivial regular solution $\Upsilon = 0$. Therefore, the embedding algebra based on the untwisted product of $\mathfrak{hs}$ and $U(\mathfrak{g})$ leads to a topological system for the particular vacuum $S_\alpha^0 = z_\alpha$.

4.6. Relation to parent system

As we have just seen the nontriviality of the Vasiliev system has to do with the nonvanishing symmetric part (4.11) of $\Omega^{\mathcal{AB}}$ in the star-product (4.7), which makes the Klein operator regular. Using the formulation in $(d+1)$ dimensions we now demonstrate that it is nevertheless possible to describe degrees of freedom using just untwisted star-product. In this case the nontriviality enters through the specific choice of the vacuum solution.

Having found that $\partial_\nu s^\nu = 0$ at $\vartheta = 0$ and hence s_α is pure gauge we observe that the second equation in (3.15) then implies $\frac{\partial}{\partial z^\nu}\bar{F}_{\alpha\beta} = 0$ and the extended system takes the form

$$\mathrm{d}W + W\star W = 0, \qquad \mathrm{d}\bar{F}_{\alpha\beta} + \left[W, \bar{F}_{\alpha\beta}\right]_\star = 0, \qquad \left[\bar{F}_{\alpha\beta}, \bar{F}_{\gamma\delta}\right]_\star = \epsilon_{\beta\gamma}\bar{F}_{\alpha\delta} + \cdots, \tag{4.26}$$

known in the literature [10]. Note that this is again of the type (2.1) with $\mathfrak{g} = sp(2)$.

According to the discussion in 3.3.1, taking as a vacuum solution $\bar{F}_{\alpha\beta} = \frac{1}{4\mathrm{i}}\{Y_\alpha^A, Y_\beta^B\}_\star\eta_{AB}$ allows to perturbatively eliminate $\bar{F}_{\alpha\beta}$. However, the system can describe degrees of freedom if the vacuum is chosen differently. Following [10] we first fix the allowed class of functions to be polynomials in Y_2^A with coefficients in formal series Y_1^A. With this choice the vacuum

$$\bar{F}_{\alpha\beta}^0 = \frac{1}{4\mathrm{i}}\left\{Y_\alpha^A, Y_\beta^B\right\}_\star \eta_{AB}\,\Big|_{Y_1^A \to Y_1^A + V^A} \tag{4.27}$$

is not equivalent to the one without shift in Y_1 (the shift is not well-defined for formal series). As a consequence, the linearized system is non-empty and was shown in [10] to describe massless fields of all integer spins at the off-shell level (i.e., equivalent to the linearized Vasiliev system before factorization). Note that in this case the argument based on Whitehead lemma does not work because $[\bar{F}_{\alpha\beta}^0, \cdot]$ is not homogeneous in Y (in particular, Δ-cohomology is nonempty in degree 1 and, in a certain sense, is precisely a configuration space of HS fields).

A closely related system makes sense in the context of conformal HS fields on the d-dimensional boundary. More precisely, replacing AdS_d with its boundary and the AdS compensator V^A satisfying $V^AV_A + 1 = 0$ with the conformal one satisfying $V^AV_A = 0$ the system (4.26) describes, in particular, off-shell conformal HS fields. More precisely, the system describes totally symmetric conformal HS gauge fields at the off-shell level provided one performs a consistent factorization, as described in 3.3.2, with respect to the $sp(2)$ algebra. Remarkably, these fields can be seen as leading boundary values for the bulk HS fields. For more details see [39].

4.7. Vasiliev theory squared

It is instructive to see what happens if instead of the Lie superalgebra $\mathfrak{g} = osp(1|2)$ one takes the Lie algebra $\mathfrak{g} = sp(2)$ while keeping the algebra $\mathfrak{A}_0$ and the vacuum as in section 4.5. The odd generators S_α of $osp(1|2)$ are now absent and the $sp(2)$ generators $T_{\alpha\beta}$ satisfy

$$\left[T_{\alpha\beta},\, T_{\gamma\delta}\right]_\star = \epsilon_{\alpha\delta}T_{\beta\gamma} + 3 \text{ terms}. \tag{4.28}$$

If we take the same vacuum $T^0_{\alpha\beta} = \frac{\mathrm{i}}{4}\{z_\alpha, z_\beta\}_\star$ as before then the second equation in (2.3) is a second order equation because of (4.14). Namely, using (4.14) one finds that global symmetries ξ are solved from

$$\left[T_{\alpha\beta},\, \xi\right]_\star = \frac{\mathrm{i}}{4}\left\{z_\alpha,\, \left[z_\beta,\, \xi\right]_\star\right\}_\star + (\alpha \leftrightarrow \beta) = \left(z_\alpha + \mathrm{i}\frac{\partial}{\partial y_\alpha}\right)\frac{\partial}{\partial z_\beta}\xi + (\alpha \leftrightarrow \beta) = 0, \tag{4.29}$$

whose general solution involves two arbitrary functions of y_α, namely,

$$\xi = \xi_0(y) + \partial^{-1}\left(\left\{\mathrm{d}z^\alpha z_\alpha,\, \xi_1(y)\right\}_\star \star \varkappa\right), \tag{4.30}$$

where $\partial = \mathrm{d}z^\alpha\partial_\alpha$. Therefore, $\xi \in \mathbb{H}^0(\mathfrak{g}, \mathfrak{A})$ turns out to contain two branches. These are parameterized by original $\xi_0(y)$ appearing in the $osp(1|2)$ case, and additional $\xi_1(y)$ in the $sp(2)$ case.

The advantage of having a bosonic oscillator realization of the superalgebra $osp(1|2)$ is that the vacuum odd generators act as $[z_\alpha, \cdot]_\star = -2\mathrm{i}\frac{\partial}{\partial z_\alpha}$ and the centralizer of the vacuum, which is the HS algebra, is independent of the auxiliary variables z_α. In the $sp(2)$ case the vacuum bosonic generators $[T_{\alpha\beta}, \cdot]_\star$ are second order operators giving rise to the centralizer bigger than the original HS algebra.

When solving the field equations one can either ignore the second branch at every order of the perturbative expansion or define a certain projector that explicitly removes these modes. One way or another, every solution in the $osp(1|2)$ case is a solution in the $sp(2)$ case as well.

It is worth noting that the above extra branch arises due to the particular choice of the star-product and the vacuum which were previously used in the standard $osp(1|2)$ case. On the other hand, in section 4.6 we showed that the extra branch in $\mathbb{H}^0(\mathfrak{g}, \mathfrak{A})$ can be avoided by taking a slightly different vacuum.

5. Algebraic structure and AKSZ form

In this section we discuss the structure of the basic system (2.1) in some more mathematical details. Starting with a Lie superalgebra $\mathfrak{g}$ and associative superalgebra $\mathfrak{A}$ let us consider the superspace of linear maps τ: $\mathfrak{g} \to \mathfrak{A}$, where $\mathfrak{A}$ is understood as a Lie superalgebra, i.e. with the Lie operation $[f, g]_\star = f\star g - (-1)^{|f||g|}g\star f$. The Grassmann degree on the space of maps originates from those on $\mathfrak{g}$ and $\mathfrak{A}$. More precisely, if $\mathfrak{g} = \mathfrak{g}_{\bar 0} \oplus \mathfrak{g}_{\bar 1}$ and $\mathfrak{A} = \mathfrak{A}_{\bar 0} \oplus \mathfrak{A}_{\bar 1}$ are the decompositions into homogeneous components then degree-0 maps sends $\mathfrak{g}_{\bar{0}}$ to $\mathfrak{A}_{\bar{0}}$ and $\mathfrak{g}_{\bar{1}}$ to $\mathfrak{A}_{\bar{1}}$ while degree-1 map sends $\mathfrak{g}_{\bar{0}}$ to $\mathfrak{A}_{\bar{1}}$ and $\mathfrak{g}_{\bar{1}}$ to $\mathfrak{A}_{\bar{0}}$. The condition that τ is a homomorphism reads as

$$\tau(a)\star\tau(b) - (-1)^{|a||b|}\tau(b)\star\tau(a) = \tau([a,\ b]), \qquad \forall a, b \in \mathfrak{g}. \tag{5.1}$$

If e_a denote a basis in $\mathfrak{g}$ then the above condition takes the form of the third equation in (2.1).

There is a natural equivalence on the superspace of homomorphisms:

$$\tau \sim \mathcal{I}_{\mathfrak{A}} \circ \tau \circ \mathcal{I}_{\mathfrak{g}}, \tag{5.2}$$

where $\mathcal{I}_{\mathfrak{A}}$ and $\mathcal{I}_{\mathfrak{g}}$ are inner automorphisms of respectively $\mathfrak{A}$ and $\mathfrak{g}$. An infinitesimal versions of the above equivalence relations read as [8]

$$\tau(a) \sim \tau(a) + [\tau(a), \xi]_\star, \qquad \tau(a) \sim \tau(a) + \tau([a,\ \beta]), \tag{5.3}$$

where $\xi \in \mathfrak{A}$ and $\beta \in \mathfrak{g}$. In the context of HS theories the equivalence $\mathcal{I}_{\mathfrak{A}}$ is interpreted as a genuine equivalence. Indeed, the above transformation is precisely the gauge symmetry (2.2) in the sector of T_a variables. At the same time $\mathcal{I}_{\mathfrak{g}}$ is treated as a physical symmetry. Note that interpreting $\mathcal{I}_{\mathfrak{g}}$ as a gauge symmetry can also be useful though we do not have meaningful examples at the moment.

It turns out that the superspace of homomorphisms subject to the equivalence relation generated by $\mathcal{I}_{\mathfrak{A}}$ completely determines the gauge invariant system (2.1). To see this it is convenient to switch to the language of Q-manifolds. First one defines a supermanifold associated to the superspace. Namely, if E_A is a basis in $\mathfrak{A}$ the components of the homomorphism are $\tau(e_a) = T_a^A E_A$. One then reinterprets T_a^A as coordinates on a supermanifold M_0 by prescribing Grassmann parity as $|T_a^A| = |E_A| + |e_a|$. Note that we assume (5.1) imposed so that M_0 is a surface in the space of coordinates T_a^A singled out by $[T_a, T_b]_\star = C_{ab}^c T_c$.

In order to take into account the gauge symmetry one promotes parameters ξ to ghost coordinates extending M_0 to M. More precisely, introducing components of the gauge parameter ξ through $\xi = \xi^A E_A$, ξ^A are promoted to coordinates W^A such that $|W^A| = |E_A| + 1$ and $\mathrm{gh}(W) = 1$; ghost degree of T_a^A is zero. Finally, M is equipped with the odd nilpotent vector field Q determined by

$$QT_a = \left[W,\ T_a\right]_\star, \qquad QW = \frac{1}{2}[W,\ W]_\star. \tag{5.4}$$

The gauge symmetry induced by Q is precisely the above equivalence $\mathcal{I}_{\mathfrak{A}}$.

Given a Q-manifold equipped with a nonnegative ghost degree one can define a free differential algebra on a given space-time manifold (see e.g. [11, 67] for more details). Namely, to each coordinate ψ^I of ghost degree p one associates a p-form field Ψ^I on the space-time manifold. Furthermore, if $p \geqslant 1$ then coordinate ψ^I also gives rise to a gauge parameter ϵ^I which is a $(p-1)$-form. The equations of motion and gauge symmetries read as

$$\mathrm{d}\Psi^I + Q^I(\Psi) = 0, \qquad \delta_\epsilon \Psi^I = \mathrm{d}\epsilon^I - \epsilon^J \frac{Q^I(\Psi)}{\Psi^J}. \tag{5.5}$$

Applying the above construction to the Q manifold M one finds 0-form fields T_a^A and 1-form fields $W^A = \mathrm{d}x^\mu W_\mu^A$ (by slight abuse of notation we use the same symbol for a coordinate on M and its associated field) along with the 0-form gauge parameters ξ^A associated to W^A. Equations and gauge transformations (5.5) are then equations (2.1) and (2.2). Note, however,

[8] Everywhere in this section $[A, B]_\star$ denotes the supercommutator $A \star B - (-1)^{|A||B|} B \star A$, where $|A|$ stands for the total Grassmann degree of A.

that the third equation in (2.2) does not arise this way but is satisfied thanks to the definition of M.

There is a natural way to encode the third equation from (2.1) by a certain extension of the Q-manifold M and by using a more general AKSZ framework. The extension amounts to introducing extra coordinates of the negative ghost degree needed to incorporate the constraints on T_a into the Q-structure. This can be done in a nice way using the BRST machinery. Indeed, introducing ghost variables c^a such that $\mathrm{gh}(c^a) = 1$ and $|c^a| = |e_a| + 1$ let us consider polynomials in c^a with values in $\mathfrak{A}$. The coordinates on the extended supermanifold $\mathcal{M}$ are components of a generic element of this algebra

$$\Psi = \sum_{k=0}^{\infty} \psi^A_{a_1 \ldots a_k} c^{a_1} \ldots c^{a_k} \, E_A,$$

$$\mathrm{gh}\left(\psi^A_{a_1 \ldots a_k}\right) = \mathrm{gh}\left(E_A\right) - k, \qquad \left|\psi^A_{a_1 \ldots a_k}\right| = \left|E_A\right| + \left|e_{a_1}\right| + \cdots + \left|e_{a_k}\right| - k \tag{5.6}$$

so that $\mathrm{gh}(\Psi) = 1$ and $|\Psi| = 1$. Note that coordinates ψ^A_a and ψ^A are precisely T^A_a and ξ^A introduced above. The Q-structure on $\mathcal{M}$ is introduced as follows

$$Q\Psi = \frac{1}{2}[\Psi, \Psi]_\star + q\Psi, \qquad q = -\frac{1}{2} c^a c^b C^c_{ab} \frac{\partial}{\partial c^c}. \tag{5.7}$$

It is easy to check that it coincides with (5.4) for $\psi^A = \epsilon^A$ and $\psi^A_a = T^A_a$ while in general acts nontrivially on $\psi^A_{a_1 \ldots a_k}$ with $k \geqslant 1$. The two terms in Q have a simple interpretation: the second term originates from the cohomology differential of the Lie superalgebra $\mathfrak{g}$ while the first one from that of $\mathfrak{A}$ understood as a Lie superalgebra. The later identification becomes clear if one considers ψ^A as a ghost variable associated to a basis element E_A.

Given a Q-manifold equipped with the not necessarily non-negative ghost degree the AKSZ procedure [68] (for a review and further details see e.g. [10, 69]) determines an associated gauge theory. In this case, in addition to equation (5.5) there are extra algebraic equations associated with coordinates of ghost degree -1:

$$Q^I(\Psi) = 0, \qquad \mathrm{gh}(\psi^I) = -1. \tag{5.8}$$

There are no fields associated to coordinates of negative ghost degree. In particular, the coordinates with negative degree are put to zero in $Q^I(\Psi)$ entering the above formula. Coordinates with ghost degree -2 and higher do not produce new equations of motion. In fact they are needed to encode identities (identities between identities, etc) between the equations in the Batalin–Vilkovisky description of the AKSZ system. To conclude, the equations of motion and gauge symmetries of the AKSZ system defined by (5.6) and (5.7) are respectively (2.1) and (2.2). In the particular case where $\mathfrak{g}$ is a Lie algebra (not superalgebra) the above AKSZ system was originally proposed in [10].

As a final remark let us mention that the consistent factorization given in section 3.3.2 can be also naturally embedded into the AKSZ framework. If c^α denote ghost variables associated to the ideal $\mathfrak{h} \subset \mathfrak{g}$ determining the factorization then in addition to ghost variables c^a one introduces the ghost momenta b_α, $\mathrm{gh}(b_\alpha) = -1$ conjugated to c^α. By allowing Ψ to depend on b_α as well this leads to extra fields including, in particular, u-fields in (3.17). The extended AKSZ system then naturally incorporates equations (3.17) and gauge transformations (3.16).

6. Conclusions

In this paper we have attempted to uniformize all known Vasiliev HS theories within a single framework given by system (2.1), whose algebraic origin is manifest. We observed that the specific features of the realization of HS algebra $\mathfrak{hs}$ do affect the choice of dynamical symmetries $\mathfrak{g}$ and the embedding algebra $\mathfrak{A}$. It would be interesting to try to avoid any specific realization of HS algebras and give an invariant description of HS theories.

More generally, system (2.1) provides a class of integrable, as we expect, models that are defined by the following data: (i) the symmetry algebra of the vacuum, which in the HS context is the HS algebra, $\mathfrak{hs}$; (ii) the algebra of dynamical symmetries, $\mathfrak{g}$, namely by the image of $U(\mathfrak{g})$ in $\mathfrak{A}$, which in the HS story is related to the Lorentz algebra. For example, in $3d$ and $4d$ theories the image is such that we get the enveloping algebra of the vacuum Lorentz algebra. In d-dimensional theory the relation to the Lorentz algebra is made somewhat implicit because of the Howe duality [7].

With these two data one can construct the embedding algebra $\mathfrak{A}$ as a twisted product of $U(\mathfrak{g})$ and $\mathfrak{hs}$ and write (2.1). The dynamics at the linearized level is determined by connections of $\mathbb{H}^0(\mathfrak{g}, \mathfrak{A}) = \mathfrak{hs}$ whose curvatures are not zero but given by $\mathbb{H}^1(\mathfrak{g}, \mathfrak{A})$. The dynamics is nontrivial if the rhs of (2.7) is non-vanishing, possibly at higher orders of the perturbation theory.

Any theory is substantially characterized by its observables. A natural class of observables for (2.1), advocated in [70–72], is given by Casimir operators of $\mathfrak{g}$, i.e. invariant polynomials of T_a. Another type of observables are Wilson loops $tr\ \mathrm{Pexp} \oint W$. Wilson loops can be generalized to decorated Wilson loops where there are insertions of any functions of T_a in the adjoint representation of $\mathfrak{A}$. It was shown in [73] that all correlation functions in Vasiliev theory can be computed in terms of such observables. It would be interesting to prove that the models described by (2.1) are integrable at least for a subset of observables in the sense of having a free-field realizations, as it happens for the holographic S-matrix in the HS theories with boundary conditions preserving full HS algebra.

A question, which we leave for further developments, is how big is the class of HS theories that are covered by the system (2.1). The known HS theories of Vasiliev type do not exhaust all possible HS fields. In dimensions higher than four the spin degrees of freedom, which are characterized by irreducible (spin)-tensors of the Wigner little group can be of more general symmetry type than just totally symmetric and the spectrum of string theory involves such fields.

Acknowledgments

We would like to thank Glenn Barnich, Xavier Bekaert, Nicolas Boulanger, Slava Didenko, Carlo Iazeolla, Per Sundell and Mikhail Vasiliev for many useful discussions. This work was initiated and partially performed during visits of K A and M G at the Albert Einstein Institute, Golm. The work of K A was supported by RFBR grant No 14-01-00489. The work of E S was supported in part by RFBR grant No 14-02-01172. The work of M G was supported by Russian Science Foundation grant 14-42-00047.

Appendix. Standard form of Vasiliev equations

2d system. In $2d$ dimensions we distinguish between two types of HS systems: first are topological models and second are models with propagating matter fields. Both of them are of the form (2.1) with $\mathfrak{g} = u(1)$. The choice of $\mathfrak{A}$ depends on the presence or absence of local degrees of freedom. It follows that the two-dimensional fields are given by 1-form $W(x)$ and 0-form $T(x)$ subjected to the BF equations of motion

$$\begin{aligned} \mathrm{d}W + W\star W &= 0, \\ \mathrm{d}T + [W, T]_\star &= 0. \end{aligned} \tag{A.1}$$

The embedding algebra $\mathfrak{A}$ can be taken either finite-dimensional [35] or infinite-dimensional [28, 36, 74]. The particular choice of infinite-dimensional $\mathfrak{A}$ yields local degrees of freedom [28]. In both cases the system (A.1) follows from $2d$ BF action functional.

3d system. The full system of equations has the form [3]

$$\mathrm{d}W + W\star W = 0, \quad \{S_\alpha, B\star\varkappa\}_\star = 0, \tag{A.2}$$

$$\mathrm{d}(B\star\varkappa) + [W, B\star\varkappa]_\star = 0, \quad \left[S_\alpha, S_\beta\right]_\star = -2\mathrm{i}\epsilon_{\alpha\beta}(1 + B\star\varkappa), \tag{A.3}$$

$$\mathrm{d}S_\alpha + \left[W, S_\alpha\right]_\star = 0. \tag{A.4}$$

Let us note that the Prokushkin–Vasiliev system, [4], can be cast into the same form as above, but $\mathfrak{A}$ is slightly different. The bosonic projection is made by the following kinematical constraints

$$[\varkappa, B]_\star = 0, \qquad [\varkappa, W]_\star = 0, \qquad \{\varkappa, S_\alpha\}_\star = 0, \tag{A.5}$$

where $\varkappa$ is the Klein operator. Alternatively, this is just a condition that W and S belong to respectively even and odd components of $\mathfrak{A}$ or in the formulation given in section 2, S_α determine a parity-even map from $\mathfrak{g}$ to $\mathfrak{A}$. These constraints imply that $\varkappa$ can be removed from the third equation, giving simply $\mathrm{d}B + [W, B]_\star = 0$.

4d system. The full system of equations has the form [2]

$$\mathrm{d}W + W\star W = 0, \tag{A.6a}$$

$$\mathrm{d}(B\star\varkappa) + [W, B\star\varkappa]_\star = 0, \tag{A.6b}$$

$$\mathrm{d}S_\alpha + \left[W, S_\alpha\right]_\star = 0, \qquad \mathrm{d}\bar{S}_{\dot\alpha} + \left[W, \bar{S}_{\dot\alpha}\right]_\star = 0, \tag{A.6c}$$

$$\left[S_\alpha, S_\beta\right]_\star = -2\mathrm{i}\epsilon_{\alpha\beta}(1 + B\star\varkappa), \left[\bar{S}_{\dot\alpha}, \bar{S}_{\dot\beta}\right]_\star = -2\mathrm{i}\epsilon_{\dot\alpha\dot\beta}\left(1 + B\star\bar\varkappa\right), \tag{A.6d}$$

$$\{S_\alpha, B\star\varkappa\}_\star = 0, \qquad \{\bar{S}_{\dot\alpha}, B\star\bar\varkappa\}_\star = 0, \tag{A.6e}$$

$$\left[S_\alpha, \bar{S}_{\dot\alpha}\right]_\star = 0, \tag{A.6f}$$

along with kinematical constraints ensuring the theory is bosonic

$$[K, B]_\star = 0, \quad [K, W]_\star = 0, \quad \{K, S_\alpha\}_\star = 0, \quad \{K, \bar{S}_{\dot\alpha}\}_\star = 0, \tag{A.7}$$

where $K = \varkappa\star\bar\varkappa$ is the total Klein operator, $K\star K = 1$. Thanks to the bosonic projection $\varkappa$ can be replaced with $\bar\varkappa$ in the second equation. Again, conditions (A.7) are equivalent to bosonic truncation introduced in section 2. Let us note that the extra constraint (3.25) we had to impose is a way to say that $\Upsilon = B\star\varkappa$ and $\bar\Upsilon = B\star\bar\varkappa$ from (A.6*d*) originate from the same B.

d-dimensional system. The full system of equations has the form [5]

$$\mathrm{d}W + W\star W = 0 \qquad \{S_\alpha, B\star\varkappa\}_\star = 0, \tag{A.8}$$

$$\mathrm{d}(B\star\varkappa) + [W, B\star\varkappa]_\star = 0, \qquad \left[S_\alpha, S_\beta\right]_\star = -2\mathrm{i}\epsilon_{\alpha\beta}(1 + B\star\varkappa), \tag{A.9}$$

$$\mathrm{d}S_\alpha + \left[W, S_\alpha\right]_\star = 0, \tag{A.10}$$

supplemented with the constraints from the $sp(2)$-factor of the coset

$$\begin{aligned}&\left[F^0_{\alpha\beta}, W\right]_\star = 0, \left[F^0_{\alpha\beta}, B\right]_\star = 0,\\ &\left[F^0_{\alpha\beta}, S_\gamma\right]_\star = \epsilon_{\alpha\gamma}S_\beta + \epsilon_{\beta\gamma}S_\alpha, \ \left[F^0_{\alpha\beta}, \varkappa\right]_\star = 0.\end{aligned} \tag{A.11}$$

Taking into account the explicit realization of $\mathfrak{A}$ and $F^0_{\alpha\beta}$ as a star product algebra the above conditions imply that bosonic truncation from section 2 is fulfilled automatically.

Let us stress that in the original paper [5] the oscillators Y^A_α were doubled by introducing Z^A_α, which have extra components z^a_α as compared to z_α we used. Correspondingly, there were more fields S^A_α introduced. However the equations for the Lorentz components S^a_α have the form of Weyl algebra $[S^a_\alpha, S^b_\beta] = -2\mathrm{i}\epsilon_{\alpha\beta}\eta^{ab}$ since there is no deformation due to B. Therefore, $S^a_\alpha = z^a_\alpha$ is an exact solution to all orders and oscillators z^a_α can be removed from the definition of $\mathfrak{A}$, as we did, while the field S^a_α can be removed from the Vasiliev equations.

References

[1] Vasiliev M A 1990 Consistent equation for interacting gauge fields of all spins in (3 + 1)-dimensions *Phys. Lett.* B **243** 378–82
[2] Vasiliev M A 1992 More on equations of motion for interacting massless fields of all spins in (3 + 1)-dimensions *Phys. Lett.* B **285** 225–34
[3] Vasiliev M A 1992 Equations of motion for $d = 3$ massless fields interacting through Chern-Simons higher spin gauge fields *Mod. Phys. Lett.* A **7** 3689–702
[4] Prokushkin S and Vasiliev M A 1998 3-d higher spin gauge theories with matter arXiv:hep-th/9812242
[5] Vasiliev M A 2003 Nonlinear equations for symmetric massless higher spin fields in (A)dS(d) *Phys. Lett.* B **567** 139–51
[6] Vasiliev M A 1999 Higher spin gauge theories: star-product and AdS space arXiv:hep-th/9910096
[7] Bekaert X, Cnockaert S, Iazeolla C and Vasiliev M A 2005 Nonlinear higher spin theories in various dimensions arXiv:hep-th/0304049
[8] Didenko V and Skvortsov E 2014 Elements of Vasiliev theory arXiv:1401.2975
[9] Vasiliev M A 1991 Properties of equations of motion of interacting gauge fields of all spins in (3 + 1)-dimensions *Class. Quantum Grav.* **8** 1387–417
[10] Grigoriev M 2012 Parent formulations, frame-like Lagrangians, and generalized auxiliary fields *J. High Energy Phys.* JHEP12(2012)048
[11] Vasiliev M A 2006 Actions, charges and off-shell fields in the unfolded dynamics approach *Int. J. Geom. Methods Mod. Phys.* **3** 37–80
[12] Vasiliev M A 1988 Equations of motion of interacting massless fields of all spins as a free differential algebra *Phys. Lett.* B **209** 491–7
[13] Vasiliev M A 1989 Consistent equations for interacting massless fields of all spins in the first order in curvatures *Ann. Phys.* **190** 59–106
[14] Vasiliev M A 2001 Cubic interactions of bosonic higher spin gauge fields in AdS(5) *Nucl. Phys.* B **616** 106–62
[15] Cap A, Schichl H and Vanzura J 1995 On twisted tensor products of algebras *Commun. Algebra* **23** 4701–35
[16] Barnich G, Grigoriev M, Semikhatov A and Tipunin I 2005 Parent field theory and unfolding in BRST first-quantized terms *Commun. Math. Phys.* **260** 147–81
[17] Barnich G and Grigoriev M 2006 Parent form for higher spin fields on anti-de Sitter space *J. High Energy Phys.* JHEP08(2006)013
[18] Grigoriev M 2006 Off-shell gauge fields from BRST quantization arXiv:hep-th/0605089

[19] Alkalaev K B, Grigoriev M and Tipunin I Y 2009 Massless Poincaré modules and gauge invariant equations *Nucl. Phys.* B **823** 509–45

[20] Alkalaev K B and Grigoriev M 2010 Unified BRST description of AdS gauge fields *Nucl. Phys.* B **835** 197–220

[21] Alkalaev K and Grigoriev M 2011 Unified BRST approach to (partially) massless and massive AdS fields of arbitrary symmetry type *Nucl. Phys.* B **853** 663–87

[22] Skvortsov E 2008 Mixed-symmetry massless fields in Minkowski space unfolded *J. High Energy Phys.* JHEP07(2008)004

[23] Boulanger N, Iazeolla C and Sundell P 2009 Unfolding mixed-symmetry fields in AdS and the BMV conjecture: I. General formalism *J. High Energy Phys.* JHEP07(2009)013

[24] Boulanger N, Iazeolla C and Sundell P 2009 Unfolding mixed-symmetry fields in AdS and the BMV conjecture: II. Oscillator realization *J. High Energy Phys.* JHEP07(2009)014

[25] Skvortsov E 2009 Gauge fields in (A)dS(d) and connections of its symmetry algebra *J. Phys. A: Math. Theor.* **42** 385401

[26] Skvortsov E 2010 Gauge fields in (A)dS(d) within the unfolded approach: algebraic aspects *J. High Energy Phys.* JHEP01(2010)106

[27] Shaynkman O V and Vasiliev M A 2000 Scalar field in any dimension from the higher spin gauge theory perspective *Theor. Math. Phys.* **123** 683–700

[28] Vasiliev M A 1995 Higher spin gauge interactions for matter fields in two-dimensions *Phys. Lett.* B **363** 51–57

[29] Prokushkin S and Vasiliev M A 1999 Higher spin gauge interactions for massive matter fields in 3-D AdS space-time *Nucl. Phys.* B **545** 385

[30] Vasiliev M A 1990 Closed equations for interacting gauge fields of all spins *JETP Lett.* **51** 503–7

[31] Konshtein S and Vasiliev M A 1989 Massless representations and admissibility condition for higher spin superalgebras *Nucl. Phys.* B **312** 402

[32] Konstein S E and Vasiliev M A 1990 Extended higher spin superalgebras and their massless representations *Nucl. Phys.* B **331** 475–99

[33] Sezgin E and Sundell P 2013 Supersymmetric higher spin theories *J. Phys. A: Math. Theor.* **46** 214022

[34] Blencowe M 1989 A consistent interacting massless higher spin field theory in $D = (2 + 1)$ *Class. Quantum Grav.* **6** 443

[35] Alkalaev K 2014 On higher spin extension of the Jackiw–Teitelboim gravity model *J. Phys. A: Math. Theor.* **47** 365401

[36] Alkalaev K 2014 Global and local properties of AdS_2 higher spin gravity *J. High Energy Phys.* JHEP10(2014)122

[37] Bergshoeff E, de Wit B and Vasiliev M A 1991 The structure of the super-$W_\infty(\lambda)$ algebra *Nucl. Phys.* B **366** 315–46

[38] Vasiliev M A 1989 Quantization on sphere and high spin superalgebras *JETP Lett.* **50** 374–7

[39] Bekaert X and Grigoriev M 2013 Higher order singletons, partially massless fields and their boundary values in the ambient approach *Nucl. Phys.* B **876** 667–714

[40] Vasiliev M 2008 private communication

[41] Skvortsov E and Vasiliev M 2006 Geometric formulation for partially massless fields *Nucl. Phys.* B **756** 117–47

[42] Alkalaev K B 2008 On manifestly sp(2) invariant formulation of quadratic higher spin Lagrangians *J. High Energy Phys.* JHEP06(2008)081

[43] Sezgin E and Sundell P 2002 Massless higher spins and holography *Nucl. Phys.* B **644** 303–70

[44] Klebanov I and Polyakov A 2002 AdS dual of the critical O(N) vector model *Phys. Lett.* B **550** 213–9

[45] Didenko V and Skvortsov E 2013 Towards higher-spin holography in ambient space of any dimension *J. Phys. A: Math. Theor.* **46** 214010

[46] Vasiliev M A 2004 Higher spin superalgebras in any dimension and their representations *J. High Energy Phys.* JHEP12(2004)046

[47] Shaynkman O, Tipunin I Y and Vasiliev M 2006 Unfolded form of conformal equations in M dimensions and o(M + 2) modules *Rev. Math. Phys.* **18** 823–86

[48] Feigin B L 1988 Lie algebras $gl(\lambda)$ and cohomologies of Lie algebras of differential operators *Russ. Math. Surv.* **43** 169–70

[49] Eastwood M G 2005 Higher symmetries of the Laplacian *Ann. Math.* **161** 1645–65

[50] Konstein S, Vasiliev M and Zaikin V 2000 Conformal higher spin currents in any dimension and AdS/CFT correspondence *J. High Energy Phys.* JHEP12(2000)018
[51] Fradkin E and Vasiliev M A 1987 Candidate to the role of higher spin symmetry *Ann. Phys.* **177** 63
[52] Maldacena J and Zhiboedov A 2013 Constraining conformal field theories with a higher spin symmetry *J. Phys. A: Math. Theor.* **46** 214011
[53] Boulanger N, Ponomarev D, Skvortsov E and Taronna M 2013 On the uniqueness of higher-spin symmetries in AdS and CFT *Int. J. Mod. Phys.* A **28** 1350162
[54] Stanev Y S 2013 Constraining conformal field theory with higher spin symmetry in four dimensions *Nucl. Phys.* B **876** 651–66
[55] Alba V and Diab K 2013 Constraining conformal field theories with a higher spin symmetry in $d = 4$ arXiv:1307.8092
[56] Fernando S and Gunaydin M 2010 Minimal unitary representation of SU(2, 2) and its deformations as massless conformal fields and their supersymmetric extensions *J. Math. Phys.* **51** 082301
[57] Boulanger N and Skvortsov E 2011 Higher-spin algebras and cubic interactions for simple mixed-symmetry fields in AdS spacetime *J. High Energy Phys.* JHEP09(2011)063
[58] Govil K and Gunaydin M 2013 Deformed twistors and higher spin conformal (super-) algebras in four dimensions arXiv:1312.2907
[59] Vasiliev M A 1980 'Gauge' form of description of massless fields with arbitrary spin *Yad. Fiz.* **32** 855–61 (in Russian)
[60] Vasiliev M A 1987 Free massless fields of arbitrary spin in the de Sitter space and initial data for a higher spin superalgebra *Fortschr. Phys.* **35** 741–70
[61] Lopatin V E and Vasiliev M A 1988 Free massless bosonic fields of arbitrary spin in d-dimensional de Sitter space *Mod. Phys. Lett.* A **3** 257
[62] Fradkin E and Linetsky V Y 1990 Infinite dimensional generalizations of simple Lie algebras *Mod. Phys. Lett.* A **5** 1967–77
[63] Wigner E 1950 Do the equations of motion determine the quantum mechanical commutation relations? *Phys. Rev.* D **77** 711
[64] Vasiliev M A 1991 Higher spin algebras and quantization on the sphere and hyperboloid *Int. J. Mod. Phys.* A **6** 1115–35
[65] Gunaydin M 1989 Singleton and doubleton supermultiplets of space-time supergroups and infinite spin superalgebras *Conf. on Supermembrane and Physics in 2 + 1 Dimensions (Trieste, 17–21 July 1989)*
[66] Iazeolla C and Sundell P 2011 Families of exact solutions to Vasiliev's 4D equations with spherical, cylindrical and biaxial symmetry *J. High Energy Phys.* JHEP12(2011)084
[67] Barnich G and Grigoriev M 2005 BRST extension of the nonlinear unfolded formalism arXiv:hep-th/0504119
[68] Alexandrov M, Kontsevich M, Schwartz A and Zaboronsky O 1997 The geometry of the master equation and topological quantum field theory *Int. J. Mod. Phys.* A **12** 1405–30
[69] Barnich G and Grigoriev M 2011 A Poincaré lemma for sigma models of AKSZ type *J. Geom. Phys.* **61** 663–74
[70] Sezgin E and Sundell P 2007 An exact solution of 4D higher-spin gauge theory *Nucl. Phys.* B **762** 1–37
[71] Sezgin E and Sundell P 2011 Geometry and observables in Vasiliev's higher spin gravity arXiv:1103.2360
[72] Colombo N and Sundell P 2012 Higher spin gravity amplitudes from zero-form charges arXiv:1208.3880
[73] Didenko V and Skvortsov E 2013 Exact higher-spin symmetry in CFT: all correlators in unbroken Vasiliev theory *J. High Energy Phys.* JHEP04(2013)158
[74] Fradkin E and Linetsky V Y 1989 Higher spin symmetry in one-dimension and two-dimensions. 1. *Mod. Phys. Lett.* A **4** 2635–47