

What is the relativistic spin operator?

Heiko Bauke,^{1,*} Sven Ahrens,¹ Christoph H. Keitel,¹ and Rainer Grobe^{1,2}

¹Max-Planck-Institut für Kernphysik, Saupfercheckweg 1, 69117 Heidelberg, Germany

²Intense Laser Physics Theory Unit and Department of Physics,
Illinois State University, Normal, IL 61790-4560 USA

(Dated: May 10, 2014)

Although the spin is regarded as a fundamental property of the electron, there is no universally accepted spin operator within the framework of relativistic quantum mechanics. We investigate the properties of different proposals for a relativistic spin operator. It is shown that most candidates are lacking essential features of proper angular momentum operators, leading to spurious Zitterbewegung (quivering motion) or violating the angular momentum algebra. Only the Foldy-Wouthuysen operator and the Pryce operator qualify as proper relativistic spin operators. We demonstrate that ground states of highly charged hydrogen-like ions can be utilized to identify a legitimate relativistic spin operator experimentally.

Keywords: spin, relativistic quantum mechanics, hydrogen-like ions

PACS numbers: 03.65.Pm, 31.15.aj, 31.30.J-

Introduction Quantum mechanics forms the universally accepted theory for the description of physical processes on the atomic scale. It has been validated by countless experiments and it is used in many technical applications. However, quantum mechanics presents physicists with some conceptual difficulties even today. In particular, the concept of spin is related to such difficulties and myths [1, 2]. Although there is consensus that elementary particles have some quantum mechanical property that is called spin, the understanding of the physical nature of the spin is still incomplete [3].

Historically, the concept of spin was introduced in order to explain some experimental findings such as the emission spectra of alkali metals and the Stern-Gerlach experiment. A direct measuring of the spin (or more precisely the electron's magnetic moment), however, was missing until the pioneering work by Dehmelt [4]. Nevertheless, spin measurement experiments [5–10] still require sophisticated methods. Pauli and Bohr even claimed that the spin of free electrons was impossible to measure for fundamental reasons [11]. Recent renewed interest in fundamental aspects of the spin arose, for example, from the growing field of (relativistic) quantum information [12–17], quantum spintronics [18], spin effects in graphene [19–21] and in light-matter interaction at relativistic intensities [22–24].

According to the formalism of quantum mechanics, each measurable quantity is represented by a Hermitian operator. Taking the experiments that aim to measure bare electron spins seriously, we have to ask the question: What is the correct (relativistic) spin operator. Although the spin is regarded as a fundamental property of the electron, a universally accepted spin operator for the Dirac theory is still missing. The pivotal question we try to tackle is: Which mathematical operator corresponds to an experimental spin measurement? This question may be answered by comparing experimental results with theoretical predictions originating from different spin operators and testing which operator is compatible with experimental data.

A relativistic spin operator may be introduced by splitting the undisputed total angular momentum operator $\hat{\mathbf{J}}$ into an external part $\hat{\mathbf{L}}$ and an internal part $\hat{\mathbf{S}}$, commonly referred to as the orbital angular momentum and the spin, viz. $\hat{\mathbf{J}} = \hat{\mathbf{L}} + \hat{\mathbf{S}}$. The question for the right splitting of the total angular momentum into an orbital part and a spin part is closely related to the quest for the right relativistic position operator [25–27]. This becomes evident by writing $\hat{\mathbf{L}} = \hat{\mathbf{r}} \times \hat{\mathbf{p}}$ with the position operator $\hat{\mathbf{r}}$ and the kinematic momentum operator $\hat{\mathbf{p}}$, which is in atomic units as used in this paper $\hat{\mathbf{p}} = -i\boldsymbol{\nabla}$. Thus, different definitions of the spin operator $\hat{\mathbf{S}}$ induce different relativistic position operators $\hat{\mathbf{r}}$.

Introducing the position vector \mathbf{r} and the operator $\hat{\Sigma} = (\hat{\Sigma}_1, \hat{\Sigma}_2, \hat{\Sigma}_3)^T$ via

$$\hat{\Sigma}_i = -i\alpha_j \alpha_k \quad (1)$$

with (i, j, k) being a cyclic permutation of $(1, 2, 3)$ and the matrices $(\alpha_1, \alpha_2, \alpha_3)^T = \boldsymbol{\alpha}$ obeying the algebra

$$\alpha_i^2 = 1, \quad \alpha_i \alpha_k + \alpha_k \alpha_i = 2\delta_{i,k}, \quad (2)$$

the operator of the relativistic total angular momentum is given by $\hat{\mathbf{J}} = \mathbf{r} \times \hat{\mathbf{p}} + \hat{\Sigma}/2$. Thus, the most obvious way of splitting $\hat{\mathbf{J}}$ is to define the orbital angular momentum operator $\hat{\mathbf{L}}_P = \mathbf{r} \times \hat{\mathbf{p}}$ and the spin operator $\hat{\mathbf{S}}_P = \hat{\Sigma}/2$, which is a direct generalization of the orbital angular momentum operator and the spin operator of the nonrelativistic Pauli theory. This naive splitting, however, suffers from several problems, e. g., $\hat{\mathbf{L}}_P$ and $\hat{\mathbf{S}}_P$ do not commute with the free Dirac Hamiltonian nor with the Dirac Hamiltonian for central potentials. Thus, in contrast to classical and nonrelativistic quantum theory the angular momenta $\hat{\mathbf{L}}_P$ and $\hat{\mathbf{S}}_P$ are not conserved. This has consequences, e. g., for the labeling of the eigenstates of the hydrogen atom. In nonrelativistic theory, bound hydrogen states may be constructed as simultaneous eigenstates of the Pauli-Coulomb Hamiltonian, the squared orbital angular momentum, the z -components of the orbital angular momentum and the spin. In the Dirac theory,

however, the squared total angular momentum $\hat{\mathbf{J}}^2$, the total angular momentum in z -direction \hat{J}_3 , and the so-called spin-orbit operator \hat{K} (or the parity) are utilized [28, 29]. In particular, it is not possible to construct simultaneous eigenstates of the Dirac-Coulomb Hamiltonian and some component of $\hat{\mathbf{S}}_P$.

Relativistic spin operators To overcome conceptual problems with the naive splitting of $\hat{\mathbf{J}}$ into $\hat{\mathbf{L}}_P$ and $\hat{\mathbf{S}}_P$, several alternatives for a relativistic spin operator have been proposed. However, there is no single commonly accepted relativistic spin operator, leading to the unsatisfactory situation that the relativistic spin operator is not unambiguously defined. We will investigate the properties of different popular definitions of the spin operator which result from different splittings of $\hat{\mathbf{J}}$ with the aim to find means that allow to identify the legitimate relativistic spin operator by experimental methods.

Table 1 summarizes various proposals for a relativistic spin operator $\hat{\mathbf{S}}$. These operators are often motivated by abstract group theoretical considerations rather than by experimental evidence. For example, Wigner showed in his seminal work [54–56] that the spin degree of freedom can be associated with irreducible representations of the sub-group of the inhomogeneous Lorentz group that leaves the four-momentum invariant. We will denote individual components of $\hat{\mathbf{S}}$ by \hat{S}_i with index $i \in \{1, 2, 3\}$. The spin operators are defined in terms of the particle's rest mass m_0 , the speed of light c , the matrix β such that

$$\beta^2 = 1, \quad \alpha_i \beta + \beta \alpha_i = 0, \quad (3)$$

the free particle Dirac Hamiltonian

$$\hat{H}_0 = c \boldsymbol{\alpha} \cdot \hat{\mathbf{p}} + m_0 c^2 \beta, \quad (4)$$

and the operator

$$\hat{p}_0 = (m_0^2 c^2 + \hat{\mathbf{p}}^2)^{1/2}. \quad (5)$$

In the nonrelativistic limit, i. e., when the plane wave expansion of a wave packet has only components with momenta which are small compared to $m_0 c$, expectation values for all operators in Tab. 1 converge to the same value. Note that the nomenclature in Tab. 1 is not universally adopted in the literature and other authors may utilize different operator names. Furthermore, the spin operators can be formulated by various different but algebraically equivalent expressions. For example, the so-called Gürsey-Ryder operator in [46, 47] is equivalent to the Chakrabarti operator of Tab. 1.

One may conclude that an operator can not be considered as a relativistic spin operator if it does not inherit the key properties of the nonrelativistic Pauli spin operator. In particular, we demand from a proper relativistic spin operator the following features:

1. It is required to commute with the free Dirac Hamiltonian.
2. A spin operator must feature the two eigenvalues $\pm 1/2$ and it has to obey the angular momentum algebra

$$[\hat{S}_i, \hat{S}_j] = i \epsilon_{i,j,k} \hat{S}_k \quad (6)$$

with $\epsilon_{i,j,k}$ denoting the Levi-Civita symbol.

The first property is required to ensure that the relativistic spin operator is a constant of motion if forces are absent, such that spurious Zitterbewegung of the spin is prevented. The second requirement is commonly regarded as the fundamental property of angular momentum operators of spin-half particles [57]. The physical quantity that is represented by the operator $\hat{\mathbf{S}}$ should not depend on the orientation of the chosen coordinate system. This can be ensured by fulfilling [57]

$$[\hat{J}_i, \hat{S}_j] = i \epsilon_{i,j,k} \hat{S}_k. \quad (7)$$

The angular momentum algebra (6) and the relation (7) determine the properties of the spin and the orbital angular momentum as well as the relationship between them. As a consequence of (7), the orbital angular momentum $\hat{\mathbf{L}} = \hat{\mathbf{J}} - \hat{\mathbf{S}}$ that is induced by a particular choice of the spin obeys $[\hat{J}_i, \hat{L}_j] = i \epsilon_{i,j,k} \hat{L}_k$. Thus, $\hat{\mathbf{L}}$ is a physical vector operator, too. As $\hat{\mathbf{L}}$ represents an angular momentum operator, it must obey the angular momentum algebra. Furthermore, we may say that the total angular momentum $\hat{\mathbf{J}}$ is split into an internal part $\hat{\mathbf{S}}$ and an external part $\hat{\mathbf{L}}$ only if internal and external angular momenta can be measured independently, i. e., $\hat{\mathbf{S}}$ and $\hat{\mathbf{L}}$ commute. Both conditions are fulfilled if, and only if, the spin operator $\hat{\mathbf{S}}$ satisfies the angular momentum algebra (6) because the commutator relations

$$[\hat{L}_i, \hat{L}_j] = i \epsilon_{i,j,k} \hat{L}_k + [\hat{S}_i, \hat{S}_j] - i \epsilon_{i,j,k} \hat{S}_k, \quad (8)$$

$$[\hat{L}_i, \hat{S}_j] = i \epsilon_{i,j,k} \hat{S}_k - [\hat{S}_i, \hat{S}_j] \quad (9)$$

follow from (7). All spin operators in Tab. 1 fulfill (7). The Czachor spin operator \hat{S}_{Cz} , the Frenkel spin operator \hat{S}_F , and the Fradkin-Good operator \hat{S}_{FG} , however, are disqualified as relativistic spin operators by violating the angular momentum algebra (6). Furthermore, the Pauli spin operator \hat{S}_P and the Chakrabarti spin operator \hat{S}_{Ch} do not commute with the free Dirac Hamiltonian, ruling them out as meaningful relativistic spin operators. According to our criteria, only the Foldy-Wouthuysen spin operator \hat{S}_{FW} and the Pryce spin operator \hat{S}_{Pr} remain as possible relativistic spin operators.

Electron spin of hydrogen-like ions The question of which of the proposed relativistic spin operators (if any) in Tab. 1 provides the correct mathematical description of spin can be answered definitely only by comparing theoretical predictions with experimental results. For this purpose one needs a physical setup that shows strong relativistic effects and is as simple as possible. Such a setup is provided by the bound eigenstates of highly charged hydrogen-like ions, i. e., atomic systems with an atomic core of Z protons and a single electronic charge. These ions can be produced at storage rings [58] or by utilizing electron beam ion traps [59, 60] up to $Z = 92$ (hydrogen-like uranium). The degenerate bound eigenstates of the corresponding Coulomb-Dirac Hamiltonian

$$\hat{H}_C = \hat{H}_0 - \frac{Z}{|\mathbf{r}|} \quad (10)$$

TABLE 1: Definitions and commutation properties of various relativistic spin operators.

Operator name	Definition	$[\hat{H}_0, \hat{S}] = 0?$	$[\hat{S}_i, \hat{S}_j] = i\epsilon_{ijk}\hat{S}_k?$	Eigenvalues $= \pm 1/2?$
Pauli [30–32]	$\hat{S}_P = \frac{1}{2}\hat{\Sigma}$	no	yes	yes
Foldy-Wouthuysen [33–37]	$\hat{S}_{FW} = \frac{1}{2}\hat{\Sigma} + \frac{i\beta}{2\hat{p}_0}\hat{p} \times \alpha - \frac{\hat{p} \times (\hat{\Sigma} \times \hat{p})}{2\hat{p}_0(\hat{p}_0 + m_0c)}$	yes	yes	yes
Czachor [38]	$\hat{S}_{Cz} = \frac{m_0^2 c^2}{2\hat{p}_0^2}\hat{\Sigma} + \frac{im_0c\beta}{2\hat{p}_0^2}\hat{p} \times \alpha + \frac{\hat{p} \cdot \hat{\Sigma}}{2\hat{p}_0^2}\hat{p}$	yes	no	no
Frenkel [39–41]	$\hat{S}_F = \frac{1}{2}\hat{\Sigma} + \frac{i\beta}{2m_0c}\hat{p} \times \alpha$	yes	no	no
Chakrabarti [42–48]	$\hat{S}_{Ch} = \frac{1}{2}\hat{\Sigma} + \frac{i}{2m_0c}\alpha \times \hat{p} + \frac{\hat{p} \times (\hat{\Sigma} \times \hat{p})}{2m_0c(m_0c + \hat{p}_0)}$	no	yes	yes
Pryce [47, 49–52]	$\hat{S}_{Pr} = \frac{1}{2}\beta\hat{\Sigma} + \frac{1}{2}\hat{\Sigma} \cdot \hat{p}(1-\beta)\frac{\hat{p}}{\hat{p}^2}$	yes	yes	yes
Fradkin-Good [46, 53]	$\hat{S}_{FG} = \frac{1}{2}\beta\hat{\Sigma} + \frac{1}{2}\hat{\Sigma} \cdot \hat{p}\left(\frac{\hat{H}_0}{c\hat{p}_0} - \beta\right)\frac{\hat{p}}{\hat{p}^2}$	yes	no	yes

are commonly expressed as simultaneous eigenstates $\psi_{n,j,m,\kappa}$ of \hat{H}_C , \hat{J}^2 , \hat{J}_3 , and the so-called spin-orbit operator $\hat{K} = \beta\{\hat{\Sigma} \cdot [\mathbf{r} \times (-i\boldsymbol{\nabla}) + 1]\}$ fulfilling the eigenequations [28, 29]

$$\hat{H}_C\psi_{n,j,m,\kappa} = \mathcal{E}(n, j)\psi_{n,j,m,\kappa} \quad n = 1, 2, \dots, \quad (11)$$

$$\hat{J}^2\psi_{n,j,m,\kappa} = j(j+1)\psi_{n,j,m,\kappa} \quad j = \frac{1}{2}, \frac{3}{2}, \dots, n - \frac{1}{2}, \quad (12)$$

$$\hat{J}_3\psi_{n,j,m,\kappa} = m\psi_{n,j,m,\kappa} \quad m = -j, (j-1), \dots, j, \quad (13)$$

$$\hat{K}\psi_{n,j,m,\kappa} = \kappa\psi_{n,j,m,\kappa} \quad \kappa = -j - \frac{1}{2}, j + \frac{1}{2}. \quad (14)$$

The eigenenergies are given with α_{el} denoting the fine structure constant by

$$\mathcal{E}(n, j) = m_0c^2 \left[1 + \left(\frac{\alpha_{el}^2 Z^2}{n - j - 1/2 + \sqrt{(j - 1/2)^2 - \alpha_{el}^2 Z^2}} \right) \right]^{-1/2} \quad (15)$$

In order to establish a close correspondence between the nonrelativistic Schrödinger-Pauli theory and the relativistic Dirac theory, one may desire to find a splitting of \hat{J} into a sum $\hat{J} = \hat{L} + \hat{S}$ of commuting operators such that both \hat{L} and \hat{S}

1. fulfill the angular momentum algebra, and
2. form a complete set of commuting operators that contains \hat{H}_C as well as \hat{S}_3 and/or \hat{L}_3 .

The latter property would ensure that *all* hydrogenic energy eigenstates are spin eigenstates and/or orbital angular momentum eigenstates, too. Such hypothetical eigenstates would be superpositions of $\psi_{n,j,m,\kappa}$ of the same energy. Consequently, these superpositions are eigenstates of \hat{J}^2 , too, because the energy (15) depends on the principal quantum number n as well as the quantum number j . Thus, any complete set of commuting operators for specifying hydrogenic quantum states necessarily includes \hat{J}^2 . As a consequence of the postulated angular momentum algebra for \hat{L} and \hat{S} , the operator \hat{J}^2 commutes with \hat{L}^2 as well as with \hat{S}^2 , but with neither \hat{S}_3 nor \hat{L}_3

[61] excluding \hat{S}_3 and \hat{L}_3 from any complete set of commuting operators for specifying relativistic hydrogenic eigenstates. In conclusion, hydrogenic energy eigenstates are generally not eigenstates of any spin operator that fulfills the angular momentum algebra.

In momentum space, the relativistic spin operators introduced in Tab. 1 are simple matrices. Thus, by employing the momentum space representation of $\psi_{n,j,m,\kappa}$, spin expectation values of the degenerate hydrogenic ground states $\psi_\uparrow = \psi_{1,1/2,1/2,1}$ and $\psi_\downarrow = \psi_{1,1/2,-1/2,-1}$ can be evaluated [62]. For simplicity, we measure spin along the z -direction for the remainder of this section. The spin expectation value of a general superposition $\psi = \cos(\eta/2)\psi_\uparrow + \sin(\eta/2)e^{i\zeta}\psi_\downarrow$ of the hydrogenic ground states ψ_\uparrow and ψ_\downarrow is given by

$$\langle \psi | \hat{S}_3 | \psi \rangle = \cos^2 \frac{\eta}{2} \langle \psi_\uparrow | \hat{S}_3 | \psi_\uparrow \rangle + \sin^2 \frac{\eta}{2} \langle \psi_\downarrow | \hat{S}_3 | \psi_\downarrow \rangle + 2 \cos \frac{\eta}{2} \sin \frac{\eta}{2} \cos \zeta \operatorname{Re} \langle \psi_\uparrow | \hat{S}_3 | \psi_\downarrow \rangle. \quad (16)$$

For all spin operators introduced in Tab. 1, the mixing term $\operatorname{Re} \langle \psi_\uparrow | \hat{S}_3 | \psi_\downarrow \rangle$ vanishes and, furthermore, $\langle \psi_\uparrow | \hat{S}_3 | \psi_\uparrow \rangle = -\langle \psi_\downarrow | \hat{S}_3 | \psi_\downarrow \rangle > 0$. Thus, the expectation value (16) is maximal for $\eta = 0$ and minimal for $\eta = \pi$ and the inequality

$$\langle \psi_\downarrow | \hat{S}_3 | \psi_\downarrow \rangle \leq \langle \psi | \hat{S}_3 | \psi \rangle \leq \langle \psi_\uparrow | \hat{S}_3 | \psi_\uparrow \rangle \quad (17)$$

holds for all hydrogenic ground states ψ .

For every proposed spin operator in Tab. 1 we get different values for the upper and lower bounds in (17). The spin expectation values $\langle \psi_\uparrow | \hat{S}_3 | \psi_\uparrow \rangle$ and implicitly $\langle \psi_\downarrow | \hat{S}_3 | \psi_\downarrow \rangle$ for the operators of Tab. 1 are displayed as a function of the atomic number Z in Fig. 1. None of the spin operators in Tab. 1 commutes with \hat{H}_C . Thus, the expectation values $\langle \psi_\uparrow | \hat{S}_3 | \psi_\uparrow \rangle$ and $\langle \psi_\downarrow | \hat{S}_3 | \psi_\downarrow \rangle$ generally do not equal one of the eigenvalues of \hat{S}_3 . For small atomic numbers ($Z < 20$), all spin operators yield about $\pm 1/2$. For larger Z , however, expectation values differ significantly from each other. In particular, spin expectation

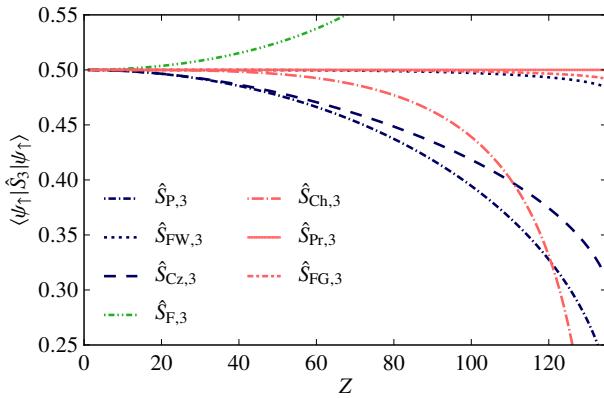


FIG. 1: Spin expectation values $\langle \psi_\uparrow | \hat{S}_3 | \psi_\uparrow \rangle$ of various relativistic spin operators for the hydrogenic ground state ψ_\uparrow as a function of the atomic number Z measured in z -direction. Spin expectation values for ψ_\downarrow follow by symmetry via $\langle \psi_\uparrow | \hat{S}_3 | \psi_\uparrow \rangle = -\langle \psi_\downarrow | \hat{S}_3 | \psi_\downarrow \rangle$.

values differ from $\pm 1/2$ even for spin operators with eigenvalues $\pm 1/2$. This means, it is possible to discriminate between different relativistic spin operator candidates. The spin expectation value's magnitude decreases with growing Z when the Pauli, the Fouldy-Wouthuysen, the Czachor, the Chakrabarti, or the Fradkin-Good spin operator is applied. The Frenkel spin operator yields spin expectation values with modulus exceeding $1/2$ which is due to the violation of the angular momentum algebra. Only the Pryce operator yields spin expectation of $\pm 1/2$ for all values of Z . In fact, calculations show that all hydrogenic states $\psi_{n,j,m,k}$ with $m = \pm j$ are eigenstates of the Pryce spin operator but not those states with $m \neq \pm j$.

An experimental test for relativistic spin operators

Theoretical considerations led to several proposals for a relativistic spin operator as illustrated in Tab. 1. The identification of the correct relativistic spin operator, however, demands an experimental test. The inequality (17) may serve as a basis for such an experimental test. More precisely, the inequality (17) allows falsification of the hypothesis that the spin measurement procedure is an experimental realization of some operator \hat{S} , where \hat{S} is one of the operators in Tab. 1. In this test the electron of a highly charged hydrogen-like ion is prepared in its ground state ψ_\uparrow first, e. g., by exposing the ion to a strong magnetic field in z -direction and turning it off adiabatically. (Preparing a superposition of ψ_\uparrow and ψ_\downarrow will reduce the sensitivity of the experimental test.) Afterwards, the spin will be measured along z -direction, e. g., by a Stern-Gerlach-like experiment, yielding the experimental expectation value s . Comparing this experimental value to each of the seven bounds shown in Fig. 1 will allow exclusion of some of the proposed spin operators. The hypothesis that the spin measurement procedure is an experimental realization of the operator \hat{S} is compatible with the experimental result s if, and only if, the inequality $\langle \psi_\downarrow | \hat{S}_3 | \psi_\downarrow \rangle \leq s \leq \langle \psi_\uparrow | \hat{S}_3 | \psi_\uparrow \rangle$ is fulfilled. Otherwise, this operator is excluded as a relativistic spin operator by exper-

imental evidence. In particular, realizing full spin-polarization, i. e., $s = \pm 1/2$, eliminates all operators in Tab. 1 except the Pryce operator.

Conclusions We investigated the properties of various proposals for a relativistic spin operator. Only the Fouldy-Wouthuysen operator and the Pryce operator fulfill the angular momentum algebra and are constants of motion in the absence of forces. While different theoretical considerations led to different spin operators, the definite relativistic spin operator has to be justified by experimental evidence. Energy eigenstates of highly charged hydrogen-like ions, in particular the ground states, can be utilized to exclude candidates for a relativistic spin operator experimentally. The proposed spin operators predict different maximal degrees of spin polarization. Only the Pryce spin operator allows for a complete polarization of spin in the hydrogenic ground state.

We have enjoyed helpful discussions with Prof. C. Müller, S. Meuren, Prof. Q. Su and E. Yakaboylu. R. G. acknowledges the warm hospitality during his sabbatical leave in Heidelberg. This work was supported by the NSF.

* heiko.bauke@mpi-hd.mpg.de

- [1] H. Nikolić, Foundations of Physics **37**, 1563 (2007).
- [2] D. Giulini, Studies In History and Philosophy of Science Part B: Studies In History and Philosophy of Modern Physics **39**, 557 (2008).
- [3] M. Morrison, Studies In History and Philosophy of Science Part B: Studies In History and Philosophy of Modern Physics **38**, 529 (2007).
- [4] H. Dehmelt, Science **247**, 539 (1990).
- [5] D. Hanneke, S. Fogwell, and G. Gabrielse, Phys. Rev. Lett. **100**, 120801 (2008).
- [6] P. Neumann, J. Beck, M. Steiner, F. Rempp, H. Fedder, P. R. Hemmer, J. Wrachtrup, and F. Jelezko, Science **329**, 542 (2010).
- [7] B. B. Buckley, G. D. Fuchs, L. C. Bassett, and D. D. Awschalom, Science **330**, 1212 (2010).
- [8] T. Close, F. Fadugba, S. C. Benjamin, J. Fitzsimons, and B. W. Lovett, Phys. Rev. Lett. **106**, 167204 (2011).
- [9] S. Sturm, A. Wagner, B. Schabinger, J. Zatorski, Z. Harman, W. Quint, G. Werth, C. H. Keitel, and K. Blaum, Phys. Rev. Lett. **107**, 023002 (2011).
- [10] J. DiSciacca and G. Gabrielse, Phys. Rev. Lett. **108**, 153001 (2012).
- [11] N. F. Mott, Proc. R. Soc. Lond. A **124**, 425 (1929).
- [12] A. Peres and D. R. Terno, Rev. Mod. Phys. **76**, 93 (2004).
- [13] W. T. Kim and E. J. Son, Phys. Rev. A **71**, 014102 (2005).
- [14] R. Wiesendanger, Rev. Mod. Phys. **81**, 1495 (2009).
- [15] N. Friis, R. A. Bertlmann, M. Huber, and B. C. Hiesmayr, Phys. Rev. A **81**, 042114 (2010).
- [16] K. D. Petersson, L. W. McFaul, M. D. Schroer, M. Jung, J. M. Taylor, A. A. Houck, and J. R. Petta, Nature **490**, 380 (2012).
- [17] P. L. Saldanha and V. Vedral, New J. Phys. **14**, 023041 (2012).
- [18] D. D. Awschalom, L. C. Bassett, A. S. Dzurak, E. L. Hu, and J. R. Petta, Science **339**, 1174 (2013).
- [19] D. A. Abanin, R. V. Gorbachev, K. S. Novoselov, A. K. Geim, and L. S. Levitov, Phys. Rev. Lett. **107**, 096601 (2011).
- [20] M. Mecklenburg and B. C. Regan, Phys. Rev. Lett. **106**, 116803 (2011).

- [21] J. Güttinger, T. Frey, C. Stampfer, T. Ihn, and K. Ensslin, Phys. Rev. Lett. **105**, 116801 (2010).
- [22] A. Di Piazza, C. Müller, K. Z. Hatsagortsyan, and C. H. Keitel, Rev. Mod. Phys. **84**, 1177 (2012).
- [23] S. Ahrens, H. Bauke, C. H. Keitel, and C. Müller, Phys. Rev. Lett. **109**, 043601 (2012).
- [24] S. Ahrens, H. Bauke, C. H. Keitel, and R. Grobe, “Electron-spin dynamics induced by photon spins,” (2014), arXiv:1401.5976.
- [25] T. D. Newton and E. P. Wigner, Rev. Mod. Phys. **21**, 400 (1949).
- [26] T. F. Jordan and N. Mukunda, Phys. Rev. **132**, 1842 (1963).
- [27] R. O’Connell and E. Wigner, Phys. Lett. A **67**, 319 (1978).
- [28] H. A. Bethe and E. E. Salpeter, *Quantum Mechanics of One- and Two-Electron Atoms* (Dover, Mineola, 2008).
- [29] B. Thaller, *Advanced Visual Quantum Mechanics* (Springer, Heidelberg, New York, 2000).
- [30] E. L. Hill and R. Landshoff, Rev. Mod. Phys. **10**, 87 (1938).
- [31] P. A. M. Dirac, Proc. R. Soc. London, Ser. A **322**, 435 (1971).
- [32] H. C. Ohanian, Amer. J. Phys. **54**, 500 (1986).
- [33] L. L. Foldy and S. A. Wouthuysen, Phys. Rev. **78**, 29 (1950).
- [34] E. de Vries, Fortschr. Phys. **18**, 149 (1970).
- [35] J. P. Costella and B. H. J. McKellar, Amer. J. Phys. **63**, 1119 (1995).
- [36] S. S. Schweber, *An introduction to relativistic quantum field theory* (Dover, Mineola, 2005).
- [37] P. Caban, J. Rembieliński, and M. Włodarczyk, Ann. Phys. **330**, 263 (2013).
- [38] M. Czachor, Phys. Rev. A **55**, 72 (1997).
- [39] J. Hilgevoord and S. A. Wouthuysen, Nucl. Phys. **40**, 1 (1963).
- [40] A. S. Wightman, “L’invariance dans la mecanique quantique relativiste,” (Herman, 1960) Chap. V: Relations de dispersion et particules élémentaires.
- [41] V. Bargmann, L. Michel, and V. L. Telegdi, Phys. Rev. Lett. **2**, 435 (1935).
- [42] A. Chakrabarti, J. Math. Phys. **4**, 1215 (1963).
- [43] F. Gürsey, Phys. Lett. **14**, 330 (1965).
- [44] F. Gürsey, in *High energy physics* (Gordon and Breach, 1965) pp. 53–88.
- [45] F. Gürsey, Phys. Lett. **14**, 330 (1965).
- [46] I. Kirsch, L. H. Ryder, and F. W. Hehl, “The Gordon decompositions of the inertial currents of the Dirac electron correspond to a Foldy-Wouthuysen transformation,” (2001), arXiv:hep-th/0102102.
- [47] L. H. Ryder, General Relativity and Gravitation **31**, 775 (1999).
- [48] T. Choi, J. Korean Phys. Soc. **62**, 1085 (2013).
- [49] M. H. L. Pryce, Proc. R. Soc. London, Ser. A **150**, 166 (1935).
- [50] M. H. L. Pryce, Proc. R. Soc. London, Ser. A **195**, 62 (1948).
- [51] A. J. Macfarlane, J. Math. Phys. **4**, 490 (1963).
- [52] R. A. Berg, J. Math. P. **6**, 34 (1965).
- [53] D. M. Fradkin and R. H. Good, Jr., Rev. Mod. Phys. **33**, 343 (1961).
- [54] E. Wigner, Annals of Mathematics **40**, 149 (1939).
- [55] V. Bargmann and E. P. Wigner, PNAS **34**, 211 (1948).
- [56] E. P. Wigner, Rev. Mod. Phys. **29**, 255 (1957).
- [57] J. J. Sakurai and J. Napolitano, *Modern Quantum Mechanics*, 2nd ed. (Pearson Education, Upper Saddle River, 2010).
- [58] T. Stöhlker, P. Mokler, K. Beckert, F. Bosch, H. Eickhoff, B. Franzke, M. Jung, Y. Kandler, O. Klepper, C. Kozhuharov, R. Moshammer, F. Nolden, H. Reich, P. Rymuza, P. Spädtke, , and M. Steck, Phys. Rev. Lett. **71**, 2184 (1993).
- [59] D. Robbins, P. Beiersdorfer, A. Faenov, T. Pikuz, D. Thorn, H. Chen, K. Reed, A. Smith, K. Boyce, G. Brown, R. Kelley, C. Kilbourne, , and F. Porter, Phys. Rev. A **74**, 022713 (2006).
- [60] H.-J. Kluge, T. Beier, K. Blaum, L. Dahl, S. Eliseev, F. Herfurth, B. Hofmann, O. Kester, S. Koszudowski, C. Kozhuharov, G. Maero, W. Nörterhäuser, J. Pfister, W. Quint, U. Ratzinger, A. Schempp, R. Schuch, T. Stöhlker, R. C. Thompson, M. Vogel, G. Vorobjev, D. F. A. Winters, and G. Werth, in *Current Trends in Atomic Physics, Advances in Quantum Chemistry*, Vol. 53, edited by S. Salomonson and E. Lindroth (Academic Press, 2008).
- [61] C. Cohen-Tannoudji, B. Diu, and F. Laloë, *Quantum Mechanics: Volume 2* (John Wiley & Sons, 1977).
- [62] A. Rubinowicz, Phys. Rev. **73**, 1330 (1948).