The electromagnetic spike solutions

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Abstract
The aim of this paper is to use the existing relation between polarized
electromagnetic Gowdy spacetimes and vacuum Gowdy spacetimes to find
explicit solutions for electromagnetic spikes by a procedure which has been
developed by one of the authors for gravitational spikes. We present new
inhomogeneous solutions which we call the EME and MEM electromagnetic
spike solutions.

PACS numbers: 04.20.−q, 04.20.Jb, 04.40.Nr, 98.80.Jk

(Some figures may appear in colour only in the online journal)

1. Introduction

According to Belinskii, Khalatnikov and Lifshitz (BKL) [1–3], a generic spacelike singularity
is characterized by asymptotic locality. Asymptotically towards the initial singularity each
spatial point evolves independently from its neighbors in an oscillatory manner that is
represented by a sequence of Bianchi type I and II vacuum models. In [4] Berger and Moncrief
studied T^3-Gowdy spacetimes numerically and observed the development of large spatial
derivatives near the singularity, which they called 'spiky features'. These structures where
found to occur in the neighborhood of isolated spatial surfaces, cf [5]. Further numerical
investigations (see [6] for an overview) seemed to indicate that the BKL conjecture is correct
generically, but certain difficulties arose in simulating these spikes. An important step was
made in [7] where a solution generating technique and Fuchsian methods developed in [8, 9]
where used to produce asymptotic expansion for spikes, which where classified in 'true' and
'false' spikes, where the latter are only a rotation artifact. Based on these transformations, in
[10] an explicit spike solution was found in terms of elementary functions. The explicit spike
solution suggests a new way to simulate spikes numerically, and in [11] it was confirmed
that the explicit spike solution indeed describes the spiky structures, and does so remarkably
accurately. The numerical results provide very strong evidence that apart from local BKL
behavior, there also exist formation of spatial structures at and in the neighborhood of certain

spatial surfaces, thus breaking asymptotic locality. Moreover the complete description of a generic spacelike singularity should involve spike oscillations [12], which are described by sequences of spike solutions and rotated Kasner solutions.

We are interested in investigating the nature of BKL behavior at spacetime singularities in the presence of electromagnetic fields. The aim of this paper is to use the known relation between polarized/diagonal electromagnetic Gowdy spacetimes and Gowdy spacetimes, as given in [13], to find spike solutions for the electromagnetic case. According to the BKL-picture a generic spacelike singularity is vacuum dominated. This project thus will help to clarify whether the introduction of a Maxwell field changes the picture or not.

The sign conventions of [14] are used. In particular, we use metric signature $- + + +$ and geometrized units. We use the sign convention $\epsilon^{0123} > 0$ for the Levi-Civita tensor, but because we will switch to using a time variable that increases towards the past, the component ϵ^{0123} with respect to that coordinate system will be negative.

2. Basic equations and metric

If $(\bar{P}, \bar{Q}, \bar{\lambda})$ is a solution of the vacuum Einstein equations with Gowdy symmetry (or orthogonally transitive G_2 isometry), i.e. with line element

$$\mathrm{d}s^2 = -\mathrm{e}^{(\bar{\lambda}-3\tau)/2}(\mathrm{d}\tau^2 + \mathrm{e}^{2\tau}\,\mathrm{d}x^2) + \mathrm{e}^{-\tau}[\mathrm{e}^{\bar{P}}(\mathrm{d}y + \bar{Q}\,\mathrm{d}z)^2 + \mathrm{e}^{\bar{P}}\,\mathrm{d}z^2], \tag{1}$$

then (P, χ, λ) with (9)–(10) of [13] [with the following correspondence $t = \mathrm{e}^{-\tau}$ and $(\theta, x, y) = (x, y, z)$]:

$$P = 2\bar{P} - \tau, \qquad \lambda = 4\bar{\lambda} + 4\bar{P} - \tau, \qquad \chi = \bar{Q}, \tag{2}$$

will be a solution of the Einstein–Maxwell equations with polarized Gowdy symmetry (or diagonal G_2 isometry) with line element:

$$\mathrm{d}s^2 = \mathrm{e}^{(\lambda-3\tau)/2}(\mathrm{d}\tau^2 + \mathrm{e}^{2\tau}\,\mathrm{d}x^2) + \mathrm{e}^{-\tau}(\mathrm{e}^P\,\mathrm{d}y^2 + \mathrm{e}^{-P})\,\mathrm{d}z^2, \tag{3}$$

and a Maxwell field described by the vector potential with only one non-zero component, namely $A_3 = \chi(x, \tau)$. We will assume from now on that we are in the second case, i.e. our metric is described via (3). The Einstein–Maxwell equations with P and χ are given by (19)–(24) of [15]:

$$P_{\tau\tau} - \mathrm{e}^{-2\tau} P_{xx} = 2\left(\chi_\tau^2\,\mathrm{e}^{P+\tau} - \chi_x^2\,\mathrm{e}^{P-\tau}\right)$$
$$\chi_{\tau\tau} - \mathrm{e}^{-2\tau}\chi_{xx} = \mathrm{e}^{-2\tau}P_x\chi_x - (P_\tau + 1)\chi_\tau$$
$$\lambda_\tau = -P_\tau^2 - \mathrm{e}^{-2\tau}P_x^2 - 4\left(\chi_\tau^2\,\mathrm{e}^{P+\tau} + \chi_x^2\,\mathrm{e}^{P-\tau}\right)$$
$$\lambda_x = -2P_\tau P_x - 8\,\mathrm{e}^{P+\tau}\chi_\tau\chi_x.$$

We will use the non-vanishing β-normalized variables [16],

$$\beta = \tfrac{1}{2}\mathrm{e}^{-\frac{\lambda-3\tau}{4}}. \tag{4}$$

We take this opportunity to clarify the sign confusion in equation (9) of [10]. β and other kinematic variables are defined with respect to the future-pointing congruence. Therefore a positive β describes expansion towards the future (and contraction towards the past). We also take this opportunity to correct the error in the β expression in equation (2) of [11].

These β-normalized variables refer to the β-normalized commutation functions associated with an orthonormal frame:

$$\Sigma_{\alpha\beta} = \frac{\sigma_{\alpha\beta}}{\beta}, \qquad N_{\alpha\beta} = \frac{n_{\alpha\beta}}{\beta}.$$

The electric and magnetic fields are similarly normalized:

$$\mathcal{E}_\alpha = \frac{E^\alpha}{\beta}, \qquad \mathcal{B}_\alpha = \frac{B^\beta}{\beta}.$$

The 3-by-3 $\Sigma_{\alpha\beta}$ and $N_{\alpha\beta}$ matrices in our case are

$$\Sigma_{\alpha\beta} = \begin{pmatrix} -2\Sigma_+ & 0 & 0 \\ 0 & \Sigma_+ + \sqrt{3}\Sigma_- & 0 \\ 0 & 0 & \Sigma_+ - \sqrt{3}\Sigma_- \end{pmatrix} \tag{5}$$

$$N_{\alpha\beta} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \sqrt{3}N_\times \\ 0 & \sqrt{3}N_\times & 0 \end{pmatrix}, \tag{6}$$

while the non-zero electric and magnetic components are

$$\mathcal{E} = \frac{E^3}{\beta}, \qquad \mathcal{B} = \frac{B^2}{\beta}.$$

The β-normalized variables $\mathbf{Y} = (\Sigma_+, \Sigma_-, N_\times, \mathcal{E}, \mathcal{B})$ are related to the partial derivatives of P, λ and χ as follows.

$$\mathbf{Y} = \left(\frac{1}{2} + \frac{1}{6}\lambda_\tau, -\frac{P_\tau}{\sqrt{3}}, -\frac{\mathrm{e}^{-\tau}P_x}{\sqrt{3}}, 2\chi_\tau\,\mathrm{e}^{\frac{1}{2}(P+\tau)}, -2\chi_x\,\mathrm{e}^{\frac{1}{2}(P-\tau)} \right). \tag{7}$$

The evolution equations for the β-normalized variables can be derived from the above Einstein–Maxwell equations and (7):

$$\partial_t\Sigma_- = \mathrm{e}^{-\tau}\partial_x(N_\times) + \frac{1}{2\sqrt{3}}(\mathcal{B}^2 - \mathcal{E}^2),$$

$$\partial_t N_\times = \mathrm{e}^{-\tau}\partial_x\Sigma_- - N_\times,$$

$$\partial_t\mathcal{E} = -\mathrm{e}^{-\tau}\partial_x\mathcal{B} + \frac{1}{2}\sqrt{3}N_\times\mathcal{B} + \frac{1}{2}(\sqrt{3}\Sigma_- - 1)\mathcal{E},$$

$$\partial_t\mathcal{B} = -\mathrm{e}^{-\tau}\partial_x\mathcal{E} - \frac{1}{2}\sqrt{3}N_\times\mathcal{E} - \frac{1}{2}(\sqrt{3}\Sigma_- + 1)\mathcal{B}.$$

We do not use the evolution equation for Σ_+, instead using the Gauss constraint to find Σ_+:

$$\Sigma_+ = \tfrac{1}{2}\left[1 - \Sigma_-^2 - N_\times^2 - \tfrac{1}{3}(\mathcal{E}^2 + \mathcal{B}^2) \right].$$

3. Explicit solutions

We now apply the vacuum-to-electromagnetic transformation (2) to the explicit solutions in section 4 of [10].

3.1. Homogeneous solutions

3.1.1. Reparameterized Kasner solution. Applying the transformation (2) to the Kasner seed solution (17) of [10] yields

$$P = v\tau + 2P_0, \qquad \chi = \chi_0, \qquad \lambda = -v^2\tau + 4(\lambda_0 + P_0), \tag{8}$$

where $v = 2w - 1$ and P_0, χ_0 and λ_0 are arbitrary constants. The result is trivial—this solution is just a re-parametrization of the Kasner solution. Nevertheless, we will use this parametrization of the Kasner solution in figure 3 later.

The β-normalized variables $\mathbf{Y_K}$ have the same form as (18) of [10]:

$$\mathbf{Y_K} = \left(\frac{1}{2} - \frac{1}{6}v^2, -\frac{v}{\sqrt{3}}, 0, 0, 0 \right). \tag{9}$$

3.1.2. Electric Rosen solution. Applying (2) to the rotated Kasner solution (22) of [10] with the simplifying choice $P_0 + \ln \chi_0 = 0$ yields

$$P = -\tau - 2\ln \text{sech}\, w\tau + 2\ln 2\chi_0 \tag{10}$$

$$\chi = -\frac{1}{2\chi_0}(1 + \tanh w\tau) \tag{11}$$

$$\lambda = -(4w^2 + 1)\tau - 4\ln \text{sech}\, w\tau + 4(\lambda_0 + \ln 2\chi_0). \tag{12}$$

In terms of the β-normalized variables,

$$\mathbf{Y_{ER}} = \left(\frac{1}{3} + \frac{2}{3}w(\tanh w\tau - w), \frac{1}{\sqrt{3}}(1 - 2w\tanh w\tau), 0, -2w\,\text{sech}\, w\tau, 0\right). \tag{13}$$

This describes an electric Bianchi I model. We refer to [17] for a presentation of this solution in a broader context. Up to the arbitrary constant and choosing $\chi_0 = 1$, $\xi = e^{-\tau}$ we have exactly the same χ as described in [17] with a minus sign where now $w = \frac{\alpha_0}{2}$ and $\gamma_0 = \lambda_0$. This solution was first found by Gerald Rosen [18] and we will refer to it as the electric Rosen solution with subscript ER.

3.1.3. Magnetic Rosen solution. Applying (2) to the Taub solution (24)–(26) of [10] yields

$$P = \tau + 2\ln \text{sech}\, w\tau - 2\ln 2\chi_0 \tag{14}$$

$$\chi = 2\chi_0 wx + \chi_1 \tag{15}$$

$$\lambda = -(4w^2 + 1)\tau - 4\ln \text{sech}\, w\tau + 4(\lambda_1 - \ln 2\chi_0). \tag{16}$$

We assume now that $\chi_1 = 0$ for simplicity. In terms of the β-normalized variables,

$$\mathbf{Y_{MR}} = \left(\frac{1}{3} + \frac{2}{3}w(\tanh w\tau - w), \frac{1}{\sqrt{3}}(2w\tanh w\tau - 1), 0, 0, -2w\,\text{sech}\, w\tau\right). \tag{17}$$

We will refer to it as the magnetic Rosen solution and use the subscript MR. In particular we will use in the following the variables $(\Sigma_-)_{MR} = \frac{1}{\sqrt{3}}(2w\tanh w\tau - 1)$ and $\mathcal{B}_{MR} = -2w\,\text{sech}\, w\tau$. The solution (17) is sometimes called the pure magnetic or magnetovacuum solution since later it has been generalized to include, e.g. a perfect fluid. It is not surprising that this solution appears here instead of the Bianchi II vacuum solution since there is a natural correspondence between heteroclinic chains consisting of Bianchi type II solutions in the vacuum case and heteroclinic chains in the case with a magnetic field which include orbits corresponding to both solutions of the vacuum Einstein equations of Bianchi type II and solutions of the Einstein–Maxwell equations of Bianchi type I. For recent work on oscillatory singularities in Bianchi models with magnetic fields we refer to [19].

3.2. Inhomogeneous solutions

We will now proceed to the derivation of the inhomogeneous solutions. It is convenient to introduce the following definitions:

$$f = xw\,e^\tau\,\text{sech}\, w\tau, \qquad s = \frac{2f}{f^2 + 1}, \qquad c = \frac{f^2 - 1}{f^2 + 1}, \tag{18}$$

where it holds that $c^2 + s^2 = 1$.

3.2.1. EME electromagnetic spike solution. Applying (2) to the rotated Taub solution (28)–(30) of [10][5] we obtain a new, inhomogeneous solution:

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[5] As pointed out on page 14 of [11], the third minus sign in equation (28) of [10] should be a plus sign, and the factor 4 in equation (34) of [10] should not be there.

$$P = -3\tau - 2\ln\,\text{sech}\,w\tau + 2\ln(f^2 + 1) + 2\ln 2\chi_0 \tag{19}$$

$$\chi = -\frac{1}{2\chi_0}\frac{f^2}{(f^2 + 1)xw} \tag{20}$$

$$\lambda = -(4w^2 + 9)\tau - 12\ln\,\text{sech}\,w\tau + 4\ln(f^2 + 1) + 4(\lambda_1 + \ln 2\chi_0). \tag{21}$$

The spike occurs at $x = 0$, and the electric field is zero there. We will refer to the solution as the EME spike solution, because (for $|w| > 1$ cases) worldlines along large x experience a sequence of three Rosen transitions: electric-magnetic-electric. Its β-normalized variables are given by

$$\mathbf{Y_{EME}} = \left[-\frac{1}{3}(1 + 2w^2 - 2c + \sqrt{3}(c - 2)\bar{\Sigma}_-), \frac{1}{\sqrt{3}} + c\bar{\Sigma}_-, s\frac{\mathcal{B}_{MR}}{\sqrt{3}}, s\sqrt{3}\bar{\Sigma}_-, c\mathcal{B}_{MR}\right] \tag{22}$$

where we have denoted $\bar{\Sigma}_- = (\Sigma_-)_{MR} - \frac{1}{\sqrt{3}}$, and the subscripts MR and EME refer to the magnetic Rosen and the EME spike solution.

3.2.2. MEM electromagnetic spike solution.   Applying (2) to the spike solution (33)–(35) of [10] yields

$$P = 3\tau + 2\ln\,\text{sech}\,w\tau - 2\ln(f^2 + 1) - 2\ln 2\chi_0 \tag{23}$$

$$\chi = -\chi_0 w[e^{-2\tau} + 2(w\tanh w\tau - 1)x^2] + \chi_2 \tag{24}$$

$$\lambda = -(4w^2 + 9)\tau - 12\ln\,\text{sech}\,w\tau + 4\ln(f^2 + 1) + 4(\lambda_2 - \ln 2\chi_0). \tag{25}$$

We will refer to it as the MEM spike solution and use the subscript MEM. Its β-normalized variables are given by

$$\mathbf{Y_{MEM}} = \left[-\frac{1}{3}(1 + 2w^2 - 2c + \sqrt{3}(c - 2)\bar{\Sigma}_-), -\frac{1}{\sqrt{3}} - c\bar{\Sigma}_-, -s\frac{\mathcal{B}_{MR}}{\sqrt{3}}, c\mathcal{B}_{MR}, s\sqrt{3}\bar{\Sigma}_-\right]. \tag{26}$$

This completes the transformation of the explicit solutions in section 4 of [10]. These spike solutions are new. The solutions above can also be generated by starting with the Kasner solution (8) above and applying the transformations in the subsection 3.2.4 successively.

3.2.3. Properties of the spike.   Both electromagnetic spike solutions have non-trivial electric and magnetic fields, and the names 'EME' and 'MEM' indicate what Rosen transitions occur at large x for $|w| > 1$. Note that the spatial dependence of the different variables lies, as in the vacuum case, in c and s, which depends on x in a spiky way. The electromagnetic spike solutions have the same radius as the vacuum spike solution. For more discussions on c and s, and on the radius of the spike, see sections 4.5 and 4.6 of [10].

In the vacuum case there is a distinction between the false spike solution and true spike solution, with the false spike solution being a rotated Taub solution. There is no such distinction here because the frame rotation transformation is absent in polarized/diagonal case. Both the EME and MEM spike solutions are two different spiky solutions.

3.2.4. Alternative derivation.   The solution-generating transformations in polarized electromagnetic Gowdy spacetimes are

$$e^{-\hat{P}/2} = \frac{e^{-P/2}}{\chi^2 + e^{-(P+\tau)}}, \qquad \hat{\chi} = -\frac{\chi}{\chi^2 + e^{-(P+\tau)}}, \tag{27}$$

and

$$\hat{P} = -P, \qquad \hat{\chi}_\tau = -e^{P-\tau}\chi_x, \qquad \hat{\chi}_x = -e^{P+\tau}\chi_\tau. \tag{28}$$

These transformations correspond to transformations (6) and (7) in [10], but curiously the role of solution-generating transformation has switched. Transformation (6) in [10] is merely a frame rotation, but here transformation (27) is a solution-generating transformation. On the other hand, transformation (7) in [10] is a solution-generating transformation, but here transformation (28) is merely a 90-degree duality rotation for the electromagnetic field (see e.g. [20] about duality rotation) followed by a switch of the coordinates y and z. Transformation (28), like transformation (7) in [10], is also very simple when expressed in β-normalized variables:

$$(\hat{\Sigma}_-, \hat{N}_\times, \hat{\mathcal{E}}, \hat{\mathcal{B}}) = (-\Sigma_-, -N_\times, \mathcal{B}, \mathcal{E}). \tag{29}$$

Thus here we do not have 'false' spikes, since both spikes represent real inhomogeneous solutions, although quite similar.

Transformation (27) maps the Kasner solution (8) to the electric Rosen solution (10)–(11); transformation (28) then maps it to the magnetic Rosen solution (14)–(15); (27) then maps it to the EME spike solution (19)–(20); and (28) maps it to the MEM spike solution (23)–(24). At each step, λ is obtained by quadrature.

4. Visualization

We have already seen that the vacuum transformations have a different interpretation in the electromagnetic case. Now we proceed to visualize the dynamics of the different solutions and we compare the dynamics of the solutions found here corresponding to polarized Gowdy spacetimes with a electromagnetic field with the ones found in [10] corresponding to Gowdy spacetimes. For this purpose we use Hubble-normalized variables [21, 22], denoted with a superscript H. For the spatially homogeneous background dynamics, it is best to use the Hubble-normalized variables which are related to the β-normalized ones via

$$(\Sigma_+, \Sigma_-, N_\times, \mathcal{E}, \mathcal{B})^H = \frac{1}{1-\Sigma_+}(\Sigma_+, \Sigma_-, N_\times, \mathcal{E}, \mathcal{B}), \tag{30}$$

and satisfy

$$\Sigma_+^{H2} + \Sigma_-^{H2} + N_\times^{H2} + \tfrac{1}{3}(\mathcal{E}^{H2} + \mathcal{B}^{H2}) = 1. \tag{31}$$

The Hubble-normalized energy density Ω of the electromagnetic field is given by

$$\Omega = \tfrac{1}{3}(\mathcal{E}^{H2} + \mathcal{B}^{H2}), \tag{32}$$

while the Hubble-normalized spatial curvature Ω_k is given by

$$\Omega_k = N_\times^{H2}. \tag{33}$$

The dynamics of the electric and magnetic Rosen solutions can be described by their orbits in the state space. When projected on the $(\Sigma_+, \Sigma_-)^H$ plane, these orbits form straight lines emanating from one corner of a triangle superscribing the Kasner circle (see figure 1). The two transition sets combine to describe the dynamics during an electromagnetic equivalence of a Kasner era, which consists of long Kasner epochs (described by the Kasner equilibrium points), punctuated by brief periods of transitions. Either the electric or the magnetic component becomes significant during these periods of transitions. Compare with figure 5 of [10].
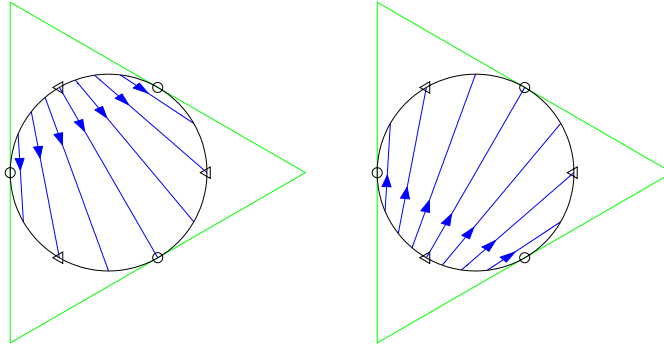
Figure 1. The electric and magnetic Rosen orbits projected on the $(\Sigma_+, \Sigma_-)^H$ plane.

The coincidence of the projected Rosen orbits with the projected Taub orbits compels one to compare the Rosen solutions with the Taub solution. Consider the Taub solution (24)–(26) of [10] with a particular value for w (call it w_T) and the corresponding magnetic Rosen solution whose orbit starts and ends at the same Kasner points as this Taub orbit. Then one finds that the w-parameter for the magnetic Rosen solution takes the value $w_{MR} = w_T/2$. Consequently the rate of change for the magnetic Rosen transition from one Kasner point to the next is only one half of that for the Taub transition.

The implication of this is that in more general models where both modes are present, gravitationally-driven Taub transitions would dominate electromagnetically-driven Rosen transitions towards the singularity, and that the Hubble-normalized energy density of the electromagnetic field would tend to zero towards the singularity, with the caveat that this occurs almost everywhere, except possibly at some 'spiky' worldlines where the Taub mode has a local zero. Along these spiky worldlines, whether (vacuum) spike transitions would dominate Rosen transitions is unknown. In figure 2 we visualize the orbits of the MEM spike solution (23)–(25) along various worldlines $x = $ const, projected on the $(\Sigma_+, \Sigma_-)^H$ plane. The projected orbits of the EME spike solution (19)–(21) differ only in the sign of Σ_-. Along worldlines far away from the spike worldline, the orbits approximate the electric and magnetic Rosen orbits. Along the spike worldline, the projected orbit is a straight line. For the case $|w| \geqslant 1$, all these orbits end at the same Kasner points. For the case $0 < |w| < 1$, the orbit along the spike worldline ends at a different Kasner point from all others. Compare with figure 6 of [10]. In figure 3 we visualize two families of the solutions (one with $w = 0.75$ and the other with $w = 1.25$). Note that the Kasner solution plotted here uses the parametrization in (8). Compare with figure 9 of [10].

We now compare the orbits of MEM spike solution with the orbits of vacuum spike solution. Call the corresponding parameters w_{MEM} and w_S. We set $w_S = 2w_{MEM} + 1$ so that both solutions start at the same Kasner point. But the solutions will not end at the same Kasner points. See figure 4.

5. Discussion and outlook

In this paper we have presented new solutions to the Einstein–Maxwell solutions with polarized Gowdy symmetry which generalize the known magnetic and electric Rosen solutions which are spatially homogeneous to the inhomogeneous case. These solutions represent spikes in the electromagnetic field as well as the gravitational field, and are building blocks of oscillatory behavior. In contrast to the vacuum case there are no false and true electromagnetic
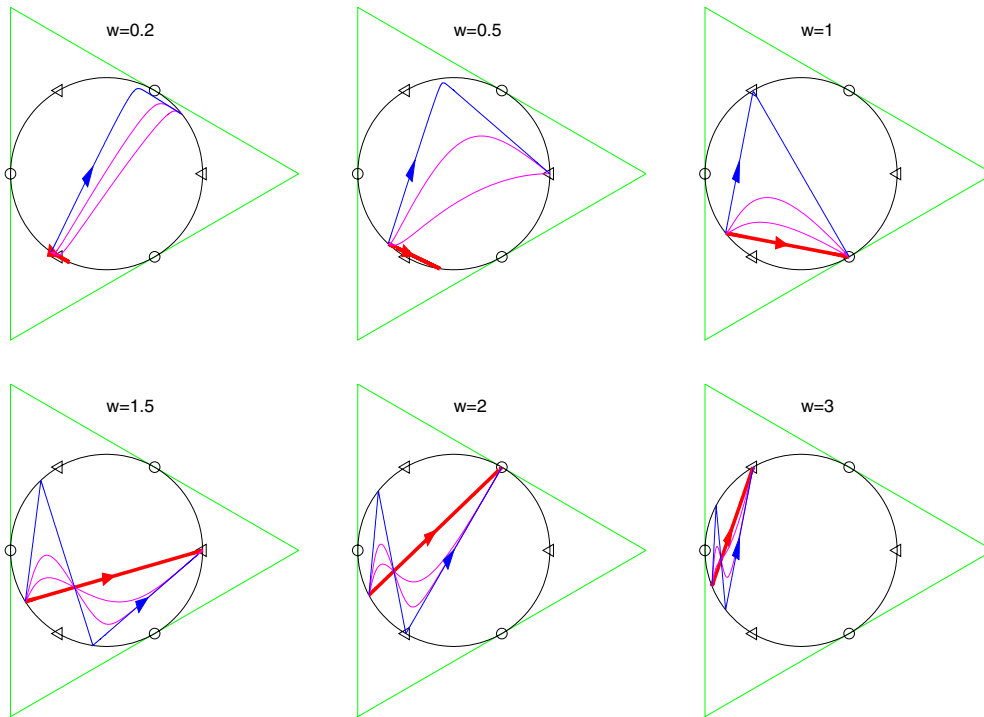
Figure 2. Orbits of the MEM spike solution projected on the $(\Sigma_+, \Sigma_-)^H$ plane. Orbits are colored red along the spike worldline $x = 0$, blue along $x = 1000$, and magenta along small values of x. $w = 0.2, 0.5, 1, 1.5, 2, 3$ respectively.
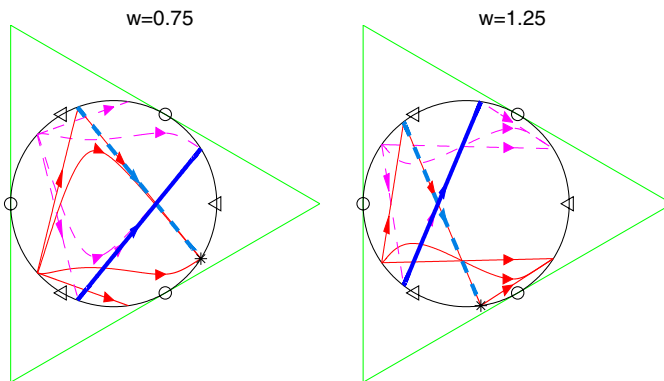

Figure 3. Two families of orbits with $w = 0.75$ and 1.25. The Kasner seed is indicated by a black *, the electric Rosen orbit in dashed light blue, the magnetic Rosen orbit in dark blue, orbits of the EME spike solution in dashed magenta, and orbits of the MEM spike solution in red.

spikes, however the solutions are now linked via a duality rotation. From the analysis one can see that it is the gravitational solutions which dominate the electromagnetic solutions. Therefore we conjecture that the (vacuum) gravitational spike solution plays a larger role than the electromagnetic spike solutions in the oscillatory regime. The analysis represents also additional support to the analysis carried out in [23], where an inhomogeneous generalization of Bianchi VI$_0$ with a pure magnetic field was considered. In fact, the latter solution can be
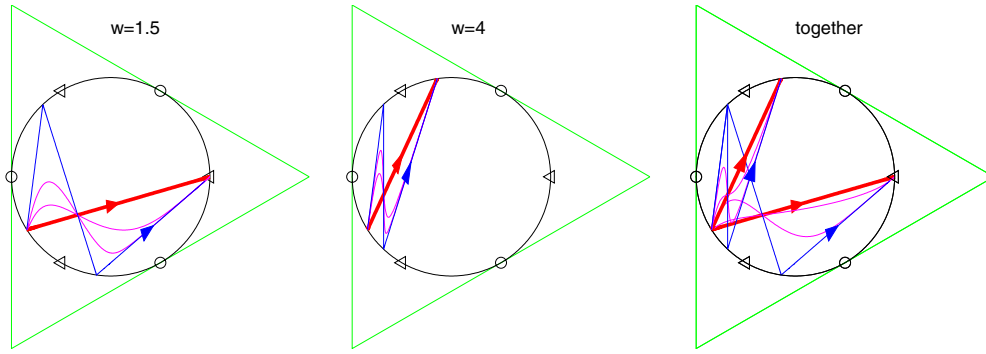
Figure 4. Comparison of orbits for the MEM spike solution with $w_{MEM} = 1.5$ and orbits for the vacuum spike solution with $w_S = 4$. Both sets of orbits start at the same Kasner point but end at two different Kasner points.

seen as coming from an electromagnetic field which does not come from a vector potential, cf [24]. In [24] an inhomogeneous generalization of Bianchi VII$_0$ was also presented. Both these inhomogeneous generalizations are called of local or twisted Gowdy symmetry. It is of interest to investigate further these cases as a pre-step to the analysis of the Einstein–Maxwell system with full Gowdy symmetry which is considerably more complicated. It remains unclear whether pure magnetic spike solutions exist. In [25] numerical and analytical evidence was presented that (gravitational) spikes can generate matter perturbations and it was argued that this phenomenon might explain the formation of structure in the early Universe. Electromagnetic spikes should have a similar effect on matter as well. It remains to be seen how gravitational spikes interact with electromagnetic spikes, and whether such interactions amplify or suppress the electromagnetic field.

Acknowledgments

Appendix

The equations in this section are taken from from [15] where $-\lambda$ instead of λ was used. Note also that we have put $\omega = 0$ and we have used a different sign convention for $\epsilon^{\alpha\beta\gamma\delta}$.

The basic quantity in electromagnetism is the *electromagnetic field tensor* $F_{\alpha\beta}$ which is antisymmetric. The *Maxwell equations* in absence of charged matter are

$$\nabla^\alpha F_{\alpha\beta} = 0$$
$$\nabla_\alpha F_{\beta\gamma} + \nabla_\gamma F_{\alpha\beta} + \nabla_\beta F_{\gamma\alpha} = 0.$$

The second Maxwell equation can be solved by introducing a four potential such that

$$F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu. \tag{A.1}$$

In the *Lorentz gauge*, where by definition

$$\nabla^\alpha A_\alpha = 0$$

holds, the Maxwell equations in curved space time can be written as:

$$\nabla^\alpha \nabla_\alpha A_\beta - R^\gamma{}_\beta A_\gamma = 0. \tag{A.2}$$

The energy-momentum tensor for an electromagnetic field is

$$T_{\alpha\beta} = \frac{1}{4\pi}\left(F_{\alpha\gamma}F_\beta{}^\gamma - \frac{1}{4}g_{\alpha\beta}F_{\delta\epsilon}F^{\delta\epsilon}\right), \tag{A.3}$$

which is trace-free. The *dual electromagnetic field tensor* is

$$^*F^{\gamma\delta} = \tfrac{1}{2}\epsilon^{\alpha\beta\gamma\delta}F_{\alpha\beta}, \tag{A.4}$$

where

$$\epsilon^{\alpha\beta\gamma\delta} = (-\det g)^{-\frac{1}{2}}\eta^{\alpha\beta\gamma\delta},$$

where $\det g$ is the determinant of the matrix $g_{\alpha\beta}$. Let n_α be a unit future-pointing vector orthogonal to a spacelike hypersurface. Then we can define the electric and the magnetic fields as follows:

$$E^\alpha = F^{\alpha\beta}n_\beta, \tag{A.5}$$

$$B^\alpha = {}^*F^{\alpha\beta}n_\beta. \tag{A.6}$$

The symmetry assumptions imply that vector potential has only the following components:

$$A_3 = \chi(\tau, x).$$

A_2 is periodic of period 2π with respect to x. Using (A.1) one can compute that the electromagnetic field tensor has the following non-trivial components:

$$F_{03} = \chi_\tau,$$
$$F_{13} = \chi_x.$$

The dual electromagnetic field tensor has according to (A.4) the following non-trivial components:

$$^*F^{02} = -\chi_x\, e^{\frac{3\tau-\lambda}{2}},$$
$$^*F^{12} = \chi_\tau\, e^{\frac{3\tau-\lambda}{2}}.$$

Choosing

$$n_\alpha = (\sqrt{-g_{00}}, 0, 0, 0)$$

as the unit future-pointing vector we compute the non-vanishing components of the electric and the magnetic field with (A.5) and (A.6):

$$E^3 = \chi_\tau\, e^{\frac{-\lambda+7\tau+4P}{4}},$$
$$B^2 = \chi_x\, e^{\frac{-\lambda+3\tau}{4}}.$$

$\mathcal{E}$ and $\mathcal{B}$ are β-normalized orthonormal frame components of the electric and magnetic fields.

$$\mathcal{E} = \frac{e^3{}_3 E^3}{\beta} = 2\chi_\tau\, e^{\frac{1}{2}(P+\tau)}$$

$$\mathcal{B} = \frac{e^2{}_2 B^2}{\beta} = -2\chi_x\, e^{\frac{1}{2}(P-\tau)},$$

where $e^3{}_3 = e^{(-P-\tau)/2}$ and $e^2{}_2 = e^{(P-\tau)/2}$ are the frame coefficients.

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