Color/kinematics duality for general abelian orbifolds of $\mathcal{N} = 4$ super Yang-Mills theory

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Abstract

To explore color/kinematics duality for general representations of the gauge group we formulate the duality for general abelian orbifolds of the $SU(N)$, $\mathcal{N} = 4$ super Yang-Mills theory in four dimensions, which have fields in the bi-fundamental representation, and use it to construct explicitly complete four-vector and four-scalar amplitudes at one loop. For fixed number of supercharges, graph-organized L-loop n-point integrands of all orbifold theories are given in terms of a fixed set of polynomials labeled by L representations of the orbifold group. In contrast to the standard duality-satisfying presentation of amplitudes of the $\mathcal{N} = 4$ super Yang-Mills theory, each graph may appear several times with different internal states. The color and R-charge flow provide a way to deform the amplitudes of orbifold theories to those of more general quiver gauge theories which do not necessarily exhibit color/kinematics duality on their own.

Based on the organization of amplitudes required by the duality between color and kinematics in orbifold theories we show how the amplitudes of certain non-factorized matter-coupled supergravity theories can be found through a double-copy construction.

We also carry out a comprehensive search for theories with fields solely in the adjoint representation of the gauge group and amplitudes exhibiting color/kinematics duality for all external states and find an interesting relation between supersymmetry and existence of the duality.

Contents

1 Introduction

Recent detailed investigations of properties of gauge theories with fields in the adjoint representation of a semi-simple Lie group have revealed that their scattering amplitudes have a surprisingly rich structure, especially in the presence of supersymmetry. While most of such structure emerges at the planar level, color/kinematics (BCJ) duality [1] relates the leading and subleading color components of scattering amplitudes and may extend to the non-planar level some of the remarkable properties of planar amplitudes such as, in the case of $\mathcal{N} = 4$ super Yang-Mills (sYM) theory, dual superconformal symmetry [2], the amplitude/Wilson loop duality [3] and the relation between Wilson loops and scattering amplitudes [4].

There is by now substantial evidence for the duality between the color and kinematic factors (not including propagators) for the graph-organized integrands of amplitudes of gauge theories with fields in the adjoint representation, both at tree-level and at loop level. Moreover, if the integrand of an amplitude is given in a form that manifestly exhibits the duality, then amplitudes in certain factorized supergravity theories can be obtained in the same graph-organized form [5] by simply replacing the amplitude's color factors with another set of kinematic factors.

While of no less importance, four-dimensional field theories with fields in other representations have been comparatively less studied from the perspective of their scattering amplitudes. Among them, quiver gauge theories – with product gauge groups and fields in the adjoint and bi-fundamental representations – are perhaps the simplest and the ones most closely related to theories with only fields in the adjoint representation[1]. Certain quiver gauge theories exhibit a special point in the space of couplings where they become (regular or non-regular) orbifolds of $\mathcal{N} = 4$ sYM theory. In this paper we will study such theories which have a $\mathbf{Z}_n$ orbifold point: we will define and test color/kinematics duality at the orbifold point and then argue that amplitudes for a general choice of couplings can be found by simply dressing the amplitudes at the orbifold point following the color and R-charge flow.

Compactification of string theory on orbifolds – smooth spaces modded out by some discrete group Γ – is a classic construction of four-dimensional matter-coupled gauge and gravity theories [8]. The spectrum of (massless) states consists of *untwisted-* and *twisted-*sector states. The former are the Γ-invariant states of ten-dimensional flat space string theory. In a closed string theory the latter are zero-length strings which are closed up to the action of the orbifold group and thus localized at the orbifold fixed points. For oriented

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[1]The three-dimensional ABJM can be formulated as a quiver gauge theory, with fields in the bi-fundamental representation. Its scattering amplitudes have been extensively studied. The formulation of color/kinematics duality, discussed and explored in [6, 7], was aided by the three-algebra formulation of this theory, in which all propagating fields formally carry a single (adjoint-like) color index.

open strings it is necessary to specify the action of the orbifold group on the Chan-Patton factors, which is most conveniently described in terms of D-branes, as it was discussed by Douglas and Moore [9]. Twisted sector states are then described by strings stretched between a stack of D-branes and their images under the orbifold group. Consequently, these states are massless provided that the D-branes are placed at a fixed point of the action of the orbifold group; the corresponding fields transform in the bi-fundamental representation of the gauge group. This construction can be realized from a field theory perspective [10] by starting with maximally-supersymmetric gauge theory and projecting onto the Γ-invariant states while allowing Γ to act both on the R-symmetry and on the gauge group indices. The result is a vast class of quiver gauge theories whose planar limits have special properties. If the action of the orbifold group on the gauge degrees of freedom is in a regular representation – which from a string theory perspective is required for the cancellation of tadpoles – then the $\mathcal{N} \geq 1$ quiver gauge theories are conformal in the multi-color limit. For non-supersymmetric theories conformal invariance is broken at one-loop level in the multi-color limit while it is present in the planar theory [11]. In the orbifold theory the couplings of the various gauge-group factors are equal and proportional to that of the parent theory; renormalizability however requires that the theory be deformed off this "natural line" to a general quiver theory with the same matter content.

Perhaps the simplest orbifolds are those with trivial action on the gauge degrees of freedom[2]; the resulting theories are $\mathcal{N} = 2$ and $\mathcal{N} = 1$ $SU(N)$ sYM theories without additional matter multiplets and $SU(N)$ gauge theories with zero, two, four or six additional scalars and specific interactions making them the dimensional reduction of $D = 6, 8, 10$ pure gauge theories. Color/kinematics-satisfying representations of four-gluon amplitudes were constructed at one loop in [13] for the former and at one and two-loops in [14] for the latter theories, and were instrumental in obtaining certain amplitudes in $\mathcal{N} \leq 4$ supergravity theories with additional matter multiplets.

In this paper we shall formulate color/kinematics duality for general abelian orbifolds of the $\mathcal{N} = 4$ sYM theory and focus on $\Gamma \simeq \mathbf{Z}_n$. An option is to seek presentations of amplitudes in which each internal line corresponds to a field in a definite representation of the gauge group; then, the commutation relations of the gauge group with generators in the appropriate representation can be interpreted as color Jacobi identities and can be used as the starting point for the definition of color/kinematics duality [15]. Alternatively, the color Jacobi relations relevant to the orbifold theory are taken to be the (appropriately-defined) image of the Jacobi relations of the parent theory through the projection [12] which truncates it to the daughter theory. In this second approach all calculations are effectively done in the parent theory for all except one arbitrarily-chosen propagator for each loop, which is acted upon by an orbifold group element; the orbifold

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[2]Since the planar inheritance discussed in [12] relies on the regularity of the representation of the orbifold group (*i.e.* $\mathrm{Tr}[g] = 1$ iff g is the trivial element of the group), these theories do not exhibit it.

theory is obtained by summing over all elements of Γ. Since the parent theory is assumed to only have fields in the adjoint representation, its Jacobi relations are the standard ones; however, graphs carrying different *inequivalent* choices of orbifold group insertion – either because of a different element of Γ or because of a different action of a fixed element on the fields running in loops – are treated independently. As we shall describe in section 4, the kinematic Jacobi relations mix the corresponding kinematic factors in a pattern determined by the R-charges of internal and external legs. While we shall adopt the second approach, in section 7 we shall argue that the two definitions of color/kinematics duality described here are equivalent for orbifold theories. Thus, for more general quiver gauge theories that do not have an orbifold point as well as for theories with fields in other representations one may use the former strategy.

In the framework above, scattering amplitudes in the orbifold theory are obtained by independently summing all graphs over all orbifold group elements inserted in each loop. As we shall see, an interesting feature of this construction is that, for some $\mathcal{N} \leq 1$ amplitudes, the resulting graphs appear to have edges corresponding to fields not present in the orbifold theory; such graphs are absent if one does not require that color/kinematics duality is present. While this may appear problematic, all cuts through the "unphysical" propagator(s) vanish. It should be possible to understand the appearance of such fields from the perspective of a putative Lagrangian whose Feynman graphs produce directly amplitudes in a form that exhibit the duality. As discussed in [16], such a Lagrangian has only cubic vertices and the vast majority of its fields are auxiliary.

We shall also attempt to classify all field theories with fields in the adjoint representation which exhibit color/kinematics duality for any choice of external states and are power-counting renormalizable (though perhaps not actually renormalizable) when reduced to four dimensions. We will find that four- and five-point matter amplitudes uniquely fix them to be either the pure $\mathcal{N}$-extended sYM theories in various dimensions, or YM-scalar theories that can be interpreted as the dimensional reduction of a pure gauge theory in higher dimension. In higher dimensions we shall find that the tree-level four-fermion amplitude of a YM theory coupled to a single fermion obeys color/kinematic duality only in dimensions $D = 3, 4, 6, 10$, *i.e.* in the dimensions in which the theory is also supersymmetric. In contrast, tree-level four-point amplitudes with at least two external gluons impose essentially no constraints as they depend only on the minimal coupling of matter fields and thus are the same in supersymmetric and non-supersymmetric theories. Our results are consistent with [17] where one-loop four-gluon amplitudes have been shown to have a color/kinematic satisfying form for general matter content. Similarly to tree-level amplitudes with at most two external matter fields, these amplitudes are insensitive to the matter self-coupling and thus do not receive contributions from the interactions which may break color/kinematics duality at tree level. It would be interesting to find ways to avoid these constraints and use the power of color/kinematics duality in theories which may not otherwise exhibit it.

The paper is organized as follows. In the next section we review the construction of field theory orbifolds, and discuss their deformation into more general quiver gauge theories. In section 3, after reviewing the color/kinematics duality in theories with fields in the adjoint representation of the gauge group and in particular for the $\mathcal{N} = 4$ sYM theory, we analyze a general $SU(N)$ gauge theory with adjoint matter, antisymmetric couplings and cubic and quartic interactions and constrain it such that the four-and five-point amplitudes obey the duality. In section 4 we formulate the duality for a general abelian orbifold at tree- and loop-level, and spell out the kinematic Jacobi relations for one-loop amplitudes. In section 5 we include examples of four-gluon and four-scalar amplitudes in $\mathcal{N} = 2$, $\mathcal{N} = 1$ and $\mathcal{N} = 0$ orbifold quiver gauge theories. Based on the construction in earlier sections and on the physical interpretation of the kinematic numerator factors we discuss in section 6 a double-copy-like construction for certain non-factorizable supergravity theories which are orbifolds of $\mathcal{N} = 8$ supergravity. We summarize our results in section 7, comment on their extension to more general (quiver) gauge theories and gauge theories with fields in other representations and prove that, for fields in the fundamental representation, our definition of color/kinematics duality reduces to using the gauge group defining commutation relations as color Jacobi identities. Two appendices contain a summary of our notations and details omitted in section 3.

2   Quiver gauge theories and field theory orbifolds

A general quiver gauge theory is specified by its gauge group factors, the coupling of each factor, and the matter content including the representations (adjoint or bi-fundamental) of matter fields under the gauge group factors and global symmetry groups. Particular quiver gauge theories exhibit an "orbifold point" – *i.e.* a particular choice of couplings for which it can be interpreted as a field theory orbifold [10, 12] of some parent theory. Orbifold field theories are obtained by consistently truncating a parent field theory to the fields and interactions that are invariant under some discrete subgroup Γ of the global symmetry group. All couplings of the resulting quiver gauge theory are equal and are said to be on the "natural line" in coupling space. It is worth mentioning that, while the truncation is consistent, the resulting theory may not be renormalizable; to carry out the renormalization program it is in principle necessary to deform the theory off the natural line and to allow for different renormalization constants for the couplings of different gauge group factors. Generically, the U(1) factor originally accompanying each gauge group acquires non-vanishing beta function [18, 19] and decouples in the IR.

The action of an element $\gamma \in \Gamma$ on the fields of the parent theory is specified by the pair (r_γ, g_γ) giving, respectively, the representation of γ in the flavor symmetry group F and in the (global part of) the gauge group G. In the following we will not write explicitly the index γ and, with a slight abuse of notation, interpret the elements of the

orbifold group as the pairs (r, g). In general these representations need not be faithful. Perhaps the simplest nontrivial example corresponds to choosing $g = 1$, $i.e.$ a trivial representation of Γ in the gauge group; in these cases, the truncation eliminates some of the fields of the parent theory while preserving the representations of the remaining ones. Pure $\mathcal{N} \leq 2$ sYM theories can be interpreted as such orbifolds of $\mathcal{N} = 4$ sYM theory. More interesting theories, with matter fields in the adjoint and bi-fundamental representations, are obtained by choosing both r and g to be nontrivial [10, 12]. While in principle one may orbifold any field theory, a judicious choice for the parent theory and of orbifold group leads to daughter theories inheriting interesting properties [12].

Well-studied examples [10] are orbifolds of $SU(|\Gamma|N)$ $\mathcal{N} = 4$ sYM theory with an orbifold group Γ of rank $|\Gamma|$ whose elements are pairs (r, g) with $r \in SU(4)$ and g taken to be a faithful and regular representation of r in $SU(|\Gamma|N)$.[3]

In the following we will assume that Γ is abelian and relax the constraints on its representations. The physical fields of the daughter theory are invariant under the action of all elements of Γ, $i.e.$

$$\Phi_{a_1 \ldots a_n} = r_{a_1}^{a_1} \ldots r_{a_n}^{a_n} \, g \Phi_{a_1 \ldots a_n} g^\dagger \, , \tag{2.1}$$

where $a_1, \ldots a_n$ are $SU(4)$ indices in the fundamental representation. Following our assumption that the orbifold group is abelian, we have written its generators as diagonal matrices. It is convenient to introduce explicitly orbifold projection operators [12] which enforce the condition (2.1) and act on a generic field as

$$\mathcal{P}_\Gamma \Phi_{a_1 \ldots a_n}^A = \frac{1}{|\Gamma|} \sum_{(r,g) \in \Gamma} r_{a_1}^{a_1} \ldots r_{a_n}^{a_n} \, g^{AB} \Phi_{a_1 \ldots a_n}^B \, ; \tag{2.2}$$

the summation is taken over all elements of Γ. In this expression the indices A and B denote an arbitrary representation; for a field in the adjoint representation they each take $(|\Gamma|^2 N^2 - 1)$ values. With the normalization $\mathrm{Tr}[T^A T^B] = \delta^{AB}$ we have,

$$g^{AB} = \mathrm{Tr}\big(T^A g T^B g^\dagger\big) = (g^\dagger)^{BA} \, . \tag{2.3}$$

The cases in which Γ acts trivially in the gauge group were discussed in detail in [13]: $\Gamma \subset SU(3) \subset SU(4)$ leads to pure sYM theories and $\Gamma \subset SU(4)$ breaks supersymmetry completely and leads to YM theory with $0, 2, 4$ or 6 complex scalar fields.

In general, if the action of the orbifold group in the (parent) gauge group is nontrivial (thought still potentially not faithful[4]) the daughter theory is a quiver gauge theory with fields transforming in bi-fundamental representations. A common technical assumption[5]

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[3]Choosing g to be an unfaithful representation of Γ leads to inclusion of orbifolds of $\mathcal{N} \leq 2$ sYM theories in this framework. However, the interesting properties discussed in [10, 12] such as planar inheritance no longer hold.

[4]Such cases may be rephrased as orbifolds of a less-than-maximally (s)YM theories.

[5]For string theory orbifold constructions regularity is necessary for tadpole cancellation.
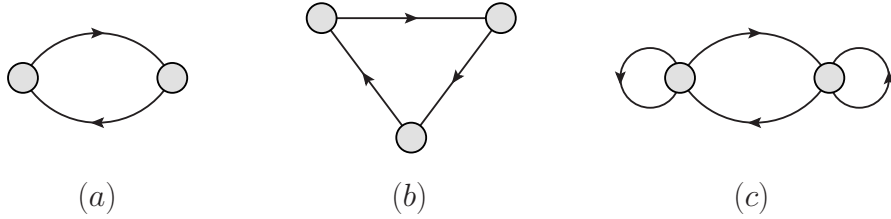
$$(a) \qquad\qquad (b) \qquad\qquad (c)$$

Figure 1: Quivers for the $\mathcal{N} = 2, 1, 0$ examples. Each node is a gauge group factor and lines joining them are fields/multiplets in bi-fundamental representation. The arrow points from $\bar{\mathbf{N}}$ to $\mathbf{N}$. Lines starting and ending at the same node represent adjoint matter fields.

is that the orbifold is *regular*, that is

$$\mathrm{Tr}\,g \neq 0 \quad \text{iif} \quad g = 1 \ . \tag{2.4}$$

It was shown in [12] that, with such an orbifold group, planar scattering amplitudes of the daughter theory are inherited from the parent to all orders in perturbation theory. We will not make this assumption, but rather consider a general representation of Γ in the gauge group; then the parent gauge group $SU(N)$ is broken to $SU(N_1) \times \ldots \times SU(N_n)$ with $N = N_1 + \cdots + N_n$. We will still observe a relation between regularity of the orbifold and absence of tadpole graphs in amplitudes.

The simplest non-trivial example, preserving $\mathcal{N} = 2$ supersymmetry, is the regular $\mathbf{Z}_2$ orbifold generated by

$$r = \mathrm{diag}(1, 1, -1, -1) \ , \qquad g = \begin{pmatrix} I_N & 0 \\ 0 & -I_N \end{pmatrix} \ . \tag{2.5}$$

This theory has gauge group $SU(N) \times SU(N) \times U(1)$ and contains one $\mathcal{N} = 2$ vector multiplet in the adjoint representation of each $SU(N)$ factor, one vector multiplet with the $U(1)$ gauge field and two hypermultiplets transforming in the $(\mathbf{N}, \bar{\mathbf{N}})$ and $(\bar{\mathbf{N}}, \mathbf{N})$ representations, respectively. This field content is summarized by the quiver in fig. 1(a).

Similarly, one can obtain an orbifold with $\mathcal{N} = 1$ supersymmetry using the generators

$$r = \mathrm{diag}(1, \omega, \omega, \omega) \ , \qquad g = \begin{pmatrix} I_N & 0 & 0 \\ 0 & \omega I_N & 0 \\ 0 & 0 & \omega^2 I_N \end{pmatrix} \quad \text{with} \quad \omega^3 = 1 \ . \tag{2.6}$$

This $\mathbf{Z}_3$ orbifold theory has gauge group $SU(N) \times SU(N) \times SU(N) \times U(1)^2$. The field content amounts to five $\mathcal{N} = 1$ vector multiplets and six chiral multiplets. Three of the vector multiplets transform in the adjoint representations of the three $SU(N)$ factors,

i.e. $(\mathbf{N}^2 - 1, 1, 1)$, $(1, \mathbf{N}^2 - 1, 1)$ and $(1, 1, \mathbf{N}^2 - 1)$; the remaining two vector multiplets contain the $U(1)$ gauge fields. The chiral multiplets transform in $(\mathbf{N}, \bar{\mathbf{N}}, 1)$, $(1, \mathbf{N}, \bar{\mathbf{N}})$ and $(\bar{\mathbf{N}}, 1, \mathbf{N})$ representations and in the conjugate representations. This field content is summarized by the quiver in fig. 1(b).

Finally, a simple $\mathbf{Z}_2$ orbifold producing an $\mathcal{N} = 0$ theory is generated by

$$r = \mathrm{diag}(-1, -1, -1, -1) , \qquad g = \begin{pmatrix} I_N & 0 \\ 0 & -I_N \end{pmatrix} . \tag{2.7}$$

As discussed in [20], this theory contains the massless modes of a stack of N electric and N magnetic $D3$-branes in type 0B string theory. The gauge group is $SU(N) \times SU(N) \times U(1)$ and the field content consists of one gluon and six adjoint scalars for each $SU(N)$ factor, one U(1) gluon and six additional scalar fields neutral under the $SU(N) \times SU(N)$, four fermions transforming in the bi-fundamental representation $(\mathbf{N}, \bar{\mathbf{N}})$ and four fermions transforming the anti-bi-fundamental representation $(\bar{\mathbf{N}}, \mathbf{N})$. This field content is summarized by the quiver in fig. 1(c).

It is not difficult to deform a quiver gauge theory off its orbifold point (if it has one). From a Lagrangian perspective one simply identifies the various gauge fields and dresses their interactions with the desired couplings. Similarly, to find the integrands of amplitudes for general couplings from those at the orbifold point it suffices to represent them in a cubic graph-based form, which reflects the flow of color and R charge. Each vertex of the graph belongs to a single gauge group and thus can be dressed with the desired coupling.

3 Color/kinematics duality for theories with adjoint fields

3.1 Review

The scattering amplitudes of any matter-coupled gauge theory with fields in the adjoint representation and antisymmetric couplings (an example of which is the $\mathcal{N} = 4$ sYM theory(can be organized in terms of graphs with only trivalent vertices (cubic graphs); assuming that all interactions are governed by the gauge coupling g, the general expression of the dimensionally-regularized L-loop m-point scattering amplitude in such a theory is

$$\mathcal{A}_m^{L-\mathrm{loop}} = i^L g^{m-2+2L} \sum_{i \in \mathcal{G}_3} \int \prod_{l=1}^{L} \frac{d^D p_l}{(2\pi)^D} \frac{1}{S_i} \frac{n_i C_i}{\prod_{\alpha_i} p_{\alpha_i}^2} . \tag{3.1}$$

The sum runs over the complete set $\mathcal{G}_3$ of m-point L-loop cubic graphs, including all permutations of external legs, the integration is over the L independent loop momenta p_l and the denominator is determined by the product of all propagators of the corresponding

graph. The coefficients C_i are the color factors, obtained by assigning to every three-vertex in the graph a factor of the antisymmetric structure constant

$$\tilde{f}^{ABC} = i\sqrt{2}f^{ABC} = \text{Tr}([T^A, T^B]T^C)\,, \tag{3.2}$$

while respecting the cyclic ordering of edges at the vertex. The symmetry factors S_i of each graph remove the potential overcount introduced by the summation over all permutations of external legs included by definition in the set $\mathcal{G}_3$, as well as any symmetries of the graph with fixed external legs. As in section 2, the gauge group generators T^A are assumed to be hermitian and are normalized as $\text{Tr}[T^A T^B] = \delta^{AB}$. The coefficients n_i are kinematic numerator factors depending on momenta, polarization vectors and spinors. For supersymmetric amplitudes in an on-shell superspace they will also contain Grassmann parameters.

An amplitude is said to exhibit color/kinematics duality [1] if the kinematic numerators of a cubic-graph representation of the amplitude satisfy antisymmetry and (generalized) Jacobi relations for each propagator, in one-to-one correspondence with the properties of color-factors. That is, for the representation in eq. (3.1), it requires that

$$C_i + C_j + C_k = 0 \qquad \Rightarrow \qquad n_i + n_j + n_k = 0\,. \tag{3.3}$$

Such representations were conjectured [5] to exist to all loop orders and to all multiplicities in $\mathcal{N} = 4$ sYM theory; they are related to other representations by generalized gauge transformations,

$$n_i \to n_i + p_i^2 f(p)\,, \quad n_j \to n_j + p_j^2 f(p)\,, \quad n_k \to n_k + p_k^2 f(p)\,, \tag{3.4}$$

which leave the amplitude invariant but reorganize contact terms associated to each graph. Here $f(p)$ can be any function with the correct dimension and p_i, p_j and p_k are the momenta of the internal lines that participate in the Jacobi relations (3.3).

Color/kinematics duality for pure sYM theories in various dimensions has been discussed extensively, especially at tree level [21, 22, 23, 16, 24, 25, 26, 27], where explicit representations of the numerator factors n_i in terms of color-ordered amplitudes are known for any number of external legs [28, 27]. Loop-level color/kinematics-satisfying four- and five-point amplitudes have been constructed through four-loops [5, 29] and two-loops [30], respectively, in $\mathcal{N} = 4$ sYM theory. In less-than-maximal supersymmetric theories four-point amplitudes have been constructed at one-loop level in $\mathcal{N} = 1$ and $\mathcal{N} = 2$ theories [13], at one and two loops in pure gauge theory in [14]. All-plus one-loop amplitudes with arbitrary multiplicity in pure gauge theory (and, through dimension shifting [31], one-loop MHV amplitudes $\mathcal{N} = 4$ sYM theory) have been constructed in [32].

In the next subsection we will identify all matter-coupled gauge theories with only massless fields in the adjoint representation of some semi-simple gauge group and anti-

symmetric couplings which can obey color/kinematics duality[6]. We will find an interesting relation with supersymmetry: whenever pure YM theory coupled to a single fermion is supersymmetric in a given dimension D, the corresponding tree-level amplitudes obey color/kinematics duality.

3.2 Color/kinematics duality for gauge theories coupled with adjoint matter: a general classification

The most general Lagrangian with n_s real adjoint scalars and n_f adjoint fermions which is power-counting renormalizable in four dimensions is

$$
\begin{aligned}
\mathcal{L} =\mathrm{Tr}\Bigg[&-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}D_\mu\phi^I D^\mu\phi^I + i\bar\psi_A \, \slashed{D}\psi^A + \frac{1}{8}\alpha^{IJKL}[\phi^I,\ \phi^J][\phi^K,\ \phi^L] \\
&+\frac{1}{6}\sigma^{IJK}[\phi^I,\phi^J]\phi^K + \frac{i}{\sqrt{2}}\lambda^I_{AB}\psi^A[\phi^I,\ \psi^B] + \frac{i}{\sqrt{2}}\bar\lambda^{IAB}\bar\psi_A[\phi^I,\ \bar\psi_B]\Bigg] \ .
\end{aligned}
\tag{3.5}
$$

Here σ^{IJK} and α^{IJKL} are constant coefficients with symmetries dictated by the combination of commutators they multiply. While the notation might suggest otherwise, we do not assume the existence of any internal global symmetry acting on scalars and fermions.

To test whether this theory can exhibit color/kinematics duality we focus on the four-point amplitudes which probe this unambiguously because there is a single Jacobi relation between its numerator factors. Since the four-point amplitudes with at least two external gluons are the same as in $\mathcal{N} = 4$ sYM theory (up to the perhaps different number of scalars and fermions)[7], the first constraints arise from the four-point amplitudes with external scalars and fermions.

3.2.1 The bosonic theory

We begin by analyzing the bosonic theory in D dimensions. With the Lagrangian (3.5), the amplitude with four *different* scalars is

$$
\begin{aligned}
\mathcal{A}_4^{\mathrm{tree}}(1^{\phi^I}2^{\phi^J}3^{\phi^K}4^{\phi^L}) = &\ \alpha^{IJKL}f^{12a}f^{34a} + \alpha^{KIJL}f^{31a}f^{24a} + \alpha^{JKIL}f^{23a}f^{14a} \\
&+\frac{1}{s_{12}}\sigma^{IJM}\sigma^{KLM}f^{12a}f^{34a} + \frac{1}{s_{13}}\sigma^{KIM}\sigma^{JLM}f^{31a}f^{24a} + \frac{1}{s_{14}}\sigma^{JKM}\sigma^{ILM}f^{23a}f^{14a} \ .
\end{aligned}
\tag{3.6}
$$

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[6]We focus on theories that are power-counting renormalizable – though not necessarily renormalizable – when reduced to four dimensions.

[7]Consequently, the generalized unitarity method implies that all cuts of one-loop four-gluon amplitudes exhibit color/kinematics duality and thus that the corresponding amplitude may exhibit it as well. This is indeed the case, as shown in [17].

The origin of each term is clear; requiring that it exhibits a duality between the color and kinematic numerators leads to

$$\sigma^{IJM}\sigma^{KLM} + \sigma^{KIM}\sigma^{JLM} + \sigma^{JKM}\sigma^{ILM} + s_{12}\alpha^{IJKL} + s_{13}\alpha^{KIJL} + s_{14}\alpha^{JKIL} = 0 \ . \quad (3.7)$$

The terms with different momentum dependence must cancel separately, implying that σ^{IJM} obey a Jacobi identity and that α^{IJKL} is cyclically invariant in the first three indices. The structure of the Lagrangian (3.5) however implies that such a coefficient is projected out by the color Jacobi identity. We may therefore set to zero α^{IJKL} with indices not equal in pairs.

With the notation $\alpha^{IJ} = \alpha^{IJIJ}$, the four-scalar amplitude with pairwise identical scalars is $(I \neq J)$

$$\begin{aligned}
\mathcal{A}_4^{\text{tree}}(1^{\phi^I}2^{\phi^I}3^{\phi^J}4^{\phi^J}) =& \frac{s_{13}-s_{14}}{2s_{12}}g^2 f^{12a}f^{34a} + \alpha^{IJ}\left(f^{13a}f^{24a} + f^{14a}f^{23a}\right) \\
&+\sigma^{IJM}\sigma^{IJM}\left(-\frac{1}{s_{13}}f^{31a}f^{24a} + \frac{1}{s_{14}}f^{23a}f^{14a}\right) \ .
\end{aligned} \quad (3.8)$$

The terms on the second line exhibit color/kinematics duality on their own (as they should, due to the dimensionful nature of σ^{IJK}) while a duality between color and kinematics for the terms on the first line requires that

$$\alpha^{IJ} = \frac{1}{2}g^2 \ , \quad (\forall) \ I, J \ . \quad (3.9)$$

Thus, the quartic scalar term of (3.5) must be such that it combines with the gauge field into the dimensional reduction of a higher-dimensional pure Yang-Mills theory.

It is possible to derive further constraints on the theory by examining the five-scalar amplitude

$$\begin{aligned}
\mathcal{A}_4^{\text{tree}}(1^{\phi^1}2^{\phi^2}3^{\phi^3}4^{\phi^3}5^{\phi^3}) =& g^2\sigma^{123}\left(\frac{(k_1+k_2-k_3)\cdot(k_4-k_5)+s_{45}}{2s_{12}s_{45}}f^{12a}f^{a3b}f^{b45}\right. \\
&+ \frac{(k_2+k_3-k_1)\cdot(k_4-k_5)+s_{45}}{2s_{12}s_{45}}f^{23a}f^{a1b}f^{b45} \\
&+ \left.\frac{(k_3+k_1-k_2)\cdot(k_4-k_5)+s_{45}}{2s_{12}s_{45}}f^{31a}f^{a2b}f^{b45}\right) + (3 \leftrightarrow 4) + (3 \leftrightarrow 5) \\
&+ (\sigma^{123})^3\frac{1}{s_{13}s_{25}}f^{13a}f^{a4b}f^{b52} + (3,4,5 \text{ permutations}) \ .
\end{aligned} \quad (3.10)$$

While the last line obeys color/kinematics duality, the terms proportional to $g^2\sigma^{123}$ do not. Thus, we must either require $g = 0$ and find the scalar theory of [33] (upon using the fact that σ^{IJK} obeys the Jacobi identity to set $\sigma^{IJK} = \sigma\tilde{f}^{IJK}$) or set $\sigma^{IJK} = 0$ and find the dimensional reduction of YM theory in $D_s = D + n_s$ dimensions.

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3.2.2 Four-dimensional theories with fermions

In the absence of additional deformations of the Lagrangian (3.5), inclusion of fermions coupling to all scalars as in (3.5) rules out the bosonic trilinear coupling. Indeed, the two-scalar-two-fermion amplitude with different scalars,

$$
\mathcal{A}_4^{\text{tree}}(1^{\phi^I}2^{\phi^J}3^{\psi^A}4^{\psi^B}) = \frac{[34]}{s_{12}}\sigma^{IJK}\lambda_{AB}^K f^{12a}f^{34a} \quad I \neq J , \tag{3.11}
$$

has a single color structure (a second color structure is forbidden by the absence of a $\langle\psi\psi\rangle$ tree-level two-point function) and thus cannot exhibit color/kinematics duality.

If a scalar ϕ^I is absent from the Yukawa couplings but interacts with gluons, then there is a one-gluon exchange four-point amplitude

$$
\mathcal{A}_4^{\text{tree}}(1^{\phi^I}2^{\phi^I}3^{\psi^A}4^{\bar{\psi}_A}) = \frac{[3|\,\slashed{k}_1 - \slashed{k}_2\,|4\rangle}{s_{12}} f^{12a}f^{34a} ; \tag{3.12}
$$

because it has a single color structure, this amplitude also cannot have color/kinematics duality. We therefore conclude that all scalars must interact at tree-level with fermions through Yukawa-type couplings.

To find the constraints on Yukawa couplings we need to examine the four-fermion and other two-scalar-two-fermion amplitudes with different scalars:

$$
\mathcal{A}_4^{\text{tree}}(1^{\psi^A}2^{\psi^B}3^{\bar{\psi}_C}4^{\bar{\psi}_D}) = -\frac{\langle 34\rangle}{\langle 12\rangle}\lambda_{AB}^I\bar{\lambda}^{ICD}f^{12a}f^{34a} + \frac{\langle 34\rangle^2}{\langle 13\rangle\langle 24\rangle}g^2\delta_A^C\delta_B^D f^{13a}f^{24a}
$$
$$
-\frac{\langle 34\rangle^2}{\langle 14\rangle\langle 23\rangle}g^2\delta_A^D\delta_B^C f^{14a}f^{23a} , \tag{3.13}
$$

$$
\mathcal{A}_4^{\text{tree}}(1^{\phi^I}2^{\phi^J}3^{\psi^A}4^{\bar{\psi}_B}) = \frac{\langle 14\rangle\langle 24\rangle}{\langle 12\rangle\langle 34\rangle}g^2\delta^{IJ}\delta_A^B f^{12a}f^{34a} + \frac{\langle 14\rangle}{\langle 13\rangle}\lambda_{AC}^I\bar{\lambda}^{JBC}f^{13a}f^{24a}
$$
$$
+\frac{\langle 24\rangle}{\langle 23\rangle}\lambda_{AC}^J\bar{\lambda}^{IBC}f^{23a}f^{14a} . \tag{3.14}
$$

Then, color/kinematics duality requires that

$$
\begin{aligned}
\lambda_{AB}^I\bar{\lambda}^{ICD} &= g^2(\delta_A^C\delta_B^D - \delta_B^C\delta_A^D) \equiv g^2\delta_{AB}^{CD} \\
\lambda_{AC}^I\bar{\lambda}^{JBC} + \lambda_{AC}^J\bar{\lambda}^{IBC} &= g^2\delta^{IJ}\delta_A^B ;
\end{aligned} \tag{3.15}
$$

in both equations the repeated indices (I and C, respectively) are summed over. To solve these equations we can consider each λ_{AB}^I for fixed A and B as a n_s dimensional complex vector; there are in all $\frac{(n_f-1)n_f}{2}$ such vectors. The first eq. (3.15) implies that each of them has norm g and they are orthogonal on each other. A solution for λ exists only if the number of components of these vectors is larger than the number of vectors, *i.e.*

$$
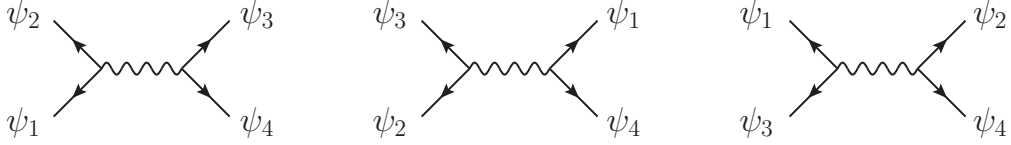n_s \geq \frac{1}{2}(n_f - 1)n_f . \tag{3.16}
$$

Figure 2: The Feynman graphs contributing to the four-fermion amplitude.

A relation between the number of scalars and fermions can be obtained by contracting the bosonic indices in the second eq. (3.15) and eliminating the left-hand side using the first eq. (3.15) with two contracted fermionic indices:

$$n_s = 2(n_f - 1) \ . \tag{3.17}$$

Equations (3.16) and (3.17) together imply that

$$0 \leq n_f \leq 4 \ . \tag{3.18}$$

Curiously, while we may define a four-dimensional field theory with an arbitrary number of fermions by dimensionally reducing D-dimensional YM theory coupled to one fermion, only for $D \leq 10$ it can exhibit color/kinematics duality. This suggests an interesting relation between this duality and supersymmetry.

For $n_f = 0, 1, 2, 4$ eqs. (3.15) can be solved explicitly and have unique solutions while for $n_f = 3$ no solution exists (see Appendix B for details). The resulting Lagrangians are those of $\mathcal{N} = n_f$ sYM theories or, equivalently, the dimensional reduction to four dimensions of minimal sYM theories in $D = 4, 6, 10$.

3.2.3 Single-fermion-coupled Yang-Mills theory in D dimensions

The results in the previous section suggest that it is interesting to explore pure gauge theories coupled to a single fermion in arbitrary dimensions D. The relevant Lagrangian is

$$L = \mathrm{Tr}\left[-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + i \bar{\psi}_A \, \slashed{D} \psi^A \right] \ . \tag{3.19}$$

It is not difficult to see that the four-gluon and two-gluon-two-fermion amplitudes obey color/kinematics duality.

The Feynman graphs contributing to the four-fermion amplitude $\mathcal{A}_4^{\mathrm{tree}}(1^\psi 2^\psi 3^\psi 4^\psi)$ are shown in fig. 2 and the amplitude is given by

$$\mathcal{A}_4^{\mathrm{tree}}(1^\psi 2^\psi 3^\psi 4^\psi) = \frac{(\bar{\eta}_1 \gamma_\mu \eta_2)(\bar{\eta}_3 \gamma^\mu \eta_4) f^{12a} f^{34a}}{s_{12}} + \frac{(\bar{\eta}_2 \gamma_\mu \eta_3)(\bar{\eta}_1 \gamma^\mu \eta_4) f^{23a} f^{14a}}{s_{14}}$$
$$+ \frac{(\bar{\eta}_3 \gamma_\mu \eta_1)(\bar{\eta}_2 \gamma^\mu \eta_4) f^{31a} f^{24a}}{s_{13}} \ , \tag{3.20}$$

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where η_i are spinor external state factors (*i.e.* solutions of the free Dirac equation).

The condition that $\mathcal{A}_4^{\text{tree}}(1^\psi 2^\psi 3^\psi 4^\psi)$ obeys color/kinematics duality constrains the external state spinors and the Dirac matrices to obey the identity

$$(\bar{\eta}_1 \gamma_\mu \eta_2)(\bar{\eta}_3 \gamma^\mu \eta_4) + (\bar{\eta}_2 \gamma_\mu \eta_3)(\bar{\eta}_1 \gamma^\mu \eta_4) + (\bar{\eta}_3 \gamma_\mu \eta_1)(\bar{\eta}_2 \gamma^\mu \eta_4) = 0 \ . \tag{3.21}$$

This is the well-known identity that appears in the supersymmetry transformation of the Lagrangian (3.19) and can be satisfied only for $D = 3, 4, 6, 10$. The appearance of this identity in the color/kinematics relation reinforces the idea that, in the presence of fermions, the duality is closely related to existence of supersymmetry.

4 Color/kinematics duality for orbifolds with bi-fundamental fields

In this section we define color/kinematics duality for orbifolds with fields in bi-fundamental representations. We will begin by discussing tree-level amplitudes and then proceed to loop-level amplitudes. We will then spell out the kinematic Jacobi relations for one-loop amplitudes. Amplitudes obeying these relations or their higher-loop counterparts are organized in terms of cubic graphs with each edge corresponding to a field with definite color and R charge; thus, each vertex in any given graph is associated to a unique gauge group factor of the orbifold theory. It is therefore straightforward to obtain the amplitudes of a quiver gauge theory which has the orbifold theory as a special point in its space of couplings by simply inspecting the color structures of various graphs and dressing each vertex with the desired coupling constant of the corresponding gauge group factor.

4.1 Tree-level amplitudes

It is well-known [12] that tree-level scattering amplitudes in orbifold field theories can be obtained directly from the amplitudes of the parent by simply attaching a projection operator (2.2) to each external line. Indeed, while a projector should formally be included for internal lines as well, their action is trivial as a consequence of the external line projection and of the symmetries of the parent theory [12] which iteratively fix all fields at each vertex.

This observation implies that for each internal line of the daughter amplitude there is a color Jacobi identity inherited from the parent by simply restricting the color indices to those present in the orbifold theory while not modifying the numerator factors. We can do this by introducing a *color-space wave functions* v_i^A for all external states; since the orbifold projection correlates the gauge group and R-symmetry (or more generally flavor-symmetry) indices, they obey

$$v_i^A = R_i \ g^{AB} v_i^B, \qquad (\forall)(r, g) \in \Gamma \ , \tag{4.1}$$

where R_i is the action/representation of a generic orbifold group element r on the i-th external particle. Without loss of generality, we assume that Γ is represented in $SU(4)$ by diagonal matrices; then, the factor R_i is given by the product of the relevant diagonal entries of r:

$$\Phi_i \equiv \Phi_{a_1 \ldots a_n} \quad \Rightarrow \quad R_i = r_{a_1}^{a_1} \ldots r_{a_n}^{a_n} \ . \tag{4.2}$$

For gauge fields, which are uncharged under R symmetry, $R_i = 1$ and v_i^A reduce to regular color space wave functions.

Let us illustrate this with a simple four-point tree-level amplitude. A color-dressed four-scalar amplitude in the $\mathcal{N} = 4$ sYM theory can be represented as

$$\mathcal{A}_4^{\text{tree}}(1^{\phi^{12}}, 2^{\phi^{23}}, 3^{\phi^{14}}, 4^{\phi^{34}}) = g^2 \left(\frac{c_s n_s}{s} + \frac{c_t n_t}{t} + \frac{c_u n_u}{u} \right) \ , \tag{4.3}$$

where g is the coupling constant, the upper indices of the arguments of $\mathcal{A}_4^{\text{tree}}$ label the three complex scalars as the representation $\mathbf{6}$ of $SU(4)$ and the color factors are

$$c_s = \tilde{f}^{A_1 A_2 B} \tilde{f}^{B A_3 A_4} \ , \qquad c_t = \tilde{f}^{A_1 A_4 B} \tilde{f}^{B A_2 A_3} \ , \qquad c_u = \tilde{f}^{A_1 A_3 B} \tilde{f}^{B A_4 A_2} \ . \tag{4.4}$$

The numerator factors are a solution of the equations

$$\frac{n_s}{s} - \frac{n_t}{t} = -i \frac{t}{s} \ , \qquad \frac{n_t}{t} - \frac{n_u}{u} = -i \frac{t}{s} \ , \qquad \frac{n_u}{u} - \frac{n_s}{s} = -i \frac{t^2}{su} \ , \tag{4.5}$$

and may be obtained through the supersymmetry Ward identities from the corresponding numerator factors of four-gluon amplitudes.

Orbifolding by a discrete group Γ with elements $(r, g) \in SU(4) \times SU(N)$, the color factors become

$$c_s = v_1^{A_1} v_2^{A_2} v_3^{A_3} v_4^{A_4} \tilde{f}^{A_1 A_2 B} \tilde{f}^{B A_3 A_4}, \qquad c_t = v_1^{A_1} v_2^{A_2} v_3^{A_3} v_4^{A_4} \tilde{f}^{A_1 A_4 B} \tilde{f}^{B A_2 A_3},$$
$$c_u = v_1^{A_1} v_2^{A_2} v_3^{A_3} v_4^{A_4} \tilde{f}^{A_1 A_3 B} \tilde{f}^{B A_4 A_2}, \tag{4.6}$$

where $v_i^{A_i}$ are solutions to the eq. (4.1) with $(r = \text{diag}(r_1^1, r_2^2, r_3^3, r_4^4))$

$$R_1 = r_1^1 r_2^2 \ , \quad R_2 = r_2^2 r_3^3 \ , \quad R_3 = r_1^1 r_4^4 \ , \quad R_4 = r_3^3 r_4^4 \quad \text{with} \quad r_1^1 r_2^2 r_3^3 r_4^4 = 1 \tag{4.7}$$

and a suitable choice of g representing the orbifold group element in $SU(N)$ (and breaking it to $SU(N_1) \times SU(N_2) \times \ldots$). The numerator factors are unchanged.

We note that it is in principle possible that some color factors vanish identically when contracted with the relevant color-space wave functions while, as we discussed, the corresponding kinematics numerator factors are unchanged. This does not imply a violation of color/kinematics duality since we can assign a non-zero kinematic numerator to the graph with vanishing color factor. A similar phenomenon occurs in the color/kinematics-satisfying representation of the four-loop $\mathcal{N} = 4$ sYM superamplitude [34], where a vanishing color factor is accompanied by a non-vanishing integrand (which makes a nontrivial contribution to the corresponding $\mathcal{N} = 8$ supergravity amplitude).

Upon projection to the orbifold invariant states the surviving color space graphs as well as the R charges of fields identify unambiguously which gauge group factor governs each cubic vertex; it is therefore straightforward to dress vertices with different couplings for each gauge group factor and thus deform the quiver theory off its orbifold point. An alternative strategy [8] with the same effect is to partition each (tree-level or more generally planar) graph into disconnected sectors which meet on the internal and external legs in bi-fundamental representation. All vertices in each such sector belong to a single gauge group and thus are dressed with the same coupling.

4.2   Loop-level amplitudes

An inspection of the Feynman rules quickly reveals that at loop level it is possible to remove all but one of the projectors acting on internal lines for each independent loop. This should be expected because, unlike tree amplitude, loop amplitudes are not in general inherited from the parent theory[9]. To construct Jacobi relations with respect to the projected internal line we begin by making two observations: (1) the position of the projection operator is not fixed and can be changed by making use of the Γ-invariance of vertices; (2) while moving the projector from one line to another, terms corresponding to different elements of Γ are mapped into each other; this is a consequence of *e.g.* R-charge conservation at each vertex.

The first observation implies that it is always possible to make sure that the three graphs related by a color Jacobi relation are such that the internal lines participating in the relation are projector-free. The second observation suggests that each color-space graph with a different insertion of the orbifold group element should be treated as a distinct graph.

In the following we will assign canonically the projector to the internal line carrying the independent loop momentum. With this labeling the amplitude has the form

$$\mathcal{A}^{(L)} = \int \prod_{k=1}^{L} \frac{d^d l_k}{(2\pi)^d} \frac{1}{|\Gamma|} \sum_{(r_k, g_k) \in \Gamma} \sum_{R_{l_k} \in \mathcal{R}} \sum_{i \in \mathcal{G}_3} \frac{1}{S_i} \frac{n_{i;R_{l_1},\ldots,R_{l_L}} c_{i;R_{l_1},\ldots,R_{l_L}}}{\prod_m p_{m,i}^2} , \qquad (4.8)$$

where as in eq. (3.1) the summation index i runs over all cubic graphs $\mathcal{G}_3$ (which includes all possible permutations of external legs) and the symmetry factor S_i removes the overcount due to the symmetries of the graph[10]. $R_{l_1}, \ldots, R_{l_L}$ are the representations of the orbifold group element $r_1, \ldots, r_L$ inside $SU(4)$ corresponding to the fields carrying the independent loop momenta $l_1, \ldots, l_L$. The set of all representations that can appear in each loop is denoted by $\mathcal{R}$. From a physical perspective, the numerator factor $n_{i;R_{l_1},\ldots,R_{l_L}}$ receives

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[8]We thank Lance Dixon for suggesting it.

[9]Inheritance is limited to planar amplitudes in theories with a regular orbifold action [12].

[10]Alternatively, one may sum only over the inequivalent cubic graphs.

contributions from the fields [11] with representations $R_{l_1}, \ldots, R_{l_L}$ running in the loop $1, \ldots, L$ while the summation over all R_{l_k} is equivalent to the summation over all the fields. Since Γ is assumed to be abelian, R_{l_k} are just phases (see eq. (4.7) for an example).

The color factors are related – but not identical – to the ones of the parent theory: as in the parent theory, to each vertex of the cubic graph is assigned a factor of the structure constant of the parent gauge group and their indices are contracted following the edges of the graph: those corresponding to the edges carrying projectors are contracted with g_k^{AB} defined in eq. (2.3), while all the others with δ^{AB}. Finally, we include an additional overall factor of $\prod_{k=1}^{L} R_{l_k}$ and a color wave function (4.1) for each external leg. It should be noted that, for each choice of orbifold group element, there are as many different color factors as elements of $\mathcal{R}$. Moreover, the numerator factors depend on the orbifold group element only through $R_{l_k} \in \mathcal{R}$, so that all the numerators corresponding to the same representation of the orbifold group are identical.

With these preparation we can now describe the construction of the kinematic Jacobi relations for an amplitude of the form (4.8):

1. Parametrize all graphs by solving momentum conservation and write out the color factors by assigning gauge group orbifold elements g_k^{AB} to the edges carrying the independent loop momenta. The R charge flow is aligned with the momentum flow.

2. Choose a graph and an edge of this graph.

3. If this edge does not carry the gauge group orbifold element g^{AB}, proceed to the next step. If it does, move it on the adjacent edges meeting the chosen one at a vertex using the identity[12],

$$g^{AA'} g^{BB'} g^{CC'} \tilde{f}^{A'B'C'} = \tilde{f}^{ABC} \quad \Leftrightarrow \quad g^{AA'} \tilde{f}^{A'B'C'} = \tilde{f}^{ABC} (g^\dagger)^{B'B} (g^\dagger)^{C'C} . \quad (4.9)$$

We note that in this equation all g matrices correspond to the same orbifold group element. To help keep track of the R factors it is useful to split the initial R into a product of two factors, each corresponding to the edges carrying the new g factors.

4. Use the Jacobi identity of the parent theory for the chosen edge and write the initial color factor as a sum of color factors associated to two other graphs.

---

[11]Without imposing color/kinematics duality there are only physical field contributions. However, when the duality is imposed, we find that we need to introduce representations which may be related to auxiliary fields; this is not surprising from the perspective of a Lagrangian that produces color/kinematics-satisfying Feynman rules. All terms in such a Lagrangian are cubic so there are many auxiliary fields. See *e.g.* [16] for a few terms in such a putative Lagrangian.

[12]This identity can be proven using (2.3) to show that $g T^A g^\dagger = g^{AB} T^B$ and expressing the structure constants as $\tilde{f}^{ABC} = \text{Tr}([T^A, T^B] T^C)$.

5. Bring the momentum assignment and the two color factors to the canonical form chosen at step 1 by repeatedly using the identity (4.9) and the defining property of the color wave-functions (4.1) (or, equivalently, R-change conservation).

6. The corresponding kinematic Jacobi relation involves the kinematic numerator factors of the original color structure as well as of the two color structures obtained at step 5.

7. Go back to step 2.

Several comments are in order regarding steps 5 and 6. In the process of rearranging the adjoint orbifold group elements at step 5 several such elements will be multiplied and it may be possible to simplify the product by using the defining relations of the orbifold group (*e.g.* for $\Gamma = \mathbf{Z}_n$ and $k < n$ we have $g^{n+k} \simeq g^k$). One may choose the numerator factor of such a graph in at least two different ways. On the one hand one can use these relations to simplify all products of orbifold group elements and simply read off the coefficient of the resulting color factor coefficient from step (1). On the other hand, one may interpret Γ as part of a (much) larger discrete group $\hat{\Gamma}$; if the rank of $\hat{\Gamma}$ is sufficiently large and we seek numerator factors which depend on $\hat{\Gamma}$ only through the representations $\mathcal{R}$ of Γ, the defining relations of the orbifold group need not be used. In the latter case, one aims to find a minimal set of representations of the orbifold group $\mathcal{R}$ for which the numerator factors are non-zero and solve the generalized kinematic Jacobi relations. We shall choose this second perspective.

It is moreover possible that there are several ways to bring to the canonical form the color factors obtained at step 4; they can differ only by elements of Γ which are trivial upon use of its defining relations. In such cases it is necessary to impose all variants of the kinematic Jacobi relations.

As at tree level, after contraction with the external color wave-functions and summation over the orbifold group elements, it is always possible to use the representations of the remaining fields and their R charges to identify the gauge group factor governing each vertex of a given graph. It is then straightforward to change the couplings off the natural line and thus find loop amplitudes of the quiver gauge theory at a generic value of its couplings.

Let us now illustrate this construction and write out the kinematic Jacobi relations for the one-loop four-point amplitudes in general abelian orbifold theories; we will use them in section 5 to construct amplitudes in $\mathcal{N} = 2$, $\mathcal{N} = 1$ and $\mathcal{N} = 0$ orbifold theories with fields in bi-fundamental representations. To this end we will require the vanishing of the numerator factors of graphs containing too high a power of the orbifold group element.
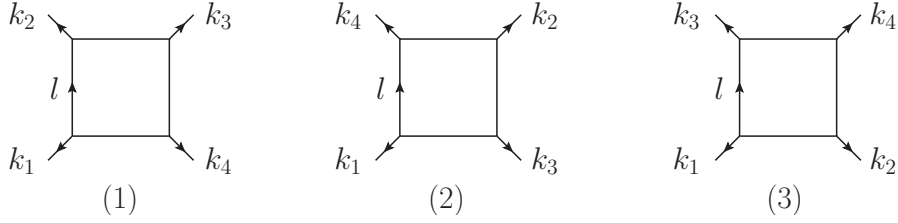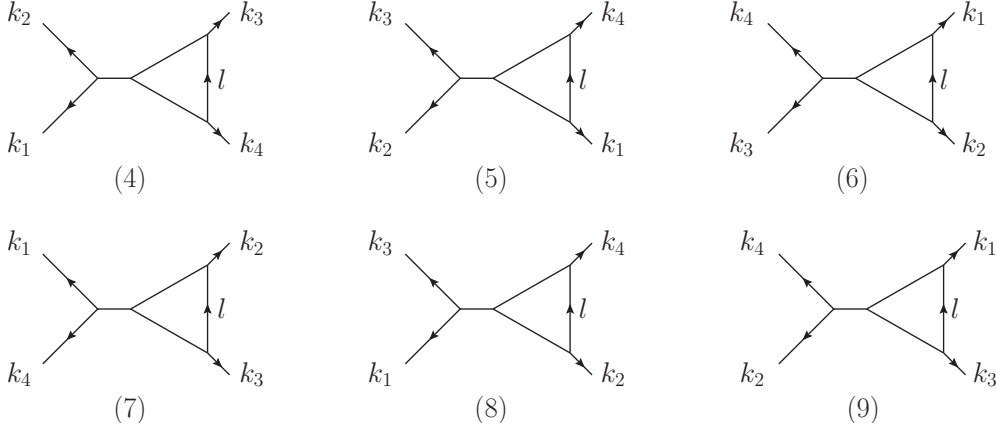
Figure 3: Basis of box integrals at one loop.

Figure 4: Basis of triangle integrals at one loop.
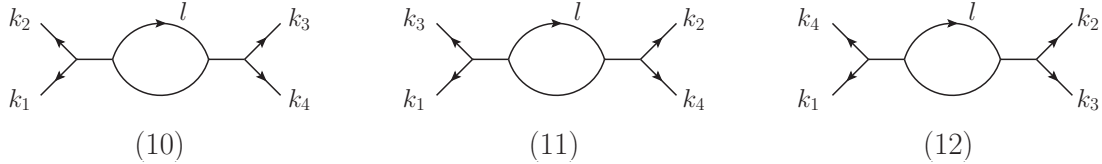
Figure 5: Basis of bubble integrals at one loop.

4.3 A detailed one-loop example

For the one-loop four-point amplitudes we can choose a basis of cubic graphs with box, triangle and bubble integrals shown in figures 3-5;[13] the figures also indicate the internal leg carrying the loop momentum. The color factor of each graph is constructed from a structure constant for each vertex contracted with δ^{AB} or g^{AB}. We list here the ones

[13]There are twelve additional bubble-on-external-line ("snail") graphs shown schematically in fig. 6 as well as fifteen tadpole graphs which we do not draw explicitly. The numerator factors of the former will appear in the kinematic Jacobi relations; they are labeled from $n_{13} - n_{24}$.

associated to the first, third and fourth graphs:

$$c_{1;R_l} = R_l\, v_1^{A_1} v_2^{A_2} v_3^{A_3} v_4^{A_4}\; \tilde{f}^{A_1 A_5 A_9} g^{A_5 A_6} \tilde{f}^{A_2 A_7 A_6} \tilde{f}^{A_3 A_8 A_7} \tilde{f}^{A_4 A_9 A_8}\ , \qquad (4.10)$$

$$c_{3;R_l} = R_l\, v_1^{A_1} v_2^{A_2} v_3^{A_3} v_4^{A_4}\; \tilde{f}^{A_1 A_5 A_9} g^{A_5 A_6} \tilde{f}^{A_3 A_7 A_6} \tilde{f}^{A_4 A_8 A_7} \tilde{f}^{A_2 A_9 A_8}\ , \qquad (4.11)$$

$$c_{4;R_l} = R_l\, v_1^{A_1} v_2^{A_2} v_3^{A_3} v_4^{A_4}\; \tilde{f}^{A_1 A_2 A_5} \tilde{f}^{A_5 A_6 A_9} \tilde{f}^{A_4 A_9 A_8} g^{A_8 A_7} \tilde{f}^{A_3 A_7 A_6}\ ; \qquad (4.12)$$

as stated in the previous section, the adjoint element of the orbifold group g has been inserted on the internal line carrying the loop momentum. Also, the direction of the R-charge flow is aligned with the momentum flow.

To illustrate the construction of the Jacobi identities let us choose the edge carrying the loop momentum in graph 1, *i.e.* the line labeled A_5 in eq. (4.10). Since this edge also carries g we must move it on the adjacent lines. Using eq. (4.9) we have

$$\begin{aligned}
c_{1;R_l} &= R_l\, v_1^{A_1} v_2^{A_2'} v_3^{A_3} v_4^{A_4}\, (g^\dagger)^{A_2' A_2} (g^\dagger)^{A_7 A_6} \tilde{f}^{A_1 A_5 A_9} \tilde{f}^{A_2 A_6 A_5} \tilde{f}^{A_3 A_8 A_7} \tilde{f}^{A_4 A_9 A_8} \\
&= \frac{R_l}{R_2}\, v_1^{A_1} v_2^{A_2} v_3^{A_3} v_4^{A_4}\; \tilde{f}^{A_1 A_5 A_9} \tilde{f}^{A_2 A_6 A_5} g^{A_6 A_7} \tilde{f}^{A_3 A_8 A_7} \tilde{f}^{A_4 A_9 A_8}\ . \qquad (4.13)
\end{aligned}$$

In the second line we have used the defining property of the color wave-functions (4.1) as well as $(g^\dagger)^{AB} = g^{BA}$ which is a consequence of the reality of the adjoint representation.

The next step is to use the Jacobi relations on the internal line which is now free from the orbifold element g:

$$\begin{aligned}
c_{1;R_l} &= \frac{R_l}{R_2}\, v_1^{A_1} v_2^{A_2} v_3^{A_3} v_4^{A_4}\, (\tilde{f}^{A_2 A_5 A_9} \tilde{f}^{A_1 A_6 A_5} + \tilde{f}^{A_1 A_2 A_5} \tilde{f}^{A_5 A_6 A_9}) g^{A_6 A_7} \tilde{f}^{A_3 A_8 A_7} \tilde{f}^{A_4 A_9 A_8} \\
&= \frac{R_l}{R_2}\, v_1^{A_1} v_2^{A_2} v_3^{A_3} v_4^{A_4}\; \tilde{f}^{A_2 A_5 A_9} \tilde{f}^{A_1 A_6 A_5} g^{A_6 A_7} \tilde{f}^{A_3 A_8 A_7} \tilde{f}^{A_4 A_9 A_8} \\
&\quad + \frac{R_l}{R_2 R_3}\, v_1^{A_1} v_2^{A_2} v_3^{A_3} v_4^{A_4}\; \tilde{f}^{A_1 A_2 A_5} \tilde{f}^{A_5 A_6 A_9} \tilde{f}^{A_4 A_9 A_8} (g^\dagger)^{A_8 A_7} \tilde{f}^{A_3 A_7 A_6} \qquad (4.14)
\end{aligned}$$

where in the triangle graph we have moved the group element past the external line with momentum k_3 in order to have it in the canonical position, on the line carrying the loop momentum, while keeping the index contraction (or equivalently the R-charge flow) as in eq. (4.12). In terms of the color factors $c_{i,R}$ eq. (4.14) is

$$c_{1;R_l} = c_{3;R_l/R_2} + c_{4;R_2 R_3/R_l}, \qquad (4.15)$$

where the inverse R factor compared to (4.14) accounts for the presence of $g^\dagger$ in that equation.

The calculation in equations (4.13), (4.14) and (4.15) is shown pictorially in fig. 7. From there or from eq. (4.15) we read off the corresponding kinematic Jacobi relation:

$$n_{1;R_l}(l) = n_{3;R_l/R_2}(l - k_2) + n_{4;R_2 R_3/R_l}(k_2 + k_3 - l)\ . \qquad (4.16)$$

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Repeating the same steps we derive the kinematic Jacobi relations for all the internal edges of all graphs in figs. 3-5; some of them involve the numerator factors of snail graphs, labeled $n_{13} - n_{24}$ and corresponding to the graphs in fig. 6:

$$- n_{1;R_2 R_3/R_l}(k_2 + k_3 - l) + n_{3;R_3/R_l}(k_3 - l) + n_{4;R_l}(l) = 0 \,, \tag{4.17}$$

$$n_{4;R_3/(R_4 R_l)}(k_3 - k_4 - l) + n_{4;R_l}(l) + n_{10;R_4 R_l}(l + k_4) = 0 \,, \tag{4.18}$$

$$n_{4;R_l}(l) + n_{4;R_3/R_l}(k_3 - l) - n_{16;R_l/R_3}(l - k_3) = 0 \,, \tag{4.19}$$

$$n_{4;R_l}(l) + n_{4;1/(R_4 R_l)}(-k_4 - l) + n_{22;1/(R_4 R_l)}(-k_4 - l) = 0 \,, \tag{4.20}$$

$$- n_{1;1/(R_1 R_l)}(-k_1 - l) + n_{2;R_l}(l) + n_{5;R_l}(l) = 0 \,, \tag{4.21}$$

$$n_{5;R_l}(l) + n_{5;R_4/(R_1 R_l)}(k_4 - k_1 - l) - n_{12;1/(R_1 R_l)}(-k_1 - l) = 0 \,, \tag{4.22}$$

$$n_{5;R_l}(l) + n_{5;R_4/R_l}(k_4 - l) - n_{21;R_l/R_4}(l - k_4) = 0 \,, \tag{4.23}$$

$$n_{5;R_l}(l) + n_{5;1/(R_1 R_l)}(-k_1 - l) + n_{15;1/(R_1 R_l)}(-k_1 - l) = 0 \,, \tag{4.24}$$

$$- n_{1;1/R_l}(-l) + n_{3;R_l/R_1}(l - k_1) + n_{6;R_l}(l) = 0 \,, \tag{4.25}$$

$$n_{6;R_1/(R_2 R_l)}(k_1 - k_2 - l) + n_{6;R_l}(l) + n_{10;1/(R_2 R_l)}(-k_2 - l) = 0 \,, \tag{4.26}$$

$$n_{6;R_l}(l) + n_{6;R_1/R_l}(k_1 - l) - n_{13;R_l/R_1}(l - k_1) = 0 \,, \tag{4.27}$$

$$n_{6;R_l}(l) + n_{6;1/(R_2 R_l)}(-l - k_2) + n_{19;1/(R_2 R_l)}(-l - k_2) = 0 \,, \tag{4.28}$$

$$- n_{1;R_2/R_l}(k_2 - l) + n_{2;R_2 R_4/R_l}(k_2 + k_4 - l) + n_{7;R_l}(l) = 0 \,, \tag{4.29}$$

$$n_{7;R_l}(l) + n_{7;R_2/(R_3 R_l)}(k_2 - k_3 - l) - n_{12;R_3 R_l}(l + k_3) = 0 \,, \tag{4.30}$$

$$n_{7;R_l}(l) + n_{7;R_2/R_l}(k_2 - l) + n_{18;R_l/R_2}(l - k_2) = 0 \,, \tag{4.31}$$

$$n_{7;R_l}(l) + n_{7;1/(R_3 R_l)}(-k_3 - l) - n_{24;1/(R_3 R_l)}(-k_3 - l) = 0 \,, \tag{4.32}$$

$$n_{2;R_4/R_l}(k_4 - l) - n_{3;R_3 R_4/R_l}(k_3 + k_4 - l) + n_{8;R_l}(l) = 0 \,, \tag{4.33}$$

$$n_{8;R_l}(l) + n_{8;R_4/(R_2 R_l)}(k_4 - k_2 - l) - n_{11;R_2 R_l}(l + k_2) = 0 \,, \tag{4.34}$$

$$n_{8;R_l}(l) + n_{8;R_4/R_l}(k_4 - l) - n_{23;R_l/R_4}(l - k_4) = 0 \,, \tag{4.35}$$

$$n_{8;R_l}(l) + n_{8;1/(R_2 R_l)}(-k_2 - l) + n_{17;1/(R_2 R_l)}(-k_2 - l) = 0 \,, \tag{4.36}$$

$$- n_{2;R_l/R_1}(l - k_1) + n_{3;1/R_l}(-l) + n_{9;R_l}(l) = 0 \,, \tag{4.37}$$

$$n_{9;R_l}(l) + n_{9;R_1/(R_3 R_l)}(k_1 - k_3 - l) + n_{11;1/(R_3 R_l)}(-k_3 - l) = 0 \,, \tag{4.38}$$

$$n_{9;R_l}(l) + n_{9;R_1/R_l}(k_1 - l) - n_{14;R_l/R_1}(l - k_1) = 0 \,, \tag{4.39}$$

$$n_{9;R_l}(l) + n_{9;1/(R_3 R_l)}(-k_3 - l) + n_{20;1/(R_3 R_l)}(-k_3 - l) = 0 \,. \tag{4.40}$$

Further identities, relating bubble to tadpole graph numerators, can also be constructed. To require that the latter numerators vanish identically it suffices to constrain the numerators of the former to obey the identities

$$n_{i;R_l}(l) - n_{i;1/R_l}(-l) = 0 \qquad i \geq 10 \,. \tag{4.41}$$

In the next section we will use the relations (4.17)-(4.40) and (4.41) for orbifold groups preserving $\mathcal{N} = 2$, $\mathcal{N} = 1$ and $\mathcal{N} = 0$ supersymmetry and for several choices of external
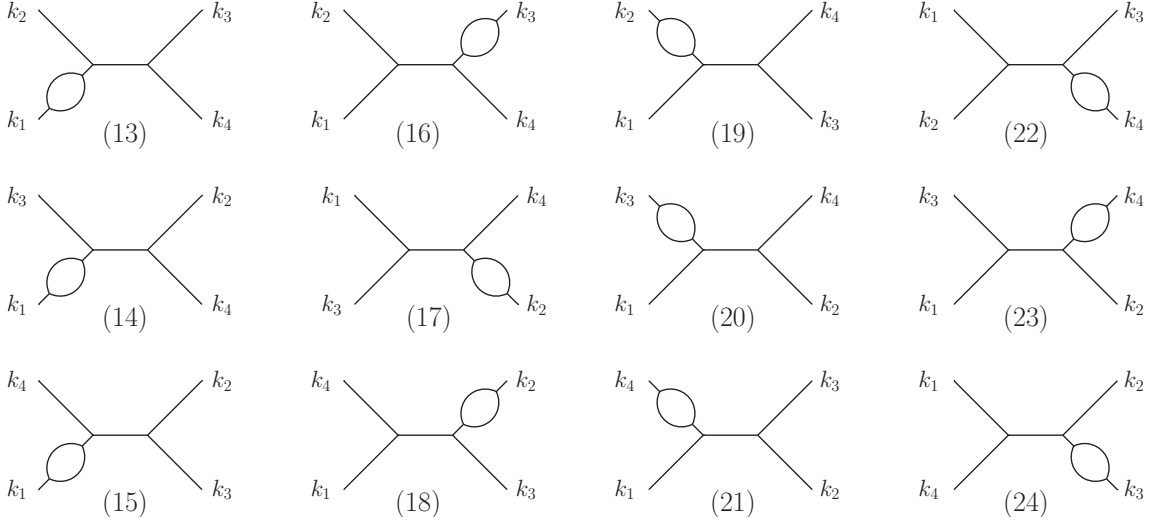
Figure 6: Bubble-on-external-line (snail) contributions to one-loop four-point amplitudes.
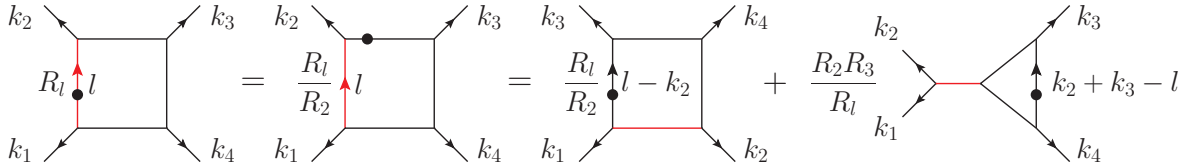
Figure 7: Example of Jacobi identity in the orbifold theory. The dot marks the internal line where the adjoint group element is inserted.

states. It should also be noted that when the external states are taken to be neutral under the orbifold group, the generalized Jacobi identities corresponding to the different particles going around the loop decouple and can be solved independently. The factors $n_{i;1}$ (and their higher-loop generalizations $n_{i;1\ldots1}$) may be of particular interest as they describe the amplitudes of the pure (s)YM theories with the same amount of supersymmetry as preserved by the orbifold group.

We note that the kinematic Jacobi relations – and consequently the kinematic numerators – have very limited information on the details of the orbifold group; in the color/kinematics-based organization of amplitudes changing the orbifold group (and thus the field content of the theory) amounts solely to changing the color factors while keeping the kinematic numerators fixed.

5 Direct computations at one loop

To construct examples of amplitudes in orbifold theories whose integrands obey the kinematic Jacobi relations constructed in the previous section it is perhaps useful to proceed as in the case of the $\mathcal{N} = 4$ theory and first solve them in terms of master graphs. It is not difficult to see that, as in the case of pure $\mathcal{N} = 2$ and $\mathcal{N} = 1$ theories, a possible choice of master graphs is given by the box integrals with all possible orbifold group insertions. This is however not the minimal set. Due to our organization of the calculation – such that the projection to the orbifold spectrum is effectively done only upon the summation over all orbifold group elements – as well as due to the fact that all information on the orbifold group is contained in the color factors, the box integral kinematic numerator factors n_{1,R_l}, n_{2,R_l} and n_{3,R_l}, are closely related to those of the parent $\mathcal{N} = 4$ sYM theory: summing them over all orbifold group elements should yield the kinematic numerator factors of a color/kinematic-satisfying representation of $\mathcal{N} = 4$ sYM amplitude. This constraint determines three box integral numerators in terms of the other ones. In some cases, the number of master graphs can be further reduced by demanding that the amplitudes of a theory with reduced supersymmetry reproduce as particular cases the known amplitudes of a theory with higher number of supersymmetries.

In each of the explicit calculations that we will discuss we use an ansatz in which the numerator factors are polynomials in the Mandelstam variables s, t and u and in the products of external and loop momenta $\tau_{il} = k_i \cdot l$. When the numerator factors are not expected to be manifestly local (*e.g.* due to the presence of polarization vectors in the spinor-helicity basis) inverse powers of the Mandelstam variables are also introduced. The degree of the polynomials and the maximum number of loop momenta depend on the residual amount of supersymmetry and on the choice of external legs. We then take the following steps to obtain[14] amplitude presentations with manifest color/kinematics duality.

1. Solve $(4.17 - 4.40)$; it turns out that imposing absence of tadpoles through eq. (4.41) is not always possible so these equations are not imposed. Construct an ansatz for the master graphs.

2. Fix the free coefficients of the ansatz by imposing that they reproduce the correct s-, t- and u-channel cuts. To evaluate (generalized) cuts it is useful to use the $\mathcal{N} = 4$ on-shell superamplitudes weighted with the appropriate orbifold group elements, as suggested by the inheritance properties of tree-level amplitudes. For example, the

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[14]The integrands we present in this section are correct up to snail integrands and, in the non-supersymmetric examples, up to rational terms. For massless external particles, the snail integrands integrate to zero in dimensional regularization and thus, for $\mathcal{N} \geq 1$, the amplitudes we find are complete.

s-channel supercut is given by

$$
\mathcal{F}_4^{1-\text{loop}}\Big|_{s_{12}} = \sum_{(r,g)\in\Gamma} \Big(\frac{\tilde{f}^{A_1 CD}\tilde{f}^{DBA_2}}{\langle k_2 l_2\rangle\langle l_1 k_1\rangle} - \frac{\tilde{f}^{A_1 BD}\tilde{f}^{DCA_2}}{\langle k_2 l_1\rangle\langle l_2 k_1\rangle} \Big) g^{BB'} \Big(\frac{\tilde{f}^{A_3 B'E}\tilde{f}^{ECA_4}}{\langle k_4 l_2\rangle\langle l_1 k_3\rangle} - \frac{\tilde{f}^{A_3 CE}\tilde{f}^{EB'A_4}}{\langle k_4 l_1\rangle\langle l_2 k_3\rangle} \Big)
$$

$$
\frac{\delta^4\big(\sum_{i=1}^4\langle l_1 i\rangle\eta_i^a\big)\delta^4\big(\sum_{i=1}^2\langle l_2 i\rangle\eta_i^a + r_a^a\sum_{i=3}^4\langle l_2 i\rangle\eta_i^a\big)}{\langle k_1 k_2\rangle\langle k_3 k_4\rangle\langle l_1 l_2\rangle^2} \,, \tag{5.1}
$$

while the other supercuts can be obtained by relabeling the external legs. The evaluation of the cut yields a polynomial in the diagonal entries r_a^a of the $SU(4)$ matrices representing the orbifold group. Since changing the orbifold group amounts to changing only r_a^a, the coefficient of each independent monomial of the cuts of the ansatz must match the corresponding coefficients in their direct evaluation.

3. Require that in D dimensions the snail integrals do not have any $1/\mu^2$ pole when an infrared regulator μ is introduced[15]. This condition is necessary to ensure that snail graphs can be included in the presentation of the amplitude (since in these graphs one of the internal lines produces a factor of $1/\mu^2$) and is implemented through the integral reduction

$$
\int \frac{l^\mu}{l^2(l+k_i)^2} \to -\frac{1}{2}\int \frac{k_i^\mu}{l^2(l+k_i)^2} \,, \quad \int \frac{l^\mu l^\nu}{l^2(l+k_i)^2} \to \frac{D}{4(D-1)}\int \frac{k_i^\mu k_i^\nu}{l^2(l+k_i)^2} + \mathcal{O}(\mu^2) \,. \tag{5.2}
$$

and using some of the free coefficients to set to zero the terms proportional to $1/\mu^2$. Alternatively, one may simply fix the contribution of snail graphs by requiring that the amplitude's UV divergence is governed by the beta function(s) of the theory and/or that the IR divergences have the expected form, cf. *e.g.* [35].

4. The remaining free parameters, which drop out upon reduction to master integrals, correspond to either redundancies of the ansatz, different representation of the amplitude related by the orbifold version of the generalized gauge transformations [5], or to parts of the ansatz that integrate to zero and are not fixed by the standard two-particle cuts. Some of them are fixed by requiring manifest Bose/Fermi symmetry of the integrand with respect to permutation of external data. Moreover the remaining free coefficients are chosen to set to zero the numerators of as many bubble and triangle graphs as possible. Last, the residual coefficients can be set to whatever values bring the integrand to one's subjectively chosen simplest form[16].

Let up now follow these steps and construct the four-gluon and four-scalar amplitudes in $\mathcal{N}=2$, $\mathcal{N}=1$ and $\mathcal{N}=0$ supersymmetric orbifold theories.

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[15]This infrared regulator should not be confused with the dimensional regularization parameter with a similar notation.

[16]Reduction to an integral basis can be used to show that the amplitude is independent of these coefficients through $\mathcal{O}(\epsilon^0)$.

5.1 Four-gluon amplitudes

5.1.1 Four-gluon amplitudes with $\mathcal{N} = 2$ supersymmetry

The theories with the simplest orbifold group action have $\mathcal{N} = 2$ supersymmetry; the case of amplitudes with external gluons is particularly easy. Without any loss of generality we take the gluons of momenta k_1 and k_2 to have negative helicity (and the other two of positive helicity) and the diagonal $SU(4)$ matrices r to have the first two entries equal to unity:

$$r = \mathrm{diag}(1, 1, r_3^3, r_4^4) \ . \tag{5.3}$$

Because gluons are uncharged under $SU(4)$, the phase factors R_i representing the orbifold group action on the R-symmetry indices of the external lines are trivial,

$$R_1 = R_2 = R_3 = R_4 = 1 \ . \tag{5.4}$$

Requiring that r is an $SU(4)$ element, the phase factors R_l that capture the action of the orbifold group on internal lines can be

$$R_l \in \mathcal{R} = \left\{ 1, r_3^3, r_4^4 \right\} \ . \tag{5.5}$$

Thus, the amplitude contains three copies of each graphs in figures 3-5, each dressed with a different color factor, cf. discussion in secs. 4.2 and 4.3.

As mentioned previously, the numerator factors of triangle and bubble graphs are determined in terms of those of box graphs through the kinematic Jacobi relations (4.17-4.40). Further constraints on the numerator factors n_{i,r_3^3} and n_{i,r_4^4} follow from the S_2 subgroup of $SU(4)$ that interchanges the R-symmetry indices 3 and 4 on all fields. The unit determinant constraint on r is invariant under this transformation, which interchanges r_3^3 and r_4^4. We thus expect that

$$n_{i;r_3^3} = n_{i;r_4^4} \tag{5.6}$$

for all $i = 1, \ldots, 12$. Moreover, the fact that n_{i,R_l} receives contributions from fields in the representation R_l of the orbifold group implies that, summing over all representations while setting $r = 1$ should lead to the numerator factor of the same graph in $\mathcal{N} = 4$ sYM theory:

$$n_{i;1} + 2n_{i;r_3^3} = n_i^{\mathcal{N}=4}, \qquad i = 1, 2, 3 \ ; \tag{5.7}$$

This brings the number of independent master graphs down to three – the three box graphs with unit orbifold group element insertion. Last but not least, due to the large amount of supersymmetry preserved by the orbifold group it turns out that it is possible to require that two triangle and one bubble diagrams have vanishing numerator factors,

$$n_{i;R_l} = 0 \quad \text{for } i = 4, 6, 10, \ (\forall) \ R_l \ . \tag{5.8}$$

We will use the following ansatz for the numerator factors of the three master graphs:

$$n_{i;R_l}(l) = i\hat{n}_{i;R_l}(l)\langle k_1 k_2\rangle^2 [k_3 k_4]^2, \quad \hat{n}_{i;R_l} = \frac{\mathcal{P}_{4;1}\big(s,t,\tau_{1l},\tau_{2l},\tau_{3l},l^2\big)}{s^2 tu} + i\frac{\mathcal{P}_2\big(s,t\big)\epsilon(k_1,k_2,k_4,l)}{s^2 tu}.$$
(5.9)

The polynomial $\mathcal{P}_{4;1}$ is of degree four in all its arguments and of degree up to one in the last four arguments, $\tau_{il} = k_i \cdot l$ and l^2, while $\mathcal{P}_2$ is a polynomial of degree two in its arguments. These polynomials are different for different graphs and orbifold phase R_l. As clarified in [14] for the case of the color/kinematics-satisfying amplitudes in pure $\mathcal{N} = 1$ and $\mathcal{N} = 2$ sYM theories found in [13], the non-locality is presumably due to the choice of helicity basis for the external gluons.

Matching the s-, t- and u-channel cuts we find that the master graphs' "reduced" numerator factors $\hat{n}_{i,1}$ are

$$\hat{n}_{1;1}(l) = 1 - \frac{2}{3}\frac{\tau_{1l} - \tau_{2l} + l^2}{s},$$
(5.10)

$$\hat{n}_{2;1}(l) = 1 + 2u\frac{\tau_{1l} + \tau_{2l}}{s^2} + \frac{u + 2\tau_{2l} - 2\tau_{3l}}{3s} - \frac{2l^2}{3s} + 4i\frac{\epsilon(k_1,k_2,k_4,l)}{s^2},$$
(5.11)

$$\hat{n}_{3;1}(l) = \frac{2}{3} - \frac{2}{3}\frac{\tau_{1l} + \tau_{2l} + l^2}{s}.$$
(5.12)

The numerator factors of the box graphs dressed with orbifold group elements follow from eq. 5.7

$$\hat{n}_{i;r_3^3} = \hat{n}_{i;r_4^4} = \frac{1}{2}(1 - \hat{n}_{i;1}) \qquad i = 1,2,3,$$
(5.13)

while the numerator factors for triangle and bubble graphs are determined from the relations (4.17)-(4.40). The snail and tadpole graphs have vanishing numerators, as expected. The resulting amplitude can be integrated without difficulty, $e.g.$ by first reducing it to an integral basis and using the known expressions for the basis elements.

To obtain the expressions above we have also imposed that the integrand respects Bose symmetry, $i.e.$ it is invariant under the exchange of the external particles of momenta k_1 and k_2. This further constrains the numerator factors to be a solution of

$$\hat{n}_{1;R_l}(l) = \hat{n}_{3;R_l}(l - k_1)\big|_{k_1 \leftrightarrow k_2}, \qquad \hat{n}_{2;R_l}(l) = \hat{n}_{2;1/R_l}(k_4 - l)\big|_{k_1 \leftrightarrow k_2}.$$
(5.14)

In general, this exchange symmetry holds for integrated amplitudes. We note certain similarities between $\hat{n}_{1;1}(l), \hat{n}_{2;1}(l)$ and $\hat{n}_{3;1}(l)$ and the corresponding numerator factors for the four-gluon amplitude in pure $\mathcal{N} = 2$ sYM theory [13][17]. The fact that Bose symmetry can also be imposed in that case was pointed out in [36] and takes the number of master graphs down to two.

---

[17]Note that, unlike the one presented in this section, the expressions of [13] were obtained setting to zero all three bubble numerators.

We note that this particular amplitude could have been obtained without any calculation. As we discussed, we choose the master integrals to be those corresponding to graphs with the insertion of the unit element of Γ. In our organization of the amplitude (4.8) these numerators receive contributions from fields in the trivial representation of Γ on $SU(4)$. These fields are simply those of $\mathcal{N}=2$ sYM theory, and thus one should have

$$\hat{n}_{i;1} = \hat{n}_i^{\mathcal{N}=2} = 1 - 2\hat{n}_i^{\text{chiral}} \qquad \text{and} \qquad \hat{n}_{i;r_3^3} = \hat{n}_{i;r_4^4} = \hat{n}_i^{\text{chiral}} , \qquad (5.15)$$

where $\hat{n}_i^{\text{chiral}}$ is the contribution of a single chiral multiplet in the loop, denoted by N_i^{chiral} in [13].

5.1.2 Four-gluons amplitudes with $\mathcal{N}=1$ supersymmetry

Four-gluon amplitudes in $\mathcal{N}=1$ orbifold theories can be constructed by following the same steps. As before, we shall choose the gluons carrying momenta k_1 and k_2 to have negative helicity and the other two, with momenta k_3 and k_4, to have positive helicity. Without loss of generality we can choose the representation of the orbifold r matrix inside the R-symmetry group, $r \in SU(4)$, to be

$$r = \text{diag}(1, r_2^2, r_3^3, r_4^4) , \qquad (5.16)$$

where the nontrivial entries are related by the unit determinant condition. The phases representing the action of the orbifold group elements on the external lines are trivial,

$$R_1 = R_2 = R_3 = R_4 = 1 ; \qquad (5.17)$$

the internal lines can be dressed with the phases

$$R_l \in \mathcal{R} = \left\{ 1, r_2^2, r_3^3, r_4^4, r_2^2 r_3^3, r_2^2 r_4^4, r_3^3 r_4^4 \right\} . \qquad (5.18)$$

Despite the reduced amount of supersymmetry, it is still possible to set to zero the snail numerator factors.

The S_3 subgroup of the $SU(4)$ R-symmetry of the parent theory which permutes the nontrivial entries of the orbifold matrix suggests that we can choose

$$n_{i;r_3^3} = n_{i;r_4^4} = n_{i;r_2^2} , \qquad n_{i;r_2^2 r_4^4} = n_{i;r_3^3 r_4^4} = n_{i;r_2^2 r_3^3} . \qquad (5.19)$$

Moreover, to recover the $\mathcal{N}=4$ sYM theory for a particular choice or orbifold ($\Gamma = \mathbf{1}$, $r_2^2 = r_3^3 = r_4^2 = 1$, etc), the numerators should obey

$$n_{i;1} + 3n_{i;r_2^2} + 3n_{i;r_2^2 r_3^3} = n_i^{\mathcal{N}=4} , \qquad i = 1, 2, 3 . \qquad (5.20)$$

There are therefore six independent numerators. We will choose the three box integrals with $R_l = 1$ and $R_l = r_2^2$ as master graphs and parametrize their numerators as

$$n_{i;R_l}(l) = i\hat{n}_{i;R_l}(l)\langle k_1 k_2 \rangle^2 [k_3 k_4]^2, \quad \hat{n}_{i;R_l}(l) = \frac{\mathcal{P}_{5;2}\left(s, t, \tau_{il}, l^2\right)}{s^3 tu} + i\frac{\mathcal{P}_{3;1}\left(s, t, \tau_{il}\right)\epsilon(k_1, k_2, k_4, l)}{s^3 tu},$$

$$(5.21)$$

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where $\mathcal{P}_{5;2}$ and $\mathcal{P}_{3;1}$ are, respectively, polynomials of degree five and three which also have at most degrees two and one in their loop-momentum-dependent arguments. As in the $\mathcal{N} = 2$ theories, they are different for different graphs and orbifold phase R_l. Matching the generalized cuts and setting the remaining coefficients as explained below leads to

$$\hat{n}_{1;1}(l) = 1 + \frac{\tau_{2l} - \tau_{1l} - l^2}{s} , \tag{5.22}$$

$$\hat{n}_{1;r_2^2}(l) = -(\tau_{1l} + \tau_{2l})\left(\frac{ut}{3s^3} + \frac{\tau_{1l} - \tau_{2l}}{s^2} + \frac{l^2}{s^2}\right) + \frac{3\tau_{1l} + 2\tau_{2l}}{3s} + \frac{l^2}{6s} , \tag{5.23}$$

$$\hat{n}_{2;1}(l) = 1 + \frac{u}{2s} + 3u\frac{\tau_{2l} + \tau_{1l}}{s^2} + \frac{\tau_{2l} - \tau_{3l}}{s} - \frac{l^2}{s} + 6i\frac{\epsilon(k_1, k_2, k_4, l)}{s^2} , \tag{5.24}$$

$$\hat{n}_{2;r_2^2}(l) = -\frac{u}{4s} + \frac{tu^2}{8s^3} + (\tau_{1l} + \tau_{2l})\left(\frac{u}{2s^2} + \frac{2}{3s} - \frac{tu}{4s^3} - \frac{l^2}{s^2} + 4i\frac{\epsilon(k_1, k_2, k_4, l)}{s^3}\right)$$
$$- \frac{\tau_{2l} + \tau_{4l}}{6s} + (\tau_{1l} + \tau_{2l})^2\frac{u - t}{s^3} + 2\frac{\tau_{1l}\tau_{4l}}{s^2} + \frac{2u - t}{6s^2}l^2 , \tag{5.25}$$

$$\hat{n}_{3;1}(l) = \frac{1}{2} - \frac{\tau_{1l} + \tau_{2l} + l^2}{s} , \tag{5.26}$$

$$\hat{n}_{3;r_2^2}(l) = \left(\frac{1}{2} + \frac{\tau_{1l} + \tau_{2l}}{s}\right)\left(\frac{1}{2} - \frac{tu}{3s^2} - \frac{\tau_{1l} + \tau_{2l} + l^2}{s}\right) + \frac{l^2}{6s} . \tag{5.27}$$

As in the $\mathcal{N} = 2$ case, the numerator factors for triangle graphs can be obtained from eqs. (4.17)-(4.40), while the numerator factors, $\hat{n}_{1;r_2^2 r_3^3} = \hat{n}_{2;r_2^2 r_3^3} = \hat{n}_{3;r_2^2 r_3^3}$, of the remaining box integrals follow from eq. (5.20). To obtain eqs. (5.22)-(5.27) some of the Ansatz' coefficients not determined by the generalized cuts have need fixed such that the integrand is manifestly Bose-symmetric under in the external lines 1 and 2, as in eq. (5.14). We have also been able to set to zero one bubble, two triangle and all snail graphs,

$$\hat{n}_{i;R_l} \equiv 0 , \qquad (\forall)\ R_l , \qquad i = 4, 6, 10 \quad \text{and} \quad i > 12 , \tag{5.28}$$

while the remaining free coefficients have been set to zero for simplicity.

It is worth mentioning that, unlike the $\mathcal{N} = 2$ case, the known four-gluon amplitudes of $\mathcal{N} = 2$ and $\mathcal{N} = 1$ pure sYM theories in BCJ form do not completely determine the master graph numerators in the orbifold theory. Indeed, using the fact that the invariant spectrum is that of $\mathcal{N} = 1$ sYM theory and that for $r_1^1 = 1$ the invariant spectrum is that of $\mathcal{N} = 2$ sYM theory we also have the relations

$$n_{i;1} = n_i^{\mathcal{N}=1} \quad \text{and} \quad n_{i;1} + n_{i;r_1^1} + n_{i;r_3^3 r_4^4} = n_i^{\mathcal{N}=2} . \tag{5.29}$$

However, $n_{i;r_1^1}$ and $n_{i;r_3^3 r_4^4}$ appear in these equations in the same way as in (5.20) (cf. eq. (5.19)) and thus only their sum is fixed.

5.1.3 Four-gluon amplitudes with no supersymmetry

As in the previous cases, the gluons carrying momenta k_1 and k_2 are taken to have negative helicity and the gluons with momenta k_3 and k_4 are taken to have positive helicity. The

orbifold r matrix is now a general diagonal element of $SU(4)$,

$$r = \text{diag}(r_1^1, r_2^2, r_3^3, r_4^4) \, , \tag{5.30}$$

with the entries are related by the unit determinant condition. The phases representing the action of the orbifold group elements on the external lines are trivial as before,

$$R_1 = R_2 = R_3 = R_4 = 1 \, ; \tag{5.31}$$

the internal lines can be dressed with the phases

$$R_l \in \mathcal{R} = \left\{ 1, r_1^1, r_2^2, r_3^3, r_4^4, r_1^1 r_2^2, r_1^1 r_3^3, r_1^1 r_4^4, r_2^2 r_3^3, r_2^2 r_4^4, r_3^3 r_4^4, r_1^1 r_2^2 r_3^3, r_1^1 r_2^2 r_4^4, r_1^1 r_3^3 r_4^4, r_2^2 r_3^3 r_4^4 \right\} \, . \tag{5.32}$$

The S_4 subgroup of the $SU(4)$ R-symmetry of the parent theory which permutes the nontrivial entries of the orbifold matrix suggests that we can choose

$$n_{i;r_3^3} = n_{i;r_4^4} = n_{i;r_2^2} = n_{i;r_1^1} \, ,$$
$$n_{i;r_2^2 r_4^4} = n_{i;r_3^3 r_4^4} = n_{i;r_2^2 r_3^3} = n_{i;r_1^1 r_3^3} = n_{i;r_1^1 r_4^4} = n_{i;r_1^1 r_2^2} \, ,$$
$$n_{i;r_1^1 r_2^2 r_4^4} = n_{i;r_1^1 r_3^3 r_4^4} = n_{i;r_2^2 r_3^3 r_4^4} = n_{i;r_1^1 r_2^2 r_3^3} \, . \tag{5.33}$$

Moreover, to recover the $\mathcal{N} = 4, 2, 1$ sYM theories for particular choices of orbifold group, the numerators should obey the following relations $(i = 1, 2, 3)$

$$n_{i;1} + 4n_{i;r_1^1} + 6n_{i;r_1^1 r_2^2} + 4n_{i;r_1^1 r_2^2 r_3^3} = n_i^{\mathcal{N}=4} \, , \tag{5.34}$$
$$n_{i;1} + 2n_{i;r_1^1} + 2n_{i;r_1^1 r_2^2} + 2n_{i;r_1^1 r_2^2 r_3^3} = n_{i,1}^{\mathcal{N}=2} \, , \tag{5.35}$$
$$n_{i;r_1^1} + n_{i;r_1^1 r_2^2} = n_{i,r_2^2}^{\mathcal{N}=1} \, . \tag{5.36}$$

Taking in account these relations, we can choose the first two boxes with $R_l = 1$ and $R_l = r_1^1$ as master graphs and parametrize their numerators as

$$n_{i;R_l}(l) = i\hat{n}_{i;R_l}(l)\langle k_1 k_2 \rangle^2 [k_3 k_4]^2, \quad \hat{n}_{i;R_l}(l) = \frac{\mathcal{P}_{6;3}\left(s, t, \tau_{il}, l^2\right)}{s^4 tu} + i\frac{\mathcal{P}_{4;2}\left(s, t, \tau_{il}\right)\epsilon(k_1, k_2, k_4, l)}{s^4 tu}, \tag{5.37}$$

where $\mathcal{P}_{6;3}$ and $\mathcal{P}_{4;2}$ are, respectively, polynomials of degree six and four which also have at most three and two powers of the loop momentum l.

Generalized unitarity fixes the numerator of the first box in fig. 3 to be,

$$\hat{n}_{1;1} = 1 + 4\frac{\tau_{1l} + 3\tau_{2l} - l^2}{3s} \, , \tag{5.38}$$

Additionally, the numerator of the second box in fig. 3 is

$$\hat{n}_{2;1} = \frac{u - t}{s} + 8\frac{u\tau_{1l}^3 - t\tau_{2l}^3}{s^4} + \frac{4\tau_{4l}}{3s} - 4\tau_{2l}^2\left(\frac{t}{s^3} + \frac{2t\tau_{4l}}{s^3 u}\right) + 4\tau_{1l}^2\left(\frac{u}{s^3} - 2\frac{t - 2u}{s^4}\tau_{2l} + \frac{2u\tau_{4l}}{s^3 t}\right)$$
$$- 4\tau_{1l}\left(\frac{t}{s^2} + \frac{t - u}{s^3}\tau_{2l} + 2\frac{2t - u}{s^4}\tau_{2l}^2 + \frac{2\tau_{4l}^2}{s^2 t}\right) + 4\tau_{2l}\left(\frac{u}{s^2} + \frac{2\tau_{4l}^2}{s^2 u}\right)$$
$$- 4l^2\left(\frac{1}{3s} + \frac{u\tau_{1l} - t\tau_{2l}}{s^3}\right) + 8i\left(\frac{1}{s^2} + \frac{\tau_{1l} + \tau_{2l}}{s^3} + 2\frac{(\tau_{1l} + \tau_{2l})^2}{s^4}\right)\epsilon(k_1, l_2, k_4, l) \, , \tag{5.39}$$

All the other numerators can be obtained from the ones shown here. Specifically:

- the numerators of the third box can be obtained using the Bose symmetry (5.14);

- the expressions for the numerators $\hat{n}_{i,r_1^1}, \hat{n}_{i,r_1^1 r_2^2}$ and $\hat{n}_{i,r_1^1 r_2^2 r_3^3}$ follow from the requirement that the amplitudes reproduce the known formulae in the $\mathcal{N} = 1$ and $\mathcal{N} = 2$ cases, as in eqs. (5.34-5.36);

- the numerators for the triangle and bubble graphs are obtained using the kinematic Jacobi relations (4.17)-(4.40).

Some of the free coefficients that are not fixed by the cut conditions have been chosen such that one bubble, two triangle and all snail graphs are set to zero,

$$\hat{n}_{i;R_l} \equiv 0 , \qquad (\forall) \ R_l , \qquad i = 4, 6, 10 \quad \text{and} \quad i > 12 . \tag{5.40}$$

The remaining coefficients have been fixed to obtain a particularly simple representation of the amplitude. We emphasize that, in the construction of the numerators we have used four-dimensional cuts and therefore some rational terms are not accounted for. We have checked that, upon integration, the graphs with numerator factors $n_{i;1}$ yield the pure YM four-gluon amplitude [37] up to such terms. One may, alternatively, turn this around and use the numerator factors of [14] of the pure YM four-gluon amplitude in BCJ form to construct $n_{i;1}$ and with it construct the amplitudes of all non-supersymmetric orbifolds.

5.2   Four-scalar amplitudes

Four-scalar amplitudes are the simplest amplitudes with external legs charged under the orbifold generators. We will focus here on the particular field configuration $\left(1^\phi 2^\phi 3^{\bar{\phi}} 4^{\bar{\phi}}\right)$ and construct the corresponding one-loop amplitude $\mathcal{A}_4^{1-\text{loop}}\left(1^\phi 2^\phi 3^{\bar{\phi}} 4^{\bar{\phi}}\right)$ for $\mathcal{N} = 2, \mathcal{N} = 1$ and $\mathcal{N} = 0$ orbifolds. An important simplification compared to gluon amplitudes comes from the expectation that these amplitudes have manifestly local numerator factors; this is a consequence of the observation [14] that the non-locality of the numerator factors of one-loop four-gluon amplitudes in pure $\mathcal{N} = 2$ and $\mathcal{N} = 1$ sYM theories is a consequence of the use of helicity states for the external fields.

5.2.1   Four-scalar amplitudes with $\mathcal{N} = 2$ supersymmetry

As in the case of gluon amplitudes with $\mathcal{N} = 2$ supersymmetry (see section 5.1.1) , without loss of generality we take the $SU(4)$ orbifold matrices matrices r to be as in eq. (5.3)

$$r = \text{diag}(1, 1, r_3^3, r_4^4) . \tag{5.41}$$

We focus on the amplitude $\mathcal{A}_4^{1-\text{loop}}\left(1^{\phi^{13}} 2^{\phi^{13}} 3^{\phi^{24}} 4^{\phi^{24}}\right)$. The phase factors associated to the external legs are then,

$$R_1 = R_2 = r_3^3, \qquad R_3 = R_4 = r_4^4 = \frac{1}{r_3^3} , \tag{5.42}$$

while the set of phase factors capturing the action of the orbifold group on (the $SU(4)$ representation of) the internal line carrying the loop momentum l is given by,

$$R_l \in \mathcal{R} = \left\{ 1, r_3^3, r_4^4 \right\} . \tag{5.43}$$

Unlike the gluon amplitude, constraints on the numerator factors are less severe here; in particular, since the external states are changed under the orbifold group, the S_2 symmetry permuting r_3^3 and r_4^4 cannot be a symmetry of this amplitude. However, the requirement that as $\Gamma = \mathbf{1}$ the amplitude reduces to that of the $\mathcal{N} = 4$ theory relates the numerator factors as

$$
\begin{aligned}
n_{j;1} + n_{j;r_3^3} + n_{j;r_4^4} &= n_j^{\mathcal{N}=4} \equiv is^2, &\quad j = 1, 2, 3 \\
n_{j;1} + n_{j;r_3^3} + n_{j;r_4^4} &= 0 &\quad , \quad j \geq 4 .
\end{aligned}
\tag{5.44}
$$

Together with the Jacobi relations (4.17)-(4.40), these equations imply that six of the nine box integrals can be chosen as master graphs.

For their numerator factors we use a manifestly local ansatz

$$n_{j,R_l}(l) = i\hat{n}_{j,R_l}(l) = i\mathcal{P}_2\big(s,t\big) \tag{5.45}$$

with a different degree-two polynomials $\mathcal{P}_2$ for each graph and phase factor R_l. The result of the unitarity cut calculation is sufficiently simple for us to list explicitly all numerator factors:

R_l \ $\hat{n}_i$	$\hat{n}_1$	$\hat{n}_2$	$\hat{n}_3$	$\hat{n}_4$	$\hat{n}_5$	$\hat{n}_6$	$\hat{n}_7$	$\hat{n}_8$	$\hat{n}_9$	$\hat{n}_{10}$	$\hat{n}_{11}$	$\hat{n}_{12}$
1	s^2	$-st$	0	0	st	0	st	su	$-su$	0	$-2su$	$-2st$
r_3^3	0	0	0	0	0	0	$-st$	0	su	0	su	st
r_4^4	0	$-su$	s^2	0	$-st$	0	0	$-su$	0	0	su	st

. (5.46)

Each entry is the numerator factor of the graph specified by the top entry of the column dressed with the orbifold phase specified by the left-most entry of the row. We note that the sum of the entries of each column gives the numerator factor of that graph in the $\mathcal{N} = 4$ sYM four-scalar amplitude, cf. eq. (5.44). We also note that, perhaps due to the large amount of supersymmetry, the Levi-Civita tensor is absent from all numerator factors.

5.2.2 Four-scalars amplitudes with $\mathcal{N} = 1$ supersymmetry

The four-scalar amplitude in $\mathcal{N} = 1$ supersymmetric orbifold theories are very interesting as they exhibit some of the features of non-supersymmetric theories while still being

relatively compact. As in the case of the gluon amplitude in these theories we choose the r matrix representing the orbifold group inside $SU(4)$ as

$$r = \text{diag}(1, r_2^2, r_3^3, r_4^4) , \qquad r_2^2 r_3^3 r_4^4 = 1 . \tag{5.47}$$

We focus on the amplitude $\mathcal{A}_4^{1-\text{loop}}(1^{\phi^{12}} 2^{\phi^{12}} 3^{\phi^{34}} 4^{\phi^{34}})$. The phases associated to the external legs are then,

$$R_1 = R_2 = r_2^2, \qquad R_3 = R_4 = r_3^3 r_4^4 = \frac{1}{r_2^2} . \tag{5.48}$$

The set of phases R_l dressing the internal leg carrying the loop momentum l is

$$R_l \in \mathcal{R} = \left\{ 1, r_2^2, r_3^3, r_4^4, r_2^2 r_3^3, r_2^2 r_4^4, r_3^3 r_4^4, (r_2^2)^2, (r_2^2)^2 r_3^3, (r_2^2)^2 r_4^4, (r_3^3)^2 r_4^4, r_3^3 (r_4^4)^2, (r_3^3)^2 (r_4^4)^2 \right\} . \tag{5.49}$$

This set is larger than the one in eq. (5.18) for the gluon amplitude in part due to the $SU(4)$-charge flow between external legs. We should also note that, unlike the previous three examples, some of the phases above (the last six elements of the set) cannot be obtained from the R-symmetry labels of a *physical* particle going around the loop. However, a solution to the kinematic Jacobi relations (4.17)-(4.40) appears to exist only if these fictitious particles are included. The appearance of these extra representations at the intermediate steps of the computation should not be a surprise because physical amplitudes are obtained only after the summation over all the orbifold elements (r, g). The two-particle cuts of such an ansatz can be correct order by order in the r_a^a only if all cuts containing at least one unphysical particle vanish. This requires that the numerator factor of an unphysical graph contains an inverse propagator for at least one unphysical particle in all cuts.

From the S_3 symmetry permuting the the nontrivial elements of r, the S_2 subgroup interchanging r_3^3 and r_4^4 preserves the external line phase factors (5.48) and thus can be used to impose the following relations:

$$n_{i;r_4^4} = n_{i;r_3^3} , \quad n_{i;r_2^2 r_4^4} = n_{i;r_2^2 r_3^3} , \quad n_{i;(r_2^2)^2 r_4^4} = n_{i;(r_2^2)^2 r_3^3} , \quad n_{i;r_3^3 (r_4^4)^2} = n_{i;(r_3^3)^2 r_4^4} . \tag{5.50}$$

Of the maximum of 39 box master integrals we are therefore left with 27. Last but not least, further numerator relations come from the requirement that as $\Gamma = \mathbf{1}$ the amplitude reduces to that of $\mathcal{N} = 4$ sYM theory. We will not write them out explicitly.

As in the $\mathcal{N} = 2$ four-scalar amplitude example, the numerator factors are expected to be local so we use the ansatz

$$n_{i,R_l}(l) = i \hat{n}_{i,R_l}(l) = i \mathcal{P}_{2;1}\left(s, t, \tau_{1l}, \tau_{2l}, \tau_{3l}, l^2\right) - c \, \epsilon(k_1, k_2, k_4, l) , \tag{5.51}$$

where c is a real constant and $\mathcal{P}_{2;1}$ is a degree-two polynomial which also has at most unit degree in its l-dependent arguments (c and $\mathcal{P}_{2;1}$ are different for different graphs). The unitarity cuts determine the numerator factors of the master graphs:

- The graph topology (1) in fig. 3

$$\hat{n}_{1;1} \;=\; s^2 - \frac{s}{3}(\tau_{1l} - \tau_{2l} + l^2) \,, \tag{5.52}$$

$$\hat{n}_{1;r_2^2} \;=\; -\frac{s}{12}(5\tau_{1l} + \tau_{2l} + 2l^2) \quad,\quad \hat{n}_{1;r_3^3} = -\frac{s}{12}(\tau_{1l} + 5\tau_{2l} - 2l^2)\,, \tag{5.53}$$

$$\hat{n}_{1;r_2^2 r_3^3} \;=\; \frac{s}{12}(5\tau_{1l} + \tau_{2l} + 2l^2) \quad,\quad \hat{n}_{1;r_3^3 r_4^4} = \frac{s}{12}(\tau_{1l} + 5\tau_{2l} - 2l^2)\,. \tag{5.54}$$

- The graph topology (2) in fig. 3

$$\hat{n}_{2;1} \;=\; \frac{5}{24}su - st - (\tau_{1l} + \tau_{2l})\left(t + \frac{2}{3}s\right) + \frac{s}{4}(\tau_{4l} + \tau_{2l} - l^2) - 2i\epsilon(k_1, k_2, k_4, l)\,, \tag{5.55}$$

$$\hat{n}_{2;r_2^2} \;=\; -\frac{s}{12}(\tau_{1l} - \tau_{4l} + l^2)\,, \tag{5.56}$$

$$\hat{n}_{2;r_3^3} \;=\; -\frac{5}{24}su + (\tau_{1l} + \tau_{2l})\left(t + \frac{7}{12}s\right) - \frac{s}{6}(\tau_{4l} + \tau_{2l} - l^2) + 2i\epsilon(k_1, k_2, k_4, l)\,, \tag{5.57}$$

$$\hat{n}_{2;r_2^2 r_3^3} \;=\; \frac{s}{12}(\tau_{1l} - \tau_{4l} + l^2)\,, \tag{5.58}$$

$$\hat{n}_{2;r_3^3 r_4^4} \;=\; -\frac{5}{6}su - (\tau_{1l} + \tau_{2l})\left(t + \frac{7}{12}s\right) + \frac{s}{4}(\tau_{4l} + \tau_{2l} - l^2) - 2i\epsilon(k_1, k_2, k_4, l)\,, \tag{5.59}$$

$$\hat{n}_{2;(r_3^3)^2 r_4^4} \;=\; \frac{s}{24}(u - 2\tau_{2l} - 2\tau_{4l} + 2l^2)\,, \tag{5.60}$$

$$\hat{n}_{2;(r_3^3 r_4^4)^2} \;=\; -\frac{s}{24}(u - 2\tau_{2l} - 2\tau_{4l} + 2l^2)\,. \tag{5.61}$$

- The graph topology (3) in fig. 3 can be obtained employing the Bose symmetry for the exchange of particles 1 and 2.

We note that, as required, the numerator factors corresponding to graphs with some unphysical fields going around the loop can be expressed as a linear combination of inverse propagators so that the corresponding integrals contribute to only one of the three two-particles cuts – the one that cuts only physical internal lines. As an example of this, we consider the numerator factor $\hat{n}_{1;r_3^3 r_4^4}$. It is easy to see that the internal line between the momenta k_2 and k_3 may be unphysical (with phase $(r_3^3 r_4^4)^2$). The numerator factor can be written as

$$\hat{n}_{1;r_3^3 r_4^4} = \frac{s}{24}\left((l + k_1)^2 - 5(l - k_2)^2\right)\,. \tag{5.62}$$

The second inverse propagator removes the propagator corresponding to the unphysical particle, while the first inverse propagator appears to be problematic. However, inspecting the numerator factors of the second and seventh graph,

$$\hat{n}_{2;(r_3^3 r_4^4)^2} = -\frac{s}{24}\left((l - k_2 - k_4)^2 + l^2\right)\,, \quad \hat{n}_{7;(r_2^2)^2} = -\frac{s}{24}\left(t - (l - k_2)^2 - (l + k_3)^2 + 4l^2\right)\,, \tag{5.63}$$

we note that the first term of the first graph, the second term of the second graph and the first term of the seventh graph are all proportional to the same triangle integral $I_3(t)$; the proportionality constant includes the combination of color factors,

$$C_{1;r_3^3 r_4^4} - C_{2;(r_3^3 r_4^4)^2} - C_{7;(r_2^2)^2} \tag{5.64}$$

34

which vanishes due to one of the color Jacobi identities. For all other numerator factors, one can see that in the box color structures all terms in which the propagator of an unphysical state is not removed vanish due to similar cancellations.

The triangle color structures leave behind terms with unphysical fields going around the loop in a "snail" integral (where the bubble is attached to the vertex). Similarly, the bubble color structures leave behind some tadpole integrals (where the bubble is attached to the internal propagator). Such snail and tadpole integrals are not constrained by the standard two-particles cuts we have employed; however, they vanish upon integration for massless external particles in $D = 4 - 2\epsilon$ dimensions.

It is also easy to verify that, by setting $r_2^2 = r_3^3 = r_4^4 = 1$ and summing all the numerators corresponding to the same labeled graph we obtain the standard numerator factors of the one-loop four-scalar amplitude $\mathcal{N} = 4$ sYM theory.

As in the previous examples we have imposed Bose symmetry for the exchange of external particles 1 and 2 (as well as 3 and 4); it requires that

$$\hat{n}_{1,R_l}(l) = \hat{n}_{3,R_l/R_1}(l - k_1)\big|_{k_1 \leftrightarrow k_2}, \qquad \hat{n}_{2,R_l}(l) = \hat{n}_{2,R_4/R_l}(k_4 - l)\big|_{k_1 \leftrightarrow k_2}. \qquad (5.65)$$

Moreover, we have required that the numerator factors of snail integrals obey the relation (5.2) and we have set to zero the numerators of one bubble and two triangle graphs,

$$\hat{n}_{i;R_l} \equiv 0 , \qquad (\forall)\ R_l , \quad i = 4, 6, 10. \qquad (5.66)$$

Finally, we have required that upon setting $r_2^2 = 1$ and summing over the numerator factors corresponding to identical graphs we reproduce $\mathcal{N} = 2$ orbifold amplitude derived in the previous section. These conditions fix all coefficients of the ansatz.

The expressions (5.52-5.61) solve all the Jacobi-like relations (4.17-4.40). However, the conditions for having vanishing tadpole numerators (4.41) are not generically satisfied. It is very interesting to note that the color factor of a tadpole graph always contains a term of the form $f^{ABC}g^{AB}$, which can be expressed in the trace basis as,

$$f^{ABC}g^{AB} = 2i\,\text{Im}\Big(\text{Tr}g\,\text{Tr}g^{\dagger}T^C\Big) . \qquad (5.67)$$

Thus, the right-hand side vanishes for any regular orbifold due to (2.4), implying that amplitudes in such theories have no tadpole graphs[18]. This mirrors the string theory result, where regularity of the orbifold guarantees tadpole cancellation.

5.2.3   Four-scalars amplitudes with no supersymmetry

The last example we discuss is the one-loop four-scalar amplitude $\mathcal{A}_4^{1-\text{loop}}\big(1^{\phi^{12}}2^{\phi^{12}}3^{\phi^{34}}4^{\phi^{34}}\big)$ in non-supersymmetric orbifold theories. The $SU(4)$ matrix r generating the orbifold

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[18]The converse is not necessarily true, and it might be possible to find examples of non-regular orbifolds with vanishing tadpole integrands.

group is now a general diagonal matrix of phases subject to the unit determinant condition:

$$r = \mathrm{diag}(r_1^1, r_2^2, r_3^3, r_4^4) \ , \quad \det(r) = 1 \ . \tag{5.68}$$

The external line phases for the amplitude $\mathcal{A}_4\big(1^{\phi^{12}} 2^{\phi^{12}} 3^{\phi^{34}} 4^{\phi^{34}}\big)$ are

$$R_1 = R_2 = r_1^1 r_2^2, \qquad R_3 = R_4 = r_3^3 r_4^4 = \frac{1}{r_1^1 r_2^2} \ , \tag{5.69}$$

while the internal line phases R_l belong to the 33-element set

$$
\begin{aligned}
R_l \in \mathcal{R} \ = \ \big\{ & 1, r_1^1, r_2^2, r_3^3, r_4^4, r_1^1 r_2^2, r_1^1 r_3^3, r_1^1 r_4^4, r_2^2 r_3^3, r_2^2 r_4^4, r_3^3 r_4^4, r_1^1 r_2^2 r_3^3, r_1^1 r_2^2 r_4^4, r_1^1 r_3^3 r_4^4, r_2^2 r_3^3 r_4^4, \\
& (r_1^1)^2 r_2^2, r_1^1 (r_2^2)^2, (r_3^3)^2 r_4^4, r_3^3 (r_4^4)^2, (r_1^1 r_2^2)^2, (r_1^1)^2 r_2^2 r_3^3, (r_1^1)^2 r_2^2 r_4^4, r_1^1 (r_2^2)^2 r_3^3, \\
& r_1^1 (r_2^2)^2 r_4^4, r_1^1 (r_3^3)^2 r_4^4, r_2^2 (r_3^3)^2 r_4^4, r_1^1 r_3^3 (r_4^4)^2, r_2^2 r_3^3 (r_4^4)^2, (r_3^3 r_4^4)^2, (r_1^1 r_2^2)^2 r_3^3, \\
& (r_1^1 r_2^2)^2 r_4^4, r_1^1 (r_3^3 r_4^4)^2, r_2^2 (r_3^3 r_4^4)^2 \big\} \ .
\end{aligned}
\tag{5.70}
$$

As explained in the case of the same amplitude in $\mathcal{N} = 1$ orbifold theories, while some of the elements of $\mathcal{R}$ cannot be obtained from the R-symmetry labels of a physical particle going around the loop, they appeared necessary for the existence of a solution of the kinematic Jacobi relations which is consistent with all unitarity cuts. The external phases (5.69) are invariant under the $S_2 \times S_2 \subset S_4 \subset SU(4)$ symmetry permuting r_1^1 and r_2^2 and, independently, r_3^3 and r_4^4. This symmetry implies that the 99 numerator factors of box integrals are related as

$$
\begin{aligned}
& n_{i;r_2^2} = n_{i;r_1^1} \ , \qquad n_{i;r_4^4} = n_{i;r_3^3} \ , \quad n_{i;r_2^2 r_4^4} = n_{i;r_2^2 r_3^3} = n_{i;r_1^1 r_4^4} = n_{i;r_1^1 r_3^3} \ , \\
& n_{i;r_1^1 r_2^2 r_4^4} = n_{i;r_1^1 r_2^2 r_3^3} \ , \quad n_{i;r_2^2 r_3^3 r_4^4} = n_{i;r_1^1 r_3^3 r_4^4} \ , \quad n_{i;r_1^1 (r_2^2)^2} = n_{i;(r_1^1)^2 r_2^2} \ , \\
& n_{i;r_3^3 (r_4^4)^2} = n_{i;(r_3^3)^2 r_4^4} \ , \quad n_{i;r_1^1 (r_2^2)^2 r_3^3} = n_{i;r_1^1 (r_2^2)^2 r_4^4} = n_{i;(r_1^1)^2 r_2^2 r_3^3} = n_{i;(r_1^1)^2 r_2^2 r_4^4} \ , \\
& n_{i;r_1^1 r_3^3 (r_4^4)^2} = n_{i;r_2^2 r_3^3 (r_4^4)^2} = n_{i;r_2^2 (r_3^3)^2 r_4^4} = n_{i;r_1^1 (r_3^3)^2 r_4^4} \ , \quad n_{i;r_2^2 (r_3^3 r_4^4)^2} = n_{i;r_1^1 (r_3^3 r_4^4)^2} \ , \\
& n_{i;(r_1^1 r_2^2)^2 r_4^4} = n_{i;(r_1^1 r_2^2)^2 r_3^3} \ .
\end{aligned}
\tag{5.71}
$$

We are therefore left with 48 master integrals (the three additional relations, which we do not write explicitly, imposing that as $r_i^i = 1$ we recover the $\mathcal{N} = 4$ sYM amplitude reduce this number down to 45).

As for the other scalar amplitude examples, the ansatz for the numerator factors of the master integrals is manifestly local:

$$n_{i,R_l}(l) = i\hat{n}_{i,R_l}(l) = i\mathcal{P}_{2;2}\big(s, t, \tau_{1l}, \tau_{2l}, \tau_{3l}, l^2\big) s - c\, \epsilon(k_1, k_2, k_4, l) \ ; \tag{5.72}$$

here, as before, c is a real constant while $\mathcal{P}_{2;2}$ is a polynomial (different for each graph) of degree two which is also up to degree two in its loop-momentum-dependent arguments. The unitarity cuts and the additional conditions explained below determine the numerator factors to be (we list only the non-vanishing ones):

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- The graph topology (1) in fig. 3

$$\hat{n}_{1;1} \;=\; s^2 + (\tau_{1l} - \tau_{2l})\Big(\tau_{1l} - \tau_{2l} + \tfrac{4}{3}l^2\Big) + \tfrac{4}{3}\tau_{1l}\tau_{2l} + \tfrac{2}{3}l^4 \;,\tag{5.73}$$

$$\hat{n}_{1;r_1^1} \;=\; \tfrac{1}{6}(\tau_{1l} + \tau_{2l})(s + \tau_{2l} + \tau_{1l}) - \tfrac{4}{3}\tau_{1l}(\tau_{1l} + l^2) + \tfrac{s}{6}(2\tau_{2l} - l^2) - \tfrac{l^4}{3} \;,\tag{5.74}$$

$$\hat{n}_{1;r_3^3} \;=\; -s\tau_{2l} - (\tau_{1l} + \tau_{2l})\Big(\tfrac{t}{2} + \tau_{4l}\Big) + \tfrac{4}{3}\tau_{1l}\tau_{2l} + l^2\Big(\tfrac{s}{2} + 2\tfrac{\tau_{2l} - \tau_{1l}}{3}\Big) - \tfrac{l^4}{3} \;,\tag{5.75}$$

$$\hat{n}_{1;r_1^1 r_2^2} \;=\; -(\tau_{1l} + \tau_{2l})\Big(\tfrac{7}{12}s + \tfrac{\tau_{2l} + \tau_{1l}}{6}\Big) + \tfrac{4}{3}\tau_{1l}(\tau_{1l} + l^2) + \tfrac{l^4}{3} \;,\tag{5.76}$$

$$\hat{n}_{1;r_1^1 r_3^3} \;=\; \tau_{4l}(\tau_{1l} + \tau_{2l}) + (\tau_{1l} - \tau_{2l})\Big(\tfrac{2}{3}l^2 - \tfrac{s}{12}\Big) - \tfrac{u}{2}\tau_{2l} + \tfrac{t}{2}\tau_{1l} - \tfrac{4}{3}\tau_{1l}\tau_{2l} - l^2\tfrac{s - l^2}{3} \;,\tag{5.77}$$

$$\hat{n}_{1;r_3^3 r_4^4} \;=\; -\tfrac{1}{6}(\tau_{1l} + \tau_{2l})^2 + \tfrac{4}{3}\tau_{2l}(\tau_{2l} - l^2) + \tfrac{7}{12}s(\tau_{1l} + \tau_{2l}) - \tfrac{s}{6}l^2 + \tfrac{l^4}{3} \;,\tag{5.78}$$

$$\hat{n}_{1;r_1^1 r_2^2 r_3^3} \;=\; -(\tau_{1l} + \tau_{2l})\Big(\tau_{4l} + \tfrac{t}{2}\Big) - (\tau_{1l} - \tau_{2l})\Big(\tfrac{2}{3}l^2 - \tfrac{s}{2}\Big) + \tfrac{4}{3}\tau_{1l}\tau_{2l} + \tfrac{s}{2}l^2 - \tfrac{l^4}{3} \;,\tag{5.79}$$

$$\hat{n}_{1;r_1^1 r_3^3 r_4^4} \;=\; \tfrac{1}{6}(\tau_{1l} + \tau_{2l})(\tau_{1l} + \tau_{2l} - s) - \tfrac{s}{3}\Big(\tau_{1l} + \tfrac{l^2}{2}\Big) - \tfrac{4}{3}\tau_{2l}(\tau_{2l} - l^2) - \tfrac{l^4}{3} \;.\tag{5.80}$$

- The graph topology (2) in fig. 3

$$\hat{n}_{2;1} \;=\; -st + (\tau_{1l} + \tau_{2l})\Big(\tfrac{7}{12}s - \tfrac{17}{12}t + \tau_{1l} + \tfrac{7}{6}\tau_{2l} + \tfrac{\tau_{4l}}{6}\Big) + 2s(\tau_{4l} - \tau_{1l})$$
$$\qquad\qquad - \tfrac{4}{3}\tau_{1l}\tau_{4l} - l^2\Big(2s - \tfrac{u}{6} + \tfrac{\tau_{2l} - \tau_{1l}}{2} + \tau_{4l}\Big) + \tfrac{l^4}{2} \;,\tag{5.81}$$

$$\hat{n}_{2;r_1^1} \;=\; \tfrac{\tau_{1l}}{6}(3s + \tau_{1l} + \tau_{2l}) + \tfrac{\tau_{4l}}{6}(3\tau_{1l} - \tau_{2l} - 3s) + \tfrac{l^2}{6}(3s + \tau_{2l} - \tau_{1l} + 2\tau_{4l}) - \tfrac{l^4}{6} \;,\tag{5.82}$$

$$\hat{n}_{2;r_3^3} \;=\; -\tfrac{su}{2} + (\tau_{1l} + \tau_{2l})\Big(t - \tfrac{\tau_{4l}}{3}\Big) + \tfrac{s}{2}(2\tau_{2l} + \tau_{4l}) - \tfrac{2}{3}\tau_{2l}\tau_{4l}$$
$$\qquad\qquad + l^2\Big(\tfrac{t}{6} - \tfrac{7}{12}s + \tfrac{\tau_{2l} - \tau_{1l}}{3} + \tfrac{2}{3}\tau_{4l}\Big) - \tfrac{l^4}{3} + 2i\epsilon(k_1, k_2, k_4, l) \;,\tag{5.83}$$

$$\hat{n}_{2;r_1^1 r_2^2} \;=\; \tau_{4l}\Big(\tfrac{7}{12}s - \tfrac{\tau_{1l}}{2} + \tfrac{\tau_{2l}}{6}\Big) - \tau_{1l}\Big(\tfrac{7}{12}s + \tfrac{\tau_{1l} + \tau_{2l}}{6}\Big) -$$
$$\qquad\qquad l^2\Big(\tfrac{7}{12}s + \tfrac{\tau_{2l} - \tau_{1l}}{6} + \tfrac{\tau_{4l}}{3}\Big) + \tfrac{l^4}{6} \;,\tag{5.84}$$

$$\hat{n}_{2;r_1^1 r_3^3} \;=\; s\Big(\tfrac{7}{12}\tau_{1l} - \tfrac{\tau_{4l}}{12}\Big) + \tau_{4l}\tfrac{\tau_{1l} + 3\tau_{2l}}{3} + \tfrac{l^2}{6}(2s - \tau_{1l} - 3\tau_{2l} - 2\tau_{4l}) + \tfrac{l^4}{6} \;,\tag{5.85}$$

$$\hat{n}_{2;r_3^3 r_4^4} \;=\; -\tfrac{5}{4}su + (\tau_{1l} + \tau_{2l})\Big(\tfrac{3}{2}t + \tfrac{7}{6}\tau_{1l} + \tau_{2l} - \tfrac{\tau_{4l}}{6}\Big) - \tfrac{2}{3}\tau_{1l}\tau_{4l}$$
$$\qquad - l^2\Big(\tfrac{t}{3} + \tfrac{9}{4}s + \tfrac{\tau_{2l} - \tau_{1l}}{2} + \tau_{4l}\Big) + s\Big(\tau_{2l} - \tfrac{11}{12}\tau_{1l} + \tfrac{23}{12}\tau_{4l}\Big) + \tfrac{l^4}{2} \;,\tag{5.86}$$

$$\hat{n}_{2;r_1^1 r_2^2 r_3^3} \;=\; -\tfrac{s}{2}\tau_{1l} - \tau_{4l}\tfrac{\tau_{1l} + 3\tau_{2l}}{3} + \tfrac{l^2}{6}\Big(-\tfrac{3}{2}s + \tau_{1l} + 3\tau_{2l} + 2\tau_{4l}\Big) - \tfrac{l^4}{6} \;,\tag{5.87}$$

$$\hat{n}_{2;r_1^1 r_3^3 r_4^4} = \frac{5}{24}su + (\tau_{1l} + \tau_{2l})\left(\frac{5}{12}t - \frac{7}{6}(\tau_{1l} + \tau_{2l})\right) + \frac{2}{3}\tau_{1l}\tau_{4l} - s\left(\tau_{2l} - \frac{\tau_{1l}}{4} + \frac{5}{4}\tau_{4l}\right)$$
$$+ l^2\left(\frac{t}{6} + \frac{17}{12}s + \frac{\tau_{2l} - \tau_{1l}}{3} + \frac{2}{3}\tau_{4l}\right) - \frac{l^4}{3} - 2i\epsilon(k_1, k_2, k_4, l) , \tag{5.88}$$

$$\hat{n}_{2;(r_3^3)^2 r_4^4} = -\frac{su}{4} + \frac{s}{2}(\tau_{2l} + \tau_{4l}) - \frac{l^2}{6}\left(2s - t + 3\tau_{1l} + \tau_{2l} - 2\tau_{4l}\right) - \frac{l^4}{6} , \tag{5.89}$$

$$\hat{n}_{2;r_1^1(r_3^3)^2 r_4^4} = \frac{7}{24}su - \frac{7}{12}s(\tau_{2l} + \tau_{4l}) + \frac{l^2}{6}\left(\frac{5}{2}s - t + 3\tau_{1l} + \tau_{2l} - 2\tau_{4l}\right) + \frac{l^4}{6} , \tag{5.90}$$

$$\hat{n}_{2;(r_3^3 r_4^4)^2} = (\tau_{1l} + \tau_{2l} - 3s)\left(\frac{u}{12} - \frac{\tau_{2l} + \tau_{4l}}{6}\right) - \frac{l^2}{6}\left(4s + t - \tau_{1l} + \tau_{2l} + 2\tau_{4l}\right) + \frac{l^4}{6} , \tag{5.91}$$

$$\hat{n}_{2;r_1^1(r_3^3 r_4^4)^2} = \left(\frac{5}{2}s - \tau_{1l} - \tau_{2l}\right)\left(\frac{u}{12} - \frac{\tau_{2l} + \tau_{4l}}{6}\right) + \frac{l^2}{6}\left(\frac{7}{2}s + t - \tau_{1l} + \tau_{2l} + 2\tau_{4l}\right) - \frac{l^4}{6} . \tag{5.92}$$

As in the previous cases, we have imposed on the numerator factors the exchange symmetry between particles 1 and 2, which yields the relations (5.65). Hence, the numerator factor of the third box can be found as

$$\hat{n}_{3,R_l}(l) = \hat{n}_{1,R_l R_1}(l + k_1)\big|_{k_1 \leftrightarrow k_2} ; \tag{5.93}$$

and will not be written explicitly. It is not difficult to check that upon setting $r_i^i = 1$ and summing the numerator factors of identical graphs (*i.e.* or the graphs that are in general different only because of the insertion of the orbifold group element in their color structure) one recovers the numerator factors of the one-loop four-scalar amplitude in $\mathcal{N} = 4$ sYM theory.

We have also required that all snail integrals obey the relation (5.2) for general D as in the $\mathcal{N} = 1$ case and we have set to zero the numerators of one bubble and two triangle graphs,

$$\hat{n}_{i;R_l} \equiv 0 , \qquad (\forall)\ R_l , \quad i = 4, 6, 10. \tag{5.94}$$

Last but not least, we have required that, with the appropriate choices for the diagonal entries of the matrix r we reproduce the numerator factors of the $\mathcal{N} = 2$ and $\mathcal{N} = 1$ one-loop four-scalar amplitudes discussed in previous sections.

As in the $\mathcal{N} = 1$ case, the numerator factors (5.73)-(5.92) solve all the generalized Jacobi relations (4.17)-(4.40), but have non-vanishing tadpole numerators unless the right-hand side of eq. (5.67) is equal to zero, as it happens for regular orbifolds.

6 On the double-copy construction of non-factorized gravity amplitudes

Whenever a pair of (supersymmetric) gauge theories coupled with matter fields in the adjoint representations can be related to a gravitational theory through Kawai-Lewellen-Tye relations [38], the amplitudes of the gravitational theory can be immediately obtained from a duality-satisfying presentation of the corresponding gauge amplitudes by replacing the color factors of one theory with the kinematic numerators of the second one corresponding the the same color factor [1, 16]. As summarized in earlier sections, this double-copy property of (super)gravity amplitudes was tested in $\mathcal{N} = 8$ supergravity as well as in supergravity theories with reduced supersymmetry and matter couplings. Many such theories can be obtained as factorized orbifolds of $\mathcal{N} = 8$ supergravity [13]. Other interesting theories – such as pure $\mathcal{N} = 3$ and $\mathcal{N} = 2$ supergravities – are however not factorized and thus it is not immediately clear how to construct their scattering amplitudes in terms of simpler gauge theories. The difficulty relates to the fact that the orbifold group acts on the two $\mathcal{N} = 4$ sYM copies making up the parent theory in a correlated fashion.

The formalism discussed in section 4.2 for the calculation of loop amplitudes in orbifold gauge theories suggests a possible approach to this problem. As we discussed at length, the numerator factors in eq. (4.8) receive contributions from fields of the parent ($\mathcal{N} = 4$ sYM theory) in representations $R_{l_1}, \ldots, R_{l_L}$ of Γ running in the $1, \ldots, L$ loop (and the summation projects out the non-invariant components of fields once the Γ-representation of the color factor is accounted for). This is very much analogous to what we need to "factorize" a non-factorizable theory. We can therefore formulate our proposal.

We consider an orbifold supergravity theory with an abelian orbifold group $\Gamma \in SU(4) \times SU(4) \subset SU(8)$ and denote by Γ_1 and Γ_2 the subgroups of Γ in the two $SU(4)$ factors. At least one of them is isomorphic to Γ, while the other is at least a subgroup of Γ. Assuming that amplitudes of the Γ_1 and Γ_2 orbifolds of $\mathcal{N} = 4$ sYM are known in a color/kinematics-satisfying representation of the form (4.8) with numerator factors $n_{i;R_{l_1},\ldots,R_{l_L}}$ and $\tilde{n}_{i;R'_{l_1},\ldots,R'_{l_L}}$ respectively, we expect that, for any number of external legs, the L-loop amplitudes of the Γ-orbifold of $\mathcal{N} = 8$ supergravity are given by

$$\mathcal{M}^{(L)} = \int \prod_{k=1}^{L} \frac{d^d l_k}{(2\pi)^d} \sum_{\substack{(R_{l_k}, R'_{l_k}) \in (\mathcal{R}_1, \mathcal{R}_2) \\ R_{l_k} R'_{l_k} = 1}} \sum_{i \in \mathcal{G}_3} \frac{1}{S_i} \frac{n_{i;R_{l_1},\ldots,R_{l_L}} \, \tilde{n}_{i;R'_{l_1},\ldots,R'_{l_L}}}{\prod_m p^2_{m,i}} \ . \tag{6.1}$$

Here $\mathcal{R}_1$ and $\mathcal{R}_2$ are the sets of representations of Γ_1 and Γ_2 on the fields of $\mathcal{N} = 4$ sYM theory. The assignment of representations to the two kinematic numerators guarantees that the supergravity fields that contribute to a numerator factor – realized as the tensor product of the fields of the two gauge theories – are neutral under Γ; this is realized either

as the tensor product of invariant fields, $R_l = 1 = R'_l$, or as the tensor product of fields with conjugate Γ-representations, $R'_l = R^*_l$.

Using the fact that at one loop each field contributes independently to amplitudes both in $\mathcal{N} = 4$ sYM and in $\mathcal{N} = 8$ supergravity, one can easily convince oneself that this proposal is manifestly true at this order. [19] To nonetheless illustrate it let us look at a $\Gamma = \mathbf{Z}_2$ orbifold which acts on the fundamental representation of $SU(8)$ as $(\mathbf{1}_2, -\mathbf{1}_2, \mathbf{1}_2, -\mathbf{1}_2)$; for this choice of Γ its two parts Γ_1 and Γ_2 are $\Gamma_1 = \Gamma_2 = \mathbf{Z}_2$. The two sets of $SU(4)$ representations are

$$\mathcal{R}_1 = \{1, r_3^3, r_4^4\} = \{1, -1, -1\} = \mathcal{R}_2 . \tag{6.2}$$

Then, since $r_i^i r_j^j = 1$ for all choices of $i, j = 3, 4$ the amplitude is

$$\mathcal{M}^{(1)} = \int \frac{d^d l}{(2\pi)^d} \sum_{i \in \mathcal{G}_3} \frac{1}{S_i} \frac{n_{i;1}\tilde{n}_{i;1} + \sum_{p,q=3,4} n_{i;r_p^p}\tilde{n}_{i;r_q^q}}{\prod_m p^2_{m,i}} . \tag{6.3}$$

It is not difficult to see that the second term in the numerator represents the contribution to the amplitude of four $\mathcal{N} = 4$ vector multiplets (e.g. by noticing that the tensor product of two $\mathcal{N} = 2$ hypermultiplets yields four $\mathcal{N} = 4$ vector multiplets). Using also the fact that the first term, $n_{i;1}\tilde{n}_{i;1}$, yields the amplitude of $\mathcal{N} = 4$ supergravity coupled to two vector multiplets [39, 13], it follows that (6.3) is the four-graviton amplitude of $\mathcal{N} = 4$ supergravity coupled to six vector multiplets.

It is not difficult to see that this is indeed the correct result. From the perspective of the states of $\mathcal{N} = 8$ supergravity one can interpret the $\mathbf{Z}_2$ orbifold acting as $(\mathbf{1}_2, -\mathbf{1}_2, \mathbf{1}_2, -\mathbf{1}_2)$ as a $\mathbf{Z}_2$ orbifold acting as $(\mathbf{1}_4, -\mathbf{1}_4)$; in this formulation the theory is factorized [13] and described as the double-copy of $\mathcal{N} = 4$ sYM and pure YM coupled to six real scalars – which is precisely $\mathcal{N} = 4$ supergravity coupled to six vector multiplets.

At higher loops the states propagating in each loop follow a similar pattern, as eq. (6.1) retains in each of them all the supergravity fields that are invariant under Γ. Thus, we expect that beyond one loop eq. (6.1) holds as long as the double-copy construction [5] holds for $\mathcal{N} = 8$ supergravity.

We note here that, while the construction described here accommodates a large class of supergravity theories with matter, it remains difficult to construct the scattering amplitudes of pure $\mathcal{N} \leq 3$ supergravities. To this end it seems necessary to enhance this double-copy construction with additional projections eliminating matter multiplets that appear together with the supergravity multiplet in the tensor product of $\mathcal{N} \leq 2$ vector multiplets.

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[19]Matter amplitudes are generically divergent in supergravity theories; their representation obtained by using eq. (6.1) with $\mathcal{N} \leq 1$ orbifold factors will contain tadpole integrals (which integrate to zero in dimensional regularization).

7 Conclusions and further comments

In this paper we discussed in detail the color/kinematics duality for general abelian orbifolds of $\mathcal{N} = 4$ sYM theory with an unitary gauge group. They have matter fields in the adjoint and bi-fundamental representation of gauge group; among them, corresponding to an unfaithful representation of the orbifold group in $SU(N)$ are also the pure $\mathcal{N} = 1$ and $\mathcal{N} = 2$ sYM theories as well as pure YM theory with 0, 2, 4, or 6 scalar fields, whose one-loop amplitudes in a color/kinematics-satisfying representation were discussed previously in [13] and [14], respectively[20]. An interesting result is that the one-loop four-gluon amplitudes of $\mathcal{N} = 2$ orbifold theories are determined by the $\mathcal{N} = 4$ and pure $\mathcal{N} = 2$ four-gluon one-loop amplitudes and the kinematic Jacobi relations. More generally, the integrands of amplitudes in all orbifolds with fixed amount of supersymmetry are described by a finite number of polynomials in external and loop momenta; differences between theories are encoded only in their color factors.

We have also carried out a comprehensive search for field theories with only massless fields in the adjoint representation and antisymmetric structure constant couplings whose amplitudes can exhibit color/kinematics duality for all external states. While tree-level four-point amplitudes with at least two gluons generically obey color/kinematics duality independently of the number of scalars and fermions in the theory, four-point amplitudes with only external scalars and/or fermions do not, unless the theory is some $\mathcal{N}$-extended pure sYM theory or the dimensional reduction of pure YM theory in some number of dimensions. We also showed that the amplitudes of pure YM theory coupled with a single fermion exhibit color/kinematics duality only in the dimensions in which the theory is supersymmetric, *i.e.* in $D = 3, 4, 6, 10$.

It would be very interesting explore the possibility of using the amplitudes of theories obeying color/kinematics duality to construct amplitudes in theories where the duality is not present; see [40] for a related discussion. A possible approach could be to start with a higher-dimensional theory with amplitudes obeying the duality and carry out a Scherk-Schwarz dimensional reduction; the numerator factors of amplitudes of the resulting lower-dimensional theory will still exhibit some form of color/kinematics duality while some fields will be massive. Taking the formal infinite mass limit at the level of the integrand

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[20]We note here again that, in general, non-supersymmetric orbifolds are on the "natural line", where the quartic scalar coupling is the same as the gauge coupling. However, such theories are not renormalizable already at one-loop unless the couplings are allowed to have different values. This holds, in particular, for the dimensional reduction of D-dimensional pure YM theory to four dimensions which, for $D \leq 10$ can be interpreted as an orbifold of $\mathcal{N} = 4$ sYM theory, as discussed here. Inspecting the four-vector amplitudes in $\mathcal{N} = 4$ supergravity with matter computed in [13] it is easy to see that their divergence originates form a one-loop divergence with box-graph color structure in the matter-coupled pure YM theory factor.

of loop amplitudes (*i.e.* the mass is assumed to be larger than any dimensionful regulator one might choose) one is left with a massless theory which, while *a priori* needs not exhibit the duality, has amplitudes whose kinematic numerators are related to each other and to the color factors by the Jacobi relations of the higher-dimensional theory.

We have also discussed the construction of scattering amplitudes in quiver gauge theories that have an orbifold point. While in general they may not obey color/kinematics duality due to different couplings for different gauge group factors, they can be obtained from the amplitudes at the orbifold point by judiciously dressing of graphs' vertices with different couplings for each gauge group, following the color and R-charge flow. It would be interesting to explore whether it is possible to endow more general quiver gauge theories (or more general gauge theories) with color/kinematics duality. A possible strategy may be to embed the theory in a larger one for which color/kinematic duality is present and decouple or project out the extra fields at the end of the calculation. To this end it would be interesting to study the interplay of color/kinematics duality and spontaneous symmetry breaking.

Certain field theories with fields in the fundamental representation can be obtained from quiver gauge theories by decoupling some of the gauge group factors. In theories with an orbifold point this can be done by first deforming them off the orbifold point and then taking to zero the coupling of the desired gauge group. As discussed in the introduction, an alternative strategy is to use the defining commutation relations of the gauge group as Jacobi identities [15]. We will argue here that this is indeed a direct consequence of the orbifold color/kinematics duality discussed in this paper for a non-regular orbifold which splits off one unit of rank from the gauge group of the parent theory. While the construction is quite general, here we illustrate it by considering a $\mathbf{Z}_2$ the orbifold generated by

$$r = \text{diag}\big(-1, -1, -1, -1\big), \qquad g = \begin{pmatrix} I_N & 0 \\ 0 & -1 \end{pmatrix}, \qquad (7.1)$$

which breaks the gauge symmetry of an $SU(N+1)$ sYM parent theory down to $SU(N) \times U(1)$. It is immediate to verify that the theory has one vector and six scalars transforming in the adjoint representation of $SU(N)$, one $U(1)$ vector and six additional scalars which are a singlets under $SU(N)$, together with four fundamental and four anti-fundamental fermions. To exhibit the consequence of color/kinematics duality we consider an amplitude with two external fermions, *e.g.* $\mathcal{A}(1^{\psi_i} 2^{\bar{\psi}_j} 3^{+a} 4^{-b})$; the fermions are labeled by their fundamental and anti-fundamental indices and the two gluons carry adjoint $SU(N)$ indices. As illustrated in section 4.1, the tree-level amplitude in the BCJ presentation has the same numerator factors as the corresponding amplitude in the parent theory while the color factors simply need to be dressed with the color wave-functions. In our example,

a solution to eq. (4.1) with r and g in eq. (7.1) is

$$v_{1,i}^A T_{i'j}^A = \delta_{i,i'}\delta_{j,N+1} , \qquad v_{2,j}^A T_{ij'}^A = \delta_{j,j'}\delta_{i,N+1} , \qquad v_{3,a}^A = v_{4,a}^A = \delta_a^A , \qquad (7.2)$$

where T_{ij}^A are $SU(N+1)$ generators. For $A < N^2$ they are also the $SU(N)$ generators and we denote the corresponding index by a. With a little algebra we can rewrite the color factors as

$$\begin{aligned}
C_s &= v_{1,i}^{A'}v_{2,j}^{B'}v_{3,a}^{C'}v_{4,b}^{D'} \tilde{f}^{A'B'X}\tilde{f}^{XC'D'} = \tilde{f}^{abX}T_{ji}^X , \\
C_t &= v_{1,i}^{A'}v_{2,j}^{B'}v_{3,a}^{C'}v_{4,b}^{D'} \tilde{f}^{A'D'X}\tilde{f}^{XB'C'} = -(T^aT^b)_{ji} , \\
C_u &= v_{1,i}^{A'}v_{2,j}^{B'}v_{3,a}^{C'}v_{4,b}^{D'} \tilde{f}^{A'C'X}\tilde{f}^{XD'B'} = (T^bT^a)_{ji} .
\end{aligned} \qquad (7.3)$$

Remarkably, the Jacobi identity of the $SU(N+1)$ gauge group of the parent theory becomes the defining commutation relation of the $SU(N)$ gauge group of the daughter theory:

$$C_s + C_t + C_u = 0 \quad \mapsto \quad [T^a, T^b] = if^{ab}{}_c T^c . \qquad (7.4)$$

The numerators factors corresponding to the three terms in the relation are unchanged.

Since the defining commutation relations of the gauge group make no reference to the initial orbifold construction, the relation (7.4) can be used as the starting point for defining color/kinematics duality in presence of field in the fundamental representation for theories which *do not* have an orbifold point as well as for theories with fields in *arbitrary* representations of the gauge group. The simplest instance is QED, where the gauge group is $U(1)$ and thus the right-hand side of eq. (7.4) vanishes. This would suggest that the numerator factors of two Feynman graphs contributing to *e.g.* $e^+e^- \to \gamma\gamma$ should be equal. This turns out to be the case,

$$A_{+-+-} = -2e^2 \left(\frac{n}{u} + \frac{n}{t} \right) \qquad \text{with} \qquad n = \frac{\langle 23\rangle\langle 13\rangle[41]^2}{s} . \qquad (7.5)$$

Here the fermion momenta are k_1 and k_2, the photon momenta are k_3 and k_4 and we choose the reference null vectors defining the photon polarization vectors as $q_3 = k_4$ and $q_4 = k_3$.[21] Using (7.4) and focusing on the transformation of fields in bi-fundamental representations under a single factor of the gauge group it may be possible to define and use color/kinematics duality for quiver gauge theories with general matter content at least on some "natural line" defined by specific relations between the couplings of various gauge group factors and matter fields.

Our formulation in section 4.2 of color/kinematics duality for orbifolds holds in principle to all orders in perturbation theory. It would of course be interesting and instructive to construct higher-loop examples of amplitudes and check whether or not they exhibit the

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[21] Alternatively, one may simply manipulate into this form the result obtained with a more standard choice of reference spinors (*e.g.* one for which one of the two Feynman graphs vanishes).

duality. As described there, the resulting integrand will be presented as a sum of terms each of which is the contribution of fields of specific representations under the $SU(4)$ part of the orbifold group running in the various loops. For pure sYM theories – *i.e.* for orbifolds with trivial action on the gauge group of the parent theory – contributions come only from invariant fields.

The details of the construction of BCJ representations of amplitudes of orbifold theories suggested a natural proposal for a double-copy construction of amplitudes in non-factorized orbifold supergravities for which the orbifold group can be embedded in an $SU(4) \times SU(4)$ subgroup of $SU(8)$. We illustrated it in a simple example and argued that it should hold as long as the amplitudes of $\mathcal{N} = 8$ supergravity are given by a double-copy construction. The field content of the theories covered by this construction is insensitive to the orbifold group action on the two gauge theory factors and is given by the neutral part of the tensor product of their fields. Thus, since the trivial R-symmetry representation is always part of the gauge theory spectrum, the matter content of these theories is always larger than the one of the corresponding factorized supergravity (with amplitudes given only in terms of $n_{i,1}$ and $\tilde{n}_{i,1}$); pure $\mathcal{N} \leq 3$ supergravities cannot be constructed this way. It may nevertheless be possible to enhance the double-copy formula (6.1) with additional projectors such that the resulting spectrum is only a subset of that of the simplest factorized supergravity with the same amount of supersymmetry.

Acknowledgments

As this paper was being written up we became aware of the upcoming paper [15] where color/kinematics duality for fields in the fundamental representation is discussed in terms of the commutation relations of the gauge group as well as a possible double-copy approach to the construction of one-loop amplitudes in pure $\mathcal{N} \leq 3$ supergravities.

A   Short summary of notation

Since the paper is notationally heavy, we give here a summary of our notation:

(r, g) generic element of the orbifold group with $r \in SU(4)$ and $g \in SU(N)$.

Γ orbifold group, assumed to be discrete and Abelian, and hence isomorphic to $\mathbf{Z}_k$.

$|\Gamma|$ rank of the orbifold group Γ.

R_i representation of Γ associated to the i-th external leg,
product of diagonal entries of r.

R_l representation of Γ associated to the internal leg carrying loop momentum l,
product of diagonal entries of r.

$\mathcal{R}$ set of possible R_l, chosen case-by-case.

$\mathcal{G}_3$ set of distinct cubic graphs.

B   Solutions to eqs. (3.15)

In this appendix we solve the equations (3.15) for the three $2 \leq n_f \leq 4$ allowed numbers of fermions[22]

For $n_f = 2$ (and $n_s = 2$, cf. eq. 3.17), the only independent equation is

$$\lambda_{12}^I \bar{\lambda}^{J12} + \lambda_{12}^J \bar{\lambda}^{I12} = g^2 \delta^{IJ} \ , \tag{B.1}$$

whose solutions are

$$\lambda_{12}^1 = \frac{g}{\sqrt{2}} e^{i\theta} \qquad , \qquad \lambda_{12}^2 = \pm i \frac{g}{\sqrt{2}} e^{i\theta} \ . \tag{B.2}$$

The phase can be eliminated by redefining the fermions, $\psi \to e^{-\theta/2}\psi$, and the two signs of λ_{12}^2 correspond to the two possible definitions of the complex scalar field:

$$\frac{i}{\sqrt{2}} \lambda_{AB}^I \psi^A [\phi^I, \ \psi^B] \to \frac{ig}{\sqrt{2}} \epsilon_{AB} \psi^A [\frac{\phi^1 \pm i\phi^2}{\sqrt{2}}, \ \psi^B] \ . \tag{B.3}$$

The resulting Lagrangian

$$\begin{aligned}
\mathcal{L}_{\mathcal{N}=2} = \mathrm{Tr}\Big[&-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - D_\mu \bar{\phi} D^\mu \phi + i\bar{\psi}_A \not{D}\psi^A - \frac{1}{2} g^2 [\phi, \ \bar{\phi}]^2 \\
&+ \frac{ig}{\sqrt{2}} \epsilon_{AB} \psi^A [\phi, \ \psi^B] + \frac{ig}{\sqrt{2}} \epsilon^{AB} \bar{\psi}_A [\bar{\phi}, \ \bar{\psi}_B] \Big]
\end{aligned} \tag{B.4}$$

is that of $\mathcal{N} = 2$ sYM theory.

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[22]$n_f = 0$ is trivial while $n_f = 1$ implies that $n_s = 0$ so $\lambda = 0$. The resulting Lagrangians are those of pure $\mathcal{N} = 0$ and $\mathcal{N} = 1$ (s)YM theories.

For $n_f = 3$, $n_s = 4$, the equation (3.15) has 6 independent components when $I = J$:

$$\lambda^I_{12}\bar{\lambda}^{I12} + \lambda^I_{31}\bar{\lambda}^{I31} = \frac{1}{2}g^2$$

$$\lambda^I_{23}\bar{\lambda}^{I23} + \lambda^I_{12}\bar{\lambda}^{I12} = \frac{1}{2}g^2$$

$$\lambda^I_{31}\bar{\lambda}^{I31} + \lambda^I_{23}\bar{\lambda}^{I23} = \frac{1}{2}g^2 \qquad (B.5)$$

$$\lambda^I_{21}\bar{\lambda}^{I31} = 0$$

$$\lambda^I_{12}\bar{\lambda}^{I32} = 0$$

$$\lambda^I_{13}\bar{\lambda}^{I23} = 0 \; ;$$

the repeated I index is not summed over. The first 3 equations fix the absolute value of λ^I_{12}, λ^I_{23} and λ^I_{31} for each $I = 1, \ldots, 4$:

$$|\lambda^I_{12}| = \frac{g}{2} = |\lambda^I_{23}| = \frac{g}{2} = |\lambda^I_{31}| \; . \qquad (B.6)$$

This is however inconsistent with the last three equations which require that at least two of them vanish. Thus, for $n_f = 3$ the equations (3.15) have no solution.

For $n_f = 4$, $n_s = 6$ the system (3.15) has the following unique solution relating λ and its conjugate:

$$\lambda^I_{AB} = \frac{\rho^I}{2}\epsilon_{ABCD}\bar{\lambda}^{ICD} \quad \text{with} \quad \rho^I = \frac{1}{4g^2}\epsilon^{ABCD}\lambda^I_{AB}\lambda^I_{CD} \; ; \qquad (B.7)$$

the index I in the definition of ρ^I is not summed over. The condition that λ and $\bar{\lambda}$ are conjugates of each other implies that ρ^I is a phase.

The remaining freedom in the choice of Yukawa couplings drops out of the Lagrangian. To see this we define the complex scalars

$$\phi^{AB} = \sum_I \frac{\sqrt{\rho^I}}{g}\bar{\lambda}^{IAB}\phi^I \qquad \bar{\phi}_{AB} = \sum_I \frac{\sqrt{\bar{\rho}^I}}{g}\lambda^I_{AB}\phi^I \; ; \qquad (B.8)$$

it is not difficult to see that the properties (B.7) of the Yukawa couplings imply that complex conjugation is the same as lowering of indices with the Levi-Civita tensor:

$$\frac{1}{2}\epsilon_{ABCD}\phi^{CD} = \frac{1}{2}\epsilon_{ABCD}\sum_I \frac{\sqrt{\rho^I}}{g}\bar{\lambda}^{IAB}\phi^I = \sum_I \frac{\sqrt{\bar{\rho}^I}}{g}\lambda^I_{AB}\phi^I = \bar{\phi}_{AB} \; . \qquad (B.9)$$

In terms of the new scalar field, the scalar-fermion interaction term becomes

$$\frac{i}{\sqrt{2}}\lambda^I_{AB}\psi^A[\phi^I, \; \psi^B] \to \frac{ig}{\sqrt{2}}\psi^A[\bar{\phi}_{AB}, \; \psi^B] \qquad (B.10)$$

while the quadratic scalar term is

$$D_\mu \bar{\phi}_{AB} D^\mu \phi^{AB} = 2 \sum_{I,J} \frac{\sqrt{\bar{\rho}^I}}{g} \frac{\sqrt{\rho^J}}{g} \delta^{IJ} g^2 D_\mu \phi^I D^\mu \phi^J = 2 \sum_I D_\mu \phi^I D^\mu \phi^I \ . \tag{B.11}$$

The resulting Lagrangian,

$$
\begin{aligned}
L_{\mathcal{N}=4} = &-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{4} D_\mu \bar{\phi}_{AB} D^\mu \phi^{AB} + \frac{g^2}{16} [\bar{\phi}_{AB},\, \bar{\phi}_{CD}][\phi^{AB},\, \phi^{CD}] \\
&+ i \bar{\psi}_A \bar{\sigma}^\mu D_\mu \psi^A + \frac{ig}{\sqrt{2}} \left(\bar{\psi}_A \left[\phi^{AB},\, \bar{\psi}_B\right] + \psi^A \left[\bar{\phi}_{AB},\, \psi^B\right] \right)
\end{aligned}
\tag{B.12}
$$

is that of $\mathcal{N} = 4$ sYM theory.

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