

A multi-dimensional numerical scheme for two-fluid Relativistic MHD

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ABSTRACT

The paper describes an explicit multi-dimensional numerical scheme for Special Relativistic Two-Fluid Magnetohydrodynamics of electron-positron plasma and a suit of test problems. The scheme utilizes Cartesian grid and the third order WENO interpolation. The time integration is carried out using the third order TVD method of Runge-Kutta type, thus ensuring overall third order accuracy on smooth solutions. The magnetic field is kept near divergence-free by means of the method of generalized Lagrange multiplier. The test simulations, which include linear and non-linear continuous plasma waves, shock waves, strong explosions and the tearing instability, show that the scheme is sufficiently robust and confirm its accuracy.

Key words: magnetic fields – plasmas – relativistic processes – MHD – waves – methods: numerical

1 INTRODUCTION

It is now well recognized that magnetic fields play a very important role in many astrophysical phenomena and in particular in those involving relativistic outflows. The magnetic fields are likely to be involved in launching, powering and collimation of such outflows. The dynamics of relativistic magnetized plasma can be studied using diverse mathematical frameworks. The most developed one so far is the single fluid ideal relativistic Magnetohydrodynamics (RMHD). During the last decade, an impressive progress has been achieved in developing robust and efficient computer codes for integration of its equations. Only within a scope of proper review one can acknowledge the numerous contributions made by many different research groups and individuals. This framework is most suitable for studying large-scale macroscopic motions as it does not require to resolve the microphysical scales for numerical stability. One of its obvious disadvantages is that it allows only numerical dissipation. While at shocks this is not of a problem, the dissipation observed in simulations at other locations may be questionable. This becomes more of an issue as ever growing body of evidence suggests the importance of magnetic dissipation associated with magnetic reconnection in dynamics of highly magnetized relativistic plasma. The framework of resistive RMHD allows to incorporate this magnetic dissipation but the inevitably phenomenological nature of its Ohm's law puts constraints on its robustness. It is not yet clear how much progress can be achieved in this direction.

At the other extreme is the particle dynamics, describing the motion of individual charges in their collective electromagnetic field. The so-called Particle-in-Cell (PIC) numerical approach has been extremely productive in studying the microphysics of relativistic collisionless shock waves and magnetic reconnection. The numerical stability considerations require PIC codes to resolve the scales of plasma oscillations. The accuracy considerations can be even more demanding, pushing towards the particle gyration scales. The accuracy of PIC simulations also depends on the number of macro-particles and rather weakly, with the numerical noise level being inversely proportional of the square root of the number. These features make this approach less suitable for studying macroscopic phenomena and only few examples of such simulations can be found in literature (e.g. Arons et al. 2005; Spitkovsky & Arons 2002).

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Somewhere in between there lays the multi-fluid approximation, where plasma is modeled as a collection of several interpenetrating charged and neutral fluids, coupled via macroscopic electromagnetic field and inter-fluid friction. Undoubtedly, this approach is not as comprehensive in capturing the microphysics of collisionless plasma as the particle dynamics (and kinetics). However, it does this better than the single fluid MHD. In particular, it captures the collective interaction between positive and negative charges which results in plasma waves with frequencies above the plasma frequency. Compared to the resistive MHD, the generalized Ohm's law of the multi-fluid approximations takes into account the finite inertia of charged particles. From the numerical prospective, the potential of this framework is not clear. By the present time, there has been only a very limited effort to explore it. Zenitani et al. (2009a,b) used this approach for studying the relativistic magnetic reconnection, Amano & Kirk (2013) for studying the termination shocks of pulsar winds, and Kojima & Oogi (2009) tried to construct two-fluid models of steady-state pulsar magnetospheres. The ability, or the lack of it, to describe accurately the magnetic reconnection is probably the most important issue. Zenitani et al. (2009a) concluded that the inertial terms make a significant contribution to the reconnection electric field, even exceeding that of the friction term, which represents the resistivity. This hints that the inertial terms can play an important role in driving the magnetic reconnection. On the other hand, they have also found that the outcome strongly depends on the model of resistivity. Thus, the reconnection issue remains widely open. Another issue is the required numerical resolution. Zenitani et al. (2009a,b) set the grid scale to be comparable to the electron gyration radius. If such high resolution is indeed required then it will be easier just to do PIC simulations. Only if much lower resolution is sufficient, the multi-fluid approach can be advantageous when it comes to macroscopic simulations.

The aforesaid works on the relativistic multi-fluid numerical simulations neither described in details the design of their numerical schemes nor presented test simulations demonstrating their robustness and accuracy. Zenitani et al. (2009a,b) state that they use a “modified Lax-Wendroff scheme”, which employs artificial viscosity, and comment that this is a “bottleneck” of their code. In particular, they find that the contribution of the artificial viscosity to the reconnection electric field is comparable to other contributions. In this paper, we present a high-order accurate Godunov-type scheme for the two-fluid RMHD which does not involve such artificial terms. As a first step, we constrain ourselves with the Special Relativistic framework and the case of pure electron-positron plasma. The scheme utilizes the third order WENO interpolation and the time integration is carried out using the third order TVD method of Runge-Kutta type, thus ensuring overall third order accuracy on smooth solutions. In order to keep the magnetic field divergence-free we employ the method of generalized Lagrange multiplier. The code is parallelized using MPI. We also present a suit of one-dimensional and multi-dimensional test problems and the results of our test simulations. The code is parallelized using MPI. It can be provided by the authors on request.

2 3+1 SPECIAL RELATIVISTIC ELECTRON-POSITRON TWO-FLUID MHD

In the multi-fluid approximation, it is assumed that plasma consists of several interacting fluids, one per each kind of particles involved, and the electromagnetic field. Charged fluids interact with each other electromagnetically and via the so-called inter-fluid friction. The coupling between a fluid of electrically neutral particles and other fluids is only via the inter-fluid friction. Given such understanding, the system of multi-fluid approximation consists of the Maxwell equations and copies of the usual single fluid equations, one per each involved fluid, augmented with the force terms describing these interactions. The fluid equations may be derived via averaging of the corresponding kinetic equations (e.g. Goedbloed & Poedts 2004).

In one form or another, the equations of relativistic two-fluid MHD were used by many researchers, going back to the work by Akhiezer & Polovin (1956). Following Zenitani et al. (2009a) we adopt the 3+1 Special Relativistic equations originated from the covariant formulation by Gurovich & Solov'ev (1986). In a global inertial frame with time-independent coordinate grid, the equations of two-fluid RMHD include the continuity equations

$$\partial_t(\rho_{\pm}\gamma_{\pm}) + \nabla_i(\rho_{\pm}u_{\pm}^i) = 0, \quad (1)$$

the energy equations

$$\partial_t(w_{\pm}\gamma_{\pm}^2 - p_{\pm}) + \nabla_i(w_{\pm}\gamma_{\pm}u_{\pm}^i) = (j_{\pm}^i E_i) \pm \sigma_{\mp} c^2 n_{+} n_{-} (\gamma_{-} - \gamma_{+}), \quad (2)$$

the momentum equations

$$\partial_t((w_{\pm}/c^2)\gamma_{\pm}u_{\pm}^s) + \nabla_i((w_{\pm}/c^2)u_{\pm}^i u_{\pm}^s + g^{is} p_{\pm}) = q_{\pm} E^s + \frac{1}{c} e^{sik} j_{\pm}^{\pm} B_k \pm \sigma_{\mp} n_{+} n_{-} (u_{-}^s - u_{+}^s), \quad (3)$$

and the Maxwell equations

$$\nabla_i B^i = 0, \quad (4)$$

$$\frac{1}{c} \partial_t B^s + e^{sik} \partial_i E_k = 0, \quad (5)$$

$$\nabla_i E^i = 4\pi(q_+ + q_-), \quad (6)$$

$$\frac{1}{c}\partial_t E^s - e^{sik}\partial_i B_k = -\frac{4\pi}{c}(j_+^s + j_-^s). \quad (7)$$

In these equations, n_{\pm} is the proper number density of species, $\rho_{\pm} = m_e n_{\pm}$ is the proper mass density, γ_{\pm} is the Lorentz factor, $u_{\pm}^i = \gamma_{\pm} v_{\pm}^i$ are the spatial components of 4-velocity, w_{\pm} is the relativistic enthalpy, p_{\pm} is the isotropic thermodynamic pressure, $q_{\pm} = \pm en_{\pm}$ is the electric charge density, $j_{\pm}^i = \pm en_{\pm} u_{\pm}^i$ is the electric current density, σ_f is the dynamic coefficient of friction between these fluids, g^{ij} is the metric tensor of space, e^{ijk} is the Levi-Civita tensor of space, and c is the speed of light. These equations have to be supplemented by equations of state (EOS), e.g. the dimensionless polytropic EOS

$$w_{\pm} = n_{\pm} m_e c^2 + \Gamma p_{\pm} / (\Gamma - 1), \quad (8)$$

where Γ is the ratio of specific heats.

Denote as B_0 the characteristic value of magnetic (and electric) field, as T_0 the characteristic time scale, as $L_0 = cT_0$ the characteristic length scale, and as n_0 the characteristic number density of particles. Then the characteristic scale for ρ is $m_e n_0$ and the characteristic scale for w and p is $m_e c^2 n_0$. The corresponding dimensionless equations of the two-fluid electron-positron RMHD are

$$\partial_t(n_{\pm}\gamma_{\pm}) + \nabla_i(n_{\pm}u_{\pm}^i) = 0, \quad (9)$$

$$\partial_t(w_{\pm}\gamma_{\pm}^2 - p_{\pm}) + \nabla_i(w_{\pm}\gamma_{\pm}u_{\pm}^i) = \pm \frac{1}{\mathcal{K}_m} n_{\pm}(u_{\pm}^i E_i) \pm \frac{1}{\mathcal{K}_f} n_+ n_- (\gamma_- - \gamma_+), \quad (10)$$

$$\partial_t(w_{\pm}\gamma_{\pm}u_{\pm}^s) + \nabla_i(w_{\pm}u_{\pm}^i u_{\pm}^s + g^{is}p_{\pm}) = \pm \frac{1}{\mathcal{K}_m} n_{\pm}\gamma_{\pm}(E^s + e^{sik}v_i^{\pm}B_k) \pm \frac{1}{\mathcal{K}_f} n_+ n_- (u_-^s - u_+^s), \quad (11)$$

$$\nabla_i B^i = 0, \quad (12)$$

$$\partial_t B^s + e^{sik}\partial_i E_k = 0, \quad (13)$$

$$\nabla_i E^i = \frac{1}{\mathcal{K}_q}(n_+\gamma_+ + n_-\gamma_-), \quad (14)$$

$$\partial_t E^s - e^{sik}\partial_i B_k = -\frac{1}{\mathcal{K}_q}(n_+u_+^s - n_-u_-^s). \quad (15)$$

The three dimensionless parameters in these equations are

$$\mathcal{K}_q = \frac{B_0}{4\pi e L_0 n_0}, \quad \mathcal{K}_m = \frac{m_e c^2}{e B_0 L_0}, \quad \mathcal{K}_f = \frac{m_e c}{\sigma_f n_0 L_0}. \quad (16)$$

The dimensionless polytropic EOS is

$$w_{\pm} = n_{\pm} + \Gamma p_{\pm} / (\Gamma - 1). \quad (17)$$

We write the dimensionless two-fluid equations in the form which focuses on their mathematical structure. Although this somewhat hides the physical origin of the system, all the important plasma parameters can be easily recovered when this is needed. Some of them are listed in Table 1. The parameters \mathcal{K}_m , \mathcal{K}_f , and \mathcal{K}_q also allow simple physical interpretation. \mathcal{K}_m is the ratio of the particle rest mass energy $m_e c^2$ to the potential drop $e B_0 L_0$ in the electric field of strength $E_0 = B_0$ over the characteristic length scale of the problem. Obviously, this is a measure of the magnetic field strength. \mathcal{K}_f is the ratio of the relaxation time scale $T_f = m_e / n_0 \sigma_f$ in plasma of number density n_0 and the light-crossing time scale $T_0 = L_0 / c$. \mathcal{K}_q is the inverse ratio of n_0 and the Goldreich-Julian number density of charged particles

$$n_{GJ} = B_0 / 4\pi e L_0. \quad (18)$$

From the expressions for plasma β and σ given in Table 1 it is easy to see how to setup two-fluid problems corresponding to single-fluid RMHD problems, e.g. the test problems of Komissarov (1999). One can use the same dimensionless values for the total thermodynamic pressure and density, but scale the electromagnetic field by the factor of $\sqrt{\mathcal{K}_m / \mathcal{K}_q}$.

Table 1. Important plasma parameters in dimensional and dimensionless form. Here the magnetic field \hat{B} is measured in the plasma rest frame, $w = w_+ + w_-$, $p = p_+ + p_-$ and $n = n_+ + n_-$.

Name	Dimensional expression	Dimensionless expression
plasma frequency	$\Omega_p^2 = \frac{4\pi e^2 n^2 c^2}{w}$	$\Omega_p^2 = \frac{\mathcal{K}_q}{\mathcal{K}_m} \left(\frac{n^2}{w}\right)$
skin depth	$\lambda_s = \frac{c}{\Omega_p}$	$\lambda_s = \left(\frac{1}{\Omega_p}\right)$
magnetization β	$\beta = \frac{8\pi p}{B^2}$	$\beta = \frac{\mathcal{K}_m}{\mathcal{K}_q} \left(\frac{2p}{B^2}\right)$
magnetization σ	$\sigma = \frac{\hat{B}^2}{8\pi w}$	$\sigma = \frac{\mathcal{K}_q}{\mathcal{K}_m} \left(\frac{\hat{B}^2}{2w}\right)$
gyration radius	$R_g = \frac{m_e c^2}{eB} \left(\left(\frac{w-p}{\rho c^2}\right)^2 - 1 \right)^{1/2}$	$R_g = \frac{\mathcal{K}_m}{B} \left(\left(\frac{w-p}{n}\right)^2 - 1 \right)^{1/2}$

By summing together the energy equations Eqs.10 and then utilizing the Maxwell equations one obtains the conservation law for the total energy,

$$\partial_t \left(\sum_{\pm} (w_{\pm} \gamma_{\pm}^2 - p_{\pm}) + \frac{\mathcal{K}_q}{\mathcal{K}_m} (B^2 + E^2) \right) + \nabla_i \left(\sum_{\pm} w_{\pm} \gamma_{\pm} u_{\pm}^i + \frac{\mathcal{K}_q}{\mathcal{K}_m} e^{ijk} E_j B_k \right) = 0. \quad (19)$$

Similarly, one finds the total momentum conservation law

$$\partial_t \left(\sum_{\pm} w_{\pm} \gamma_{\pm} u_{\pm}^s + \frac{\mathcal{K}_q}{\mathcal{K}_m} e^{sjk} E_j B_k \right) + \nabla_i \left(\sum_{\pm} (w_{\pm} u_{\pm}^i u_{\pm}^s + p_{\pm} g^{is}) + \frac{\mathcal{K}_q}{\mathcal{K}_m} \left(-E^i E^s - B^i B^s + \frac{1}{2} (B^2 + E^2) g^{is} \right) \right) = 0. \quad (20)$$

It is advantageous to discretize these equations, when constructing a numerical scheme, as they allow to conserve the total energy-momentum down to the rounding error.

By subtracting the momentum equations for the positron and electron fluids one obtains the so-called generalized Ohm law of the two-fluid MHD:

$$\partial_t \left(\sum_{\pm} \pm (w_{\pm} \gamma_{\pm} u_{\pm}^s) \right) + \nabla_i \left(\sum_{\pm} \pm (w_{\pm} u_{\pm}^i u_{\pm}^s + p_{\pm}) g^{is} \right) = \frac{1}{\mathcal{K}_m} \tilde{n} (E^s + e^{sik} v_i B_k) + \frac{2}{\mathcal{K}_f} n_+ n_- (u_-^s - u_+^s). \quad (21)$$

In this equation, $\tilde{n} = n_+ \gamma_+ + n_- \gamma_-$ is the total number density of charged particles as measured in the laboratory frame and $v^i = (n_+ \gamma_+ v_+^i + n_- \gamma_- v_-^i) / \tilde{n}$ is their average velocity in this frame. Similarly, one finds

$$\partial_t \left(\sum_{\pm} \pm (w_{\pm} \gamma_{\pm}^2 - p_{\pm}) \right) + \nabla_i \left(\sum_{\pm} \pm (w_{\pm} \gamma_{\pm} u_{\pm}^i) \right) = \frac{1}{\mathcal{K}_m} \tilde{n} (v^i E_i) + \frac{2}{\mathcal{K}_f} n_+ n_- (\gamma_- - \gamma_+). \quad (22)$$

3 NUMERICAL METHOD

3.1 Augmented Maxwell Equations

In order to keep the magnetic field divergence free and to ensure that the Gauss law is satisfied as well, we use the so-called Generalized Lagrange Multiplier method (Munz et al. 2000; Dedner et al. 2002; Komissarov 2007). To this end, the Maxwell equations are modified in such a way that all four of them become dynamic. Namely,

$$\partial_t \Phi + \nabla_i B^i = -\kappa \Phi, \quad (23)$$

$$\partial_t B^s + e^{sik} \partial_i E_k + \nabla^s \Phi = 0, \quad (24)$$

$$\partial_t \Psi + \nabla_i E^i = \frac{1}{\mathcal{K}_q} (n_+ \gamma_+ + n_- \gamma_-) - \kappa \Psi, \quad (25)$$

$$\partial_t E^s - e^{sik} \partial_i B_k + \nabla^s \Psi = -\frac{1}{\mathcal{K}_q} (n_+ u_+^s - n_- u_-^s), \quad (26)$$

where the scalars Φ and Ψ are two additional dynamic variables. Obviously, the resultant equations are of the same structure as the fluid equations and hence they can be discretized in exactly the same way. This property greatly simplifies computational algorithms and coding.

3.2 Finite-difference conservative scheme

In addition to the augmented Maxwell equations, our system of integrated equations includes equations (9,19,20, 21,22). All these equations are conservation laws, some of them with source terms. Taken together, they can be written as a single phase vector equation of the form

$$\frac{\partial \mathcal{Q}}{\partial t} + \frac{\partial \mathcal{F}^m}{\partial x^m} = \mathcal{S}, \quad (27)$$

where \mathcal{Q} is the vector of conserved variables, \mathcal{F}^m is the vector of fluxes in the x^m direction and \mathcal{S} is the vector of source terms. When $\{x^m\}$ are Cartesian coordinates, and this is the only case we have coded so far, one can simply replace ∇_m with ∂_m and hence read the components of \mathcal{Q} , \mathcal{F}^m and \mathcal{S} directly from the aforesaid equations. Otherwise, these expressions will also involve components of the metric tensor.

Introduce a uniform coordinate grid such that x_i^m is the coordinate of the center of the i th cell, and $x_{i\pm 1/2}^m$ are the coordinates of its right and left interfaces. Replace, the continuous function Q with a discrete function defined at the cell centers and the continuous functions F^m with discrete functions defined at the centers of x^m cell interfaces. Finally, replace $\partial_m \mathcal{F}^m$ with the finite difference approximations:

$$\mathcal{DF}_{i,j,k} = \frac{\mathcal{F}_{i+1/2,j,k}^1 - \mathcal{F}_{i-1/2,j,k}^1}{\Delta x^1} + \frac{\mathcal{F}_{i,j+1/2,k}^2 - \mathcal{F}_{i,j-1/2,k}^2}{\Delta x^2} + \frac{\mathcal{F}_{i,j,k+1/2}^3 - \mathcal{F}_{i,j,k-1/2}^3}{\Delta x^3}. \quad (28)$$

As the result Eq.27 becomes a set of coupled ODEs

$$\frac{d\mathcal{Q}_{i,j,k}}{dt} = \mathcal{L}_{i,j,k}(\mathcal{Q}) \quad (29)$$

where

$$\mathcal{L}_{i,j,k}(\mathcal{Q}) = -\mathcal{DF}_{i,j,k}(\mathcal{Q}) + \mathcal{S}(\mathcal{Q}_{i,j,k}). \quad (30)$$

In this equation, the notation $\mathcal{DF}_{i,j,k}(\mathcal{Q})$ means that, strictly speaking, the argument of $\mathcal{DF}_{i,j,k}$ is the whole of the discrete function \mathcal{Q} (exactly the same applies to $\mathcal{L}_{i,j,k}(\mathcal{Q})$). However in reality, only few of its values, in the cells around the (i, j, k) -cell, are involved – this defines the scheme stencil.

It is easy to see that, when the equations of the system (29) are integrated simultaneously, the quantities

$$\sum_{i,j,k} \mathcal{Q}_{i,j,k} \quad (31)$$

are conserved down to rounding error. Any larger variation may only be due to a non-vanishing source term in (29) or a non-vanishing total flux of the quantity through the boundary of the computational domain. In this sense, this finite difference scheme is conservative.

3.3 Third-order WENO interpolation

In order to calculate fluxes at the cell interfaces, and hence $\mathcal{DF}_{i,j,k}(\mathcal{Q})$, we use the third-order weighted essentially non-oscillatory (WENO) interpolation developed by Liu et al. (1994), in the improved form proposed by Yamaleev & Carpenter (2009). However, instead of interpolating the conserved variables we prefer to interpolate the primitive ones, $\mathcal{P} = (n_{\pm}, p_{\pm}, u_{\pm}^s, E^s, B^s, \Phi, \Psi)$. The benefit of this is twofold. Firstly, this reduces by the factor of two the number of required conversions of conserved variables into the primitive ones. Indeed, in order to compute the flux vector from the vector of conserved variables one would have to convert the latter into the vector of primitive variables first. This would have to be done twice per each interface. Secondly, this ensures physically meaningful interpolated phase states.

The aim of the interpolation is to provide us with phase states immediately to the left and right of each cell interface given those at the cell centers. In the third order WENO interpolation, the interface values are found via a convex linear combination of two linear interpolants, which provides third order accuracy on smooth solutions. For the i -th cell the interpolants are

$$u_+(x) = u_i + \frac{\Delta u_+}{\Delta x}(x - x_i) \quad \text{and} \quad u_-(x) = u_i + \frac{\Delta u_-}{\Delta x}(x - x_i),$$

where $\Delta u_+ = u_{i+1} - u_i$ and $\Delta u_- = u_i - u_{i-1}$. Here u is the interpolated variable (We consider only one of the coordinate directions and omit indexes for others.). Their convex combination is

$$u = w_+ u_+ + w_- u_-,$$

where $w_{\pm} > 0$ and $w_+ + w_- = 1$. These conditions on the weights are satisfied when

$$w_+ = \frac{\alpha_+}{\alpha_+ + \alpha_-} \quad \text{and} \quad w_- = \frac{\alpha_-}{\alpha_+ + \alpha_-},$$

where $\alpha_{\pm} > 0$. Removing the arbitrary small constant ϵ , artificially introduced in Yamaleev & Carpenter (2009) in order to avoid division by zero during the calculations of α_{\pm} , and somewhat rearranging their equations, we find the interpolated values of u at the left and right interfaces of the cell

$$u_{i\pm 1/2} = u_i \pm (a_{\pm} \Delta u_{\pm} + \frac{1}{2} a_{\mp} \Delta u_{\mp}) / (2a_{\pm} + a_{\mp}), \quad (32)$$

where

$$a_{\pm} = \Delta u_{\mp}^2 (\Delta u_{\pm}^2 + (\Delta u_+ + \Delta u_-)^2). \quad (33)$$

It is easy to verify that Eq.32 yields $u_{i\pm 1/2} = u_i$ when either $\Delta u_- = 0$ or $\Delta u_+ = 0$. Thus, one can put

$$u_{i\pm 1/2} = u_i \quad \text{if} \quad \Delta u_- \Delta u_+ = 0. \quad (34)$$

This allows us to avoid the 0/0 indeterminacy implied by Eqs.32,33, via a simple inspection of data.

3.4 Riemann solver

Using the interpolation described in the previous section, one obtains two phase states, \mathcal{P}_l and \mathcal{P}_r , to the left and right of each cell interface respectively. This defines 1D Riemann problems whose solutions can be used to compute the interface fluxes required in Eq.28. Finding exact and even linearized solutions of Riemann problems for complicated systems of equations can be rather expensive. In such cases, it makes sense to utilize methods which use minimum amount of information about their characteristic properties, like the Lax-Friedrich flux, which requires to know only the highest wave speed of the system and purely for the stability consideration. Since in the relativistic framework this speed is limited from above by the speed of light, the latter can replace the former in the Lax-Friedrich flux, leading to the equation

$$\mathcal{F} = \frac{1}{2}(\mathcal{F}_l + \mathcal{F}_r) - \frac{1}{2}(\mathcal{Q}_r - \mathcal{Q}_l), \quad (35)$$

where $\{\mathcal{F}_l, \mathcal{F}_r\}$ and $\{\mathcal{Q}_l, \mathcal{Q}_r\}$ are the vectors of fluxes and conserved variables corresponding to the left and the right phase states respectively. The second term in the right-hand side of this equation introduces diffusion, which stabilizes numerical solutions. Although this diffusion may be excessive, its negative effect is significantly diminished in smooth regions, when schemes with highly accurate reconstruction are employed.

3.5 Time integration

For the integration of the coupled ODEs given by Eq.29 we use the popular third-order total variation diminishing (TVD) Runge-Kutta scheme by Shu & Osher (1988). The one time-step calculations proceed as follows.

$$\begin{aligned} \mathcal{Q}^{(1)} &= \mathcal{Q}^n + \Delta t \mathcal{L}(\mathcal{Q}^n), \\ \mathcal{Q}^{(2)} &= \mathcal{Q}^n + \frac{\Delta t}{4} (\mathcal{L}(\mathcal{Q}^{(1)}) + \mathcal{L}(\mathcal{Q}^n)), \\ \mathcal{Q}^{n+1} &= \mathcal{Q}^n + \frac{\Delta t}{6} (\mathcal{L}(\mathcal{Q}^n) + \mathcal{L}(\mathcal{Q}^{(1)}) + 4\mathcal{L}(\mathcal{Q}^{(2)})), \end{aligned} \quad (36)$$

where \mathcal{Q}^n and \mathcal{Q}^{n+1} are the values of \mathcal{Q} at the times t^n and $t^{n+1} = t^n + \Delta t$, and $\mathcal{Q}^{(1)}$ and $\mathcal{Q}^{(2)}$ are two auxiliary vectors. For stability, the time step must satisfy the Courant condition. Since the highest speed of communication in our case is the speed of light, this condition reads $\Delta t = c_n \min(\Delta x^i)$, where $c_n < 1$. Even smaller time step is required when the source terms are large.

3.6 Conversion of conservative into primitive variables

The recovering of the hydrodynamic primitive variables is carried out in three steps. First we use updated \mathbf{E} and \mathbf{B} to remove the electromagnetic contributions from the updated total energy and momentum. Second, via adding and subtracting the corresponding conserved variables we recover the energies and momenta of the electron and positron fluids. In the last step, we apply an iterative procedure to convert these conserved variables into the primitive variables of both fluids. This last step is the same as in a purely hydro code.

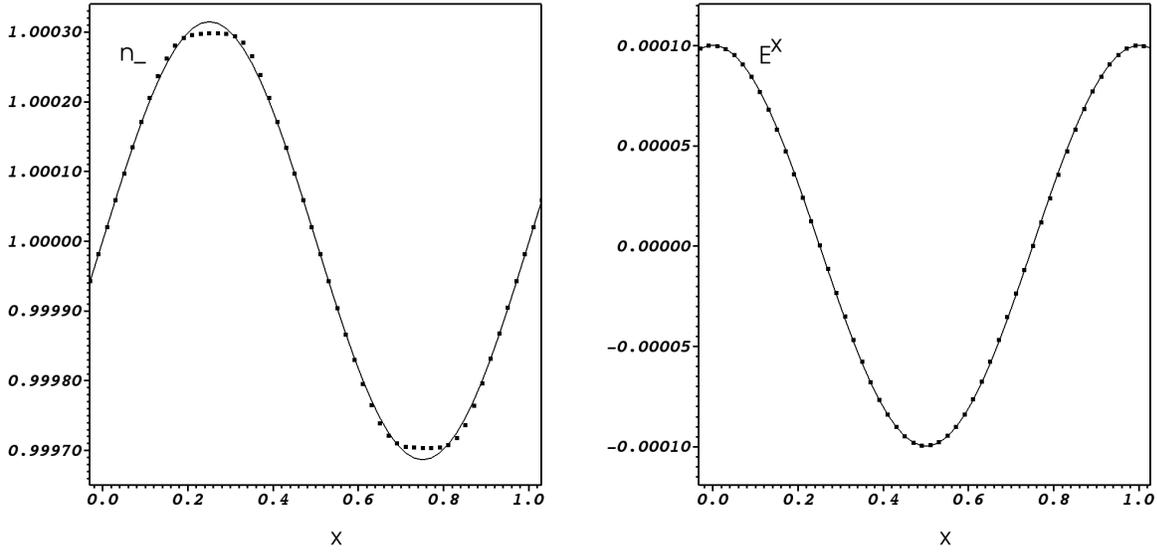


Figure 1. Propagation of small amplitude electrostatic waves. The number density of electrons (left panel) and the electric field (right panel). The continuous lines show the initial solution and the dots show the numerical solution after one period.

4 TEST SIMULATIONS

Many of our test simulations are based on slab-symmetric one-dimensional problems with known exact analytic or semi-analytic solutions. Such problems allow to test not only the 1D version of the code but also the multi-dimensional ones. The latter is achieved simply via using the multi-dimensional codes to solve one-dimensional problems aligned with one of the grid directions. In addition, we used truly multi-dimensional problems for which there is no exact solutions but which have been found useful for testing one-fluid RMHD codes. The simulations have been carried out either on a laptop or/and a workstation with multi-core processors.

4.1 Electrostatic waves

The two-fluid equations allow additional types of waves compared to the single fluid MHD. The most basic new waves are the electrostatic waves. In relativistic two-fluid MHD with $\mathcal{K}_f = \infty$, the properties of linear electrostatic waves were first studied by Sakai & Kawata (1980). Their background solution describes uniform state with zero velocity and electric field. The waves propagate along the magnetic field and perturb only the particle density and the components of velocity and electric field along the magnetic field. Choosing the x axis aligned with the magnetic field, putting $A = \tilde{A} \exp(i(\Omega t - kx))$ for the perturbation and assuming the same thermodynamic parameters for electron and positron fluids, one finds the dispersion relation

$$\Omega^2 = \Omega_p^2 (1 + (k/k_b)^2), \quad (37)$$

where Ω_p is the electron plasma frequency of the uniform background and

$$k_b^2 = \frac{1}{\mathcal{K}_m} \left(\frac{2n_b^2}{\Gamma p_b} \right), \quad (38)$$

where the n_b and p_b are the background electron (and positron) number density and pressure respectively. For the Fourier amplitudes one has

$$\tilde{n}_\pm = n_b \left(\frac{k}{\Omega} \right) \tilde{v}_\pm^x \quad (39)$$

and

$$\tilde{v}_\pm^x = \pm i \frac{1}{\mathcal{K}_m} \frac{n_b \Omega}{(\Gamma p_b k^2 - w_b \Omega^2)} \tilde{E}^x. \quad (40)$$

Figure 1 shows the results of 1D test simulations based on this solution. The parameters used in this test are $\mathcal{K}_q = \mathcal{K}_m = 1$, $\mathcal{K}_f = \infty$, $k = 2\pi$, $\Gamma = 4/3$, $n_b = p_b = 1$ and $\tilde{E}^x = 10^{-4}$. The uniform computational grid has 50 cells covering the domain $[0, 1]$. One can see that in spite of the relatively low spatial resolution, the numerical solution is quite accurate, particularly

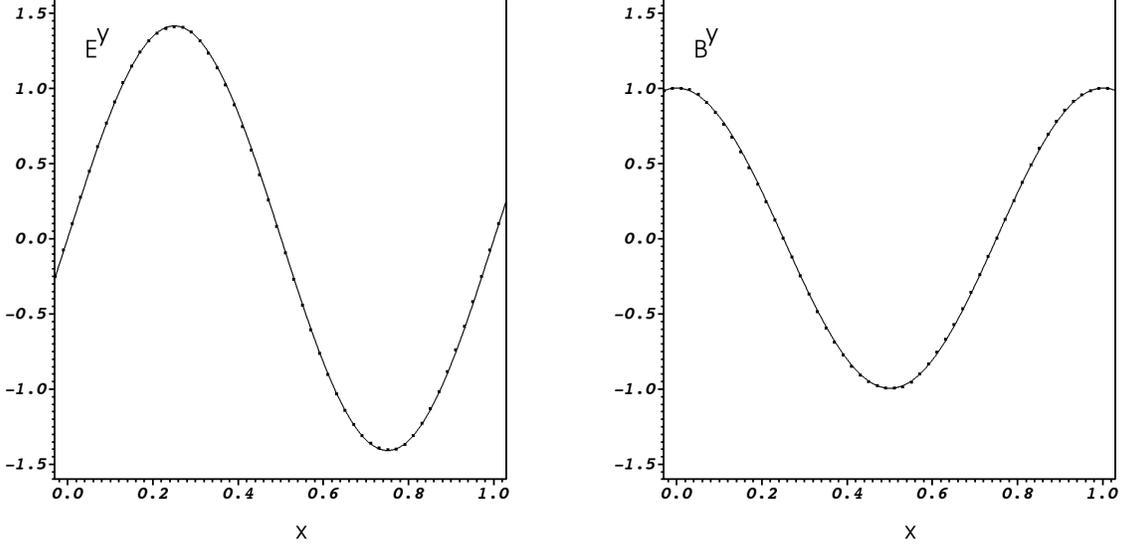


Figure 2. Propagation of circularly polarized “superluminal” waves. The left panel shows E^y and the right panel shows B^y . The continuous lines show the initial solution and the dots show the numerical solution after one period.

Table 2. L1 norm errors in the problem of circularly polarized wave.

N	p_-	B	u_-
50	9.94×10^{-2}	2.73×10^{-3}	3.56×10^{-1}
100	1.07×10^{-2}	4.23×10^{-4}	4.55×10^{-2}
200	1.23×10^{-3}	5.81×10^{-5}	5.32×10^{-3}
400	1.46×10^{-4}	7.57×10^{-6}	6.34×10^{-4}

for the electromagnetic field. Noticeable errors are observed only in the gas parameters near the turning points of the wave curve.

4.2 Circularly polarized “superluminal” waves.

These are nonlinear harmonic solutions of the two-fluid equations with $\mathcal{K}_f = \infty$ (e.g. Max 1973; Amano & Kirk 2013). Their background solution describes uniform state with vanishing velocity, magnetic and electric fields and equal densities and pressures of the electron and positron fluids. These waves do not perturb particle density and pressure, as well as the velocity component along the wave vector. The components of vectors normal to the wave vector display pure rotation, whose direction depends on the wave polarization. Choosing the x-axis aligned with the wave vector and putting $A = \tilde{A} \exp(i(\Omega t - kx))$ for the wave of left polarization one finds the dispersion relation

$$\Omega = \Omega_p (1 + (k/k_b)^2)^{1/2}, \quad (41)$$

where Ω_p is the plasma frequency and $k_b = 1/\Omega_p$. The corresponding Fourier amplitudes are related via

$$\tilde{u}_{\perp, \pm} = \pm \frac{1}{\mathcal{K}_m} \frac{n_b}{kw_b} \tilde{B}_{\perp}, \quad (42)$$

$$\tilde{E}_{\perp} = -i \frac{\Omega}{k} \tilde{B}_{\perp}, \quad (43)$$

where as before n_b and w_b are the unperturbed background electron density and enthalpy respectively. For the right polarization, $A = \tilde{A} \exp(-i(kx - \Omega t))$, Eq.42 is replaced with

$$\tilde{u}_{\perp, \pm} = \mp \frac{1}{\mathcal{K}_m} \frac{n_b}{kw_b} \tilde{B}_{\perp}. \quad (44)$$

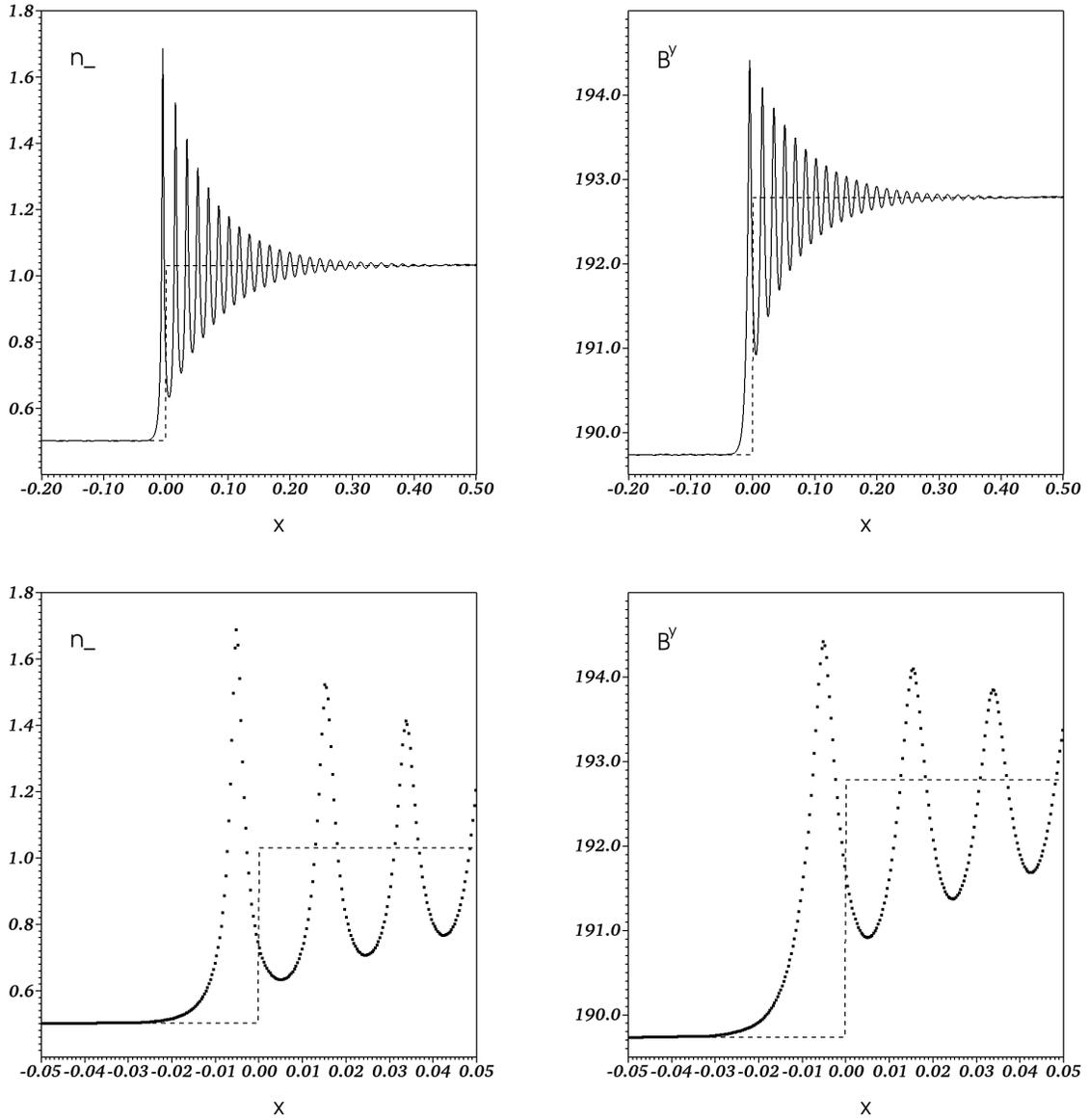


Figure 3. Fast magnetosonic shock case 1. The dashed line shows one-fluid RMHD shock solution. The solid lines in the top panels and the dots in the bottom panels show the numerical two-fluid solution.

Figure 2 shows the results of 1D test simulations, based on the solution for the left polarization wave. The parameters of the problem are $\mathcal{K}_q = 1$, $\mathcal{K}_m = 1/(2\pi)^2$, $\mathcal{K}_f = \infty$, $p_b = 1/4$, $n_b = 1$, $\Gamma = 4/3$, $v_\pm^x = 0$, $B^x = 0$ and $k = 2\pi$. The corresponding wave phase speed is $v_\phi = \sqrt{2}$. The uniform computational grid has 50 cells and covers the domain $[0, 1]$. In this figure the continuous lines show the initial solution and the dots show the numerical solution after one wave period, at $t = 1/\sqrt{2}$. Table 2 show the L1 norm of the numerical errors for p_- and the magnitudes of B , and u_- , all of which should remain constant. The data confirm that the scheme is third order accurate.

4.3 Perpendicular fast magnetosonic shocks.

In these tests, we utilized single fluid ideal RMHD solutions for fast magnetosonic shock waves which were obtained by solving the relevant shock equations. Although these solutions are discontinuities the corresponding two-fluid solutions could have continuous structure with or without sub-shocks. In any case, the asymptotic left and right states must be the same as in the single fluid solutions and this explains the virtue of such tests.

Here we present only the case of perpendicular shock, where the magnetic field is tangent to the shock front and the flow velocity is normal to it. The parameters of the one-fluid solutions in the shock frame are given in table 3. They were used to setup discontinuities in the initial data (Riemann problems). In both cases, $\mathcal{K}_m = \mathcal{K}_f = 0.01$ and $\mathcal{K}_q = 0.001$. In the

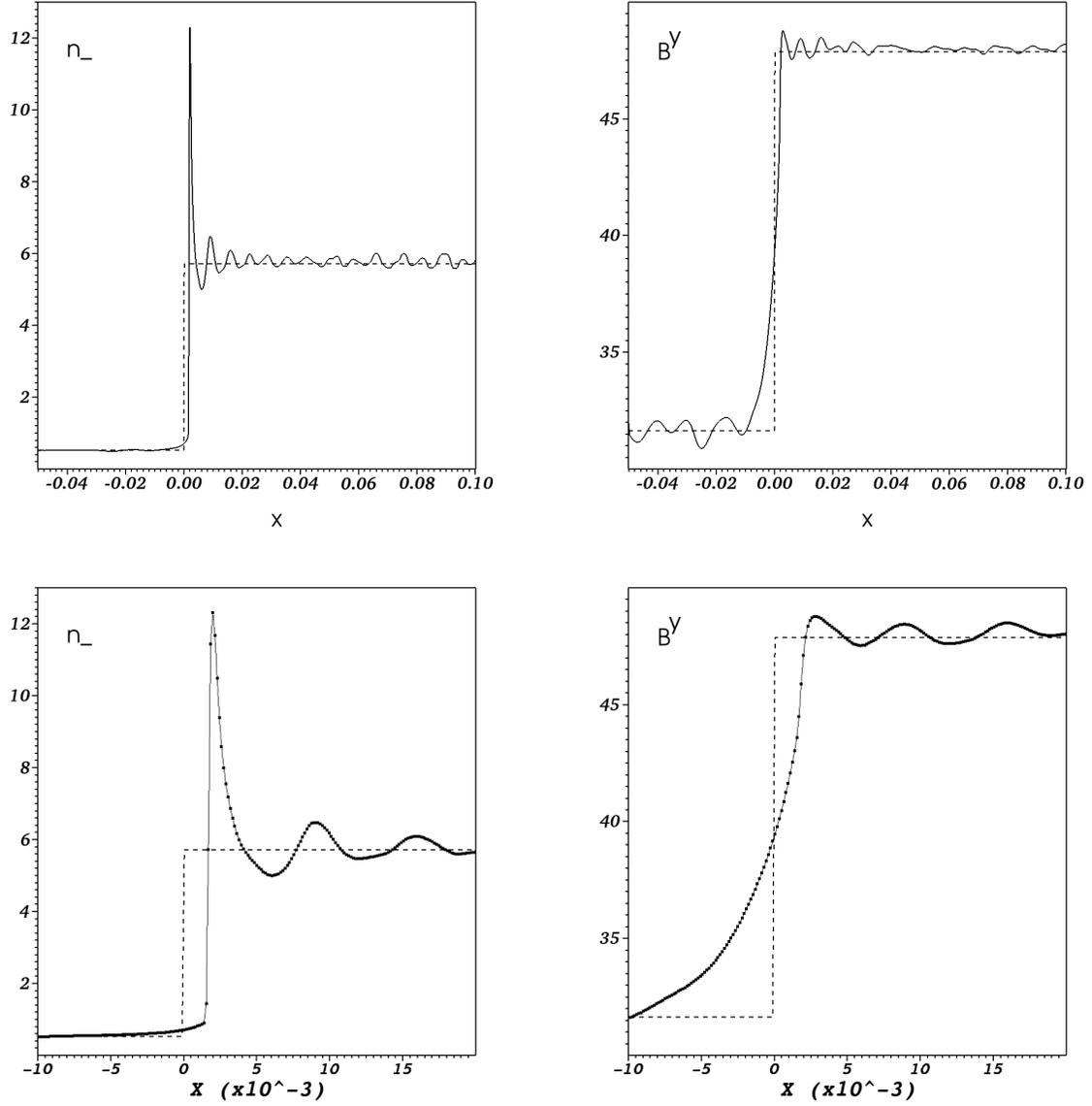


Figure 4. Fast magnetosonic shock case 2. The dashed line shows one-fluid RMHD shock solution. The solid lines in the top panels and the dots in the bottom panels show the numerical two-fluid solution.

Table 3. Perpendicular fast magnetosonic shock solutions used in test simulations. The flow Lorentz factor and magnetic field strength are given in the shock frame.

Case	Left state	Right state
1	$n_-, n_+ = 0.5$ $p_-, p_+ = 0.05$ $\gamma_-, \gamma_+ = 10$ $B = 189.728$	$n_-, n_+ = 1.02977$ $p_-, p_+ = 0.171023$ $\gamma_-, \gamma_+ = 4.93331$ $B = 192.779$
2	$n_-, n_+ = 0.5$ $p_-, p_+ = 0.05$ $\gamma_-, \gamma_+ = 10$ $B = 31.6214$	$n_-, n_+ = 5.70015$ $p_-, p_+ = 8.09570$ $\gamma_-, \gamma_+ = 1.32728$ $B = 47.8497$

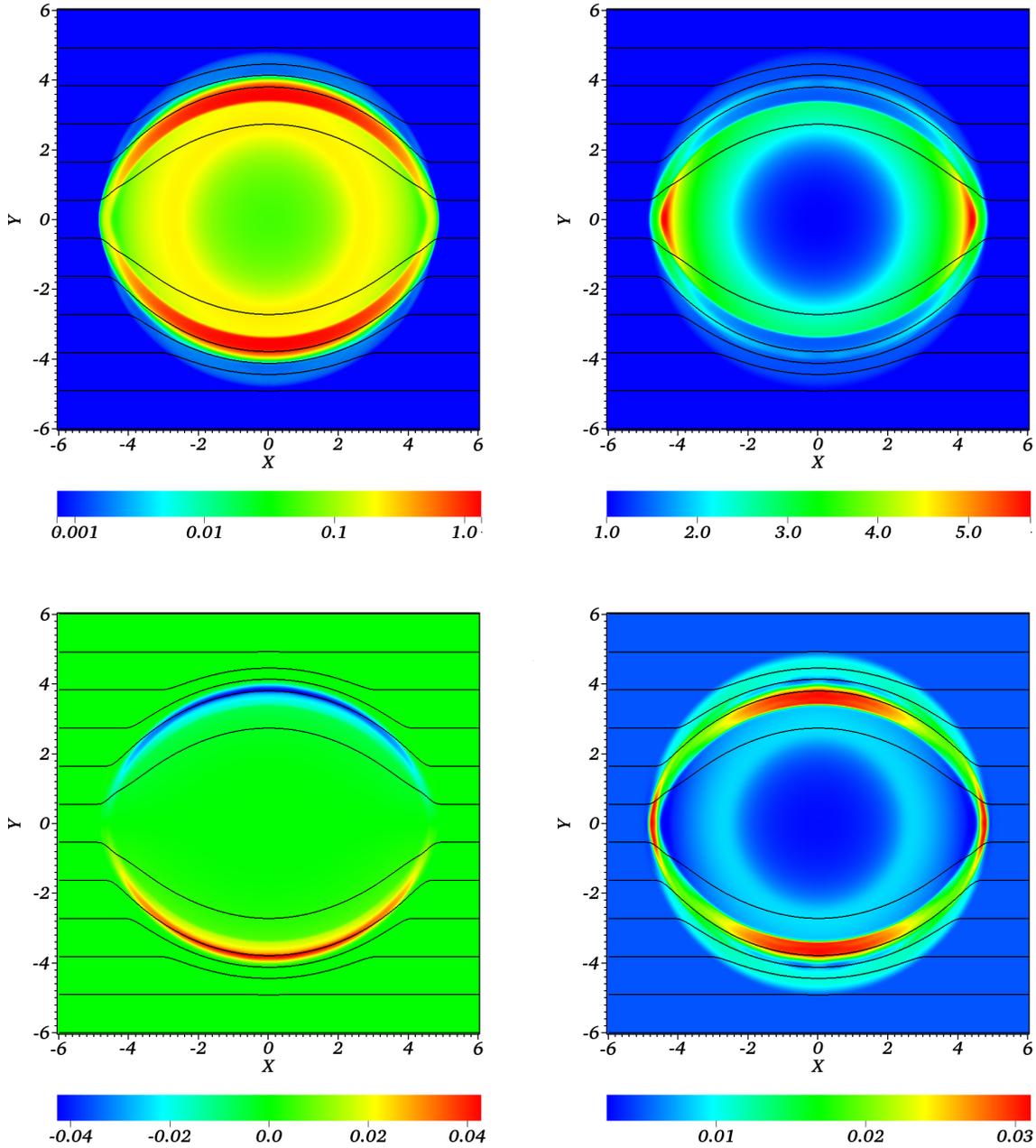


Figure 5. Cylindrical Blast Wave 2D test. Time $t = 4$. The top left panel shows the gas pressure, $\log_{10} p_+ = \log_{10} p_-$, The top right panel shows the Lorentz factor $\gamma_- = \gamma_+$, the maximum value being $\gamma_{\max} = 5.685$. The bottom left panel shows the z component of electric current, $j^z = n_+ u_+^z - n_- u_-^z$. The bottom right panel shows the gas density $n_- = n_+$. All plots show the magnetic field lines.

case 1, the numerical grid contained 2000 cells covering the domain $[-0.2, 0.5]$, whereas in the case 2, it had 1000 cells and the domain $[-0.05, 0.1]$. As to the boundary conditions, we first used the zero gradient ones (also known as the “free flow” conditions) at both the boundaries, just like in the similar test simulations for our one-fluid RMHD code. However, various waves were emitted from the position of the initial discontinuity in both directions, noticeably upsetting both the boundary states. This resulted in slightly different solutions compared to the reference ones. A somewhat better outcome is obtained when the upstream boundary state is fixed at the reference state. In any case, both boundaries reflect incident waves and the reflected waves interact with the shock, so a certain level of noise is present in the numerical solutions.

Figure 3 shows that in the case 1 the two-fluid solution does not involve a sub-shock – it is continuous in all variables. The downstream part of the solution is a wave-train of gradually diminishing amplitude. The rate of decay is determined by the dimensionless parameter \mathcal{K}_f , higher \mathcal{K}_f yielding slower decay. This suggests that for finite \mathcal{K}_f , the fixed point of the shock

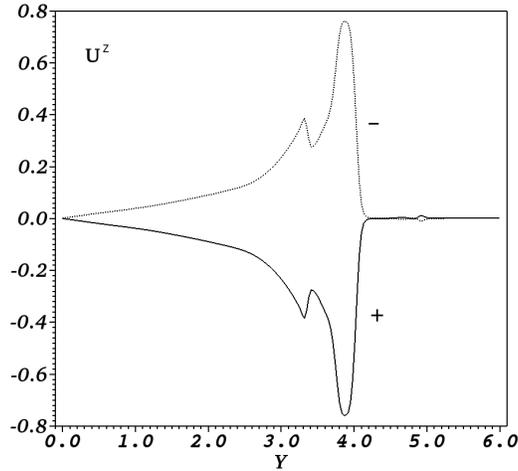


Figure 6. Cylindrical Blast Wave 2D test. Time $t = 4$. The solid line shows u_+^z and the dotted line u_-^z along the line $x = 0$.

structure equations corresponding to the right state is a stable focus. For $\mathcal{K}_f = \infty$, the fixed point must be encircled by a limit cycle. This structure is in general agreement with the analysis of nonrelativistic shocks by Sagdeev (1966). However, the wavelength of the wavetrain is significantly smaller than the skin-depth, which is $\lambda_s \simeq 3$, probably due to the relativistic effects. On the contrary, the generalized gyration radius, $R_g \simeq 7 \times 10^{-4}$, is small compared to the wavelength. When the wavelength is not properly resolved, the numerical solution develops strong oscillations on the cell scale, leading to a crash, unless these oscillations are dumped by high resistivity.

In contrast, the case 2 solution involves a hydro sub-shock – the density, pressure and velocity of the fluids are discontinuous, whereas the electric and magnetic fields are not (see figure 4). Downstream of the sub-shock, one can also see a wave-train similar to that of the case 1 but of smaller amplitude. In both cases, the two-fluid solution agrees with the single fluid solution at $x \rightarrow \pm\infty$. In this problem, the generalized gyration radius $R_g \simeq 4 \times 10^{-3}$ and the skin depth is $\lambda_s \simeq 3$.

In our test problems, the flow magnetization $\sigma \simeq 13$ and $\sigma \simeq 0.36$ for the cases 1 and 2 respectively. Thus, our results are in agreement with those of Haim et al. (2012), who analyzed the structure equations of one-dimensional nonlinear perpendicular waves and concluded that continuous soliton-type solutions of two-fluid RMHD do not exist in the low σ limit.

4.4 Cylindrical and Spherical Blast Waves

All the test problems described so far are one-dimensional with the slab symmetry. We used them to test not only our 1D scheme but also the 2D and 3D schemes, once per each basis direction of the corresponding Cartesian grid. The disadvantage of such approach is that in other directions the scheme performance and accuracy is not tested. In order to complement these problems with truly multidimensional ones, we also carried out the following two tests which involve nonlinear waves propagating at all possible angles to the magnetic field.

The first test is the so-called cylindrical explosion problem, as introduced in Komissarov (1999). This is a 2D problem with the slab symmetry and it describes the evolution of a cylindrical fireball in a uniform perpendicular magnetic field. The domain is $[-6, 6] \times [-6, 6]$ with a uniform grid of 400×400 cells. The magnetic field is $B = (1, 0, 0)$ throughout the domain. The fireball is centered on the origin and its initial radius is $r_b = 1$. In the surrounding, for $r > r_b$, the plasma parameters are $2n_+ = 2n_- = 0.01$ and $2p_+ = 2p_- = 0.003$. Initially, inside the fireball $2n_+ = 2n_- = 1$ and $2p_+ = 2p_- = 100$ for $r < 0.8r_b$, and these decay exponentially down to the external plasma values for $r_b < r < 0.8r_b$. The ratio of specific heats $\Gamma = 4/3$. Initially, the velocity vanishes everywhere. $\mathcal{K}_m = \mathcal{K}_q = 0.01$ and $\mathcal{K}_f = 1$. The dimensionless generalized gyration radius is $R_g = 0.022$ and 4.0 outside and inside of the fireball respectively and the corresponding magnetization $\sigma = 23$, and 1.2×10^{-3} .

Figure 5 illustrates the solution obtained with the 2D code at $t = 4$. It is very similar to the corresponding single fluid RMHD solution of Komissarov (1999), with all the key features present and having similar parameters. The highest Lorentz factor on the grid is $\gamma_{\max} = 5.685$, which is somewhat higher than in the single-fluid data. This is likely to be due to the higher accuracy of the two-fluid simulations. What is qualitatively different from the single-fluid data is the unequal velocities of the electron and positron fluids. This is illustrated in Figure 6 which shows the z component of the four-velocity along the y axis. It is this difference in velocities which yields the electric current shown in bottom left panel of Figure 5.

This 2D problem was also used to test the 3D code, namely we carried out three test runs with the symmetry axis of the fireball aligned with one of the three basis directions of the 3D Cartesian grid.

The second multi-dimensional test problem deals with a spherical blast wave. The initial setup is the same as in the case of the cylindrical blast wave problem in all respect except the shape of the fireball, which is now a sphere of radius r_b . The computational domain is $[-4, 4] \times [-4, 4] \times [-4, 4]$ with a uniform grid of 200^3 cells. Figure 7 illustrates the solution at $t = 3$.

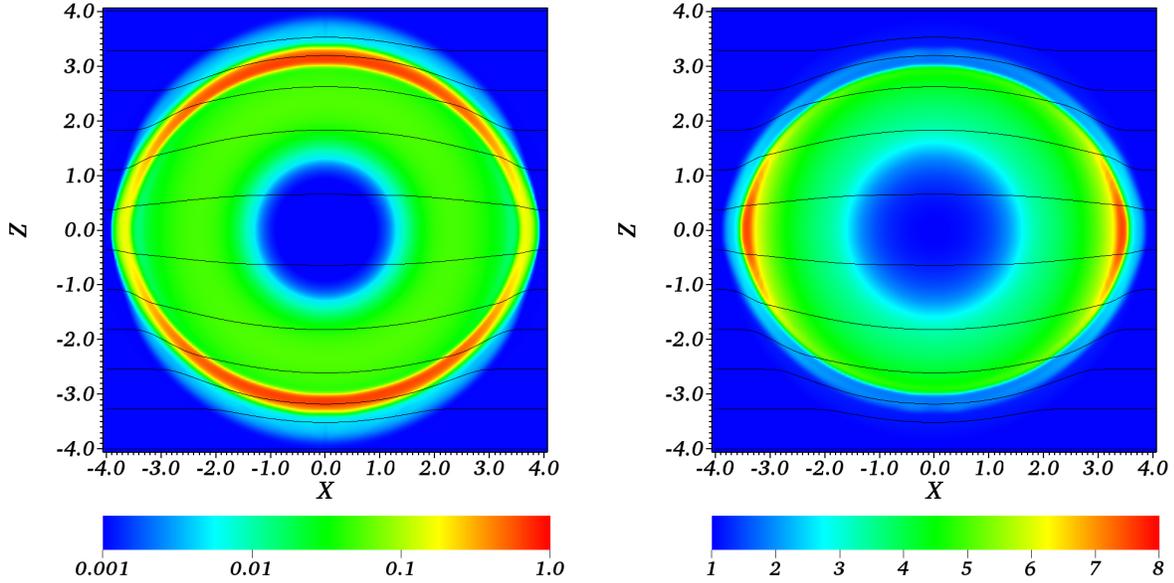


Figure 7. Spherical Blast Wave 3D test. Time $t = 3$. The left panel shows the total gas pressure $\log_{10}(p_+ + p_-)$, and the magnetic field lines in the XOZ plane. The right panel shows the Lorentz factor $\gamma_- = \gamma_+$, the maximum value being $\gamma_{\max} = 7.5$.

One can see that it is similar to the cylindrical case, with somewhat higher Lorentz factors. There are no features which are grid related and hence artificial in nature.

4.5 Tearing Instability

With the possible exception of the perpendicular shock problems, none of the above simulations test the inter-fluid friction term, which is analogous to resistivity. In the absence of exact analytic solutions of equations with non-vanishing inter-fluid friction term, we are forced simply to try problems where this term is expected to be of critical significance. For example, it is well known that, at least in the single fluid framework, it is the resistivity what drives the tearing instability (e.g. Priest & Forbes 2000; Lyutikov 2003; Komissarov et al. 2007). This important problem merits a separate comprehensive study, which we are planning to carry out in the near future. Here we present only the results of two runs for one particular set of parameters.

Following Komissarov et al. (2007) we consider a 2D current sheet with the rotating force-free magnetic field

$$\mathbf{B} = B_0[\tanh(y/l)\mathbf{i}_x + \text{sech}(y/l)\mathbf{i}_z], \quad (45)$$

and a uniform static gas distribution. This configuration is perturbed via modifying the magnetic field, $\mathbf{B} \rightarrow \mathbf{B} + \mathbf{b}$, where

$$\mathbf{b} = b_0 \sin(\pi x/\lambda)\mathbf{i}_y. \quad (46)$$

The computational domain for this simulation is $[-2, +2] \times [-1, +1]$, with the periodic boundary conditions at the x boundaries and the free flow boundary conditions at the y boundaries. The other parameters are $B_0 = 3.0$, $b_0 = 10^{-3}B_0$, $\lambda = 2$, $l = 0.1$, $n_+ = n_- = 1$, $p_+ = p_- = 0.1$, $\gamma = 4/3$, $\mathcal{K}_m = \mathcal{K}_q = 0.01$ and $\mathcal{K}_f = 0.3$.

Figure 8 shows the evolution of the maximum value of B^y on the grid for two models which differ only by their numerical resolution, one with 200×200 and another with 300×300 cells. One can see the difference between the two solutions is very small, which indicates that the simulations have almost converged. This figure also shows that the perturbation grows approximately exponentially, with gradually decreasing growth rate. The left panels of Figure 9 show the instability leads to tearing of the current sheet in the non-linear phase. Its right panels show in more details the structure of the higher resolution solution by the end of the simulation, at $t = 30$. On the whole, the solution shows expected features, which suggests that the numerical treatment of the inter-fluid friction term is bug-free.

5 CONCLUSION

In this paper we considered the equations of relativistic two-fluid Magnetohydrodynamics for electron-positron plasma and described a high-order explicit finite-difference scheme for their numerical integration. This work was carried out as a part

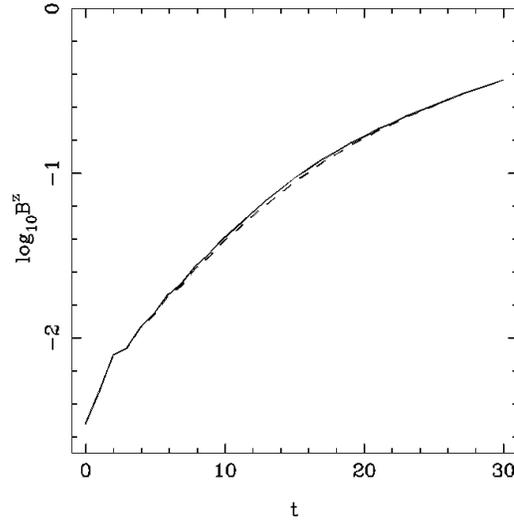


Figure 8. Convergence of the tearing instability solutions. The plots show $\log_{10} B^y$ as a function of time for the models with 200×200 (dashed line) and 300×300 (solid line) cells.

of the search for the most adequate mathematical model for the macroscopic dynamics of relativistic magnetized plasma, with astrophysical applications in mind. There are reasons to believe that this model has advantages compared to both ideal and resistive RMHD. Indeed, the ideal RMHD assumes $\mathbf{E} \cdot \mathbf{B} = 0$ and $E^2 < B^2$ and hence allows magnetic dissipation only by slow shocks. The resistive RMHD allows resistive magnetic dissipation but utilizes a simplistic phenomenological Ohm's law. Both do not capture plasma waves above the plasma frequency. The two-fluid approach allows more realistic Ohm's law via introduction of some microphysics into the system. Yet, it seems to remain oriented more towards the large-scale (macroscopic) dynamics. While other frameworks, like particle dynamics and kinetic models, capture the plasma microphysics much better, they are not practical for simulating phenomena with characteristic length scales greatly exceeding the gyration radius. Our scheme will help to explore the practicality of the two-fluid framework.

The scheme integrates dimensionless equations which include three parameters, \mathcal{K}_m , \mathcal{K}_q , and \mathcal{K}_f , introduced by analogy with the Reynolds number. In some applications, they may be of order unity. In others, much lower or much higher than unity. Our numerical scheme has its limitations and will not always be able to cope with extreme values. However, if the solution reaches asymptotic regimes rapidly it may not be necessary to deal with such extremes. Systematic studies of the parameter space will be required to establish if this is the case.

The design of our numerical scheme is based on standard modern techniques developed for systems of hyperbolic conservation laws. In particular, we use third-order WENO interpolation to setup Riemann problems at the cell interfaces, the hyperbolic fluxes are computed using the Lax-Friedrich formula and the explicit time integration is carried out with third order TVD Runge-Kutta scheme. The results of test simulations presented in Sec.4 show that the numerical scheme is sufficiently robust and confirm its third-order accuracy.

The test simulations revealed that in some problems including strong shocks the two-fluid shock structure may have oscillations on the length scale below the skin depth. When not fully resolved, these are problematic and can lead to code crashing. However, they can be damped with high localized resistivity. A more serious limitation concerns the plasma magnetization. Similarly to the existing explicit conservative schemes for single fluid RMHD, this scheme fails when $\sigma \gg 1$. The reason for this is the very large magnetic terms compared to the gas ones in Eqs.19,20. Equivalently, the source terms in Eqs.10,11 corresponding to the Lorentz force become very stiff in this limit. A different approach will be required to overcome this limitation. One possible solution is an IMEX (implicit-explicit) type of scheme (e.g. Kumar & Misra 2012).

Here we considered only the case of electron-positron plasma. Although, this is already of significant interest in many astrophysical problems, in other applications, protons and even neutrons need to be included. This is another potential direction for continuation of this work.

The current version of our code, named JANUS, will be provided by the authors on request.

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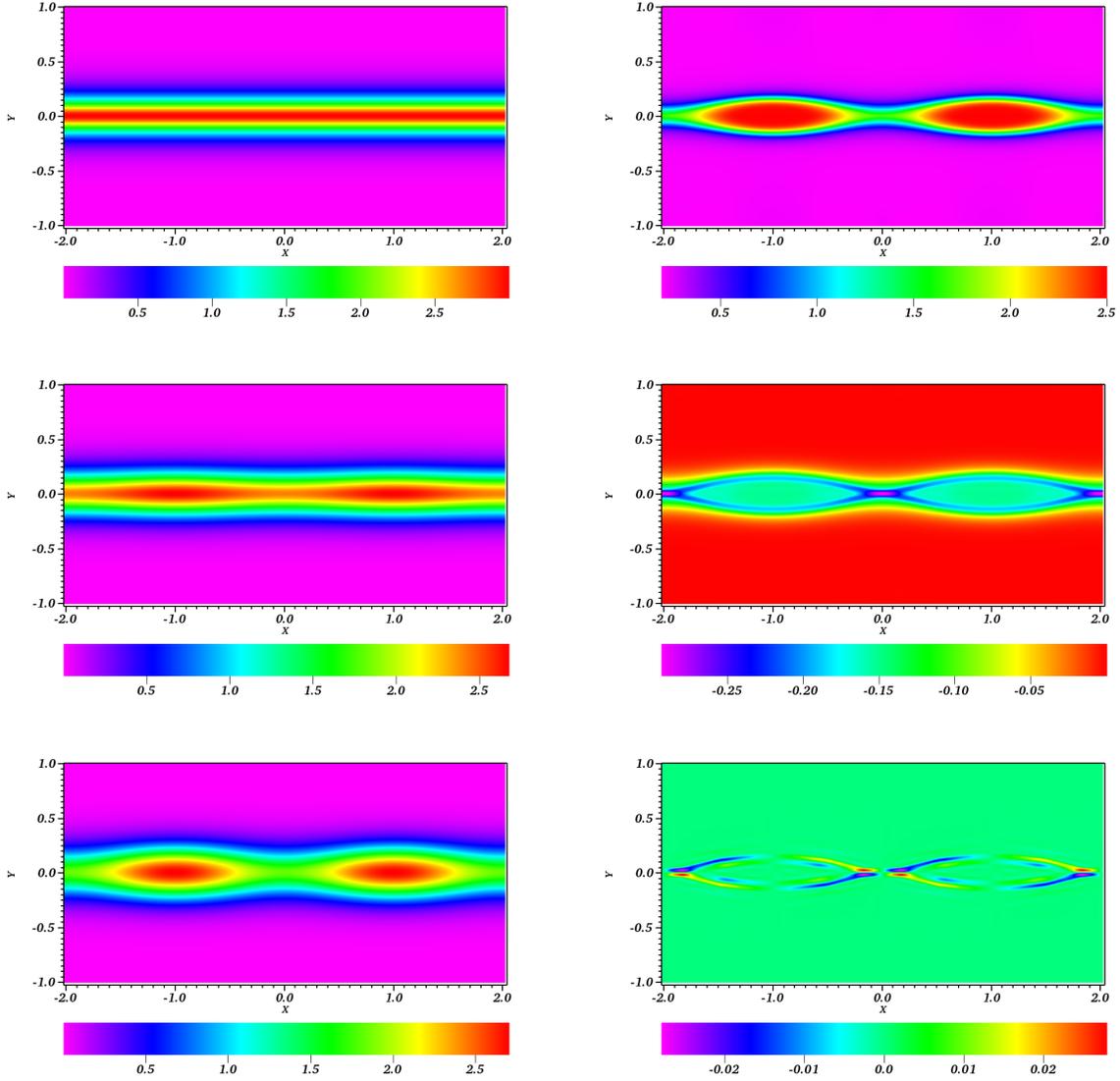


Figure 9. Tearing instability 2D test. The left panels show the B^z component of the magnetic field at times $t = 0, 15$ and 30 , from top to bottom. The right panels show $p_{\text{mat}} = p_- + p_+$ (top), $j^z = n_+ u_{\perp}^z - n_- u_{\perp}^z$ (middle) and $q = n_+ \gamma_+ - n_- \gamma_-$ (bottom) at $t = 30$.

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