

What the characters of irreducible subrepresentations of Jordan cells can tell us about LCFT

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REVIEW

What the characters of irreducible subrepresentations of Jordan cells can tell us about LCFT

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Abstract

In this article, we review some aspects of logarithmic conformal field theories (LCFTs) which can be inferred from the characters of irreducible submodules of indecomposable modules. We will mainly consider the $\mathcal{W}(2, 2p - 1, 2p - 1, 2p - 1)$ series of triplet algebras and a bit logarithmic extensions of the minimal Virasoro models. Since in all known examples of LCFTs the vacuum representation of the maximally extended chiral symmetry algebra is an irreducible submodule of a larger, indecomposable module, its character provides a lot of non-trivial information about the theory such as a set of functions which spans the space of all torus amplitudes. Despite such characters being modular forms of inhomogeneous weight, they fit in the *ADET*-classification of fermionic sum representations. Thus, they show that LCFTs naturally have to be taken into account when attempting to classify rational conformal field theories.

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1. Introduction

Suppose you have constructed a \mathcal{W} -algebra, i.e. a maximally extended symmetry algebra of a two-dimensional conformal field theory (CFT). Suppose you study its vacuum sector and count the states on each of the first N levels. Thus, you compute the first N orders of the character of its vacuum representation. Assuming N large enough, what can you infer from this?

The point is that the \mathcal{W} -algebra and its vacuum sector are purely algebraic objects which can, in principle, be computed with computer algebra systems. But a lot more of the CFT is already encoded in these structures which, by comparison, are relatively easy to come by.

If we have reason to believe that the CFT under consideration is rational, knowledge of the vacuum character to a sufficient high order N is enough to find all other admissible Virasoro highest weights and, furthermore, a set of functions which span the space of the vacuum torus amplitudes of the theory. In particular, you can decide from the vacuum character whether the theory is logarithmic or not. The reason behind this is that the vacuum torus amplitudes of a rational (logarithmic) CFT form a representation of the modular group, and that a modular form is already uniquely determined by a certain, finite, number of terms of its q -expansion.

The characters of irreducible modules, which are submodules of larger, indecomposable modules, tell us even more. Since these characters turn out to be modular forms which consist of parts with different modular weights, i.e. they are not modular forms of homogeneous modular weight, their modular transforms give rise to functions which are not characters of other representations, but can be understood as torus vacuum amplitudes.

It has been conjectured that the characters of rational CFTs, which are modular forms, all have fermionic sum representations which can all be found within the so-called *ADET*-classification of fermionic sums (cf [108]). As it turns out, this holds true even for the characters of such irreducible submodules of indecomposable modules. And that means that rational logarithmic conformal field theories (LCFTs) seem to naturally fit into these classification schemes of rational CFTs.

Our considerations are mainly based on the relatively well-understood $\mathcal{W}(2, 2p - 1, 2p - 1, 2p - 1)$ series of triplet algebras. However, as long as the vacuum representation is an irreducible submodule of an indecomposable module, our reasoning and strategies will continue to work. In particular, we expect that augmented minimal models can be explored along the lines set out in our review. Especially, the space of torus vacuum amplitudes can be determined from which, with some further assumptions or some more data from explicitly constructed representations, the characters and fusion rules could be derived.

Bosonic-fermionic q-series identities. The zero mode L_0 of the infinite-dimensional Virasoro symmetry algebra of two-dimensional conformal quantum field theory admits a natural gradation of the representation space into subspaces of fixed L_0 eigenvalue. The character of a representation displays the number of linearly independent states in each of these subspaces as a series expansion in some formal variable q , where the dimension k of the subspace with fixed L_0 eigenvalue n is indicated by a summand kq^n in this series. Non-unique realizations of the state spaces in two-dimensional CFTs imply the existence of several alternative character formulae. The Bose–Fermi correspondence indicates that the characters of two-dimensional quantum field theories can be expressed in a *bosonic* as well as *fermionic* way, leading to *bosonic-fermionic q-series identities*. The existence of such identities can be traced back to the famous *Rogers–Ramanujan identities* [119, 120, 124]

$$\sum_{n=0}^{\infty} \frac{q^{n(n+a)}}{(q)_n} = \prod_{n=1}^{\infty} \frac{1}{(1 - q^{5n-1-a})(1 - q^{5n-4+a})} \tag{1.1}$$

for $a \in \{0, 1\}$. For the notations, see section 3, equations (3.1) ff. These identities later turned out to coincide with the two characters of the $\mathcal{M}(5, 2)$ minimal model [28] (up to an overall factor q^α for some $\alpha \in \mathbb{Q}$). By using Jacobi’s triple product identity (see e.g. [7]), the right-hand side (rhs) of (1.1) can be transformed to give a simple example of a *bosonic-fermionic q-series identity*:

$$\sum_{n=0}^{\infty} \frac{q^{n(n+a)}}{(q)_n} = \frac{1}{(q)_\infty} \sum_{n=-\infty}^{\infty} (q^{n(10n+1+2a)} - q^{(5n+2-a)(2n+1)}). \tag{1.2}$$

The bosonic expression on the rhs of (1.2) corresponds to two special cases of the general character formula for minimal models $\mathcal{M}(p, p')$ [117]:

$$\hat{\chi}_{r,s}^{p,p'} = q^{\frac{c}{24} - h_{r,s}^{p,p'}} \chi_{r,s}^{p,p'} = \frac{1}{(q)_\infty} \sum_{n=-\infty}^{\infty} (q^{n(np' + pr - p's)} - q^{(np+s)(np'+r)}), \tag{1.3}$$

with $\hat{\chi}_{r,s}^{p,p'}$ being the normalized character. It has been termed bosonic in [87] because it is computed by eliminating null states from a Verma module over the Virasoro algebra. The signature of bosonic character expressions is the alternating sign, which reflects the subtraction of null vectors, whereas each summand of a fermionic character formula is manifestly positive. Furthermore, the factor $(q)_\infty$ keeps track of the free action of the Virasoro ‘creation’ modes.

Quasi-particle interpretation for fermionic character expressions. On the other hand, the fermionic sum representation for a character possesses a remarkable interpretation in terms of an underlying system of *quasi-particles*. These expressions first occurred on the left-hand side of the Rogers–Ramanujan identities (1.1). Generalizations have been obtained by Andrews and Gordon [9, 69] and later on by Lepowsky and Primc [94]. The most general fermionic expression is regarded to be a linear combination of fundamental fermionic forms. A *fundamental fermionic form* [16, 25, 135] is³

$$\sum_{\substack{\vec{m} \in (\mathbb{Z}_{\geq 0})^r \\ \text{restrictions}}} \frac{q^{\vec{m}^t A \vec{m} + \vec{b}^t \vec{m} + c}}{\prod_{i=1}^j (q)_i} \prod_{i=j+1}^r \left[\begin{matrix} g(\vec{m}) \\ m_i \end{matrix} \right]_q \tag{1.4}$$

with $A \in M_r(\mathbb{Q})$, $\vec{b} \in \mathbb{Q}^r$, $c \in \mathbb{Q}$, $0 \leq j \leq r$, g a certain linear, algebraic function in the m_i , $1 \leq i \leq r$, and the q -deformed binomial coefficient (the *q-binomial coefficient*) defined as

$$\left[\begin{matrix} n \\ m \end{matrix} \right]_q = \begin{cases} \frac{(q)_n}{(q)_m (q)_{n-m}} & \text{if } 0 \leq m \leq n \\ 0 & \text{otherwise.} \end{cases} \tag{1.5}$$

³ The constant c is not to be confused with the central charge $c_{p,p'}$.

The sum over \vec{m} is an abbreviation and implies that each component m_i of \vec{m} is to be summed over independently.

It is sometimes also called *universal chiral partition function* for exclusion statistics [15, 123]. Remarkably, in most cases the matrix A is given by the *Cartan matrix* of a simple Lie algebra or by its inverse. It turns out that (1.4) can be interpreted in terms of a system of r different species of fermionic quasi-particles with non-trivial momentum restrictions. The bosonic representations are in general unique, whereas there is usually more than one fermionic expression for the same character, giving rise to different quasi-particle interpretations for the same CFT which are conjectured to correspond to different integrable massive extensions of the theory, see section 5. There are cases for which such correspondences are known. Thus, the different interpretations in terms of quasi-particle systems may be a guide for experimental research. Note that in general, the existence of quasi-particles has been experimentally demonstrated, namely in the case of the fractional quantum Hall effect [121]. They turned out to be of charge $e/3$, as predicted by Laughlin [93, 129].

Dilogarithm identities. Furthermore, knowledge of a fermionic character expression and either the theory's effective central charge or a product form of the character results in dilogarithm identities (see also [108, 110, 136]) of the form

$$\frac{1}{L(1)} \sum_{i=1}^N L(x_i) = d, \quad (1.6)$$

where L is the Rogers dilogarithm [99, 100, 118] and x_i and d are rational numbers. It is conjectured [109] that all values of the effective central charges occurring in non-trivial rational CFTs can be expressed as one of those rational numbers that consist of a sum of an arbitrary number of dilogarithm functions evaluated at algebraic numbers from the interval $(0, 1)$. Besides their intriguing relation to wall-crossing formulae in string theory [5], dilogarithm identities are well known to arise from thermodynamic Bethe ansatz. Conversely, there is also a conjecture [128] that dilogarithm identities corresponding to Bethe ansatz equations $x_i = \prod_{j=1}^k (1 - x_j)^{2A_{ij}}$, where A is the inverse Cartan matrix of one of the *ADET* series of simple Lie algebras (see section 5), imply fermionic character expressions of rational CFT characters. Thus, the study of dilogarithm identities arising from CFTs gives further insight into the classification of all rational theories.

Fermionic expressions as evidence for rationality of extended \mathcal{W} -symmetry theories. In addition to the minimal models of the Virasoro algebra, there exist other theories endowed with more symmetries. They are generated by modes of currents different from the energy-momentum tensor. Possible extensions, which contain the Virasoro algebra as a subalgebra, lead to free fermions, Kac-Moody algebras [77], superconformal algebras [68] or more generally \mathcal{W} -algebras [20, 107]. Amongst others, the $\mathcal{W}(2, 2p - 1, 2p - 1, 2p - 1)$ series of CFTs with extended triplet algebra symmetry [82] is investigated in this review, comprising the best understood examples of LCFT models. These models correspond to central charges $c_{p,1} = 1 - 6\frac{(p-1)^2}{p}$, $p \geq 2$. In fact, the description of various physical processes in two dimensions requires logarithmic divergences in the correlation functions, as in the case of, e.g., magnetohydrodynamics [126], turbulence [114], dense polymers [111] or percolation [54, 115]. For other interesting applications, also see [55, 56] for a work concerning instantons, where a non-diagonalizable Hamiltonian may occur, and also the recent results of [13] about three-dimensional tricritical gravity.

For some values of the central charge (when there are fields with integer-spaced dimensions), the existence of fields that lead to logarithmic divergences in four-point functions

is unavoidable [70]. Conventionally, if the state space of states of a CFT decomposes into a finite sum of irreducible representations, then the theory is said to be *rational*. However, the investigation of certain classes of LCFTs pointed toward loosening this strict definition of rationality to also include reducible, but indecomposable representations. Recently, an attempt at organizing logarithmic theories into families alongside related rational theories has been started [33, 40, 112]. For LCFTs, almost all of the basic notions and tools of (rational) CFTs, such as null vectors, (bosonic) character functions, partition functions, fusion rules, modular invariance, have been generalized by now [50], the main difference from ordinary rational CFTs such as the minimal models remaining the occurrence of indecomposable representations.

In this light, the existence of a complete set of fermionic sum representations for the characters of the $\mathcal{W}(2, 2p - 1, 2p - 1, 2p - 1)$ LCFT models with $p \geq 2$ (which are also referred to as the $c_{p,1}$ models) provides further evidence that these models, although outside the usual classification scheme of rational CFTs, are nonetheless *bona fide* theories [52].

2. Character expressions for conformal field theories with \mathcal{W} -symmetry

2.1. Virasoro characters

Throughout this review, we are only interested in one chiral half of the CFT. Thus, we consider representations $\mathcal{H}(h, c)$ of the chiral symmetry algebra, which may either be irreducible representations (often denoted \mathcal{V}_h or $\mathcal{W}(h)$), or indecomposable ones (for example denoted \mathcal{R}_h or $\mathcal{R}(h, h_1, \dots)$). We stress that if there are indecomposable representations involved, the state space will not factorize into a direct sum of tensor products of holomorphic and corresponding anti-holomorphic sectors.

The *character* of an indecomposable Vir-module V of the Virasoro algebra Vir is defined by

$$\chi_V(\tau) := \text{Tr} e^{2\pi i\tau(L_0 - \frac{c}{24})}. \tag{2.1}$$

with c being the central charge and L_0 the Virasoro zero mode. The characters of the representations are an essential ingredient for a CFT. Since L_0 corresponds to the Hamiltonian of the (chiral half of the) system, the energy spectrum (at least certain sectors) is encoded in the character. The trace is usually taken over an irreducible highest weight representation and the factor $q^{-\frac{c}{24}}$ guarantees the needed linear behavior under modular transformations. By setting $q := e^{2\pi i\tau}$, (2.1) leads to

$$\chi_V(q) = q^{-\frac{c}{24}} \sum_h q^h \dim \text{eigenspace}(L_0^d, h), \tag{2.2}$$

where L_0^d is the diagonalizable summand of the possibly non-diagonalizable L_0 . To compute the character of a Verma module $V(h, c)$, we have to compute the number of linearly independent states at a given level k . We can grade the Verma module by its L_0 eigenvalue:

$$V(h, c) = \bigoplus_N V_N(h, c) \tag{2.3}$$

with

$$V_N(h, c) = \left\langle \left\{ L_{-n_1} L_{-n_2} \cdots L_{-n_k} |h\rangle \mid k \in \mathbb{Z}_{\geq 0}, n_{i+1} \geq n_i, \sum_{i=1}^k n_i = N \right\} \right\rangle. \tag{2.4}$$

Thus, the number of distinct, linear independent states at level N is given by the number $p(N)$ of additive partitions of the integer N . The generating function for the number of partitions is

$$\frac{1}{\phi(q)} \equiv \prod_{n=1}^{\infty} \frac{1}{1 - q^n} = \sum_{n=0}^{\infty} p(N) q^n, \tag{2.5}$$

where $\phi(q)$ is the Euler function. Hence the character is given by

$$\chi_{V(h,c)} = q^{h-\frac{c}{24}} \sum_{k=0}^{\infty} p(k)q^k. \tag{2.6}$$

Dedekind's η function $\eta(q) = q^{1/24}\phi(q)$ is conventionally used because it simplifies the analysis of a character function under modular transformations:

$$\chi_{V(h,c)} = \frac{q^{\frac{1-c}{24}}}{\eta(q)} q^h. \tag{2.7}$$

This series is convergent if $|q| < 1$, i.e. $\tau \in \mathbb{H}$ (upper half-plane). If $V(h, c)$ already is an irreducible representation, i.e. is non-degenerate, then this is its character. If not, the characters $\chi_{r,s}$ of irreducible representations $M(h_{r,s}, c)$ can be read off the corresponding embedding structure [39, 46].

2.2. Θ - and η -functions and their modular transformation properties

The *Jacobi–Riemann Θ -functions* and the *affine Θ -functions* are defined by

$$\Theta_{\lambda,k}(\tau) = \sum_{n \in \mathbb{Z}} q^{\frac{(2kn+\lambda)^2}{4k}} \tag{2.8}$$

and

$$(\partial\Theta)_{\lambda,k}(\tau) = \sum_{n \in \mathbb{Z}} (2kn + \lambda) q^{\frac{(2kn+\lambda)^2}{4k}} \tag{2.9}$$

with $q = e^{2\pi i\tau}$. $\lambda \in \frac{\mathbb{Z}}{2}$ is called the *index* and $k \in \frac{\mathbb{Z}_{\geq 1}}{2}$ the *modulus*. The Θ -functions satisfy the symmetries

$$\Theta_{\lambda,k} = \Theta_{-\lambda,k} = \Theta_{\lambda+2k,k} \tag{2.10}$$

and

$$(\partial\Theta)_{-\lambda,k} = -(\partial\Theta)_{\lambda,k}. \tag{2.11}$$

The *Dedekind η -function* is defined as

$$\eta(\tau) = q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n). \tag{2.12}$$

The *Jacobi–Riemann Θ -functions* and the *Dedekind η -function* are modular forms of weight $1/2$, while the *affine Θ -functions* have modular weight $\frac{3}{2}$. A *modular form of weight k* is defined by the relation

$$f\left(\frac{a\tau + b}{c\tau + d}\right) = \epsilon(a, b, c, d)(c\tau + d)^k f(\tau) \tag{2.13}$$

for $\tau \in \mathbb{C}$ and $|\epsilon(a, b, c, d)| = 1$ and with $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \equiv \text{PSL}(2, \mathbb{Z})$ and f being a holomorphic function on the upper half-plane which is also holomorphic at the *cusp*, i.e. is holomorphic as $\tau \rightarrow i\infty$. The modular transformation properties of the Θ - and η -functions for those cases of λ and k that are needed in this report are

$$\Theta_{\lambda,k}\left(-\frac{1}{\tau}\right) = \sqrt{\frac{-i\tau}{2k}} \sum_{\lambda'=0}^{2k-1} e^{i\pi\frac{\lambda\lambda'}{k}} \Theta_{\lambda',k}(\tau) \quad \text{for } \lambda \in \mathbb{Z}, \tag{2.14}$$

$$\Theta_{\lambda,k}(\tau + 1) = e^{i\pi\frac{\lambda^2}{2k}} \Theta_{\lambda,k}(\tau) \quad \text{for } \lambda - k \in \mathbb{Z}, \tag{2.15}$$

$$\eta\left(-\frac{1}{\tau}\right) = \sqrt{-i\tau}\eta(\tau), \tag{2.16}$$

$$\eta(\tau + 1) = e^{\frac{\pi i}{12}}\eta(\tau). \tag{2.17}$$

The functions $\chi_{\lambda,k} = \frac{\Theta_{\lambda,k}}{\eta}$, which often turn up as summands in character functions in section 2.1, are thus modular forms of weight zero with respect to the principal congruence subgroup $\Gamma(N)$ of the modular group $\text{PSL}(2, \mathbb{Z})$. The principal congruence subgroup is defined such that the diagonal matrix entries are congruent to 1 mod N , and the off-diagonal entries to 0 mod N . Many details about Θ -functions may be found in [4, 75].

2.3. Torus amplitudes

In a seminal work [138], Zhu proved several general facts about CFTs or vertex operator algebras, whose characters form finite-dimensional representations of the modular group. Of particular interest are theories which satisfy the C_2 condition, but we shall not assume that they define *rational* CFTs. As it is somehow beyond the scope of this review to discuss how a mathematical rigorous definition of rationality of a CFT should incorporate the case of LCFTs, it will be sufficient to appeal to the standard CFT lore of rationality. Thus, for the purpose of this review, we call a CFT rational if its chiral symmetry algebra possesses finitely many irreducible representations, each of which has finite-dimensional L_0 eigenspaces, such that their fusion products decompose into direct sums of just these irreducible representations. The C_2 condition states that the quotient space $\mathcal{H}_0/C_2(\mathcal{H}_0)$ is finite dimensional, where \mathcal{H}_0 is the vacuum representation of the CFT and $C_2(\mathcal{H}_0)$ is the space spanned by the states

$$V_{-h(\psi)-1}(\psi)\chi, \quad \text{for } \psi, \chi \in \mathcal{H}_0. \tag{2.18}$$

The C_2 condition implies that Zhu’s algebra $A(\mathcal{H}_0)$ is finite dimensional, and therefore that the CFT has only finitely many irreducible highest weight representations (see [58] for an introduction to these matters and what is meant by highest weight representations in the case of extended chiral symmetry algebras). However, it does not imply that the theory is rational, neither in the above sense, nor in the sense given by a mathematical rigorous definition of rationality. Indeed, the prime examples of logarithmic theories, the triplet algebras at $c = c_{p,1}$ [82], do satisfy the C_2 condition [23], yet are not rational since they possess indecomposable representations [60].

As shown by Zhu, if the CFT satisfies the C_2 condition, then every highest weight representation for which L_0 is diagonalizable with finite-dimensional eigenspaces gives rise to a torus amplitude. In particular, the vacuum amplitude is just given by the usual character

$$\chi_{\mathcal{H}_j}(\tau) = \text{Tr}_{\mathcal{H}_j}(q^{L_0 - \frac{c}{24}}), \quad q = e^{2\pi i\tau}, \tag{2.19}$$

which converges absolutely for $0 < |q| < 1$. Again, if the C_2 condition is satisfied, the space of torus amplitudes is finite dimensional, and it carries a representation of $\text{PSL}(2, \mathbb{Z})$.

Furthermore, Zhu points out that if the CFT satisfies the C_2 condition, then there exists a positive integer s so that for a certain redefined algebra

$$L[-2]^s\Omega + \sum_{r=0}^{s-1} g_r(q)L[-2]^r\Omega \in O_q(\mathcal{H}_0), \tag{2.20}$$

where the modes $L[-2]$ are defined in equation (4.2.3) of Zhu’s work [138], proof of his theorem 4.4.1, and $O_q(\mathcal{H}_0)$ denotes the subspace of \mathcal{H}_0 whose one-point torus functions

vanish. Here $g_r(q)$ are polynomials in the Eisenstein series $E_4(q)$ and $E_6(q)$; we shall choose the convention that the Eisenstein series are defined by

$$E_k(q) = 1 - \frac{2k}{B_k} \sum_{n=1}^{\infty} \sigma_{k-1}(n)q^n, \tag{2.21}$$

$$\sigma_k(n) = \sum_{d|n} d^k, \tag{2.22}$$

where B_k is the k th Bernoulli number. Thus, the q -expansion of the Eisenstein series reads $E_2 = 1 - 24q - 72q^2 - 96q^3 - \dots$, $E_4 = 1 + 240q + 2160q^2 + 6720q^3 + \dots$ and $E_6 = 1 - 504q - 16\,632q^2 - 122\,976q^3 - \dots$ in our normalization.

Given Zhu's definition of $\tilde{\omega} *_{\tau} \psi$ (see equation (5.3.1) in [138, p 292], together with $L[n] = \tilde{\omega}[n + 1]$), one can rewrite (2.20) as

$$\tilde{\omega}^s *_{\tau} \Omega + \sum_{r=0}^{s-1} h_r(q) \tilde{\omega}^r *_{\tau} \Omega \in O_q(\mathcal{H}_0), \tag{2.23}$$

where now $h_r(q)$ are polynomials in the Eisenstein series $E_2(q)$, $E_4(q)$ and $E_6(q)$, and we have used the notation

$$\tilde{\omega}^r *_{\tau} \Omega \equiv \underbrace{\tilde{\omega} *_{\tau} \dots *_{\tau} \tilde{\omega}}_r *_{\tau} \Omega. \tag{2.24}$$

This has a few important consequences. Firstly, it therefore follows that every torus vacuum amplitude $T(q)$ must satisfy the differential equation

$$\left[\left(q \frac{d}{dq} \right)^s + \sum_{r=0}^{s-1} h_r(q) \left(q \frac{d}{dq} \right)^r \right] T(q) = 0. \tag{2.25}$$

Secondly, it furthermore follows that the functions h_r are such that

$$\left(L[0] - \frac{c}{24} \right)^s + \sum_{r=0}^{s-1} h_r(0) \left(L[0] - \frac{c}{24} \right)^r = 0 \tag{2.26}$$

in Zhu's algebra $A(\mathcal{H}_0)$ (that is defined in section 2 of [138]). As we shall argue later, the differential equation (2.25) can be identified with the modular differential equation that was first considered in [103, 104].

If the CFT is in addition rational in the above sense Zhu showed that the space of torus amplitudes is spanned by the characters of the irreducible representations, and therefore that their characters transform into one another under the action of the modular group. In the case of logarithmic triplet neither of these two statements is correct. In fact, the space of torus vacuum amplitudes turns out to be larger than the space spanned by the characters. Only the former forms a closed representation of the modular group. Within the latter, there exists a smaller subset of characters, which forms a smaller representation of the modular group, if one omits all characters of submodules of indecomposable modules. However, restricting to this smaller set of characters precisely loses all information about the theory that would show that it is logarithmic. It is plausible that these considerations, which are explicitly proven for the triplet algebras, are generally valid for C_2 cofinite LCFTs.

2.4. The modular differential equation

We will now show how the vacuum character determines the spectrum of a conformal field theory to a very high extent. In fact, a (sufficiently high) finite order of the q -expansion of the vacuum character is enough, and can often be computed by explicitly constructing a basis of

states of the vacuum representation space up to this finite level. Beyond this, almost nothing else needs to be known. The key is that in order to be a C_2 cofinite theory, the vacuum character, as all other torus amplitudes, must satisfy a modular invariant differential equation. The equation with the smallest possible degree n is uniquely determined by the vacuum character, as long as we know more than the n first terms of it. If we do not know n , i.e. if we do not know the dimension of the space of the torus amplitudes (in the case of a rational CFT, this is just the number of inequivalent highest weight representations), we might be able to estimate it, if we can compute the dimensions of the graded subspaces of the vacuum representation up to some (possibly large) level.

We will make use of the prime example of LCFT, the $c = -2$ triplet model, in order to demonstrate this approach.

2.4.1. The $c = -2$ triplet theory. Let us briefly recall some of the properties of the triplet theory with $c = c_{2,1} = -2$, which can be found, e.g., in [47, 60, 82, 83]. The chiral algebra for this CFT is generated by the Virasoro modes L_n , and the modes of a triplet of weight 3 fields W_n^a . The commutation relations are

$$[L_m, L_n] = (m - n)L_{m+n} - \frac{1}{6}m(m^2 - 1)\delta_{m+n}, \quad (2.27a)$$

$$[L_m, W_n^a] = (2m - n)W_{m+n}^a, \quad (2.27b)$$

$$\begin{aligned} [W_m^a, W_n^b] = & g^{ab}(2(m - n)\Lambda_{m+n} + \frac{1}{20}(m - n)(2m^2 + 2n^2 - mn - 8)L_{m+n} \\ & - \frac{1}{120}m(m^2 - 1)(m^2 - 4)\delta_{m+n}) \\ & + f_c^{ab}(\frac{5}{14}(2m^2 + 2n^2 - 3mn - 4)W_{m+n}^c + \frac{12}{5}V_{m+n}^c) \end{aligned} \quad (2.27c)$$

where $\Lambda = :L^2: - \frac{3}{10}\partial^2L$ and $V^a = :LW^a: - \frac{3}{14}\partial^2W^a$ are quasi-primary normal ordered fields. g^{ab} and f_c^{ab} are the metric and structure constants of $su(2)$. In an orthonormal basis we have $g^{ab} = \delta^{ab}$, $f_c^{ab} = i\epsilon^{abc}$.

The triplet algebra (at $c = -2$) is only associative, because certain states in the vacuum representation (which would generically violate associativity) are null. The relevant null vectors are

$$N^a = (2L_{-3}W_{-3}^a - \frac{4}{3}L_{-2}W_{-4}^a + W_{-6}^a)\Omega, \quad (2.28)$$

$$\begin{aligned} N^{ab} = & W_{-3}^a W_{-3}^b \Omega - g^{ab}(\frac{8}{9}L_{-2}^3 + \frac{19}{36}L_{-3}^2 + \frac{14}{9}L_{-4}L_{-2} - \frac{16}{9}L_{-6})\Omega \\ & - f_c^{ab}(-2L_{-2}W_{-4}^c + \frac{5}{4}W_{-6}^c)\Omega. \end{aligned} \quad (2.29)$$

We shall only be interested in representations which respect these relations, and for which the spectrum of L_0 is bounded from below. Evaluating the constraint coming from (2.29), we find (see [60] for more details)

$$(W_0^a W_0^b - g^{ab}\frac{1}{9}L_0^2(8L_0 + 1) - f_c^{ab}\frac{1}{5}(6L_0 - 1)W_0^c)\psi = 0, \quad (2.30)$$

where ψ is any highest weight state, while the relation coming from the zero mode of (2.28) is satisfied identically. Furthermore, the constraint from $W_1^a N_{-1}^{bc}$, together with (2.30), implies that $W_0^a(8L_0 - 3)(L_0 - 1)\psi = 0$. Multiplying with W_0^a and using (2.30) again, this implies that

$$0 = L_0^2(8L_0 + 1)(8L_0 - 3)(L_0 - 1)\psi. \quad (2.31)$$

For irreducible representations, L_0 has to take a fixed value h on the highest weight states, and (2.31) then implies that h has to be either $h = 0, -1/8, 3/8$ or $h = 1$. However, it also follows from (2.31) that a logarithmic highest weight representation is allowed since we only have to

have that $L_0^2 = 0$ but not necessarily that $L_0 = 0$. Thus, in particular, a two-dimensional space of highest weight states with relations

$$L_0\omega = \Omega \qquad L_0\Omega = 0. \tag{2.32}$$

satisfies (2.31). This highest weight space gives rise to the ‘logarithmic’ (indecomposable) representation \mathcal{R}_0 (see [60] for more details). Note that quite some effort is needed to find relation (2.31), as the \mathcal{W} -algebra null vectors must be computed explicitly. As we will see shortly, we can derive the same relation solely from knowledge of the vacuum character, which can be found without explicitly knowing the null vectors. In fact, as shown in [23], the vacuum characters as well as C_2 cofiniteness of the whole series of $c_{p,1}$ triplet models can be established without explicit knowledge of the triplet \mathcal{W} -algebra null vectors, which, for larger p , would be impossible to get.

It follows from the above analysis (and a similar analysis for the W^a modes; see for example [60]) that the $c = -2$ triplet theory has only finitely many indecomposable highest weight representations. This suggests that it satisfies the C_2 condition, and this can be confirmed by a computer calculation (first done by Kausch [139]). Indeed, the space $\mathcal{H}_0/C_2(\mathcal{H}_0)$ has dimension 11, and it can be taken to be spanned by the vectors

$$\begin{aligned} L_{-2}^s\Omega, & \qquad \text{where } s = 0, 1, 2, 3, 4 \\ L_{-2}^s W_{-3}^a\Omega, & \qquad \text{where } s = 0, 1 \text{ and } a \in \text{adj}(\text{su}(2)). \end{aligned} \tag{2.33}$$

2.4.2. The modular differential equation. The above calculation leading to (2.31) implies that in Zhu’s algebra we have the relation

$$L_0^2(8L_0 + 1)(8L_0 - 3)(L_0 - 1) = 0, \tag{2.34}$$

where L_0 denotes the operator corresponding to the stress–energy tensor, and the product is to be understood as the product in Zhu’s algebra (see, for example, [58] for an explanation of this construction). Given (2.25) and (2.26) this therefore suggests that there should be a fifth-order differential equation that characterizes the vacuum torus amplitudes for the triplet theory, and furthermore, that the leading order (2.26) should precisely reduce to (2.34).⁴ Furthermore, since the space of vacuum torus amplitudes is invariant under the action of the modular group $\text{PSL}(2, \mathbb{Z})$, the differential equation must be modular invariant as well. The most general modular invariant differential equation of degree 5 is

$$\left[D^5 + \sum_{r=0}^4 f_r(q) D^r \right] T(q) = 0, \tag{2.35}$$

where each $f_r(q)$ is a polynomial in $E_4(q)$ and $E_6(q)$ of modular weight $10 - 2r$, and

$$D^i = \text{cod}_{(2i)} \cdots \text{cod}_{(2)} \text{cod}_{(0)}, \tag{2.36}$$

with cod_s being the modular covariant derivative on weight s modular functions

$$\text{cod}_{(s)} = q \frac{\partial}{\partial q} - \frac{1}{12}(s - 2)E_2(q), \tag{2.37}$$

which increments the weight of a modular form by 2. Here E_2 is the second Eisenstein series, and $\text{cod}_{(0)}f = f$. A differential equation of this type can always be found if the space of vacuum torus amplitudes forms a finite-dimensional representation of the modular group. For

⁴ Strictly speaking, the above argument only implies that there should be a differential equation characterizing the vacuum torus amplitudes whose order is at least 5. We shall assume in the following that the order is precisely 5; as we shall see later on, this assumption leads to a consistent description of the vacuum torus amplitudes.

the case of rational CFTs, this differential equation was first considered by [103, 104] (see also [34, 35] for further developments). It is often called the *modular differential equation*.

The first few of the D^i read to first order in q , i.e. where $E_2(q)$ is only taken as $1 - 24q + \mathcal{O}(q^2)$, and with the notation $D_q = q \frac{\partial}{\partial q}$, simply

$$\begin{aligned}
 D^0 &= 1, \\
 D^1 &= D_q, \\
 D^2 &= D_q^2 - \frac{1}{6}D_q + q4D_q, \\
 D^3 &= D_q^3 - \frac{1}{2}D_q^2 + \frac{1}{18}D_q + q\left(12D_q^2 + \frac{4}{3}D_q\right), \\
 D^4 &= D_q^4 - D_q^3 + \frac{11}{36}D_q^2 - \frac{1}{36}D_q + q\left(24D_q^3 + \frac{4}{3}D_q^2 + \frac{4}{3}D_q\right), \\
 D^5 &= D_q^5 - \frac{5}{3}D_q^4 + \frac{35}{36}D_q^3 - \frac{25}{108}D_q^2 + \frac{1}{54}D_q + q\left(40D_q^4 - \frac{20}{3}D_q^3 + \frac{20}{3}D_q^2\right), \tag{2.38}
 \end{aligned}$$

where all expressions are up to $\mathcal{O}(q^2)$. Of course, D^0 and D^1 are exact to all orders.

The most general ansatz for the differential equation (2.35) is therefore

$$\sum_{k=0}^5 \sum_{\substack{r,s \\ 4r+6s=10-2k}} a_{r,s}(E_4)^r (E_6)^s \left(\prod_{m=0}^k \text{cod}_{(2m)} \right) T(q) = 0. \tag{2.39}$$

This differential equation must reduce to (2.34) (in the sense of (2.25) and (2.26)) as $q \rightarrow 0$, and it must furthermore be satisfied for the characters of the irreducible highest weight representations of the triplet algebra. As we have explained before, there are four irreducible highest weight representations with conformal weights $h = 0, -1/8, 3/8$ and $h = 1$, and their corresponding characters are known [47, 48, 83]. In terms of the functions of section 2.2, they are given as

$$\chi_{-\frac{1}{8}}(q) = \theta_{0,2}(q)/\eta(q), \tag{2.40}$$

$$\chi_0(q) = (\theta_{1,2}(q) + (\partial\theta)_{1,2}(q))/\eta(q), \tag{2.41}$$

$$\chi_{\frac{3}{8}}(q) = \theta_{2,2}(q)/\eta(q), \tag{2.42}$$

$$\chi_1(q) = (\theta_{1,2}(q) - (\partial\theta)_{1,2}(q))/\eta(q). \tag{2.43}$$

Putting these pieces of information together we find that (up to an overall normalization constant) (2.39) is uniquely determined to be the differential equation

$$\begin{aligned}
 0 = & \left[\frac{143}{995328}E_4(q)E_6(q) + \frac{121}{82944}(E_4(q))^2\text{cod}_{(2)} + \frac{65}{2304}E_6(q)\text{cod}_{(4)}\text{cod}_{(2)} \right. \\
 & \left. - \frac{163}{576}E_4(q)\text{cod}_{(6)}\text{cod}_{(4)}\text{cod}_{(2)} + \text{cod}_{(10)}\text{cod}_{(8)}\text{cod}_{(6)}\text{cod}_{(4)}\text{cod}_{(2)} \right] T(q). \tag{2.44}
 \end{aligned}$$

It is instructive to look at the leading order of the above equation. If we expand the Eisenstein series $E_n = 1 + g_{n,1}q + \mathcal{O}(q^2)$ with $g_{n,1}$ given by $g_{2,1} = -24, g_{4,1} = 240, g_{6,1} = -504$, we obtain

$$\begin{aligned}
 0 = & \left(D_q^5 - \frac{5}{3}D_q^4 + \frac{397}{576}D_q^3 - \frac{427}{6912}D_q^2 - \frac{37}{82944}D_q + \frac{143}{995328} \right) T(q) \\
 & + q \left(40D_q^4 - \frac{895}{12}D_q^3 + \frac{2209}{96}D_q^2 - \frac{209}{216}D_q - \frac{1573}{41472} \right) T(q) + \mathcal{O}(q^2). \tag{2.45}
 \end{aligned}$$

The zero-order term in q can be factorized as

$$\frac{1}{995328}(24D_q - 11)(12D_q - 13)(24D_q + 1)(12D_q - 1)^2. \tag{2.46}$$

Recalling that D_q has to be replaced by $L_0 - \frac{c}{24} = L_0 + \frac{1}{12}$ in order to relate (2.25) to (2.26), this therefore reduces, as required, to (2.34). If we make the ansatz

$$T(q) = q^{h+\frac{1}{12}} (1 + c_1q + c_2q^2 + c_3q^3 + \mathcal{O}(q^4)), \quad (2.47)$$

the above differential equation becomes, up to third order,

$$\begin{aligned} 0 = \frac{q^{h+1/12}}{64} [q^0(h^2(h-1)(8h+1)(8h-3)) + q^1(c_1(h+1)^2h(8h+9)(8h+5) \\ + 2h(32h-45)(40h^2-5h-1)) + q^2(c_2(h+2)^2(h+1)(8h+17)(8h+13) \\ + 2c_1(32h-13)(h+1)(40h^2+75h+34) \\ + 2(3840h^4+2840h^3-17331h^2+706h-442)) \\ + q^3(c_3(h+3)^2(h+2)(8h+25)(8h+21) \\ + 2c_2(h+2)(32h+19)(40h^2+155h+149) \\ + 2c_1(3840q^4+18200q^3+14229q^2-10076q-10387) \\ + 4(2560h^4+28880h^3-66574h^2-9772h-12281)) + \mathcal{O}(q^4)]. \quad (2.48) \end{aligned}$$

2.4.3. Solving the modular differential equation. As we have seen, the modular differential equation is of fifth order for the triplet theory, and the space of vacuum torus amplitudes is therefore five dimensional. On the other hand, we have only got four irreducible representations that give rise, via their characters, to four vacuum torus amplitudes (that solve the differential equation). Let us now analyze how to obtain a fifth, linearly independent, vacuum torus amplitude. First, let us try to find a solution of the form (2.47). Because of the lowest order equation (2.46), this will only give rise to a solution provided that $h = -\frac{1}{8}, \frac{3}{8}, 0$ or $h = 1$. For each fixed h , one then finds that there is only one such solution, which therefore agrees with the corresponding character of the irreducible representation (i.e. with (2.40)–(2.43)). By the way, this conclusion was not automatic *a priori*, since there exist cases where the modular differential equation has two linearly independent solutions with the same conformal weight, both of which are of the power series form. The simplest example is provided by the two $h = 0$ characters of the $c = 1 - 24k$ series of rational CFTs, $k \in \mathbb{N}$, with extended symmetry algebra $\mathcal{W}(2, 8k)$. One of these solutions belongs to the vacuum representation, the other to a second $h = 0$ representation which, however, has a non-vanishing eigenvalue w of the W_0 zero mode [46].

The character of any highest weight representation always gives rise to a torus amplitude as in (2.47), and thus we have shown that *the space of vacuum torus amplitudes for the triplet theory is not spanned by the characters of the (irreducible) highest weight representations*. In fact, we find that the missing, linearly independent solution can be taken to be

$$T_5(q) = \log(q)(\partial\theta)_{1,2}(q)/\eta(q). \quad (2.49)$$

It is tempting to associate this vacuum torus amplitude with the logarithmic (indecomposable) highest weight representation \mathcal{R}_0 whose ground state conformal weight is $h = 0$ [47]. However, as we have just explained, this identification can only be formal. Furthermore, $T_5(q)$ is not uniquely determined by the above considerations, since we can (obviously) replace $T_5(q)$ by $T_5'(q) = T_5(q) + \alpha_0\chi_0(q) + \alpha_1\chi_1(q)$ for any (real) α_i , $i = 1, 2$. Additionally, it is not clear how $T_5(q)$ can be connected to the character of the logarithmic representation \mathcal{R}_0 , as $T_5(q)$ is not even a q -series.

Moreover, it is not difficult to compute the full character of the indecomposable representation with $h = 0$, and it turns out that it is linearly dependent on the solutions we already have found. In fact,

$$\chi_{\mathcal{R}_0} = \chi_0(q) + \chi_1(q) = 2\theta_{1,2}(q)/\eta(q). \quad (2.50)$$

To summarize, the modular differential equation has provided us with four linear independent solutions which can all be interpreted as characters of irreducible representations. The characters of the two indecomposable representations $\mathcal{R}_0 \cong \mathcal{R}_1$ turn out to be linear combinations of these four solutions.

It therefore seems that, unlike the case of a rational CFT where a (canonical) basis for the space of vacuum torus amplitudes is given in terms of the characters of the irreducible representations, the space of vacuum torus amplitudes does not possess a canonical basis in our case. In particular, Verlinde’s formula therefore cannot make sense since it presupposes such a canonical basis. (The Verlinde formula involves the matrix elements of the S -modular transformation; these matrix elements are only defined once a basis for the space of torus amplitudes has been chosen.) This is in nice agreement with the fact that the fusion rules of the triplet theory cannot be diagonalized [60], and therefore that they cannot be described by a Verlinde formula.

However, there exists a generalization of the Verlinde formula which works on the space of all torus amplitudes and then projects out all contributions from torus amplitudes which are not contained in the space of characters by limit procedures *after* the fusion rules have been computed [48]. The equivalence of the fusion rules computed via this generalized Verlinde formula with the correct fusion rules obtained by more direct methods (e.g. [45]) is shown in [53].

2.4.4. A more general analysis. For any (logarithmic) CFT which satisfies the C_2 condition, the mere existence of a finite-order differential equation allows us to derive some relations and bounds for the highest weights. As argued in the previous sections for the special case of the $c = -2$ triplet algebra, the torus amplitudes of such a theory have to satisfy an n th order holomorphic modular invariant differential equation of the form (2.35),

$$\left[D^n + \sum_{r=0}^{n-1} f_r(q) D^r \right] T(q) = 0, \tag{2.51}$$

where the $f_r(q) \in \mathbb{C}[E_4, E_6]$ are modular functions of weight $2(n - r)$. These coefficient functions may be expressed in terms of a set of n linearly independent solutions $T_1(q), \dots, T_n(q)$ of the differential equation (2.51). Note, however, that in contrast to [103, 104], these solutions cannot in general be identified with the characters of representations. In particular, we cannot therefore assume that the $T_i(q)$ have a good power series expansion in q up to a common fractional power $h_i - c/24 \pmod 1$.⁵ As we have seen, in LCFTs, torus amplitudes are not all elements in $\mathbb{C}((q))$, but may be in $\mathbb{C}((q))[\tau]$, i.e. they are power series in q times a polynomial in $\tau \equiv \frac{1}{2\pi i} \log(q)$.

The following analysis along the lines set out in [103, 104] has to take into account this fact. Therefore, we will *not* assume in the following that the highest weights are all different, $h_i \neq h_j$ for $i \neq j$, but only that $T_i(q) \neq T_j(q)$ for $i \neq j$. Note that the asymptotic behavior of two functions $T_i(q)$ and $T_j(q)$ in the limit $q \rightarrow 0$ (or $\tau \rightarrow +i\infty$) is the same whenever $T_j(q) = p(\tau)T_i(q)$ for a polynomial p , provided $T_i(q) \sim q^\alpha$ with $\alpha \neq 0$. The case $\alpha = 0$ occurs precisely when $h_i - c/24 = 0$. It is interesting to note that all known LCFTs, except for $c = 0$, share the fact that there is at most one irreducible representation with $h = c/24$. Our analysis suggests that this should be generally true such that we only need one function $T(q)$ with asymptotic behavior $T(q) \sim 1 + \mathcal{O}(q)$ for $q \rightarrow 0$. We will treat the case $c = 0$ separately below.

⁵ We will in the following always speak of power series expansions in q with the silent understanding that a common fractional power is allowed, i.e. that the functions can be expanded as $T(q) = q^\alpha \sum_{k=0}^\infty a_k q^k$, $\alpha \in \mathbb{Q}$.

With this in mind, we can express the coefficients of the modular differential equation in terms of the Wronskian of a set of n linearly independent solutions as

$$f_r(q) = (-1)^{n-r} W_r(q)/W_n(q),$$

$$W_r(q) = \det \begin{pmatrix} T_1(q) & \dots & T_n(q) \\ D^1 T_1(q) & \dots & D^1 T_n(q) \\ \vdots & & \vdots \\ D^{r-1} T_1(q) & \dots & D^{r-1} T_n(q) \\ D^{r+1} T_1(q) & \dots & D^{r+1} T_n(q) \\ \vdots & & \vdots \\ D^n T_1(q) & \dots & D^n T_n(q) \end{pmatrix}. \quad (2.52)$$

The fact that $f_r(q) \in \mathbb{C}[E_4, E_6]$ puts severe constraints on the possible polynomials $p_i(\tau)$ occurring in $T_i(q) = p_i(\tau)q^{h_i - \frac{c}{24}}(1 + a_{i,1}q + a_{i,2}q^2 + \dots)$. However, the explicit investigation of these constraints is beyond the scope of this review. For us, it is enough to note that known examples such as the torus amplitudes of the $c_{p,1}$ LCFTs (with $c_{2,1} = -2$ being the triplet model considered so far) do yield the correct coefficient functions in this way. The $c_{p,1}$ models, as all LCFTs with indecomposable representations spanning Jordan blocks of rank 2, involve only torus amplitudes $T(q)$ whose polynomial parts are either constant or at most of degree 1 in τ . In general, indecomposable representations with Jordan cells of rank R will involve torus amplitudes with polynomials in τ of degree $d < R$.

The torus amplitudes considered as functions in τ are non-singular in \mathbb{H} . As a consequence, the same applies for the W_r . Therefore, the coefficients f_r can have singularities only at the zeroes of W_n . We will see that the total number of zeroes of W_n can be expressed in terms of the number n of linearly independent torus amplitudes, the central charge c and the conformal weights h_i associated with the torus amplitudes $T_i(q)$. In order to do so, we note that in the $\tau \rightarrow +i\infty$ limit, the torus amplitudes behave as $\exp(2\pi i(h_i - \frac{c}{24})\tau)$. With the above caveat concerning the case $h = c/24$, this applies to all torus amplitudes independently of whether they are pure power series in q , or whether they have a τ -polynomial as an additional factor. This implies that $W_n \sim \exp(2\pi i(\sum_i h_i - n\frac{c}{24})\tau)$, which says that W_n has a pole of order $n\frac{c}{24} - \sum_i h_i$ at $\tau = i\infty$. Now, W_n involves precisely $\frac{1}{2}n(n-1)$ derivatives meaning that it transforms as a modular form of weight $n(n-1)$. Both facts together allow us to compute the total number of zeroes of W_n , which is

$$\frac{1}{6}\ell \equiv -\sum_{i=1}^n h_i + \frac{1}{24}nc + \frac{1}{12}n(n-1) \geq 0. \quad \ell \in \mathbb{Z}_+ - \{1\}. \quad (2.53)$$

This number cannot be negative since W_n must not have a pole in the interior of moduli space. We note that (2.53) is always a multiple of $\frac{1}{6}$ since W_n , as a single-valued function in Teichmüller space, may have zeroes at the ramification points $\exp(\frac{1}{3}\pi i)$ and $\exp(\frac{1}{2}\pi i)$ of order $\frac{1}{3}$ and $\frac{1}{2}$, respectively. Equation (2.53) provides a simple bound on the sum of the conformal weights.

For example, for the case of the $c = -2$ triplet theory, we have

$$- \left[\left(-\frac{1}{8}\right) + (0) + (0) + \left(\frac{3}{8}\right) + (1) \right] + \frac{1}{24}(5)(-2) + \frac{1}{12}(5)(4) = 0. \quad (2.54)$$

This is now sufficient to exclude that the $c = -2$ triplet theory is governed by a modular differential equation of order 6. For if it were of order 6, the additional sixth conformal weight would have to be of the form $h = \frac{3}{4} - \frac{1}{6}\ell$. This is obviously always a rational number with denominator either 4 or 12. However, this weight does not belong to the list of admissible weights of the $c = -2$ triplet theory, not even modulo 1, and thus we have a contradiction.

One can also check this directly: assuming that we have a modular differential equation of order 6 and starting with the four irreducible characters (2.40)–(2.43) one obtains an ansatz for the modular differential equation depending on the choice of the conformal weights of the two remaining representations. One finds that no consistent choice is possible, and thus no six-dimensional representation of the modular group containing the characters (2.40)–(2.43) can be found.

2.4.5. The other triplet theories. The analysis presented so far can in principle be generalized to all members of the $c_{p,1}$ series of triplet models. In practice, however, we have not found it possible to give uniform explicit expressions. The pattern which emerges in the treatment of the $c = -2$ case, i.e. the case $p = 2$, however, seems to be of a generic nature. Indeed, all the $c_{p,1}$ models are C_2 cofinite [23] and the characters of their irreducible representations are all known. They close under modular transformations provided that a certain number of ‘logarithmic vacuum torus amplitudes’ (the analogues of $T_5(q)$) is added to the set. In fact, the characters of the irreducible representation, together with additional torus amplitudes which we may again associate with the indecomposable representations, read [47]

$$\chi_{0,p}(q) = \frac{1}{\eta(q)} \Theta_{0,p}(q), \tag{2.55}$$

$$\chi_{p,p}(q) = \frac{1}{\eta(q)} \Theta_{p,p}(q), \tag{2.56}$$

$$\chi_{\lambda,p}^+(q) = \frac{1}{p\eta(q)} [(p - \lambda)\Theta_{\lambda,p}(q) + (\partial\Theta)_{\lambda,p}(q)], \tag{2.57}$$

$$\chi_{\lambda,p}^-(q) = \frac{1}{p\eta(q)} [\lambda\Theta_{\lambda,p}(q) - (\partial\Theta)_{\lambda,p}(q)], \tag{2.58}$$

$$\tilde{\chi}_{\lambda,p}(q) = \frac{1}{\eta(q)} [2\Theta_{\lambda,p}(q) - i\alpha \log(q)(\partial\Theta)_{\lambda,p}(q)], \tag{2.59}$$

where $0 < \lambda < p$ and where we made use of definitions (2.8), (2.9) and (2.12). As before, the ‘logarithmic’ torus amplitudes $\tilde{\chi}_{\lambda,p}$ are not uniquely determined by these considerations since α is a free constant; the form given above is convenient for constructing modular invariant partition functions. One should note, however, that for LCFTs the complete space of states of the full non-chiral theory is not simply the direct sum of tensor products of chiral representations (see, for example, [61]). It is therefore not clear how the full torus amplitude has to be constructed out of these generalized characters.

The congruence subgroup for the $c_{p,1}$ model is $\Gamma(2p)$. There are $2p$ characters corresponding to irreducible representations, and $(p - 1)$ ‘logarithmic’ torus amplitudes, giving rise to a $(3p - 1)$ -dimensional representation of the modular group. In particular, we therefore expect that the order of the modular differential equation is $(3p - 1)$. Furthermore, we expect that the dimension of Zhu’s algebra is $6p - 1$: it follows from the structure of the above vacuum torus amplitudes that p of the irreducible representations have a one-dimensional ground state space, while the other p irreducible representations have a ground state multiplicity 2; as above one may furthermore expect that each of the $(p - 1)$ logarithmic representations probably leads to one additional state, thus giving altogether the dimension $p + 4p + (p - 1) = 6p - 1$. Indeed, this was proved in [3]. In addition, the same authors obtained a modular differential equation of order $3p - 1$ for all p , satisfied by all the torus amplitudes [1].

While it is not possible to write down a general expression for the modular differential equation for all p , we can give support for these conjectures by analyzing the $p = 3$ triplet model with $c = -7$. The vacuum character of this theory is $\chi_{2,3}^+(q)$. Under the assumption that the modular differential equation is in fact of order $3p - 1 = 8$, we can determine it uniquely by requiring it to be solved by this vacuum character. Explicitly, we find

$$0 = \left[\left(\frac{833}{53747712} E_4(q) (E_6(q))^2 - \frac{990437}{36691771392} (E_4(q))^4 \right) - \frac{40091}{143327232} (E_4(q))^2 E_6(q) \text{cod}_{(2)} \right. \\ + \left(\frac{115}{746496} (E_6(q))^2 + \frac{53467}{47775744} (E_4(q))^3 \right) \text{cod}_{(4)} \text{cod}_{(2)} \\ - \frac{5897}{124416} E_4(q) E_6(q) \text{cod}_{(6)} \text{cod}_{(4)} \text{cod}_{(2)} + \frac{10889}{55296} (E_4(q))^2 \text{cod}_{(8)} \text{cod}_{(6)} \text{cod}_{(4)} \text{cod}_{(2)} \\ + \frac{157}{432} E_6(q) \text{cod}_{(10)} \text{cod}_{(8)} \text{cod}_{(6)} \text{cod}_{(4)} \text{cod}_{(2)} \\ - \frac{21}{16} E_4(q) \text{cod}_{(12)} \text{cod}_{(10)} \text{cod}_{(8)} \text{cod}_{(6)} \text{cod}_{(4)} \text{cod}_{(2)} \\ \left. + \text{cod}_{(16)} \text{cod}_{(14)} \text{cod}_{(12)} \text{cod}_{(10)} \text{cod}_{(8)} \text{cod}_{(6)} \text{cod}_{(4)} \text{cod}_{(2)} \right] T(q). \tag{2.60}$$

If we make the ansatz that $T(q)$ is of the form

$$T(q) = q^{h-\frac{c}{24}} \sum_{n=0}^{\infty} \sum_{k=0}^1 c_{k,n} \tau^k q^n, \tag{2.61}$$

we obtain, to lowest order, the polynomial condition

$$0 = \frac{1}{2304} (1 + 4h)^2 h^2 (h - 1)(12h - 5)(3h + 1)(4h - 7)(c_{1,0} \tau + c_{0,0}) \\ + \frac{1}{1152} (1 + 4h)h(2304h^5 - 5280h^4 + 2160h^3 + 870h^2 - 229h - 35)c_{1,0} \\ + \mathcal{O}(q). \tag{2.62}$$

As expected, we can read off from this expression the allowed conformal weights: if the character does not involve any powers of τ ($c_{1,0} = 0$), then h needs to be from the set $h \in \{0, -1/4, 1, 5/12, -1/3, 7/4\}$. Furthermore, we have two ‘logarithmic’ torus amplitudes with $h = 0$ and $h = -1/4$. This then fits nicely together with the fact that there are in fact two indecomposable highest weight representations with these conformal weights [59].

We see that the modular differential equation is a powerful tool to investigate a specifically given CFT where it may greatly help to understand its representation theory from a comparably small amount of pre-knowledge. However, it is less suitable to derive general statements about (certain series of) rational CFTs. But as we will see in the next section, we can draw some general conclusions from the rather restrictive conditions the spectrum of a rational CFT must satisfy, such that a modular differential equation can exist (which it must, if the theory is indeed rational).

2.4.6. Augmented minimal models. The diophantine equation (2.53) is very powerful. It allows us to estimate whether a CFT with given central charge c and a certain set $\{h_i : i \in I\}$ of known weights of representations has a chance to be rational in the sense that the associated torus amplitudes $T_i(q)$ can form a finite-dimensional representation of the modular group. To illustrate this, let us consider a CFT with degenerate representations in the sense of BPZ [12]. Let us further, for the sake of simplicity, restrict ourselves to the theories in the Kac table

$$c = c_{p,q} = 1 - 6 \frac{(p-q)^2}{pq}, \quad h_{r,s}(c_{p,q}) = \frac{(pr - qs)^2 - (p-q)^2}{4pq}. \tag{2.63}$$

However, we do not at this stage restrict ourselves to minimal models, so we do not prescribe finite ranges for r, s . It is clear that the standard (not truncated) fusion rules

$$[r, s] * [r', s'] = \sum_{\substack{\rho = |r-r'|+1 \\ r+r'+\rho \equiv 1 \pmod 2}}^{r+r'-1} = \sum_{\substack{\sigma = |s-s'|+1 \\ s+s'+\sigma \equiv 1 \pmod 2}}^{s+s'-1} [\rho, \sigma] \tag{2.64}$$

imply that any possible finite subset of the infinite Kac table must be a rectangle starting at $[1, 1]$ and ending at $[b, a]$ such that $1 \leq r \leq b$, $1 \leq s \leq a$. In the minimal models we have $a = p - 1$ and $b = q - 1$. Summing up all these $n = a \cdot b$ weights in formula (2.53) yields

$$-\frac{ab}{24pq}((2b + 1)p - (a + 1)q)((b + 1)p - (2a + 1)q). \quad (2.65)$$

This generically can only be a solution of the form $\ell/6$, $\ell \in \mathbb{Z}_+$, if $a = \alpha p - 1$, $b = \beta q - 1$ with $\alpha, \beta \in \mathbb{N}$ or if $a = \alpha p$, $b = \beta q$. We discard the second type of solution, since it generically leads to $\ell < 0$. Thus, the only possible finite subsets of the Kac table are formally obtained as the conformal grids of $c_{\alpha p, \beta q}$ instead of $c_{p, q}$. Assuming this, yields

$$\frac{(\alpha p - 1)(\beta q - 1)}{24}(p(2\alpha - \beta) - 1)(q(2\beta - \alpha) - 1). \quad (2.66)$$

A further constraint comes from the fact that we have naively counted all weights in a sub-rectangle of the Kac table. This yields too many representations, since $h_{r,s}(c_{p,q}) = h_{\delta q - r, \delta p - s}(c_{p,q})$ for all $\delta \in \mathbb{N}$.⁶ This leaves us with the only sensible case $\alpha = \beta$, which means that only multiples of the standard conformal grid are possible. Thus, instead of the conformal grid for $c_{p,q}$ there might exist a finite-dimensional representation of the modular group with functions $T_i(q)$, whose asymptotics is determined by an enlarged conformal grid, formally obtained by considering $c_{\alpha p, \alpha q}$ instead of $c_{p,q}$, thus allowing for a grid size determined by non-coprime numbers $p' = \alpha p$, $q' = \alpha q$. We hence find the diophantine condition

$$\frac{\ell}{6} = \frac{1}{24}(\alpha p - 1)^2(\alpha q - 1)^2. \quad (2.67)$$

Therefore, either αp or αq must be odd in order to kill a factor of at least 4 from the denominator. Since p and q are assumed to be coprime, one of them must be odd. Thus, the diophantine condition can indeed be satisfied with an $\ell \in \mathbb{Z}_+ - \{1\}$ for all α odd.

There is still one subtlety. When we consider conformal grids of this form, all weights within this grid appear at least twice. Thus, we should attempt to only take half of the weights. Assuming now from the beginning a conformal grid of size $(\alpha p - 1)(\alpha q - 1)$, α odd, but only taking half of the weights, we find that the condition (2.53) is satisfied with $\ell = 0$. This is a particularly nice result, since it means that the determinant W_n has no zeroes in the interior of moduli space. Thus, there are no values τ or q , respectively, for which the torus amplitudes become linearly dependent. Taking this as a guideline, one can do a bit more refined analysis.

In order to achieve $\ell = 0$, we should take into account the symmetry $h_{r,s} = h_{\alpha q - r, \alpha p - s}$ right from the beginning. The diophantine condition then reduces to

$$\frac{1}{6}\ell \equiv -\frac{1}{48} \frac{ab[(a + 1)q - (b + 1)p][(2a + 1)q - (2b + 1)p]}{pq} \quad \ell \in \mathbb{Z}_+ - \{1\}. \quad (2.68)$$

Obviously, the numerator has two series of non-trivial but generally valid solutions to make it vanish, namely $(a, b) = (\alpha p - 1, \alpha q - 1)$ and $(a, b) = ((\alpha p - 1)/2, (\alpha q - 1)/2)$. The second series is only valid when both $\alpha q - 1$ and $\alpha p - 1$ are even integers implying that p and q must both be odd then. Here, $\alpha \in \mathbb{N}$, of course. This result suggests that any extension of the Kac table by a common integer factor α could yield a set of conformal weights such that the diophantine condition is satisfied and therefore a modular differential equation may exist. However, we have to be a bit more careful here. When counting the number of representations

⁶ If we do not assume the representations to be irreps, we might be tempted to relax from identifying the representations according to the symmetry of the weights. For instance, this could be the case, if we do not presuppose that all representations possess more than one null vector (as in the minimal models). Concretely, it is common in the literature to associate all the various representations of the $(1, p)$ triplet models or of logarithmically extended minimal models, irreps as well as indecomposable representations, with entries from extended Kac tables. see e.g. [33, 50]. However, we still cannot neglect the symmetry of the weights, as it reflects, for example, in the fusion rules.

given by the Kac table of size $a \cdot b$, one typically has to take only one half of them, since the other half is naturally identified via the Virasoro module embedding structure. If $a \cdot b$ is odd, then this cannot be done without one remaining field which appears only once in the Kac table. Thus, there is another condition such that our generic solutions might work, namely

$$(\alpha p - 1)(\alpha q - 1) \in 2\mathbb{N} \quad \text{or} \quad (\alpha p - 1)(\alpha q - 1)/4 \in 2\mathbb{N}. \quad (2.69)$$

The second solution is thus more restrictive than the first, so we can concentrate on the easier case. We see immediately that the case $\alpha = 2$ is ruled out as a possible solution since this automatically produces an odd number. Therefore, the next possible case after the generic solution $\alpha = 1$ is $\alpha = 3$. Note that $\alpha = 1$ always works since p, q are coprime by assumption and thus cannot be both even. In fact, the case $\alpha = 2\alpha'$ even does not solve the diophantine equation at all, if the number of representations is counted correctly. Namely, the entry $h_{\alpha'q, \alpha'p}(c_{p,q})$ exactly in the center of the conformal grid appears only once. Taking this into account properly leads to a different result. Instead of $\ell = 0$, we now find that

$$\ell = pq(p + q)\alpha'^3 - (pq)^2\alpha'^4 + \frac{1}{4}(1 - \alpha'[p(\alpha'p + 1) + q(\alpha'q + 1)]). \quad (2.70)$$

Since ℓ must be integer, we immediately see that α' must be odd. But even then we cannot solve this condition, since then $n(\alpha'n + 1) \equiv n(n + 1) \equiv 0$ modulo 2, such that there is no chance to cancel the remaining $1/4$.

Summarizing, the diophantine equation implies that augmented minimal models might exist, if the weights are taken from the enlarged Kac table of size $(\alpha p - 1, \alpha q - 1)$ for any odd α . The logarithmic triplet theories are an example, since the admissible representations are taken from the conformal grid of $c = c_{3p,3}$, i.e. from the enlarged Kac table with $\alpha = 3$. Similarly, as shown in [32, 33] for the explicit cases $c = c_{9,6}$ and $c_{15,6}$, logarithmic extensions of minimal models do exist for $\alpha = 3$.

One should note that we can only manage $\ell = 0$, since the result for the appropriate sum of conformal weights from the Kac table typically turns out to be a negative rational number. Thus, solutions with $\ell > 1$ are presumably not to be found within Virasoro minimal models.

Similar considerations could be conducted for generic, rational central charges $c \neq c_{p,q}$ such that the entries from the Kac table are all rational.

On the other hand, if for a given central charge we can compute the vacuum character of the chiral symmetry algebra, or at least a sufficiently high order of it, then we only need an estimate of the expected number of admissible representations to check whether this is compatible with the given vacuum character and to identify the possible conformal weights. In principle, one could attempt to set up modular differential equations for the case $\ell = 0$ and of degree n , $n = 1, 2, 3, \dots$, increasing n until the vacuum character turns out to be a solution without contradictions. However, although the choice $\ell = 0$ is plausible, determining the degree of the modular differential equation without any further knowledge proves difficult even for today's computers as soon as n gets large.

Finally, for augmented minimal models with $c = c_{3p,3q}$, we do have some knowledge about the vacuum character and a good guess about the degree of the modular differential equation is satisfied, as well as of the asymptotics of the other torus amplitudes (i.e. the conformal weights from the augmented Kac table). From this we at least get a set of modular functions which span the space of torus amplitudes. One may then exploit the strategy of [48] to find the correct linear combinations of these functions which form the actual characters of the augmented minimal model. This works by seeking such linear combinations of the basis of torus amplitude which bring the S -matrix into a form suitable for the computation of fusion rules via the limit procedure of the generalized Verlinde formula proposed in that paper.

2.4.7. *The case $c = 0$.* A particularly interesting case is a CFT of vanishing central charge. There are some highly interesting problems in two dimensions, especially percolation, where CFT with central charge $c = 0$ seems to play an important role. In this case, the critical value for a conformal weight, such that the asymptotic behavior of amplitudes is not independent of a polynomial factor $p(\tau)$, is just $h = 0$. Standard lore in LCFT, however, holds that the vacuum representation is always part of an indecomposable structure. Moreover, the extended Kac table for $c_{2,3} = 0$ contains several entries with vanishing conformal weight.

It is therefore tempting to apply the technique of the modular differential equation to this case and see whether we can relax the condition that a torus amplitude $T(q) = p(\tau)q^{h-c/24} \sum_{k=0}^{\infty} (a_k q^k)$, where $a_0 \neq 0$, must have $p(\tau) \equiv 1$ for $h = c/24$. A CFT with vanishing central charge is a difficult object. In case of unitarity, a vanishing central charge implies that the theory is trivial, i.e. its field content is given by the identity field alone. The second possibility is, as known from string theory, that a unitary theory with $c > 0$ is tensored with a ghost theory with central charge $-c$, such that the total central charge vanishes. However, we might be interested in non-trivial (and therefore non-unitary) theories with a vanishing central charge, which cannot be decomposed into factors with non-zero central charges.

We claim that the Virasoro minimal model with a central charge $c = 0$, which is trivial, can be extended to a non-unitary logarithmic model with finitely many representations. According to the general reasoning of the previous section (see also [47]), we expect that these representations are labeled by the formal conformal grid one obtains by considering $c_{3p,3q}$ instead of $c_{p,q}$. We know that the conformal weights from this enlarged Kac table do satisfy all diophantine conditions necessary to set up a modular differential equation. Let us therefore assume that there exists a rational, possibly logarithmic, CFT such that its characters must satisfy a modular differential equation, generalized to the case of LCFTs with finitely many representations. We may then expect that this theory also fulfils Zhu's C_2 condition. Again, not all solutions can be interpreted as characters, i.e. traces over irreducible modules, but they should be understood as vacuum amplitudes on the torus.

Thus, in the case $c_{2,3} = c_{6,9}$ at hand, the extended conformal grid has the size $5 \times 8 = 40$ and thus the modular differential equation is of the order $40/2 = 20$, its solutions span the space of potential vacuum amplitudes of this theory. Whether this 20-dimensional space and the related CFT have something to do with the problem of percolation will be investigated in future work. The equation itself is extremely lengthy, so we give it here by referring to its general form (2.39) and merely listing the coefficients:

$$\begin{aligned}
 a_{4,1} &= -3857/144, \\
 a_{6,0} &= 27\,455/1728, \\
 a_{8,2} &= 11\,448\,089/55\,296, \\
 a_{10,1} &= -81\,132\,109/331\,776, \\
 a_{12,0} &= 287\,917\,355/23\,887\,872, \\
 a_{12,3} &= -24\,118\,392\,235/47\,775\,744, \\
 a_{14,2} &= 2145\,063\,995/2359\,296, \\
 a_{16,1} &= -1007\,439\,963\,335/6879\,707\,136, \\
 a_{16,4} &= 16\,235\,170\,093\,025/110\,075\,314\,176, \\
 a_{18,0} &= -7342\,258\,067\,105/82\,556\,485\,632, \\
 a_{18,3} &= -242\,834\,836\,836\,605/330\,225\,942\,528, \\
 a_{20,2} &= 422\,788\,707\,478\,865/2641\,807\,540\,224,
 \end{aligned}$$

$$\begin{aligned}
 a_{20,5} &= 124\,711\,173\,611\,453/293\,534\,171\,136, \\
 a_{22,1} &= 7535\,975\,797\,080\,575/47\,552\,535\,724\,032, \\
 a_{22,4} &= -1070\,013\,044\,796\,865/5283\,615\,080\,448, \\
 a_{24,0} &= -59\,176\,258\,416\,552\,175/2282\,521\,714\,753\,536, \\
 a_{24,3} &= 93\,314\,657\,139\,863\,735/1141\,260\,857\,376\,768, \\
 a_{24,6} &= -429\,817\,670\,835\,475\,919/2282\,521\,714\,753\,536, \\
 a_{26,2} &= -4260\,482\,854\,319\,725/71\,328\,803\,586\,048, \\
 a_{26,5} &= 2786\,843\,679\,881\,165/23\,776\,267\,862\,016, \\
 a_{28,1} &= 3016\,483\,131\,491\,075/41\,085\,390\,865\,563\,648, \\
 a_{28,4} &= -266\,846\,149\,383\,639\,005/20\,542\,695\,432\,781\,824, \\
 a_{28,7} &= 97\,023\,065\,226\,363\,863/41\,085\,390\,865\,563\,648, \\
 a_{30,0} &= 291\,326\,860\,458\,440\,185/246\,512\,345\,193\,381\,888, \\
 a_{30,3} &= 458\,410\,214\,497\,514\,285/123\,256\,172\,596\,690\,944, \\
 a_{30,6} &= -657\,895\,207\,190\,269\,363/246\,512\,345\,193\,381\,888, \\
 a_{32,2} &= 1039\,516\,789\,897\,710\,575/986\,049\,380\,773\,527\,552, \\
 a_{32,5} &= -56\,214\,246\,975\,189\,865/18\,260\,173\,718\,028\,288, \\
 a_{32,8} &= 1735\,787\,603\,351\,876\,711/986\,049\,380\,773\,527\,552, \\
 a_{34,1} &= -310\,946\,513\,309\,707\,165/2958\,148\,142\,320\,582\,656, \\
 a_{34,4} &= -941\,732\,882\,831\,629\,265/4437\,222\,213\,480\,873\,984, \\
 a_{34,7} &= 109\,979\,8540\,316\,599\,171/2958\,148\,142\,320\,582\,656, \\
 a_{36,0} &= 17\,980\,722\,135\,844\,325/35\,497\,777\,707\,846\,991\,872, \\
 a_{36,3} &= 437\,118\,904\,830\,824\,975/53\,246\,666\,561\,770\,487\,808, \\
 a_{36,6} &= 589\,451\,531\,309\,053\,765/35\,497\,777\,707\,846\,991\,872, \\
 a_{36,9} &= -26\,032\,177\,604\,788\,465/986\,049\,380\,773\,527\,552, \\
 a_{38,2} &= -138\,927\,688\,403\,253\,475/638\,959\,998\,741\,245\,853\,696, \\
 a_{38,5} &= 99\,983\,800\,440\,166\,775/35\,497\,777\,707\,846\,991\,872, \\
 a_{38,8} &= -57\,253\,337\,340\,701\,425/23\,665\,185\,138\,564\,661\,248, \\
 a_{40,1+3k} &= 0, \quad k = 0, 1, 2, 3.
 \end{aligned}
 \tag{2.71}$$

Many of its solutions will have the same asymptotic behavior modulo 1, i.e. $T_i(q) \sim T_j(q) \pmod 1$ for $T_i(q) = p_i(\tau)q^{h_i-c/24}(1 + \mathcal{O}(q))$ and $T_j(q) = p_j(\tau)q^{h_j-c/24}(1 + \mathcal{O}(q))$, if $h_i - h_j \in \mathbb{Z}$. Thus, the modular differential equation only gives us a basis in the space of vacuum amplitudes, but cannot tell us what the physically correct linear combinations are. What we obtain is therefore only a 20-dimensional set of modular forms out of which the characters of the irreducible and indecomposable representations as well as some further functions, which can only be interpreted as vacuum amplitudes, should be constructed.

Our modular forms are all constructed from the building blocks $\eta(q)$, $\Theta_{\lambda,k}(q)$, $(\partial\Theta)_{\lambda,k}(q)$ defined in section 2.2, plus

$$(\nabla\Theta)_{\lambda,k}(q) = \log(q) \sum_{n \in \mathbb{Z}} (2kn + \lambda) q^{\frac{1}{4k}(2kn + \lambda)^2},
 \tag{2.72}$$

$$E_r(q) = 1 - \frac{2r}{B_r} \sum_{n=0}^{\infty} \sigma_{r-1}(n) q^n,
 \tag{2.73}$$

with $\lambda, k \in \mathbb{Z}_+$ and $r > 0$ even, and where $\sigma_p(n)$ is the sum of the p th powers of all divisors of n and B_r is the r th Bernoulli number. Of course, the E_r are the Eisenstein series. Note, however, that E_2 is actually not a good modular form, because it has a cusp. However, E_2 plays an important role as it appears, for example, in the modular covariant derivative.

Let us now study the special case of $c = c_{2,3} = 0$. Besides the trivial solution, where the spectrum contains only the identity operator, the next smallest modular differential equation one can write down for this central charge and conformal weights from the Kac table has the order 20. There is no equation of smaller order where the corresponding diophantine conditions on the central charge and the conformal weights can be satisfied with weights taken from the Kac table. The solutions of this equation factorize into three sets which are separately closed under modular transformations. However, the character of the irreducible vacuum representation (which is part of a larger indecomposable representation) is presumably a linear combination of some of the 20 solutions in such a way that these three independent sets are intermixed. It is not yet known what the precise vacuum character is, but in a work on alternating sign matrices by Pearce *et al*, two different characters for two inequivalent $h = 0$ representations of a $c = 0$ Virasoro model are computed up to level 5 [113]. Taking these as ansatz supports our conjectures that firstly there is more than one $h = 0$ representation, and secondly, the characters of the $h = 0$ representations are non-trivial linear combinations of our solution set intermixing the three factors. The three factors of the solution are of dimensions 6, 10 and 4, respectively. They read

$$\begin{aligned} \mathcal{M}_1 &= \text{span} \left\langle \frac{1}{\eta(q)} (\Theta_{1,6}(q) + \Theta_{5,6}(q)), \frac{1}{\eta(q)} \Theta_{\lambda,6}(q) : \lambda \in \{0, 2, 3, 4, 6\} \right\rangle, \\ \mathcal{M}_2 &= \text{span} \left\langle \frac{1}{\eta(q)} (\partial \Theta)_{\lambda,6}(q) : \lambda \in \{1, 2, 3, 4, 5\}, \frac{1}{\eta(q)} (\nabla \Theta)_{\lambda,6}(q) : \lambda \in \{1, 2, 3, 4, 5\} \right\rangle, \\ \mathcal{M}_3 &= \text{span} \left\langle \frac{1}{\eta(q)} (\Theta_{1,6}(q) - \Theta_{5,6}(q)), E_2(q), \log(q) E_2(q), \right. \\ &\quad \left. \log^2(q) E_2(q) + 12 \log(q) \frac{1}{\eta(q)} (\Theta_{1,6}(q) - \Theta_{5,6}(q)) \right\rangle. \end{aligned} \tag{2.74}$$

Note the appearance of a $\log^2(q)$ term which indicates a more involved indecomposable structure of the modules than a simple rank 2 Jordan cell. Note also that a particular linear combination of otherwise very benign Theta-functions shows up in \mathcal{M}_3 in order to make this set close under modular transformations. This linear combination is removed from set \mathcal{M}_1 . As written down, these sets close under the S -transformation $\tau \mapsto -\frac{1}{\tau}$. Under the T -transformation $\tau \mapsto \tau + 1$, the sets \mathcal{M}_1 and \mathcal{M}_2 map into themselves, while \mathcal{M}_3 maps to $\mathcal{M}_3 \cup \mathcal{M}_1$. The S -matrices for the first two sets are standard. For the third set, it reads

$$S^{(3)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{4\pi^2} \\ -12 & 0 & -1 & 0 \\ 0 & -4\pi^2 & 0 & 0 \end{pmatrix}, \tag{2.75}$$

which has determinant 1. It comes as a surprise that the Eisenstein series E_2 appears as a solution to a modular differential equation.

The corresponding conformal weights are as follows. The functions in \mathcal{M}_1 have asymptotics compatible with conformal weights $h \in \{-1/24, 0, 1/8, 1/3, 5/8, 35/24\}$. The functions in \mathcal{M}_2 are compatible with $h \in \{0, 1/8, 1/3, 5/8, 1\}$, and finally the functions in

\mathcal{M}_3 are all compatible with $h = 0$. The simplest candidates for physical $h = 0$ characters which could be compared to the work by Pearce *et al* [113] are given by the expression

$$\chi_{h=0}^{(j)}(q) = \frac{1}{\eta(q)} \sum_n (-1)^n q^{h(\frac{3}{2}n + \frac{1}{2}j)} \tag{2.76}$$

for $j = 0, 1, 2$ and $h(k) = \frac{1}{3}k(2k - 1) + \frac{1}{24}$. These functions agree with their results up to the orders provided. But $j = 0, 1$ cannot yield the true characters of the physical (rank 1) $h = 0$ representations. The reason is that they are not really linearly combined out of functions of all three sets \mathcal{M}_i . Namely,

$$\chi_{h=0}^{(0)}(q) = \frac{1}{\eta(q)} \left(\Theta_{1,6}(q) - \frac{7}{12} \Theta_{5,6}(q) - \frac{1}{12} (\partial\Theta)_{5,6}(q) \right), \tag{2.77}$$

$$\chi_{h=0}^{(1)}(q) = \frac{1}{\eta(q)} \left(\Theta_{1,6}(q) - \frac{5}{12} \Theta_{5,6}(q) + \frac{1}{12} (\partial\Theta)_{5,6}(q) \right), \tag{2.78}$$

$$\chi_{h=-1/3}^{(j=2)}(q) = \frac{1}{6\eta(q)} (3\Theta_{3,6}(q) + (\partial\Theta)_{3,6}(q)) \tag{2.79}$$

only involve $\frac{1}{\eta}(\Theta_{1,6} - \Theta_{5,6}) \equiv 1$ from \mathcal{M}_3 , which forms a trivial irreducible subrepresentation under the action of the modular group, as can be read off (2.75). However, the characters of the CFT should form a basis for a representation of the modular group, in which this representation does not decompose into smaller ones. This is only possible if the physical $h = 0$ representations have characters which involve modular forms from all three sets \mathcal{M}_i in such a way that E_2 must be a part of the linear combinations. This requirement together with the condition that the q -series must have non-negative integer coefficients greatly limits the possible linear combinations.

Of course, by now much more is known about logarithmic extensions of minimal models and especially the minimal model at $c = 0$. See for example [62, 63, 116, 130] and references therein. In particular, the characters of all the representations of the logarithmically extended minimal model with $c = 0$ have been computed. It turns out that one needs the $h = 0$ characters up to order 20 to uniquely fix them in terms of the modular functions from our sets \mathcal{M}_i . This is to be expected, as the modular differential equation has order 20, which was first given in [2, 49]. Using the notation from [62, 116], we find for the characters of the irreps

$$\chi_{\mathcal{W}(0)} = \frac{1}{\eta} (\Theta_{1,6} - \Theta_{5,6}) \equiv 1, \tag{2.80}$$

$$\chi_{\mathcal{W}(1)} = \frac{1}{\eta} \left(\frac{-11}{720} \Theta_{1,6} + \frac{251}{720} \Theta_{5,6} + \frac{10}{720} (\partial\Theta)_{1,6} + \frac{70}{720} (\partial\Theta)_{5,6} \right) + \frac{1}{720} E_2, \tag{2.81}$$

$$\chi_{\mathcal{W}(2)} = \frac{1}{\eta} \left(\frac{-11}{720} \Theta_{1,6} + \frac{131}{720} \Theta_{5,6} + \frac{10}{720} (\partial\Theta)_{1,6} - \frac{50}{720} (\partial\Theta)_{5,6} \right) + \frac{1}{720} E_2, \tag{2.82}$$

$$\chi_{\mathcal{W}(5)} = \frac{1}{\eta} \left(\frac{71}{720} \Theta_{1,6} + \frac{169}{720} \Theta_{5,6} - \frac{70}{720} (\partial\Theta)_{1,6} - \frac{10}{720} (\partial\Theta)_{5,6} \right) - \frac{1}{720} E_2, \tag{2.83}$$

$$\chi_{\mathcal{W}(7)} = \frac{1}{\eta} \left(\frac{-49}{720} \Theta_{1,6} + \frac{169}{720} \Theta_{5,6} + \frac{50}{720} (\partial\Theta)_{1,6} - \frac{10}{720} (\partial\Theta)_{5,6} \right) - \frac{1}{720} E_2, \tag{2.84}$$

$$\chi_{\mathcal{W}(1/3)} = \frac{1}{\eta} \left(\frac{3}{6} \Theta_{3,6} + \frac{1}{6} (\partial\Theta)_{3,6} \right), \tag{2.85}$$

$$\chi_{\mathcal{W}(10/3)} = \frac{1}{\eta} \left(\frac{3}{6} \Theta_{3,6} - \frac{1}{6} (\partial\Theta)_{3,6} \right), \tag{2.86}$$

$$\chi_{\mathcal{W}(1/8)} = \frac{1}{\eta} \left(\frac{4}{6} \Theta_{2,6} + \frac{1}{6} (\partial \Theta)_{2,6} \right), \tag{2.87}$$

$$\chi_{\mathcal{W}(5/8)} = \frac{1}{\eta} \left(\frac{2}{6} \Theta_{4,6} + \frac{1}{6} (\partial \Theta)_{4,6} \right), \tag{2.88}$$

$$\chi_{\mathcal{W}(21/8)} = \frac{1}{\eta} \left(\frac{4}{6} \Theta_{4,6} - \frac{1}{6} (\partial \Theta)_{4,6} \right), \tag{2.89}$$

$$\chi_{\mathcal{W}(33/8)} = \frac{1}{\eta} \left(\frac{2}{6} \Theta_{2,6} - \frac{1}{6} (\partial \Theta)_{2,6} \right), \tag{2.90}$$

$$\chi_{\mathcal{W}(-1/24)} = \frac{1}{\eta} (\Theta_{0,6}), \tag{2.91}$$

$$\chi_{\mathcal{W}(35/24)} = \frac{1}{\eta} (\Theta_{6,6}). \tag{2.92}$$

This shows that E_2 is indeed a part of all non-trivial irreps with integer conformal weights. In particular, it contributes to the two non-trivial and inequivalent physical $h = 0$ rank 1 representations \mathcal{W} and \mathcal{Q} , $\chi_{\mathcal{W}} = \chi_{\mathcal{W}(0)} + \chi_{\mathcal{W}(1)}$ and $\chi_{\mathcal{Q}} = \chi_{\mathcal{W}(0)} + \chi_{\mathcal{W}(2)}$. These, as all characters of the indecomposable representations, are linear combinations of the characters of the irreps given above.

In conclusion, it is satisfying to see that the modular differential equation and some pre-knowledge on the vacuum character can tell us quite a few things about a CFT, even in the notoriously difficult case $c = 0$.

3. Bosonic and fermionic expressions for characters

Characters can be written in a closed form in more than one way. In what follows, we focus on the ‘bosonic’ and on the ‘fermionic’ character representations. The character of a free, chiral boson with momenta $p \in \mathbb{Z}_{\geq 1}$ is given by $\frac{1}{(q)_\infty}$. Truncations of the representation space through subsequent subtraction and addition of singular vectors will be encoded in the numerator. Thus, character expressions obtained in this way have been termed bosonic. In addition to the bosonic character expressions, there exists the concept of fermionic quasi-particle sum representations for a character, also called fermionic expressions, which first appeared under this name in [88]. The fermionic expressions are interesting from both a mathematical and a physical point of view and first occurred in an especially simple form in the context of the Rogers–Schur–Ramanujan identities [119, 120, 124] (for $a \in \{0, 1\}$):

$$\sum_{n=0}^{\infty} \frac{q^{n(n+a)}}{(q)_n} = \prod_{n=1}^{\infty} \frac{1}{(1 - q^{5n-1-a})(1 - q^{5n-4+a})} \tag{3.1}$$

with the q -analogues

$$(x; q)_n := \prod_{i=0}^{n-1} (1 - q^i x) \quad \text{and} \quad (q)_n := (q; q)_n = \prod_{i=1}^n (1 - q^i) \tag{3.2}$$

of the Pochhammer symbol and the classical factorial function, respectively, and by definition

$$(q)_0 := 1 \quad \text{and} \quad (q)_\infty := \lim_{n \rightarrow \infty} (q)_n, \tag{3.3}$$

the latter being the q -analogues of the classical gamma function. Note that $(q)_\infty$ is up to a factor of $q^{\frac{1}{24}}$ the modular form $\eta(\tau)$ with $q = e^{2\pi i \tau}$, the Dedekind η -function. These identities

coincide with the two characters of the minimal model $\mathcal{M}(2, 5)$ with central charge $c = -\frac{22}{5}$, which represents the Yang–Lee model (up to an overall factor of q^α for some $\alpha \in \mathbb{C}$). It is the smallest minimal model and contains only two primary operators: the identity $\mathbb{1}$ of dimension $(h, \bar{h}) = (0, 0)$ and another operator Φ of dimension $(-\frac{1}{5}, -\frac{1}{5})$. By using Jacobi’s triple product identity [76], defined for $z \neq 0$ and $|q| < 1$ (see [7]) as

$$\sum_{n=-\infty}^{\infty} z^n q^{n^2} = \prod_{n=1}^{\infty} (1 - q^{2n})(1 + zq^{2n-1})(1 + z^{-1}q^{2n-1}), \tag{3.4}$$

the rhs of (3.1) can be transformed to give a simple example of what is called a *bosonic–fermionic q-series identity*:

$$\sum_{n=0}^{\infty} \frac{q^{n(n+a)}}{(q)_n} = \frac{1}{(q)_\infty} \sum_{n=-\infty}^{\infty} (q^{n(10n+1+2a)} - q^{(5n+2-a)(2n+1)}). \tag{3.5}$$

For an instructive proof of the Jacobi triple product identity, see [78], which employs a comparison of the characters computed from a fermionic basis of the irreducible vacuum representation of charged free fermion system with the character computed from a bosonic basis of the same representation obtained by bosonization.

In general, it is always possible to write a minimal model character in a product form and thus to obtain a Rogers–Ramanujan-type identity if $p = 2s$ or $p' = 2r$, as has been demonstrated by Christe in [24]. To see this, one employs the Jacobi triple product identity (3.4) with the replacements $q \mapsto q^{\frac{pp'}{2}}$ and $z \mapsto -q^{r s - \frac{pp'}{4}}$. Product forms are also possible in the case $p = 3s$ or $p' = 3r$, but to show this, the *Watson identity* [134] (see also [65, example 5.6]) has to be used instead of the Jacobi identity. Christe also proved in the same article that for other minimal model characters, no product forms of this type exist.

The *bosonic expressions* on the rhs of (3.5) correspond to two special cases of the general character formula for minimal models by Rocha-Caridi [117]. Explicitly, they are given by

$$\begin{aligned} \chi_{1,1}^{5,2} &= 1 + q^2 + q^3 + q^4 + q^5 + 2q^6 + 2q^7 + 3q^8 + 3q^9 + 4q^{10} + 4q^{11} + 6q^{12} + 6q^{13} \\ &\quad + 8q^{14} + 9q^{15} + 11q^{16} + 12q^{17} + 15q^{18} + 16q^{19} + 20q^{20} + \dots \end{aligned} \tag{3.6}$$

and

$$\begin{aligned} \chi_{1,2}^{5,2} &= 1 + q + q^2 + q^3 + 2q^4 + 2q^5 + 3q^6 + 3q^7 + 4q^8 + 5q^9 + 6q^{10} + 7q^{11} + 9q^{12} + 10q^{13} \\ &\quad + 12q^{14} + 14q^{15} + 17q^{16} + 19q^{17} + 23q^{18} + 26q^{19} + 31q^{20} + \dots \end{aligned} \tag{3.7}$$

Note that the coefficient of q is zero because the vacuum is invariant under $L_n, n \in \{-1, 0, 1\}$. Since the rhs of (3.5) is computed by eliminating null states from the state space of a free chiral boson [39], it is referred to as bosonic form. Its signature is the alternating sign, which reflects the subtraction of null vectors. The factor $(q)_\infty$ keeps track of the free action of the Virasoro ‘raising’ modes. Furthermore, it can be expressed in terms of Θ -functions (cf 2.2), which directly point out the modular transformation properties of the character.

On the other hand, the left-hand side of (3.1) has a direct fermionic quasi-particle interpretation for the states and hence is called fermionic sum representation. In the first systematic study of fermionic expressions [87], sum representations for all characters of the unitary Virasoro minimal models and certain non-unitary minimal models were given. The list of expressions was augmented to all p and p' and certain r and s in [16]. Eventually, the fermionic expressions for the characters of all minimal models were summarized in [135]. Such a fermionic expression, which is a generalization of the left-hand side of (3.1), is a linear combination of fundamental fermionic forms, defined in (1.4).

Fermionic character expressions in CFT have various origins. Aside from the Rogers–Ramanujan identities, they first appeared in the representation theory of Lie algebras as the

Lepowsky–Primc formulae [37, 43, 94–98] and Andrews–Gordon identities [9, 109]. They also arise from thermodynamic Bethe ansatz analysis of integrable perturbations of CFT [90, 91] resulting in dilogarithm identities (cf section 1) which may be lifted back [128] to fermionic expressions, from the scaling limit of spin chains and ADE generalizations of Lepowsky–Primc [25, 88] and from spinon bases for WZW models [17–19]. All these different origins will be discussed in detail in the following sections.

4. Nahm’s conjecture

The question of how *q-hypergeometric series* (i.e. series of the form $\sum_{n=0}^{\infty} A_n(q)$ where $A_0(q)$ is a rational function and $A_n(q) = R(q, q^n)A_{n-1}(q) \forall n \geq 1$ for some rational function $R(x, y)$ with $\lim_{x \rightarrow 0} \lim_{y \rightarrow 0} R(x, y) = 0$) are related to modular forms or modular functions is an almost completely unsolved problem. But there is a conjecture by Werner Nahm (see e.g. [108]), which involves dilogarithms and torsion elements of the *Bloch group* as well as rational CFTs. If $j = r$ in (1.4), then the fundamental fermionic form reduces (with a rescaling $A \mapsto \frac{1}{2}A$)⁷ to the *q-hypergeometric series*

$$f_{A, \vec{b}, c}(\tau) = \sum_{\substack{\vec{m} \in (\mathbb{Z}_{\geq 0})^r \\ \text{restrictions}}} \frac{q^{\frac{1}{2}\vec{m}^t A \vec{m} + \vec{b} \vec{m} + c}}{(q)_{\vec{m}}}. \tag{4.1}$$

Nahm’s conjecture has no complete answer to this, but it makes a prediction which matrices $A \in M_r(\mathbb{Q})$ can occur such that (4.1) is a modular function, i.e. whether there exist suitable $\vec{b} \in \mathbb{Q}^r$ and $c \in \mathbb{Q}$ for a given matrix A . In particular, such a function can only be modular when all solutions to a certain system of algebraic equations depending on the coefficients of A , namely

$$1 - x_i = \prod_{j=1}^r x_j^{A_{ij}} \iff \sum_{j=1}^r A_{ij} \log(x_j) = \log(1 - x_i) \tag{4.2}$$

(the same we will also encounter in section 1), yield elements $\sum_i [x_i]$ of finite order in the Bloch group of the algebraic numbers [108].

The physical significance of this is that one expects that all the *q-hypergeometric series* which are modular functions are characters of rational CFTs. Given a matrix A , the modular forms for the predicted possible combinations of vectors and constants span a finite-dimensional vector space that is invariant under $\text{PSL}(2, \mathbb{Z})$ for bosonic CFTs (or under $\Gamma_0(2) = \{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{PSL}(2, \mathbb{Z}) \mid c \in 2\mathbb{Z} \}$ for fermionic CFTs), i.e. the set of characters generated in this way forms a finite-dimensional representation of the modular group, which is just the definition of rationality of a CFT. Indeed, this is just what we will find in the subsequent analysis in this report: the admissible matrices of rank 1 and 2 correspond to rational theories, most of them to the minimal models.

In general, there exist fermionic expressions for all characters of the minimal models. However, all but a finite number are not known to be of the type (4.1). Instead, they consist of finite linear combinations of fundamental fermionic forms (1.4), i.e. they involve finite *q-binomial coefficients*. But nevertheless, it is usually possible to express all characters of a given minimal model in terms of the same matrix A , albeit the choice of the matrix for that given model is in general not unique. We will comment more on that in the subsequent sections.

⁷ This rescaling has only been done in this section, since it makes the discussion of matrices A and their inverses easier. For the rest of this review, this rescaling is not necessary.

Note furthermore that the series of triplet \mathcal{W} -algebras, which are LCFTs to be discussed later in this review, was shown to be rational (in a broader sense to be defined later) with respect to its extended \mathcal{W} -symmetry algebra. These theories are not rational with respect to the Virasoro algebra alone as the symmetry algebra. By presenting fermionic sum representations of Nahm type (4.1) for the characters of the whole series of $\mathcal{W}(2, 2p-1, 2p-1, 2p-1)$ triplet algebras ($p \geq 2$), thus leading to a new infinite set of bosonic–fermionic q -series identities, we further support Nahm’s conjecture and provide further evidence that the triplet algebra series are well-defined new animals in the zoo [50] of rational CFTs.

There are also fermionic expressions for characters of other theories than those mentioned above, including for example the Kac–Peterson characters of the affine Lie algebra $A_1^{(1)}$ [85].

A lot of matrices A , among them in particular one infinite series, have been found for which Nahm’s conjecture suggests that they should lead to modular forms. However, a complete search has only been achieved for rank 1 and 2 matrices. Some of them are related to the *Dynkin diagrams* of the type A, D, E or T , corresponding to the simple Lie algebras⁸. These diagrams have r vertices if they are called X_r , where X is to be replaced by A, D, E or T . In many cases, the matrix A in the quadratic form in (4.1) is just twice the inverse Cartan matrix of a Dynkin diagram. On the other hand, it may also be half the Cartan matrix itself. These two cases are to be regarded as the special cases of another class, namely $A = C_{X_r} \otimes C_{Y_s}^{-1}$, where $X_r, Y_s \in \{A, D, E, T\}$. Note that the *Cartan matrix* is in one-to-one correspondence with a Dynkin diagram: for each vertex i that is connected to a vertex j ($i, j \in \{1, \dots, r\}$), set $A_{ij}A_{ji}$ equal to the number of lines connecting these two vertices with the restriction that $A_{ij}, A_{ji} \in \mathbb{Z}_{\leq 1}$, set $A_{ii} = 0 \forall i \in \{1, \dots, r\}$. All the other entries are zero. An exception to this is C_{T_r} , which is equal to C_{A_r} in all components but in the lower right one: $(C_{T_r})_{rr} = 1$. T_r is the *tadpole graph* corresponding to A_{2r} folded in the middle such that vertices are pairwise identified. Many *ADE*-related matrices of rank greater than 2 have also been found to correspond to rational CFTs, especially the inverse Cartan matrices. Examples of this kind will be discussed in section 5. Modular forms with matrices of the second class can be found, e.g., in Kac–Peterson characters later in this review, while the first class is common to minimal models. But there are also other types, some of which do not seem to fit this pattern.

5. ADET classification

The possibility of classifying fermionic character expressions according to simple Lie algebras is investigated further in this section.

All possible simple Lie algebras have been classified by Dynkin. Geometric constraints imply that there are only four infinite families and five exceptional cases. The infinite families are labeled by A_n, B_n, C_n or D_n and the exceptional cases by G_2, F_4, E_6, E_7 and E_8 , where n is the number of nodes of the corresponding *Dynkin diagram*. Each of the above Lie algebras is assigned a Dynkin diagram, and the set of Dynkin diagrams is in one-to-one correspondence with the set of *Cartan matrices*. If one labels the nodes of a Dynkin diagram by $a \in \{1, 2, \dots, n\}$, one can construct its Cartan matrix by setting $(C_{X_n})_{ab}(C_{X_n})_{ba}$ to the number of connecting lines between the nodes a and b of the Dynkin diagram to the Lie algebra X_n and demanding that $(C_{X_n})_{ab} \leq 0$ and integer, and furthermore $(C_{X_n})_{aa} = 2$.

The *ADE* graphs play an important role in many places in mathematics and physics. In CFT, for example, they can be used to classify modular invariant partition functions [22] and,

⁸ Watch the notation problem: the matrix in the exponent of the fermionic character expression is conventionally labeled A . This is not to be confused with the A series of Dynkin diagrams.

furthermore, the Cartan matrices also appear in quadratic form in the exponent of the fermionic character expressions.

In the following, the CFTs whose fermionic character expressions correspond to the *ADE* graphs are reported as well as the additional artificial series of *tadpole graphs*, which also appear in fermionic expressions. The corresponding Dynkin diagrams can be found in, e.g., [57]. When the matrix A in the quadratic form is mentioned, the reader is always referred to (1.4).⁹

The A_n series corresponds to the unitary \mathbb{Z}_{n+1} parafermionic theories with central charge $c_n = c_{\text{eff}}^n = \frac{2n}{n+3}$ [36, 37], where the effective central charge is defined by $c_{\text{eff}} = c - 24h_{\text{min}}$. Lepowsky and Primc [94, 95] found fermionic expressions with sum restrictions for the \mathbb{Z}_{n+1} characters, the latter consisting of the A_1 string functions of level k by Kac and Peterson [85]. Moreover, A_1 corresponds (trivially) to characters of the Ising model, namely,

$$\chi_{1,1}^{4,3} = \sum_{m=0 \pmod{2}}^{\infty} \frac{q^{\frac{m^2}{2} - \frac{1}{48}}}{(q)_m} = \frac{\Theta_{1,12} - \Theta_{7,12}}{\eta} \tag{5.1}$$

$$\chi_{2,1}^{4,3} = \sum_{m=1 \pmod{2}}^{\infty} \frac{q^{\frac{m^2}{2} - \frac{1}{48}}}{(q)_m} = \frac{\Theta_{5,12} - \Theta_{11,12}}{\eta} \tag{5.2}$$

and A_2 corresponds to the characters of $\mathcal{M}(6, 5)$, namely,

$$\chi_{1,1}^{6,5} + \chi_{1,5}^{6,5} = \sum_{\substack{\vec{m} \in (\mathbb{Z}_{\geq 0})^2 \\ m_1 + 2m_2 = 0 \pmod{3}}} \frac{q^{\vec{m}^t C_{A_2}^{-1} \vec{m} - \frac{1}{30}}}{(q)_{\vec{m}}} \tag{5.3}$$

and

$$\chi_{1,3}^{6,5} = \sum_{\substack{\vec{m} \in (\mathbb{Z}_{\geq 0})^2 \\ m_1 + 2m_2 = a \pmod{3}}} \frac{q^{\vec{m}^t C_{A_2}^{-1} \vec{m} - \frac{1}{30}}}{(q)_{\vec{m}}}, \quad a \in \{-1, 1\}. \tag{5.4}$$

Additionally, via $A = C_{A_n} \otimes C_{T_1}^{-1} = C_{A_n}$, this series also corresponds to the unitary series of minimal models via the unitary minimal model $\mathcal{M}(p+1, p)$ characters

$$q^a \chi_{1,1}^{p+1,p} = \sum_{\substack{\vec{m} \in (\mathbb{Z}_{\geq 0})^{p-2} \\ m_i \text{ even}}} \frac{q^{\frac{1}{4} \vec{m}^t C_{A_{p-2}} \vec{m}}}{(q)_{m_1}} \prod_{i=2}^{p-2} \left[\left(\left(\mathbb{1} - \frac{1}{2} C_{A_{p-2}} \right) \vec{m} \right)_i \right]_{m_i} \Big|_q. \tag{5.5}$$

This can be computed by the methods of Welsh [135] for all possible combinations of r and s , but for simplicity, only the vacuum character is given here.

The D_n series corresponds to the unitary theory of a free boson compactified on a torus of radius $R = \sqrt{\frac{n}{2}}$ with central charge $c = c_{\text{eff}} = 1$. This theory has characters $\frac{\Theta_{\lambda,k}}{\eta}$ for $\lambda \in \{-k+1, \dots, k\}$ with $\lambda = 0$ denoting the vacuum character. The fermionic expressions for these characters can all be written with the inverse Cartan matrix of D_n in the quadratic form. We will discuss this later on, when we derive the fermionic expressions for the $c_{p,1}$ series of LCFTs, where we will see that the whole $c_{p,1}$ series corresponds to the D_n series, i.e. the quadratic form in the fermionic character expressions is

$$\vec{m}^t C_{D_p}^{-1} \vec{m} \tag{5.6}$$

⁹ Upon comparing (1.4) with (4.1), one notices that the exponents differ by a factor of $\frac{1}{2}$ in the definition of the matrix A . Strictly speaking, one should write $\frac{1}{2}A$ in the exponent of every fermionic form, since in general $A = C_{X_r} \otimes C_{Y_s}^{-1}$ for some $X_r, Y_s \in \{A, D, E, T\}$ (as discussed in section 4) and in most cases occurring in this review $X_r = A_1$ and thus $A = 2C_{Y_s}^{-1}$. Therefore, since in most cases the factors 2 and $\frac{1}{2}$ cancel, the general fundamental fermionic form (1.4) is referred to in this report except in the single section 4 and where explicitly stated.

for all characters of the $c_{p,1}$ model. The sum restrictions state that the sum $m_{n-1} + m_n$ has to be either even or odd, depending on the chosen character of the model. For the subset of characters that are of the form $\frac{\Theta_{\lambda,k}}{\eta}$, both restrictions admit a realization. Note furthermore that due to the coincidence $D_3 = A_3$, some character functions corresponding to these two series are related.

The exceptional algebra E_6 corresponds to the unitary minimal model $\mathcal{M}(7, 6)$ that is the tricritical three-state Potts model [38] with central charge $c = \frac{6}{7}$, namely,

$$\chi_{1,1}^{7,6} + \chi_{5,1}^{7,6} = \sum_{\substack{\vec{m} \in (\mathbb{Z}_{\geq 0})^6 \\ m_1 - m_2 + m_4 - m_5 \equiv 0 \pmod{3}}} \frac{q^{\vec{m}^T C_{E_6}^{-1} \vec{m} - \frac{c}{24}}}{(q)_{\vec{m}}} \tag{5.7}$$

and

$$\chi_{3,1}^{7,6} = \sum_{\substack{\vec{m} \in (\mathbb{Z}_{\geq 0})^6 \\ m_1 - m_2 + m_4 - m_5 \equiv a \pmod{3}}} \frac{q^{\vec{m}^T C_{E_6}^{-1} \vec{m} - \frac{c}{24}}}{(q)_{\vec{m}}}, \quad a \in \{-1, 1\} \tag{5.8}$$

with

$$C_{E_6}^{-1} = \begin{pmatrix} \frac{4}{3} & \frac{2}{3} & 1 & \frac{4}{3} & \frac{5}{3} & 2 \\ \frac{2}{3} & \frac{4}{3} & 1 & \frac{5}{3} & \frac{4}{3} & 2 \\ 1 & 1 & 2 & 2 & 2 & 3 \\ \frac{4}{3} & \frac{5}{3} & 2 & \frac{10}{3} & \frac{8}{3} & 4 \\ \frac{5}{3} & \frac{4}{3} & 2 & \frac{8}{3} & \frac{10}{3} & 4 \\ 2 & 2 & 3 & 4 & 4 & 6 \end{pmatrix}. \tag{5.9}$$

By adding a suitable vector \vec{b} to the exponent in the fermionic expression or by changing the sum restrictions, the other characters of $\mathcal{M}(7, 6)$ might also be found to have fermionic representations of this type, but so far, none are known.

The exceptional algebra E_7 corresponds to the tricritical Ising unitary model $\mathcal{M}(5, 4)$ with central charge $c = \frac{7}{10}$ mentioned in the previous section. Here,

$$\chi_{1,1}^{5,4} = \sum_{\substack{\vec{m} \in (\mathbb{Z}_{\geq 0})^7 \\ m_1 + m_3 + m_6 \equiv 0 \pmod{2}}} \frac{q^{\vec{m}^T C_{E_7}^{-1} \vec{m} - \frac{c}{24}}}{(q)_{\vec{m}}} \tag{5.10}$$

and

$$\chi_{3,1}^{5,4} = \sum_{\substack{\vec{m} \in (\mathbb{Z}_{\geq 0})^7 \\ m_1 + m_3 + m_6 \equiv 1 \pmod{2}}} \frac{q^{\vec{m}^T C_{E_7}^{-1} \vec{m} - \frac{c}{24}}}{(q)_{\vec{m}}}. \tag{5.11}$$

The exceptional algebra E_8 corresponds to the Ising model $\mathcal{M}(4, 3)$ and, as also mentioned in the previous section,

$$\chi_{1,1}^{4,3} = \sum_{\vec{m} \in (\mathbb{Z}_{\geq 0})^8} \frac{q^{\vec{m}^T C_{E_8}^{-1} \vec{m} - \frac{1}{48}}}{(q)_{\vec{m}}}. \tag{5.12}$$

The T_n series, often called $\frac{A_{2n}}{\mathbb{Z}_2}$, corresponds to the series of non-unitary Virasoro minimal models $\mathcal{M}(2n + 3, 2)$ with effective central charge $c_{\text{eff}}^k = \frac{2n}{2n+3}$. Their characters admit a product form [6, 43], which is one side of the Andrews–Gordon identities [7, 9, 21, 69]

$$\sum_{m_1, \dots, m_n=0}^{\infty} \frac{q^{M_1^2 + \dots + M_n^2 + M_a + \dots + M_n}}{(q)_{m_1} \cdots (q)_{m_n}} = \prod_{\substack{m \neq 0 \pmod{2n+3} \\ m \neq \pm a \pmod{2n+3}}} (1 - q^m)^{-1} \tag{5.13}$$

with $M_k := m_1 + \dots + m_k$. Gordon gave the combinatorial and Andrews the analytical proof. The other side consists of the fermionic sum representations for the characters of $\mathcal{M}(2n+3, 2)$. This original formulation can be rewritten in order to match the fermionic forms as

$$\chi_{1,j}^{2n+3,2}(q) = q^{h_{1,j}^{2n+3,2} - \frac{c_{2n+3,2}}{24}} \sum_{\vec{m} \in (\mathbb{Z}_{\geq 0})^n} \frac{q^{\vec{m}^T C_{T_n}^{-1} \vec{m} + \vec{b}_{T_n}^T \vec{m}}}{(q)_{\vec{m}}} \tag{5.14}$$

with C_{T_n} being the Cartan matrix of the tadpole graph which differs from C_{A_n} only by a 1 instead of a 2 in the component $(C_{T_n})_{nn}$ and $\vec{b}_{T_n} = (\underbrace{0, \dots, 0}_{n \text{ times}}, 1, 2, \dots, k - n)$. Note that

the Andrews–Gordon identities reduce to the Rogers–Ramanujan identities for $n = 1$ and $a \in \{1, 2\}$, i.e. for $\mathcal{M}(5, 2)$.

Most of these expressions were found and verified by using Mathematica, and explicit proofs were lacking for most of them [86]. But that situation changed during the following years. A particular example is the fermionic character expression for $\chi_{1,1}^{4,3}$ related to E_8 , for which Warnaar and Pearce found a proof based on the dilute A_3 model [133]. A different direction that allowed for many of the identities to be proven was found by Melzer [106]. He observed that Virasoro characters have a natural *finitized* version in terms of *path spaces* or *corner-transfer matrix sums* in the *rough solid-on-solid model* [8]. This method of proving the identities has been extended in [14] and references therein. The fact that there are different fermionic expressions for a single character (in the sense that the matrix A in the quadratic form is different) is demonstrated impressively by the vacuum character $\chi_{1,1}^{4,3}$ of the Ising model. There is a sum representation related to A_1 and a sum representation related to E_8 . Let us discuss this. In [90], Klassen and Melzer investigated integrable massive scattering theories. There, the *ADE* algebras describe certain perturbations of coset CFTs [67] related to *ADE*. These algebras are the same. For example, the energy perturbation of the Ising model, which is called *Ising field theory*, corresponds to A_1 and to the conformal limit of Kaufman’s representation of the general Ising model in the absence of a magnetic field in terms of a single, free fermion [81], while the magnetic perturbation corresponds to a scattering theory of eight different particle species [137]. Later on in this report, when we will demonstrate the quasi-particle interpretation of the fermionic character expressions, we will see that the E_8 character corresponds also to a system of eight quasi-particle species with exactly the same charges in [137] reproduced by the sum restrictions. This is another example that different fermionic expressions for the same character point to different integrable perturbations of the CFT in consideration.

Furthermore, symmetries of the character $\chi_{r,s}^{p,p'}$ with respect to its parameters add to the non-uniqueness of a fermionic character expression. For instance,

$$\chi_{\alpha r, \alpha s}^{p, \alpha p'} = \chi_{r, \alpha s}^{\alpha p, p'} \quad \alpha \in \mathbb{Z}_{\geq 1}, \quad \langle p, \alpha p' \rangle = \langle \alpha p, p' \rangle = 1 \tag{5.15}$$

implies that the characters of $\mathcal{M}(6, 5)$ are related to those of $\mathcal{M}(10, 3)$.

6. Characters of the triplet algebras $\mathcal{W}(2, 2p - 1, 2p - 1, 2p - 1)$

6.1. Characters in bosonic form

The triplet \mathcal{W} -algebras are rational CFTs [23, 47, 60], i.e. the number of highest weight representations of the \mathcal{W} -algebra is finite, and the generalized character functions span a finite-dimensional representation of the modular group. They are rational in the generalized sense discussed in section 2.4.1, since indecomposable representations occur. Knowing the vacuum character is sufficient in proving rationality of the theory.

One can calculate the \mathcal{W} character of the vacuum representation by summing up all the Virasoro characters of the highest weight representations corresponding to integer values of h , the latter being given by

$$h_{2k+1,1} = k^2 p + kp - k. \tag{6.1}$$

All the corresponding primary fields belong to degenerate conformal families. By means of a standard free-field construction [12, 29–31], it turns out that the representations with these highest weights $h_{2k+1,1}$ correspond to a set of relatively local chiral vertex operators $\Phi_{2k+1,1}$. It follows that the local chiral algebra can be extended by them. The conditions for the existence of well-defined chiral vertex operators [83, 84] result in abstract fusion rules which imply that the local chiral algebra generated by only the stress–energy tensor and the field $\Phi_{3,1}$ closes. Repeated application of the *screening charge operator* Q [44] on $\Phi_{3,1}$ generates a multiplet structure. Thus, one also has to take care of the $\mathfrak{su}(2)$ symmetry of the triplet of fields, which results in the multiplicity of the Virasoro representation with highest weight state $|h_{2k+1,1}\rangle$ being $2k + 1$. For example, since $h_{3,1} = 2p - 1$ and its multiplicity is 3, it matches the fact that we have a triplet of fields in the algebra $\mathcal{W}(2, 2p - 1, 2p - 1, 2p - 1)$. The vacuum representation of the \mathcal{W} -algebra can then be written as the following decomposition of the state space:

$$\mathcal{H}_{|0\rangle} = \bigoplus_{k \in \mathbb{Z}_{\geq 0}} (2k + 1) \mathcal{H}_{|h_{2k+1,1}\rangle}^{\text{Vir}}. \tag{6.2}$$

The embedding structure of Feigin and Fuks [39] in the case of $p' = 1$ implies that the Virasoro characters corresponding to $h_{2k+1,1}$ —these are the only integer-valued weights for all p —are given by

$$\chi_{2k+1,1}^{\text{Vir}} = \frac{q^{\frac{1-c_{p,1}}{24}}}{\eta(q)} (q^{h_{2k+1,1}} - q^{h_{2k+1,-1}}). \tag{6.3}$$

It is thus possible to compute the vacuum character as

$$\begin{aligned} \chi_0^{\mathcal{W}}(q) &= \sum_{k \in \mathbb{Z}_{\geq 0}} (2k + 1) \chi_{2k+1,1}^{\text{Vir}}(q) \\ &= \frac{q^{\frac{1-c_{p,1}}{24}}}{\eta(q)} \left(\sum_{k \in \mathbb{Z}_{\geq 0}} (2k + 1) q^{h_{2k+1,1}} - \sum_{k \in \mathbb{Z}_{\geq 0}} (2k + 1) q^{h_{2k+1,-1}} \right) \\ &= \frac{q^{\frac{(p-1)^2}{4p}}}{\eta(q)} \left(\sum_{k \in \mathbb{Z}_{\geq 0}} (2k + 1) q^{h_{2k+1,1}} + \sum_{k \in \mathbb{Z}_{\leq -1}} (2k + 1) q^{h_{2k+1,1}} \right) \\ &= \frac{q^{\frac{(p-1)^2}{4p}}}{\eta(q)} \left(\sum_{k \in \mathbb{Z}} (2k + 1) q^{h_{2k+1,1}} \right) = \frac{1}{\eta(q)} \left(\sum_{k \in \mathbb{Z}} (2k + 1) q^{pk^2 + kp - k + \frac{(p-1)^2}{4p}} \right) \\ &= \frac{1}{p\eta(q)} \left(\sum_{k \in \mathbb{Z}} (2pk + p) q^{\frac{(2pk + (p-1))^2}{4p}} \right) = \frac{1}{p\eta(q)} ((\partial\Theta)_{p-1,p}(q) + \Theta_{p-1,1}(q)), \tag{6.4} \end{aligned}$$

where the symmetry property $h_{r,s} = h_{-r,-s}$ has been used and the Θ -functions as defined in section 2.2. The h -values of a given \mathcal{W} -algebra can be calculated by the use of the free-field construction, using Jacobi identities and null field constraints (cf section 2.4.1). The corresponding characters may be calculated as follows: the modular differential equation (see e.g. [47]) may be used to compute as many terms of the q -expansion of the character as are necessary to unambiguously identify the corresponding function, because the requirement of that function to be a modular form implies strong restrictions on that function. It turns out that

if we assume that $c_{3p,3} = c_{p,1}$ corresponds to a minimal model, which of course it does not since $3p$ and 3 are not coprime, it is possible to read the resulting h -values of the given $c_{p,1}$ theory off that enlarged Kac table.

$\frac{\Theta_{\lambda,k}(\tau)}{\eta(\tau)}$ is a modular form of weight zero with respect to the generators $\mathcal{T} : \tau \mapsto \tau + 1$ and $\mathcal{S} : \tau \mapsto -\frac{1}{\tau}$ of the modular group $\text{PSL}(2, \mathbb{Z})$. But since $\frac{(\partial\Theta)_{\lambda,k}(\tau)}{\eta(\tau)}$ is a modular form of weight 1 with respect to \mathcal{S} (cf section 2.2), some of the above character functions are of inhomogeneous modular weight, thus leading to S -matrices with τ -dependent coefficients. However, adding

$$(\nabla\Theta)_{\lambda,k}(\tau) = \frac{\log q}{2\pi i} \sum_{n \in \mathbb{Z}} (2kn + \lambda) q^{\frac{(2kn+\lambda)^2}{4k}}, \quad (6.5)$$

one finds a closed finite-dimensional representation of the modular group with constant S -matrix coefficients.

After all, it turns out that a complete set of character functions for the $c_{p,1}$ models that is closed under modular transformations [48] is given by

$$\chi_{0,p} = \frac{\Theta_{0,p}}{\eta} \quad \text{of representation } \mathcal{W}(h_{1,p}^{p,1}) \quad (6.6a)$$

$$\chi_{p,p} = \frac{\Theta_{p,p}}{\eta} \quad \mathcal{W}(h_{1,2p}^{p,1}) \quad (6.6b)$$

$$\chi_{\lambda,p}^+ = \frac{(p-\lambda)\Theta_{\lambda,p} + (\partial\Theta)_{\lambda,p}}{p\eta} \quad \mathcal{W}(h_{1,p-\lambda}^{p,1}) \quad (6.6c)$$

$$\chi_{\lambda,p}^- = \frac{\lambda\Theta_{\lambda,p} - (\partial\Theta)_{\lambda,p}}{p\eta} \quad \mathcal{W}(h_{1,3p-\lambda}^{p,1}) \quad (6.6d)$$

$$\tilde{\chi}_{\lambda,p}^+ = \frac{\Theta_{\lambda,p} + i\alpha\lambda(\nabla\Theta)_{\lambda,p}}{\eta} \quad \mathcal{R}(h_{1,p+\lambda}^{p,1}) \quad (6.6e)$$

$$\tilde{\chi}_{\lambda,p}^- = \frac{\Theta_{\lambda,p} - i\alpha(p-\lambda)(\nabla\Theta)_{\lambda,p}}{\eta} \quad \mathcal{R}(h_{1,p+\lambda}^{p,1}) \quad (6.6f)$$

where $0 < \lambda < k$, $k = pp' = p$, $\lambda = pr - p's = pr - s$ and with the *Jacobi–Riemann* Θ -function and the *affine* Θ -function defined as in section 2.2.

Note that (6.6e) and (6.6f) are not characters of representations in the usual sense. Actually, these are regularized character functions and the α -dependent part has an interpretation as torus vacuum amplitudes [51]. In the limit $\alpha \rightarrow 0$, they become the characters of the full reducible but indecomposable representations.

6.2. Fermionic character expressions for $\mathcal{W}(2, 3, 3, 3)$

Fermionic sum representation for the $c_{p,1}$ models have been presented in [52] and proven in [132]. In this section, the derivation of the fermionic formulae for the case of $p = 2$ is reviewed, while the case of $p > 2$ is postponed to the next section.

In the case of $p = 2$, the bosonic characters read

$$\chi_{1,2}^+ = \frac{\Theta_{1,2} + (\partial\Theta)_{1,2}}{2\eta} \quad \text{vacuum irrep } \mathcal{W}(0) \text{ to } h_{1,1} = 0 \quad (6.7a)$$

$$\chi_{0,2} = \frac{\Theta_{0,2}}{\eta} \quad \text{irrep to } h_{1,2} = -\frac{1}{8} \quad (6.7b)$$

$$\chi_{1,2} = \frac{\Theta_{1,2}}{\eta} \quad \text{indecomp. rep } \mathcal{R}(0) \supset \mathcal{W}(0) \text{ to } h_{1,3} = 0 \quad (6.7c)$$

$$\chi_{2,2} = \frac{\Theta_{2,2}}{\eta} \quad \text{irrep to } h_{1,4} = \frac{3}{8} \quad (6.7d)$$

$$\chi_{1,2}^- = \frac{\Theta_{1,2} - (\partial\Theta)_{1,2}}{2\eta} \quad \text{irrep to } h_{1,5} = 1. \quad (6.7e)$$

When $\alpha \rightarrow 0$, the general forms (6.6e) and (6.6f) lead to the character expression (6.7c) [48, 83]. Actually, there exist two indecomposable representations, \mathcal{R}_0 and \mathcal{R}_1 (cf section 2.4.1), which, however, share the same character.

In what follows, the fermionic expressions for $\frac{\Theta_{\lambda,2}(\tau)}{\eta(\tau)}$, $0 \leq \lambda \leq 2$, are being calculated at first. In this case, the bosonic expressions can be straightforward transformed to the fermionic ones: at first,

$$\begin{aligned} \frac{\Theta_{\lambda,k}}{(q)_\infty} &= \sum_{n=-\infty}^{+\infty} \frac{q^{\frac{(2kn+\lambda)^2}{4k}}}{(q)_\infty} \\ &= \frac{1}{(q)_\infty} \left(q^{\frac{\lambda^2}{4k}} + \sum_{n=1}^{\infty} q^{\frac{(2kn-\lambda)^2}{4k}} + \sum_{n=1}^{\infty} q^{\frac{(2kn+\lambda)^2}{4k}} \right). \end{aligned}$$

Then, an identity

$$\sum_{n=0}^{\infty} \frac{q^{n^2+nk}}{(q)_n (q)_{n+k}} = \frac{1}{(q)_\infty} \quad (6.9)$$

that can be proven using Durfee squares or the q -analogue of Kummer's theorem (see e.g. [7, pp 21, 28]) is employed to turn (6.8) into

$$\sum_{m=0}^{\infty} \frac{q^{m^2}}{(q)_m^2} + \sum_{n_1=1}^{\infty} \sum_{m_1=0}^{\infty} \frac{q^{m_1^2+m_1(2n_1)+\frac{(k(2n_1)-\lambda)^2}{4k}}}{(q)_{m_1} (q)_{m_1+2n_1}} + \sum_{n_2=1}^{\infty} \sum_{m_2=0}^{\infty} \frac{q^{m_2^2+m_2(2n_2)+\frac{(k(2n_2)+\lambda)^2}{4k}}}{(q)_{m_2} (q)_{m_2+2n_2}}. \quad (6.10)$$

Setting $n_1 = \frac{m_2-m_1}{2}$ and $n_2 = \frac{m_1-m_2}{2}$ leads to

$$\sum_{m=0}^{\infty} \frac{q^{m^2}}{(q)_m^2} + \sum_{\substack{0 \leq m_1 < m_2 = 0 \\ m_1+m_2 \equiv 0 \pmod{2}}}^{\infty} \sum_{\substack{0 \leq m_2 < m_1 = 0 \\ m_1+m_2 \equiv 0 \pmod{2}}}^{\infty} \frac{q^{\frac{k}{4}(m_1^2+m_2^2) + \frac{2-k}{2}m_1m_2 + \frac{\lambda}{2}(m_1-m_2) + \frac{\lambda^2}{4k}}}{(q)_{m_1} (q)_{m_2}}. \quad (6.11)$$

On the other hand,

$$\begin{aligned} \frac{\Theta_{\lambda,k}}{(q)_\infty} &= \sum_{n=-\infty}^{+\infty} \frac{q^{\frac{(2kn+\lambda)^2}{4k}}}{(q)_\infty} = \sum_{n_1=1}^{\infty} \sum_{m_1=0}^{\infty} \frac{q^{m_1^2+m_1(2n_1-1)+\frac{-2k(k-\lambda)(2n_1-1)+(k-\lambda)^2+k^2(2n_1-1)^2}{4k}}}{(q)_{m_1} (q)_{m_1+2n_1-1}} \\ &\quad + \sum_{n_2=1}^{\infty} \sum_{m_2=0}^{\infty} \frac{q^{m_2^2+m_2(2n_2-1)+\frac{2k(k-\lambda)(2n_2-1)+(k-\lambda)^2+k^2(2n_2-1)^2}{4k}}}{(q)_{m_2} (q)_{m_2+2n_2-1}}. \end{aligned} \quad (6.12)$$

Setting $n_1 = \frac{m_2-m_1+1}{2}$ and $n_2 = \frac{m_1-m_2+1}{2}$ implies

$$\sum_{\substack{0 \leq m_1 < m_2 = 0 \\ m_1+m_2 \equiv 1 \pmod{2}}}^{\infty} \sum_{\substack{0 \leq m_2 < m_1 = 0 \\ m_1+m_2 \equiv 1 \pmod{2}}}^{\infty} \frac{q^{\frac{k}{4}(m_1^2+m_2^2) + \frac{2-k}{2}m_1m_2 + \frac{k-\lambda}{2}(m_1-m_2) + \frac{(k-\lambda)^2}{4k}}}{(q)_{m_1} (q)_{m_2}}. \quad (6.13)$$

Thus, from (6.11) and (6.13),

$$\Delta_{\lambda,k}(\tau) = \frac{\Theta_{\lambda,k}(\tau)}{\eta(\tau)} = \sum_{\substack{\bar{m}=0 \\ m_1+m_2 \equiv 0 \pmod{2}}}^{\infty} \frac{q^{\frac{1}{4}\bar{m}^t \binom{k}{2-k} \binom{2-k}{k} \bar{m} + \frac{1}{2} \binom{\lambda}{-\lambda}^t \bar{m} + \frac{\lambda^2}{4k} - \frac{1}{24}}}{(q)_{\bar{m}}} \tag{6.14a}$$

$$= \sum_{\substack{\bar{m}=0 \\ m_1+m_2 \equiv 1 \pmod{2}}}^{\infty} \frac{q^{\frac{1}{4}\bar{m}^t \binom{k}{2-k} \binom{2-k}{k} \bar{m} + \frac{1}{2} \binom{-(k-\lambda)}{k-\lambda}^t \bar{m} + \frac{(k-\lambda)^2}{4k} - \frac{1}{24}}}{(q)_{\bar{m}}}. \tag{6.14b}$$

This twofold q -hypergeometric series has been given without explicit proof in [89]. (Note that (6.14) is not unique just as (2.8): according to (2.10) the vector may be changed in certain ways along with the constant.) These are fermionic expressions for (6.7b)–(6.7d). We obtain the fermionic expressions of the remaining two characters as follows [52]: note that $\frac{(\partial\Theta)_{1,2}}{\eta^3(q)} = 1$ and hence

$$\chi_{1,2}^{\pm} = \frac{\Theta_{1,2}}{2\eta} \pm \frac{1}{2}\eta^2. \tag{6.15}$$

We then use an identity

$$\eta(q) = q^{\frac{1}{24}} \sum_{n=0}^{\infty} \frac{(-1)^n q^{\binom{n+1}{2}}}{(q)_n} \tag{6.16}$$

by Euler (cf [136] for a simple proof). This identity may be squared, leading to

$$\eta^2(q) = \tilde{\eta}^2(q, -1) \quad \text{with} \quad \tilde{\eta}^2(q, z) = \sum_{\bar{m}=0}^{\infty} \frac{q^{\frac{1}{2}\bar{m}^t \binom{1}{0} \binom{0}{1} \bar{m} + \frac{1}{2} \binom{1}{1}^t \bar{m} + \frac{1}{12} z^{m_1+m_2}}}{(q)_{\bar{m}}}. \tag{6.17}$$

It is possible to transform the fermionic expression of $\chi_{1,2}$ which was obtained in (6.14) into

$$\begin{aligned} & \sum_{\substack{\bar{m}=0 \\ m_1+m_2 \equiv 0 \pmod{2}}}^{\infty} \frac{q^{\frac{1}{2}\bar{m}^t \binom{1}{0} \binom{0}{1} \bar{m} + \frac{1}{2} \binom{1}{-1}^t \bar{m}}}{(q)_{\bar{m}}} \\ &= \sum_{\substack{\bar{m}=0 \\ m_1+m_2 \equiv 0 \pmod{2}}}^{\infty} \frac{q^{\frac{1}{2}\bar{m}^t \binom{1}{0} \binom{0}{1} \bar{m} + \frac{1}{2} \binom{1}{-1}^t \bar{m}}}{2(q)_{\bar{m}}} + \sum_{\substack{\bar{m}=0 \\ m_1+m_2 \equiv 1 \pmod{2}}}^{\infty} \frac{q^{\frac{1}{2}\bar{m}^t \binom{1}{0} \binom{0}{1} \bar{m} + \frac{1}{2} \binom{1}{-1}^t \bar{m}}}{2(q)_{\bar{m}}} \\ &= \left(\sum_{m_1=0}^{\infty} \frac{q^{\frac{m_1^2-m_1}{2}}}{(q)_{m_1}} \right) \left(\sum_{\substack{m_2=0 \\ m_2 \equiv m_1 \pmod{2}}}^{\infty} \frac{q^{\frac{m_2^2+m_2}{2}}}{(q)_{m_2}} + \sum_{\substack{m_2=0 \\ m_2 \equiv m_1+1 \pmod{2}}}^{\infty} \frac{q^{\frac{m_2^2+m_2}{2}}}{(q)_{m_2}} \right). \end{aligned} \tag{6.18}$$

By using

$$\sum_{m=0}^{\infty} \frac{q^{\frac{m^2+m_2}{2}}}{(q)_m} = \frac{1}{2} \sum_{m=0}^{\infty} \frac{q^{\frac{m^2+m_2}{2}}}{(q)_m}, \tag{6.19}$$

which holds because

$$\begin{aligned} \sum_{m=0}^{\infty} \frac{q^{\frac{m^2+m_2}{2}}}{(q)_m} &= \sum_{m=0}^{\infty} \frac{q^{\frac{m^2+m_2}{2}} (1 - q^{m+1})}{(q)_{m+1}} \\ &= \sum_{m=0}^{\infty} \frac{q^{\frac{m^2+m_2}{2}}}{(q)_{m+1}} - \sum_{m=0}^{\infty} \frac{q^{\frac{m^2+m_2}{2} + m + 1}}{(q)_{m+1}} = \sum_{m=1}^{\infty} \frac{q^{\frac{m^2-m_2}{2}}}{(q)_m} - \sum_{m=1}^{\infty} \frac{q^{\frac{m^2+m_2}{2}}}{(q)_{m+1}}, \end{aligned} \tag{6.20}$$

$\chi_{1,2}$ may be written as

$$q^{\frac{1}{24}} \chi_{1,2} = \sum_{\vec{m}=0}^{\infty} \frac{q^{\frac{1}{2}\vec{m}' \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \vec{m} + \frac{1}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix}' \vec{m}}{(q)_{\vec{m}}}, \tag{6.21}$$

finally leading to

$$\begin{aligned} q^{\frac{1}{24}} \chi_{1,2}^{\pm} &= \frac{\Theta_{1,2}}{2\eta} \pm \frac{\eta^2}{2} \\ &= \frac{1}{2} \sum_{\vec{m}=0}^{\infty} \frac{q^{\frac{1}{2}\vec{m}' \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \vec{m} + \frac{1}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix}' \vec{m}}{(q)_{\vec{m}}} \pm \frac{1}{2} \sum_{\vec{m}=0}^{\infty} \frac{q^{\frac{1}{2}\vec{m}' \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \vec{m} + \frac{1}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix}' \vec{m} + \frac{1}{12} (-1)^{m_1+m_2}}{(q)_{\vec{m}}} \\ &= \sum_{\substack{\vec{m}=0 \\ m_1+m_2 \equiv a \pmod{2}}}^{\infty} \frac{q^{\frac{1}{2}\vec{m}' \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \vec{m} + \frac{1}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix}' \vec{m}}{(q)_{\vec{m}}} \end{aligned} \tag{6.22}$$

with $a = 0$ if the plus sign is chosen and $a = 1$ if the minus sign is chosen.

Thus, also the remaining two characters yield expressions which consist of only one fundamental fermionic form.

The following is a list of the fermionic expressions for all five characters of the LCFT model corresponding to central charge $c_{2,1} = -2$ [52]:

$$\chi_{1,2}^+ = \sum_{\substack{\vec{m} \in (\mathbb{Z}_{\geq 0})^2 \\ m_1+m_2 \equiv 0 \pmod{2}}} \frac{q^{\frac{1}{2}\vec{m}' \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \vec{m} + \frac{1}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix}' \vec{m} + \frac{1}{12}}{(q)_{\vec{m}}} \tag{6.23a}$$

$$\chi_{0,2} = \sum_{\substack{\vec{m} \in (\mathbb{Z}_{\geq 0})^2 \\ m_1+m_2 \equiv 0 \pmod{2}}} \frac{q^{\frac{1}{2}\vec{m}' \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \vec{m} - \frac{1}{24}}{(q)_{\vec{m}}} \tag{6.23b}$$

$$\chi_{1,2} = \sum_{\substack{\vec{m} \in (\mathbb{Z}_{\geq 0})^2 \\ m_1+m_2 \equiv 0 \pmod{2}}} \frac{q^{\frac{1}{2}\vec{m}' \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \vec{m} + \frac{1}{2} \begin{pmatrix} -1 \\ 1 \end{pmatrix}' \vec{m} + \frac{1}{12}}{(q)_{\vec{m}}} \tag{6.23c}$$

$$\chi_{2,2} = \sum_{\substack{\vec{m} \in (\mathbb{Z}_{\geq 0})^2 \\ m_1+m_2 \equiv 0 \pmod{2}}} \frac{q^{\frac{1}{2}\vec{m}' \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \vec{m} + \begin{pmatrix} -1 \\ 1 \end{pmatrix}' \vec{m} + \frac{1}{24}}{(q)_{\vec{m}}} \tag{6.23d}$$

$$\chi_{1,2}^- = \sum_{\substack{\vec{m} \in (\mathbb{Z}_{\geq 0})^2 \\ m_1+m_2 \equiv 1 \pmod{2}}} \frac{q^{\frac{1}{2}\vec{m}' \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \vec{m} + \frac{1}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix}' \vec{m} + \frac{1}{12}}{(q)_{\vec{m}}} \tag{6.23e}$$

and also

$$\chi_{1,2} = \sum_{\vec{m} \in (\mathbb{Z}_{\geq 0})^2} \frac{q^{\frac{1}{2}\vec{m}' \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \vec{m} + \frac{1}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix}' \vec{m}}{(q)_{\vec{m}}}. \tag{6.24}$$

Using the equality to the bosonic representation of the characters, these give *bosonic–fermionic q-series identities* generalizing the left-hand side and rhs of (3.5). In (6.23b)–(6.23d), also the last line of (6.14) may be used, where $m_1 + m_2 \equiv 1 \pmod{2}$.



Figure 1. The Dynkin diagram of $D_2 = A_1 \times A_1$.

It is remarkable that although two of the characters have inhomogeneous modular weight, there is a uniform representation for all five characters with the same matrix A in every case. But on the other hand, this is a satisfying result, since this is also the case for all other models for which fermionic character expressions are known: their different modules are only distinguished by the linear term in the exponent, not by the quadratic one. Note that the fact that the quadratic form is diagonal goes well with the description of the $c = -2$ model in terms of symplectic fermions [83, 84], see section 7.3.

The results are also in agreement with Nahm’s conjecture (see section 4), which predicts that for a matrix of the form $A = \begin{pmatrix} \alpha & 1-\alpha \\ 1-\alpha & \alpha \end{pmatrix}$ with rational coefficients, there exist a vector $\vec{b} \in \mathbb{Q}^r$ and a constant $c \in \mathbb{Q}$ such that $f_{A,\vec{b},c}(\tau) = \sum_{\vec{m} \in (\mathbb{Z}_{\geq 0})^r} \frac{q^{\frac{1}{2}\vec{m}^T A \vec{m} + \vec{b} \cdot \vec{m} + c}}{(q)_{\vec{m}}}$ is a modular function.

6.3. Fermionic character expressions for $\mathcal{W}(2, 2p - 1, 2p - 1, 2p - 1)$

The matrix $\frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ was found in the quadratic form of the fermionic expressions for the $\mathcal{W}(2, 3, 3, 3)$ model at $c = -2$ in the previous section. A generalization to $\mathcal{W}(2, 2p - 1, 2p - 1, 2p - 1)$ is possible by recognizing that the matrix in the case of $p = 2$ is just the inverse of the Cartan matrix of the degenerate case $D_2 = so(4) = A_1 \times A_1$ of the $D_n = so(2n)$ series of simple Lie algebras, where the corresponding Dynkin diagram consists just of two disconnected nodes, as shown in figure 1. Consequently, one may try the inverse Cartan matrices

$$C_{D_p}^{-1} = \begin{pmatrix} 1 & 1 & \dots & 1 & \frac{1}{2} & \frac{1}{2} \\ 1 & 2 & \dots & 2 & 1 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 1 & 2 & \dots & p-2 & \frac{p-2}{2} & \frac{p-2}{2} \\ \frac{1}{2} & 1 & \dots & \frac{p-2}{2} & \frac{p}{4} & \frac{p-2}{4} \\ \frac{1}{2} & 1 & \dots & \frac{p-2}{2} & \frac{p-2}{4} & \frac{p}{4} \end{pmatrix} \tag{6.25}$$

of $D_p = so(2p)$, $p > 2$, for the fermionic expressions of the characters of the $c_{p,1}$ models in the case of $p > 2$. The first thing we noticed by comparing expansions when we tried these matrices in (4.1) is that $\vec{b} = 0$ leads to a fermionic expression for $\frac{\Theta_{0,p}}{\eta(q)}$. However, the restriction $m_1 + m_2 \equiv 0 \pmod{2}$ has to be changed to $m_{p-1} + m_p \equiv 0 \pmod{2}$ implying that particles of the two species corresponding to the two nodes labeled by $n - 1$ and n in the D_n Dynkin diagram (see figure 2), which are both connected to the node labeled by $n - 2$, may only be created in pairs, as will be shown in detail in section 7. These expressions coincide with those found in [86] (but only those with $\vec{b} = 0$), since the characters of the free boson with central charge $c = 1$ and compactification radius $r = \sqrt{\frac{p}{2}}$ [66] equal some of the characters of the $c_{p,1}$ models. The expressions for $\frac{\Theta_{\lambda,p}}{\eta(q)}$ have $+\frac{\lambda}{2}$ and $-\frac{\lambda}{2}$ in the last two entries of \vec{b} and zero in the other components as is the case for the other, strictly two-dimensional fermionic expression for $\frac{\Theta_{\lambda,p}}{\eta(q)}$ given earlier in (6.14).

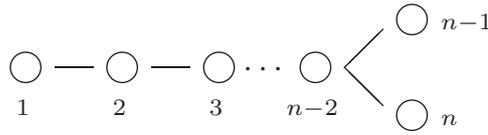


Figure 2. The Dynkin diagram of D_n .

Still missing now are fermionic expressions for those characters whose bosonic form is of inhomogeneous modular weight, i.e. which consist of theta and affine theta functions. For the vacuum character of $c = -2$ the vector is $\vec{b} = \frac{1}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Based on experience with fermionic expressions for other models, one may guess that the vector for the inhomogeneous characters of any $c_{p,1}$ model will have $+\frac{\lambda}{2}$ in both its last components, while the rest of the $k - 2$ components will increase in integer steps from top to bottom, starting with zero at the component number i . The value of i depends on the values of λ and k . All components above the component number i are zero, too. The detailed description of this vector in dependence on λ and k is given below. In this way, expressions for all characters of all $c_{p,1}$ models can be found and thus a whole new, infinite set of bosonic–fermionic q -series identities, also given below. In sections 7.3 and 7.4, we will propose a physical interpretation in terms of quasi-particles. Expanding the new fermionic character expressions in q , one may convince oneself that all coefficients match those of the bosonic character expressions. The proof of these identities can be found in [132].

In short, the fermionic sum representations for all characters of the $\mathcal{W}(2, 2p - 1, 2p - 1, 2p - 1)$, $p \geq 2$, series of triplet algebras corresponding to central charge $c_{p,1}$ can be expressed as follows and indeed equal the bosonic ones (cf (6.6a)–(6.6f)), the latter being redisplayed on the rhs for convenience [52]:

$$\chi_{\lambda,k} = \sum_{\substack{\vec{m} \in (\mathbb{Z}_{\geq 0})^k \\ m_{k-1} + m_k \equiv 0 \pmod{2}}} \frac{q^{\vec{m}^t C_{D_k}^{-1} \vec{m} + \vec{b}_{\lambda,k} \vec{m} + c_{\lambda,k}^*}}{(q)_{\vec{m}}} = \frac{\Theta_{\lambda,k}}{\eta} \tag{6.26a}$$

$$\chi_{\lambda',k}^+ = \sum_{\substack{\vec{m} \in (\mathbb{Z}_{\geq 0})^k \\ m_{k-1} + m_k \equiv 0 \pmod{2}}} \frac{q^{\vec{m}^t C_{D_k}^{-1} \vec{m} + \vec{b}_{\lambda',k}^+ \vec{m} + c_{\lambda',k}^*}}{(q)_{\vec{m}}} = \frac{(k - \lambda') \Theta_{\lambda',k} + (\partial \Theta)_{\lambda',k}}{k \eta} \tag{6.26b}$$

$$\chi_{\lambda',k}^- = \sum_{\substack{\vec{m} \in (\mathbb{Z}_{\geq 0})^k \\ m_{k-1} + m_k \equiv 1 \pmod{2}}} \frac{q^{\vec{m}^t C_{D_k}^{-1} \vec{m} + \vec{b}_{\lambda',k}^- \vec{m} + c_{\lambda',k}^*}}{(q)_{\vec{m}}} = \frac{\lambda' \Theta_{\lambda',k} - (\partial \Theta)_{\lambda',k}}{k \eta} \tag{6.26c}$$

for $0 \leq \lambda \leq k$ and $0 < \lambda' < k$, where $k = p$ since $p' = 1$ and $(\vec{b}_{\lambda,k})_i = \frac{\lambda}{2} (\pm \delta_{i,k-1} \mp \delta_{i,k})$ for $1 \leq i \leq k$, $(\vec{b}_{\lambda',k}^+)_i = \max\{0, \lambda' - (k - i - 1)\}$ for $1 \leq i < k - 1$ and $(\vec{b}_{\lambda',k}^+)_i = \frac{\lambda'}{2}$ for $k - 1 \leq i \leq k$, $(\vec{b}_{\lambda',k}^-)_i = (\vec{b}_{k-\lambda',k}^+)_i$ and $c_{\lambda,k}^* = \frac{\lambda^2}{4k} - \frac{1}{24}$. Note that this means the characters and not the torus vacuum amplitudes (6.6e) and (6.6f). Note that $\lim_{\alpha \rightarrow 0} \tilde{\chi}_{\lambda,k}^+ = \lim_{\alpha \rightarrow 0} \tilde{\chi}_{\lambda,k}^- = \chi_{\lambda,k}$ for $0 < \lambda < k$. Note also that in (6.26a), also $m_{k-1} + m_k \equiv 1 \pmod{2}$ may be used as restriction, but then the vector and the constant change to $\vec{b}_{k-\lambda,k}$ and $c_{k-\lambda,k}^*$, respectively (cf (6.14)). Thus, as in the previous section, the $p \times p$ matrix $A = C_{D_p}^{-1}$ is the same for all characters corresponding to a fixed p , i.e. for a fixed model. This is in agreement with previous results

on fermionic expressions, since it is known to also be the case for the characters of a given minimal model (see e.g. [135]).

For example, the fermionic expression of the vacuum character of the theory corresponding to central charge $c_{5,1} = -18.2$ would be

$$\chi_{4,5}^+ = \frac{\Theta_{4,5} + (\Theta)_{4,5}}{5\eta} = \sum_{\substack{\vec{m} \in (\mathbb{Z}_{\geq 0})^5 \\ m_4 + m_5 \equiv 0 \pmod{2}}} q^{\frac{\vec{m} \begin{pmatrix} 1 & 1 & 1 & \frac{1}{2} & \frac{1}{2} \\ 1 & 2 & 2 & 1 & 1 \\ 1 & 2 & 3 & 3 & 3 \\ \frac{1}{2} & 1 & 1 & \frac{3}{2} & \frac{3}{2} \\ \frac{1}{2} & 1 & 1 & \frac{3}{2} & \frac{3}{2} \end{pmatrix} \vec{m} + \frac{91}{120}}}{(q)_{\vec{m}}}. \tag{6.27}$$

7. Quasi-particle interpretation of the triplet \mathcal{W} -algebras

7.1. Quasi-particle interpretation

Non-unique realizations of the state spaces in two-dimensional CFTs establish the existence of several alternative character formulae.

The original formula, the bosonic representation (cf section 3), which traces back to Feigin and Fuks [39] and Rocha-Caridi [117], is directly based upon the structure of null vectors, i.e. the invariant ideal is divided out. The occurrence of a factor $(q)_{\infty}$ in the denominator arises naturally in the construction of Fock spaces using bosonic generators. Indeed, the character of a free chiral boson is given by

$$\chi_B = \sum_{n=0}^{\infty} p(n)q^n = \prod_{n=1}^{\infty} (1 - q^n)^{-1} = \frac{1}{(q)_{\infty}}, \tag{7.1}$$

where $p(n)$ is the number of additive partitions of the integer n into integer parts greater than zero which do not have to be distinct. Encoded by the numerator, these spaces are then truncated in a particular way in the general bosonic character expression. The interpretation as partition functions requires these expressions to be modular covariant, which is easily checked when expressing the characters in terms of Θ -functions (cf section 2.2).

In contrast, the fermionic representations possess a remarkable interpretation in terms of quasi-particles for the states, obeying Pauli's exclusion principle. The character of a free chiral fermion with periodic or anti-periodic boundary conditions is given respectively by

$$\chi_{F,P} = \sum_{n=0}^{\infty} \frac{q^{\frac{1}{2}m^2 - \frac{1}{2}m}}{(q)_m} \tag{7.2a}$$

or

$$\chi_{F,A} = \sum_{n=0}^{\infty} \frac{q^{\frac{1}{2}m^2}}{(q)_m}. \tag{7.2b}$$

In the following section, it will be discussed how this comes about.

The bosonic representations are in general unique, since a natural level gradation in terms of the eigenvalue of L_0 is induced by the operators L_n . Although the fermionic ones are also obviously graded by their L_0 eigenvalue, there is in general more than one fermionic expression for the same character, since different types of generalized exclusion statistics may be imposed which might force different quasi-particle systems to lead to the same fermionic character expression.

7.2. Quasi-particle representation of fundamental fermionic forms

The general fermionic character expression is a linear combination of fundamental fermionic forms. The characters of various series of rational CFTs, including the $c_{p,1}$ series, can be represented as a single fundamental fermionic form [16, 25, 135]. For simplicity, we would not deal with the most general case here, but with a certain specialization. This specialization is also called fundamental fermionic form in [16].

Fermionic sum representations for characters admit an interpretation in terms of fermionic quasi-particles, as shown in [88] (see also [86]). This can be easily seen from the fundamental fermionic form

$$\chi(q) = \sum_{\substack{\vec{m} \in (\mathbb{Z}_{\geq 0})^r \\ \text{restrictions}}} q^{\vec{m}^T A \vec{m} + \vec{b}^T \vec{m}} \prod_{a=1}^r \left[\begin{matrix} ((\mathbb{1} - 2A)\vec{m} + \vec{u})_a \\ m_a \end{matrix} \right]_q, \tag{7.3}$$

with the help of combinatorics: the number of additive partitions $P_M(N, N')$ of a positive integer N into M distinct non-negative integers which are smaller than or equal to N' is stated by [127, p 23]

$$\sum_{N=0}^{\infty} P_M(N, N') q^N = q^{\frac{1}{2}M(M-1)} \left[\begin{matrix} N' + 1 \\ M \end{matrix} \right]_q, \tag{7.4}$$

which in the limit $N' \rightarrow \infty$ takes the form

$$\lim_{N' \rightarrow \infty} \sum_{N=0}^{\infty} P_M(N, N') q^N = q^{\frac{1}{2}M(M-1)} \frac{1}{(q)_M}. \tag{7.5}$$

(A possible constant c has been omitted, since it would just result in an overall shift of the energy spectrum of the resulting quasi-particles.) This formula is tailored to our needs, because the requirement of distinctiveness expresses the fermionic nature of the quasi-particles, i.e. Pauli's exclusion principle. To make use of (7.5), (7.3) can be reformulated to

$$\begin{aligned} \chi(q) &= \sum_{\substack{\vec{m} \in (\mathbb{Z}_{\geq 0})^r \\ \text{restrictions}}} q^{\frac{1}{2} \sum_{i=1}^r (m_i^2 - m_i) + \sum_{i=1}^r (b_i + \frac{1}{2}) m_i + \sum_{i,j=1}^r A_{ij} m_i m_j - \frac{1}{2} \sum_{i=1}^r m_i^2} \prod_{a=1}^r \left[\begin{matrix} ((\mathbb{1} - 2A)\vec{m} + \vec{u})_a \\ m_a \end{matrix} \right]_q \\ &= \prod_{i=1}^r \left(\sum_{\substack{m_i \\ \text{restrictions}}} q^{\frac{1}{2} \sum_{i=1}^r (m_i^2 - m_i) + (b_i + \frac{1}{2}) m_i + \sum_{j=1}^r A_{ij} m_i m_j - \frac{1}{2} m_i^2} \right) \prod_{a=1}^r \left[\begin{matrix} ((\mathbb{1} - 2A)\vec{m} + \vec{u})_a \\ m_a \end{matrix} \right]_q. \end{aligned} \tag{7.6}$$

Applying (7.5) to the fundamental fermionic form (7.3) leads to

$$\prod_{i=1}^r \left(\sum_{\substack{m_i \\ \text{restrictions}}} \sum_{N=0}^{\infty} P_{m_i}(N, ((\mathbb{1} - 2A)\vec{m} + \vec{u})_a - 1) q^{N + (b_i + \frac{1}{2}) m_i + \sum_{j=1}^r A_{ij} m_i m_j - \frac{1}{2} m_i^2} \right). \tag{7.7}$$

We can then make use of the relation

$$\sum_{N=0}^{\infty} P_M^0(N, N') q^{N+kM} = \sum_{N=0}^{\infty} P_M^k(N, N' + k) q^N, \tag{7.8}$$

where we defined $P_M^k(N, N')$ like $P_M(N, N')$ but with the additional requirement that all the integers that make up a partition have to be greater than or equal to k . This relation is obvious since it is a one-to-one mapping of partitions and thus nothing more than just a mere shift of

the partitions: each part of a given partition of N into M distinct parts is increased by k , which turns N into $N + Mk$. Equation (7.8) allows us to rewrite (7.7) into

$$\prod_{i=1}^r \left(\sum_{\substack{m_i \\ \text{restrictions}}}^{\infty} \sum_{N=0}^{\infty} P_{m_i}^{b_i + \frac{1}{2} + (A - \frac{1}{2} \mathbb{1}) \vec{m}} \left(N, - \left(A - \frac{1}{2} \mathbb{1} \right) \vec{m} \right)_a + \left(\vec{b}_a - \frac{1}{2} + \vec{u}_a \right) q^N \right). \quad (7.9)$$

For the quasi-particle interpretation, the characters are regarded as partition functions Z for left-moving excitations with the ground-state energy scaled out:

$$\chi \sim Z = \sum_{\text{states}} e^{-\frac{E_{\text{states}}}{kT}} = \sum_{l=0}^{\infty} P(E_l) e^{-\frac{E_l}{kT}} \quad (7.10)$$

with T being the temperature, k being the Boltzmann constant, E_l being the energy and $P(E_l)$ being the degeneracy of the particular energy level l .

The energy spectrum consists of all the excited state energies (minus the ground state energy) that are given by

$$E_l = E_{\text{ex}} - E_{\text{GS}} = \sum_{i=1}^r \sum_{\substack{\alpha=1 \\ \text{restrictions}}}^{m_i} e_i(p_{\alpha}^i), \quad (7.11)$$

and the corresponding momenta of the states are given by

$$P_{\text{ex}} = \sum_{i=1}^r \sum_{\substack{\alpha=1 \\ \text{restrictions}}}^{m_i} p_{\alpha}^i, \quad (7.12)$$

where r denotes the number of different species of particles, m_i the number of particles of species i in the state, $e_i(p_{\alpha}^i)$ the single-particle energy of the quasi-particle α of species i and the subscript ‘restrictions’ indicates possible rules under which the excitations may be combined. Equation (7.11) is referred to as a quasi-particle spectrum in statistical mechanics (see e.g. [105]). In this context quasi means that for example magnons or phonons have other properties than real particles such as protons or electrons. And in addition, the spectrum above may contain single-particle energy levels that are different from the form in relativistic quantum field theory $e_{\alpha}(p) = \sqrt{M_{\alpha}^2 + p^2}$. This means that if we assume massless single-particle energies

$$e_i(p_{\alpha}^i) = e(p_{\alpha}^i) = v p_{\alpha}^i \quad (7.13)$$

(v referred to as the Fermi velocity, spin-wave velocity, speed of sound or speed of light), where p_{α}^i denotes the quasi-particle α of ‘species’ i ($1 \leq i \leq r$), and if in (7.9) we set

$$q = e^{-\frac{v}{kT}}, \quad (7.14)$$

we can read off that the partition function corresponds to a system of quasi-particles that are of r different species and which obey the Pauli exclusion principle

$$p_{\alpha}^i \neq p_{\beta}^i \quad \text{for } \alpha \neq \beta \quad \text{and all } i, \quad (7.15)$$

in order to satisfy Fermi statistics, but whose momenta p_{α}^i are otherwise freely chosen from the sets

$$P_i = \{ p_{\min}^i, p_{\min}^i + 1, p_{\min}^i + 2, \dots, p_{\max}^i \} \quad (7.16)$$

with minimum momenta

$$p_{\min}^i(\vec{m}) = \left[\left(A - \frac{1}{2} \right) \vec{m} \right]_i + b_i + \frac{1}{2} \quad (7.17)$$

and with the maximum momenta

$$p_{\max}^i(\vec{m}) = - \left(A - \frac{1}{2} \right) \vec{m} \Big|_i + (\vec{b})_i - \frac{1}{2} + (\vec{u})_i = - p_{\min}^i(\vec{m}) + 2(\vec{b})_i + (\vec{u})_i. \quad (7.18)$$

Thus, p_{\max}^i is either infinite if $(\vec{u})_i$ is infinite or finite and dependent on \vec{m} , A , $(\vec{b})_i$ and $(\vec{u})_i$. Note that if $(\vec{u})_i$ is infinite for all $i \in \{1, \dots, r\}$, then (7.3) reduces to the form (4.1) of Nahm's conjecture. Of course, (7.3) is only an often encountered specialization of the most general fundamental fermionic form, since the components of the q -binomial coefficient may be of a different shape from that given in (7.3), but the generalization of the previous steps is obvious. To sum up, this means that a multi-particle state with energy E_l may consist of exactly those combinations of quasi-particles of arbitrary species i , whose single-particle energies $e(p^i)$ add up to E_l and where Pauli's principle holds for any two quasi-particles of that combination which belong to the same species. Possible sum restrictions then result in the requirement that certain particles may only be created in conjunction with certain others. Thus, the characters (7.2a) and (7.2b) of the free chiral fermion with respectively periodic or anti-periodic boundary conditions are obtained in the case of $r = 1$, $p_{\max} = \infty$, $p_{\min}^{(P)} = 0$ and $p_{\min}^{(A)} = \frac{1}{2}$. On the other hand, the character (7.1) of a free chiral boson is obtained by setting $r = 1$, $p_{\min} = 1$ and $p_{\max} = \infty$ and simply not imposing any exclusion rules, i.e. not using (7.4).

Although the upper momentum boundaries may seem artificial, the phenomenon that the momenta $p_{i\alpha}^a$ for $2 \leq \alpha \leq n$ are restricted to take only a finite number of values for given \vec{m} is a common occurrence in quantum spin chains.

7.3. The $c = -2$ model

In section 6.3, we reviewed the fermionic character expressions for the series of triplet \mathcal{W} -algebras [52]. In this and in the following section, we discuss the quasi-particle content, which one can derive from the fermionic expressions, of the $c_{p,1}$ LCFTs, which have $\mathcal{W}(2, 2p - 1, 2p - 1, 2p - 1)$ as symmetry algebras.

We start with the case $p = 2$, i.e. $c_{2,1} = -2$. In contrast to the characters for, e.g., the minimal models, these characters are the traces over the representation modules of the triplet \mathcal{W} -algebra, instead of the Virasoro algebra only. However, although highest weight states are labeled by two highest weights in this case, h and w as the eigenvalues of L_0 and W_0 respectively, we consider only the traces of the operator $q^{L_0 - \frac{c}{24}}$. It turns out that these \mathcal{W} -characters are given as infinite sums of Virasoro characters, for example [47]

$$\chi_{|0\rangle} = \sum_{k=0}^{\infty} (2k + 1) \chi_{|h_{2k+1,1}\rangle}^{\text{Vir}}. \tag{7.19}$$

Let us now come to the vacuum character (6.23a) for the $c_{2,1}$ model, which features the interesting sum restriction $m_1 + m_2 \equiv 0 \pmod{2}$ expressing the fact that particles of types 1 and 2 must be created in pairs. Thus, the existence of one-particle states for either particle species is prohibited. Therefore, the single-particle energies must be extracted out of the observed multi-particle energy levels.

Applying (7.5) to the fermionic sum representation (cf also (6.23a))

$$q^{-\frac{1}{12}} \chi_{1,2}^+ = \sum_{\substack{\vec{m} \in (\mathbb{Z}_{\geq 0})^2 \\ m_1 + m_2 \equiv 0 \pmod{2}}} \frac{q^{\frac{1}{2}\vec{m} \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \vec{m} + \frac{1}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \vec{m}}}{(q)_{\vec{m}}} \tag{7.20}$$

of the vacuum character leads to

$$\chi_{1,2}^+ = \left(\sum_{m_1=0}^{\infty} \sum_{N=0}^{\infty} P_{m_1}(N) q^{N+m_1} \right) \left(\sum_{\substack{m_2=0 \\ m_2 \equiv m_1 \pmod{2}}}^{\infty} \sum_{N=0}^{\infty} P_{m_2}(N) q^{N+m_2} \right), \tag{7.21}$$

where the constant c has been omitted, since it would just result in an overall shift of the energy spectra. Using massless single-particle energies (7.13) and setting (7.14) in (7.9) then results in the partition function (7.10) corresponding to a system of two quasi-particle species, with both species having the momentum spectrum $\mathbb{N}_{\geq 1}$, i.e. a multi-particle state with energy E_l may consist of exactly those combinations of an even number of quasi-particles, having momenta p_α^i ($i \in \{1, 2\}$), whose single-particle energies $e(p_\alpha^i)$ add up to E_l and where the momenta $p_\alpha^i \in \mathbb{N}_{\geq 1}$ of each two of the quasi-particles in that combination are distinct unless they belong to different species, i.e. they respect the exclusion principle. Formally, these spectra belong to two free chiral fermions with periodic boundary conditions. Note in this context the physical interpretations in [83, 84], in which the CFT for $c_{2,1} = -2$ is generated from a symplectic fermion, a free two-component fermion field of spin 1.

7.4. The $p > 2$ relatives

Besides the best understood LCFT with central charge $c_{2,1} = -2$, we now have a look at the quasi-particle content of its $c_{p,1}$ relatives.

The restrictions $m_{p-1} + m_p \equiv Q \pmod{2}$ (Q can be thought of as denoting the total charge of the system) in (6.26a)–(6.26c) imply that the quasi-particles of species $p - 1$ and p are charged under a \mathbb{Z}_2 subgroup of the full symmetry of the D_p Dynkin diagram [86], while all the others are neutral. This charge reflects the $\mathfrak{su}(2)$ structure carried by the triplet \mathcal{W} -algebra such that all representations must have ground states, which are either $\mathfrak{su}(2)$ singlets or $\mathfrak{su}(2)$ doublets. In comparison to the $c_{2,1} = -2$ model, there exist p quasi-particles in each member of the $c_{p,1}$ series, exactly two of which can only be created in pairs, while the others do not have this restriction. These observations suggest the following conjecture: the $c_{p,1}$ theories might possess a realization in terms of free fermions such that they are generated by one pair of symplectic fermions and $p - 2$ ordinary fermions. Realizations of that kind are unknown so far, except for the well-understood case $p = 2$, and might constitute an interesting direction of future research.

Contrary to the $p = 2$ case, the quasi-particles do not decouple here: the minimal momenta for the quasi-particle species can be read off (7.17) and are given by

$$p_{\min}^i(\vec{m}) = \begin{cases} -\frac{1}{2}m_i + \sum_{j=1}^i jm_j + \sum_{j=i+1}^{p-2} im_j + \frac{i}{2}(m_{p-1} + m_p) + i + \frac{1}{2} & \text{for } 1 \leq i \leq p-2 \\ -\frac{1}{2}m_i + \sum_{j=1}^{p-2} \frac{j}{2}m_j + \frac{p-1}{2} + \frac{1}{2} + \begin{cases} \left(\frac{p}{4}m_{p-1} + \frac{2-p}{4}m_p\right) & \text{for } i = p-1 \\ \left(\frac{2-p}{4}m_{p-1} + \frac{p}{4}m_p\right) & \text{for } i = p. \end{cases} & \text{for } i = p-1, p. \end{cases} \quad (7.22)$$

Hence, they depend on the numbers of quasi-particles of the different species in the state. But as in the $p = 2$ case, the momentum spectra are not bounded from above.

8. Summary and outlook

We mainly focused on the characters of the irreducible subrepresentations contained in the indecomposable Jordan cell representations of LCFTs. Our aim was to highlight the crucial role of these particular characters for our understanding of LCFT and for how LCFT fits into the larger landscape of (rational) CFTs.

Firstly, the characters of the irreducible subrepresentations of indecomposable representations are the only characters in a LCFT, that allow by studying their properties under modular transformations to deduce that the corresponding CFT is in fact logarithmic, and not an ordinary one. In fact, the character of the vacuum representation of a LCFT generally is of this type.

Secondly, the modular differential equation can be constructed from the input of these particular characters alone. Its solutions yield a complete, finite-dimensional representation of the modular group including the characters of the irreducible subrepresentations of indecomposable representations and the necessary generalized character functions (which depend on both q and $\log(q)$, formally). The latter have an interpretation as torus vacuum amplitudes. The set of torus amplitudes yields, by a Verlinde-like formula, fusion rules in a limit, where the generalized character functions are projected out at the end. Equivalence of these fusion rules with rules computed by direct means has been established in [53].

Thirdly, in common LCFTs, the sector of the theory with $h = 0$ is indecomposable with respect to the action of L_0 . Thus, the character of the irreducible subrepresentation of the indecomposable $h = 0$ sector is an important tool to understand the vacuum structure of the corresponding LCFT. For example, the character for the vacuum representation is needed to prove the C_2 cofiniteness of the triplet series in the general approach taken in [23]. The vacuum character will also play a key role in the study of logarithmically extended minimal models. We showed that the modular differential equation predicts that augmented minimal models with the central charge $c_{p,q}$ are only possible for conformal grids of size $(\alpha p - 1) \times (\alpha q - 1)$ with α odd. Furthermore, we computed the modular differential equation and its space of solutions for $c = c_{6,9} = 0$, yielding a conjecture for the vacuum character of this theory, which still is only known to small finite order.

Finally, the characters of irreducible subrepresentations of indecomposable representations are precisely the characters with which the *ADET*-classification and the Nahm conjecture extend in a natural way to LCFTs, when their fermionic sum representations are considered. Actually, this has been proven only for the $c_{p,1}$ series, but it presumably holds more generally for logarithmically extended minimal models.

We reviewed the derivation of fermionic expressions for all characters of each $c_{p,1}$ model [52], the existence of which provides (in line with Nahm's conjecture) further evidence for the well-definedness of the LCFTs corresponding to central charge $c_{p,1}$ as being rational CFTs. Furthermore, we explained how these models admit an interpretation in terms of p fermionic quasi-particle species. We reviewed how the obtained character expressions also lead to an infinite set of new bosonic–fermionic q -series identities generalizing (3.5), see section 6.3. The case at hand is special due to the inhomogeneous structure of the bosonic character expressions in terms of modular forms. Notwithstanding, there exist fermionic quasi-particle sum representations with the same matrix A (cf (4.1)) for each of the characters of each $c_{p,1}$ model. Namely, the matrix A turns out to be the inverse of the Cartan matrix of the simply-laced Lie algebra D_p . Therefore, those expressions fit well into the known scheme of fermionic character expressions for other (standard) CFTs.

There are also numerous other avenues towards finding fermionic expressions. Among these are Bailey's lemma [11], thermodynamic Bethe ansatz [92], Kostka polynomials and Hall–Littlewood functions [10, 27, 101, 102, 122, 131]. A specific new technique involves quantum groups, crystal bases and finite paths [26, 71–73]. The *quantum groups* [74, 79, 80, 125] deepen the understanding of symmetry in systems with an infinite number of degrees of freedom. In general, the connection of quantum groups to CFT is given by the *Kazhdan–Lusztig correspondence*. In particular, the Kazhdan–Lusztig-dual quantum group to the logarithmic \mathcal{W} -algebras is known (see e.g. [40–42, 64]).

Fermionic character expressions imply a realization of the underlying theory in terms of systems of fermionic quasi-particles. We detailed this for the case of the $\mathcal{W}(2, 2p - 1, 2p - 1, 2p - 1)$ series of triplet algebras. In the $c_{2,1} = -2$ model, i.e. $p = 2$, the quasi-particle interpretation implies that there exist two fermionic quasi-particle species whose members may only be created in pairs, i.e. either a pair of particles from the same species or one particle from each species. This coincides with the realization of the $c = -2$ theory in terms of a pair of symplectic fermions [83, 84], which is a free two-component fermion field of spin 1. In the general $c_{p,1}$ model, there is a set of $p - 2$ fermionic quasi-particle species, the members of which may—aside from Pauli’s exclusion principle—be combined freely in building an arbitrary multi-particle state, and additionally a set of two species, the members of which may only be created in pairs. This interpretation suggests that the $c_{p,1}$ theories might possess a realization in terms of free fermions such that they are generated by $p - 2$ ordinary fermions and one pair of symplectic fermions. Such realizations had been unknown, except for $p = 2$, and constitute a possible direction for further research.

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