Polylogarithms, multiple zeta values and superstring amplitudes

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A formalism is provided to calculate tree amplitudes in open superstring theory for any multiplicity at any order in the inverse string tension. We point out that the underlying world-sheet disk integrals share substantial properties with color-ordered tree amplitudes in Yang-Mills field theories. In particular, we closely relate world-sheet integrands of open-string tree amplitudes to the Kawai-Lewellen-Tye representation of supergravity amplitudes. This correspondence helps to reduce the singular parts of world-sheet disk integrals – including their string corrections – to lower-point results. The remaining regular parts are systematically addressed by polylogarithm manipulations.

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1   Introduction

Perturbative string theory has undergone important conceptual and technical advances, such as the analysis of world-sheet and target-space descriptions, the use of modular properties and more importantly of the pure-spinor formalism. The application of those methods has considerably simplified the computation of both tree-level and multi-loop superstring scattering amplitudes.

The structure of superstring scattering amplitudes remarkably resembles the form and organization of field-theory amplitudes. Moreover, some features of the latter are encoded in and derived from the properties of the string world-sheet. Striking field-theory relations such as Kleiss-Kuijf (KK) [1], Bern-Carrasco-Johansson (BCJ) [2] or Kawai-Lewellen-Tye (KLT) [3] relations can be easily derived from and understood in string theory by tracing these identities back to the monodromy properties of the string world-sheet [4,5]. Furthermore, recent mathematical concepts such as the appearance of twistor space, motivic aspects, symbols and the coproduct structure for Feynman integrals have radically changed our viewpoint and strategy of how to quickly obtain, reorganize and express results for amplitudes in field-theory. It appears fruitful to further investigate and understand the natural appearance of those concepts in string theory. One particular feature of quantum field theories – the appearance of transcendental functions at loop-level – already occurs at tree-level scattering in string theory as a consequence of the underlying string world-sheet. Since generically the complete perturbative tree-level contributions to string amplitudes are easier to access than fully-fledged loop corrections, they provide a convenient testing ground for transcendental structures.

This article is devoted to tree-level open superstring amplitudes (or disk amplitudes). Tree-level interactions of open strings are described by integrals over the boundary of world-sheets with disk topology. The entire polarization dependence of open superstring tree amplitudes was shown in [6,7] to reside in tree amplitudes of the underlying Yang-Mills (YM) field theory, which emerges in the point-particle limit $\alpha' \to 0$. String corrections to the YM amplitude resemble formfactors promoting the kinematic factors of field theory into those of string theory. They enter through generalized Euler or Selberg integrals which involve multiple zeta values (MZVs) in their power-series expansion w.r.t. α' as was discussed in e.g. [8–11]. We shall show that generalized Selberg integrals involving different cyclic combinations of world-sheet Green's functions share essential symmetry properties of color-stripped YM amplitudes. In addition to cyclic symmetry in the arguments of the Green's functions, the KK and BCJ relations between field theory amplitudes literally carry over to disk integrals. This paves our way towards casting the world-sheet integrand of disk amplitudes into a form resembling the KLT representation of supergravity tree amplitudes.

One copy of YM subamplitudes therein is replaced by its world-sheet correspondent – a Selberg integral involving an appropriate cyclic product of Green's functions. This provides a field theoretic intuition for the structure of the disk integrand explicitly computed in [6, 7].

The systematics of the appearance of MZVs in disk amplitudes has been analyzed in [12]. In this work, the α'-expansion was shown to take a compact and elegant form once the contributions from different classes of MZVs are disentangled. The idea is to lift MZVs to their motivic versions endowed with a Hopf algebra structure. The latter induces an isomorphism which allows to cast the amplitudes into a very symmetric form. In analogy with the symbol of a transcendental function, this isomorphism automatically builds in all relations between MZVs. However, its invertibility overcomes the loss of information inherent to the symbol approach. In spite of this beautiful organization of disk amplitudes, the explicit calculation of the α'-dependent "seeds" accompanying the single zeta values has not yet been addressed systematically. Their exact momentum dependence is hidden within world-sheet integrals whose complexity increases drastically with multiplicity N and the order in α'. While the world-sheet integrals at five points can still be reduced to a set of single (variable) Gaussian hypergeometric functions $_3F_2$, which has been thoroughly exploited in [12] to probe weights $w \leq 16$, their six- and higher-point versions comprise multiple Gaussian hypergeometric functions [10], whose expansions in α' are much more involved. Though computing some of these expansions has been accomplished at the six- [10, 13–15] and seven-point level [7, 16] a systematic approach is still lacking.

In this article, we will provide a method to completely evaluate the world-sheet integrals, which is – in principle – applicable at any multiplicity and to any order in α'. Once a suitable set of basis integrals is identified, their singular parts can be rewritten in a form recycling regular parts from lower-point superstring amplitudes. That is, the complete N-point superstring amplitude including poles can be written in terms of regular parts of the world-sheet integrals at multiplicities smaller or equal to N. The only remaining task is the calculation of the regular parts. Employing the language of multiple polylogarithms, we provide a method to iteratively solve all integrals occurring.

The α'–expansion of the integral bases for the multiplicity $N = 5$ has been computed up to the weight $w = 16$ in [12]. The complete set up to the order $w = 22$ is now public at [17]. Nevertheless, these matrices have also recently been determined up to $w = 21$ in [18]. For the multiplicities $N = 6$ and $N = 7$ the α'–expansions of the integral bases have been determined up to weights $w = 9$ and $w = 7$, respectively and are explicitly given at [17]. An alternative expansion method for the four-point and five-point integrals has been recently developed in [18].

The article is organized as follows: after reviewing the structure of the α'-expansion in open-string tree-level amplitudes in Sect. 2, we will point out a close relation between world-sheet integrals and color-ordered subamplitudes in YM theories in Sect. 3. This correspondence inspires the calculation of world-sheet integrals: Sect. 4 is devoted to exploring the pole structure of the integrals as well as the recursive nature of the residues, while the integration of the regular parts is taken care of in Sect. 5 using polyloga-rithm manipulations. In Sect. 6, the symmetry of disk amplitudes under cyclic shifts and world-sheet parity is exploited in order to increase the efficiency of the setup.

Several appendices contain more detailed information on various aspects of this work: In appendix A we recapitulate the Hopf algebra structure of motivic MZVs. In order to support and extend the discussion in Sect. 3 on field-theory structures in world-sheet integrals, we supply derivations for some of the identities in appendix B. Examples at multiplicities $4 \leq N \leq 7$ on how the massless poles of disk integrals can be quickly read off from their integrand are gathered in appendix C. In addition, we provide further examples for six-point residues and their recursively organized α'-expansion in appendix D. The general discussion of polylogarithm identities in Sect. 5 is supplemented by the examples in appendix E. Finally, some of the intermediate steps for seven-point α'-expansions are outsourced to appendix F.

2 String tree-level amplitudes and their α'-expansion

The N-point tree-level open superstring amplitude was computed in references [6, 7] based on pure-spinor cohomology methods [19]. Remarkably, the entire polarization dependence of the amplitude is carried by color-ordered tree amplitudes A_{YM} of the underlying YM field theory. In terms of the $(N-3)!$-element basis[1] $A_{\text{YM}}(1, \sigma(2, \ldots, N-2), N-1, N)$ with $\sigma \in S_{N-3}$ [2], the N-point superstring amplitude reads

$$A_{\text{open}}(\Pi, \alpha') = \sum_{\sigma \in S_{N-3}} F_\Pi{}^\sigma(\alpha')\, A_{\text{YM}}(1, \sigma(2, \ldots, N-2), N-1, N)\,. \tag{2.1}$$

In the above equation, $A_{\text{open}}(\Pi)$ with $\Pi \in S_N$ denotes the string amplitude associated with an ordering of vertex operator positions z_i on the disk boundary according to $z_{\Pi(i)} < z_{\Pi(i+1)}$. This amounts to calculating the partial string amplitude for the particular color-ordering Π. The objects $F_\Pi{}^\sigma(\alpha')$ originate in the string world-sheet integrals and encode the string-theory modifications to the field theory amplitude[2].

In analogy to the YM situation, not all color-orderings Π of the string amplitude $A_{\text{open}}(\Pi)$ are independent [4, 5]. Choosing a basis amounts to singling out three legs: we will choose to fix legs 1, $N-1$ and N leading to the basis $A_{\text{open}}(1, \Pi(2, \ldots, N-2), N-1, N)$ of string amplitudes, with $\Pi \in S_{N-3}$. The common $(N-3)!$-structure of the basis for open-string amplitudes A_{open} and YM amplitudes A_{YM} suggests to rewrite Eq. (2.1) in matrix notation, which relates the two vectors comprising the basis amplitudes to a $(N-3)! \times (N-3)!$-matrix F [7]:

$$\mathbf{A}_{\text{open}} = F\, \mathbf{A}_{\text{YM}}\,. \tag{2.2}$$

Explicitly, one finds

$$\begin{pmatrix} A_{\text{open}}(1, \Pi_1, N-1, N) \\ \vdots \\ A_{\text{open}}(1, \Pi_{(N-3)!}, N-1, N) \end{pmatrix} = \begin{pmatrix} F_{\Pi_1}{}^{\sigma_1} & \cdots & F_{\Pi_1}{}^{\sigma_{(N-3)!}} \\ \vdots & \ddots & \vdots \\ F_{\Pi_{(N-3)!}}{}^{\sigma_1} & \cdots & F_{\Pi_{(N-3)!}}{}^{\sigma_{(N-3)!}} \end{pmatrix} \begin{pmatrix} A_{\text{YM}}(1, \sigma_1, N-1, N) \\ \vdots \\ A_{\text{YM}}(1, \sigma_{(N-3)!}, N-1, N) \end{pmatrix}, \tag{2.3}$$

where σ_i and Π_i with $i \in \{1, \ldots, (N-3)!\}$ denote the permutations of $(2, 3, \ldots, N-2)$, respectively[3]. The matrix F, however, contains redundant information: Knowing the functions $F_{\Pi_1}{}^{\sigma_1}, \ldots, F_{\Pi_1}{}^{\sigma_{(N-3)!}}$ in the first line of F allows to obtain all other entries by a suitable relabeling.

After having organized the open-string amplitude in the form of Eq. (2.2), let us now turn to the entries in the first line of the matrix F. The dependence of the disk amplitude on the inverse string tension α' is incorporated in the set of $(N-3)!$ functions $\{F_{\Pi_1}{}^\sigma(\alpha'), \ \sigma \in S_{N-3}\}$. The latter represent multiple Gaussian hypergeometric functions originating from generalized Euler or Selberg integrals [10, 14]. The origin of their $(N-3)$-fold integrations stems from vertex operator positions along the boundary of the disk world-sheet. Taylor expanding the string amplitudes (2.1) w.r.t. to small α' reproduces their low-energy behavior. Each order in α' appears with a rational function in the dimensionless Mandelstam variables

$$s_{i_1 i_2 \ldots i_p} = \alpha'(k_{i_1} + k_{i_2} + \ldots + k_{i_p})^2 \tag{2.4}$$

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[1] Throughout this work, expressions of the form $\sigma(2, 3, \ldots, N-2)$ with permutation σ have to be understood as $\sigma(2), \sigma(3), \ldots, \sigma(N-2)$.

[2] As a consequence of the manifestly supersymmetric derivation in [6] the labels $1, 2, \ldots, N$ in the subamplitude Eq. (2.1) may denote any state from the $\mathcal{N}=1$ super YM multiplet in $D=10$ space–time dimensions without any particular reference to its polarization. Moreover, pure gluon tree-level amplitudes are not affected by dimensional reduction and supersymmetry breaking. Thus, Eq. (2.1) remains valid for external gluons in any $D < 10$ superstring compactification which allows for a CFT description with the field theory subamplitudes A_{YM} adapted to the appropriate spacetime dimension.

[3] We enumerate S_{N-3} permutations in canonical order.

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supplemented by some MZVs

$$\zeta_{n_1,\ldots,n_r} \; := \; \zeta(n_1,\ldots,n_r) \; = \; \sum_{0<k_1<\ldots<k_r} k_1^{-n_1}\ldots k_r^{-n_r}\,, \quad n_l \in \mathbb{N}^+\,, \quad n_r \geq 2\,, \quad (2.5)$$

or products thereof. In (2.5) the number r of the arguments[4] and their sum $w = \sum_{i=1}^r n_i$ are called depth and weight of the MZV, respectively. The overall weight of MZV products matches the corresponding power in $s_{i_1 i_2 \ldots i_p}$. The α'-expansion of these iterated integrals in terms of MZVs has been discussed in both mathematics [8, 9, 20] and physics literature [7, 11, 12] extensively.

There are numerous relations over rational numbers $\mathbb{Q}$ between different MZVs, all of which preserve the weight w. A convenient way to automatically take all these relations into account in the α'-expansion of F, is to endow the MZVs (2.5) with a Hopf algebra structure and map F to a non-commutative algebra comodule[5] with a more transparent basis [12, 21]. The result is a sum over all non-commutative words[6] in cogenerators $f_3, f_5, \ldots$, supplemented by a commutative element f_2 [12],

$$\phi(\mathbf{A}_{\text{open}}^m) = \left(\sum_{k=0}^{\infty} f_2^k\, P_{2k} \right) \left\{ \sum_{p=0}^{\infty} \sum_{\substack{i_1,\ldots,i_p \\ \in 2\mathbb{N}^++1}} f_{i_1} f_{i_2} \ldots f_{i_p}\, M_{i_p} \ldots M_{i_2} M_{i_1} \right\} \mathbf{A}_{\text{YM}}\,. \quad (2.7)$$

The $(N-3)! \times (N-3)!$-matrices P_{2k} and M_{2k+1} are defined to be the coefficients of Riemann zeta values,

$$M_{2k+1} \; := \; F\Big|_{\zeta_{2k+1}}\,, \quad P_{2k} \; := \; F\Big|_{(\zeta_2)^k}\,, \quad (2.8)$$

with respect to a particular $\mathbb{Q}$ basis of MZVs, see e.g. table 1 in appendix A.1. The non-commutative monomial $f_2^k f_{i_1} f_{i_2} \ldots f_{i_p}$ is accompanied by a matrix product $P_{2k} M_{i_p} \ldots M_{i_2} M_{i_1}$. Appendix A gives a more detailed account on the aforementioned Hopf algebra structure, the isomorphism ϕ and the motivic version ζ^m of MZVs (2.5) to which the superscript in (2.7) alludes.

Thus, for our choice of basis expansion (2.8), the whole information about the open superstring amplitude, which is not fixed by the structure in Eq. (2.7), is contained in the matrices P_w and M_w associated with the single zeta values ζ_w in (2.8). For the multiplicities $N = 5, 6$ and $N = 7$ these matrices are available up to weights $w = 21, 9$ and $w = 7$, respectively at [17]. In the following we shall be concerned with the computation of these matrices, composed from the set of the $(N-3)!$ functions $F^\sigma := F_{\Pi_1}{}^\sigma$ and relabelings thereof. Prior to this, let us consider some of their properties yielding a simplification of their world-sheet integrals.

3 Field-theory patterns in world-sheet integrals

In this section, we establish a connection between color-ordered subamplitudes A_{YM} in YM theories and world-sheet disk integrals. In particular, we show that in the open-string amplitude (2.1) the role of the YM subamplitudes and world-sheet integrals can be swapped. This feature is reminiscent of the observation

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[4]   If there is only one argument n_1, the above definition reduces to the usual definition of the single Riemann zeta function $\zeta(n_1)$.

[5]   A discussion of the mathematical concepts is provided in appendix A.

[6]   As pointed out by Don Zagier, one can rewrite (2.7) as a geometric series

$$\phi(\mathbf{A}_{\text{open}}^m) = \left(\sum_{k=0}^{\infty} f_2^k\, P_{2k} \right) \left\{ \left(1 - \sum_{k=1}^{\infty} f_{2k+1} M_{2k+1} \right)^{-1} \right\}^t \mathbf{A}_{\text{YM}} \quad (2.6)$$

where $\{\ldots\}^t$ reverses the order of the f_{i_k}'s enclosed.

of Bern et al. [2, 22–24] (and references therein) that color and kinematic contributions to gauge-theory amplitudes share their algebraic properties and are freely interchangeable. We refer to the string-theory property as a correspondence between disk integrals and YM subamplitudes A_{YM}.

This correspondence is manifested by employing the KLT relations [3] and supplemented by a world-sheet analogues of KK [1, 25] and BCJ relations [2]. Furthermore, different ways of writing the KLT relations [26] translate into integration-by-part identities between world-sheet integrals. The properties and relations derived in this section will set the stage to get a convenient handle on the $(N-3)!$ basis of disk integrals[7] and to correctly identify their singular behavior in Sect. 4.

3.1 Basic definitions

According to the calculation in [6, 7], the set of functions $F_\Pi{}^\sigma(\alpha')$ introduced in Eq. (2.1) reads

$$F_\Pi{}^\sigma = (-1)^{N-3} \prod_{i=2}^{N-2} \int_{D(\Pi)} dz_i \prod_{i<j}^{N-1} |z_{ij}|^{s_{ij}} \tag{3.1}$$

$$\times \sigma \left\{ \frac{s_{12}}{z_{12}} \left(\frac{s_{13}}{z_{13}} + \frac{s_{23}}{z_{23}} \right) \cdots \left(\frac{s_{1,N-2}}{z_{1,N-2}} + \ldots + \frac{s_{N-3,N-2}}{z_{N-3,N-2}} \right) \right\}$$

$$= (-1)^{N-3} \prod_{i=2}^{N-2} \int_{D(\Pi)} dz_i \prod_{i<j}^{N-1} |z_{ij}|^{s_{ij}} \sigma \left\{ \prod_{k=2}^{N-2} \sum_{m=1}^{k-1} \frac{s_{mk}}{z_{mk}} \right\} . \tag{3.2}$$

We are always working in coordinates z mapping the boundary of the disk world-sheet to the real axis such that $z \in \mathbb{R}$. The integration domain $D(\Pi)$ in (3.2) is then defined by $z_{\Pi(i)} < z_{\Pi(i+1)}$ corresponding to a cyclic ordering of vertex operators along the world-sheet boundary. Conformal invariance guarantees that world-sheet positions only enter through the differences $z_{ij} := z_{i,j} := z_i - z_j$, and three of them are fixed as[8]

$$z_1 = 0, \quad z_{N-1} = 1, \quad z_N \to \infty \tag{3.3}$$

in order to mod out the redundancy of the conformal Killing group (CKG) $SL(2, \mathbb{R})$ of the disk topology. The momentum dependence of the functions $F_\Pi{}^\sigma$ is carried by the dimensionless Mandelstam variables (2.4).

The functions (3.2) can be expressed as linear combinations of certain disk integrals

$$Z_\Pi(1, 2, 3, \ldots, N-1, N) := \frac{1}{\mathcal{V}_{CKG}} \prod_{i=1}^{N} \int_{D(\Pi)} dz_i \frac{\prod_{i<j}^{N} |z_{ij}|^{s_{ij}}}{z_{12} z_{23} \ldots z_{N-1,N} z_{N,1}}, \tag{3.4}$$

whose integrand is characterized by a cycle of N world-sheet propagators $(z_i - z_j)^{-1} := z_{ij}^{-1}$, which result from the superstring CFT computation[9] on the disk. In order to keep manifest the cyclic symmetry $Z_\Pi(1, 2, 3, \ldots, N-1, N) = Z_\Pi(2, 3, \ldots, N-1, N, 1)$, we do not fix the insertion points of vertex operators as in Eq. (3.3). At any rate, the inverse volume $\mathcal{V}_{CKG}$ of $SL(2, \mathbb{R})$ can be respected anytime by

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[7]   The structure of the $(N-3)!$ dimensional basis of N-point disk integrals and the formal similarity between the underlying partial-fraction and integration-by-parts identities and the KK and BCJ relations, respectively has already been investigated to some extent in [7].

[8]   Note that once we set $z_N = \infty$, the $|z_{iN}|^{s_{iN}}$ contributions to the Koba-Nielsen factor will converge to 1 by virtue of momentum conservation.

[9]   In contrast to bosonic string theory, there are no closed subcycles of z_{ij}^{-1} in the world-sheet integrand of the superstring which reflects the absence of tachyonic propagators [7, 27].

fixing three positions z_i, z_j, z_k and inserting the Jacobian $|z_{ij}z_{ik}z_{jk}|$. In Eq. (3.4) the cyclic products of world-sheet Green's functions z_{ij}^{-1} on the disk share several properties with the corresponding field-theory subamplitudes $A_{\rm YM}$. This fact, which will become clear below, explains the reason for their appearance in the $F_\Pi{}^\sigma$.

3.2 Gravity tree amplitudes versus superstring disk amplitudes

In Eq. (3.2) the integrand of the function $F_\Pi{}^\sigma$, in particular the expression in parenthesis, looks pretty complicated and requires a more intuitive understanding. For this purpose, one notes the resemblance to the structure of the KLT relations [3], which allow to write the N-graviton tree amplitudes $\mathcal{M}$ in perturbative gravity in terms of bilinears in gauge theory subamplitudes $A_{\rm YM}, \tilde{A}_{\rm YM}$:

$$\mathcal{M}(1, 2, \ldots, N) = (-1)^{N-3} \sum_{\sigma \in S_{N-3}} A_{\rm YM}(1, \sigma(2, 3, \ldots, N-2), N-1, N) \tag{3.5}$$

$$\times \sum_{\rho \in S_{N-3}} S[\,\rho(2, \ldots, N-2)\,|\,\sigma(2, \ldots, N-2)\,]_1 \, \tilde{A}_{\rm YM}(1, \rho(2, 3, \ldots, N-2), N, N-1)\,.$$

In the above equation, $S[\rho|\sigma]_1$ is the field-theory limit of the momentum kernel[10] [26], a $(N-3)! \times (N-3)!$-matrix[11] homogeneous of degree $(N-3)$ in the Mandelstam variables s_{ij}. The subscript 1 refers to the reference momentum k_1 which shows up in the entries

$$S[\,2_\rho, \ldots, (N-2)_\rho \,|\, 2_\sigma, \ldots, (N-2)_\sigma\,]_1 \;:=\; \prod_{j=2}^{N-3} \left(s_{1,j_\rho} + \sum_{k=2}^{j-1} \theta(j_\rho, k_\rho)\, s_{j_\rho, k_\rho} \right). \tag{3.6}$$

with shorthand $i_\rho := \rho(i)$. The object $\theta(j_\rho, k_\rho)$ equals 1 if the ordering of the legs j_ρ, k_ρ is the same in the ordered sets $\rho(2, \ldots, N-2)$ and $\sigma(2, \ldots, N-2)$, and zero if the ordering is opposite. In other words, it keeps track of labels which swap their relative positions in the two permutations ρ and σ.

After substituting Eq. (3.6) into Eq. (3.5) and applying partial-fraction identities in the integrand of Z_Π such as

$$\frac{1}{z_{ij}z_{ik}} + \frac{1}{z_{ji}z_{jk}} + \frac{1}{z_{ki}z_{kj}} = 0\,, \tag{3.7}$$

one finds the second line of Eq. (3.5) to match the formula (3.2) for the functions[12] $F_\Pi{}^\sigma$,

$$F_\Pi{}^\sigma = (-1)^{N-3} \sum_{\rho \in S_{N-3}} S[\,\rho(2, \ldots, N-2)\,|\,\sigma(2, \ldots, N-2)\,]_1 \, Z_\Pi(1, \rho(2, 3, \ldots, N-2), N, N-1)\,, \tag{3.8}$$

once we perform the $SL(2, \mathbb{R})$ fixing (3.4)

$$Z_\Pi(1, \rho(2, 3, \ldots, N-2), N, N-1) = \prod_{i=2}^{N-2} \int_{D(\Pi)} {\rm d}z_i \; \frac{\prod\limits_{i<j}^{N-1} |z_{ij}|^{s_{ij}}}{z_{1\rho(2)} z_{\rho(2),\rho(3)} \cdots z_{\rho(N-3),\rho(N-2)}} \tag{3.9}$$

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[10] For convenience we reversed the first permutation in comparison to the original reference [26], which turns $S[\rho|\sigma]_1$ into a symmetric matrix.

[11] In a string theory context, the momentum kernel gathers monodromy phases due to complex contour deformations on a genus-zero world-sheet which are used to derive closed string tree amplitudes relations [3, 26] and open-string subamplitudes relations [4, 5]. The $\alpha' \to 0$ limit relevant to our discussion amounts to replacing factors $\sin(\pi s_{ij})$ by the arguments of the sin functions.

[12] Note that the extended set of $(N-2)!$ functions $F_\Pi{}^{\sigma(23...N-1)}$ considered in Subsect. 2.5 of [7] accordingly follows by the action of $\sigma \in S_{N-2}$ on $\sum_{\rho \in S_{N-3}} S[2, \ldots, N-2|\rho]_1 Z_\Pi(1, \rho(2, 3, \ldots, N-2), N, N-1)$ or any integration-by-parts equivalent representation thereof.

and identify the functions Z_Π with $\tilde{A}_{\rm YM}$.

Note, that the computation of the $D = 4$ maximally helicity-violating (MHV) superstring disk amplitude quite naturally arrives at the basis (3.9) after choosing appropriate reference spinors and performing partial-fraction decompositions [27]. According to this reference, a tree-graph can be associated to the rational-function part of the integrand (3.4) and partial fractions (3.7) can be graphically described yielding the functions (3.9) as a basis. In the language of graphs the set of $(N-3)!$ integrals (3.9) represent the same Hamilton basis as introduced in [28].

In the KLT formula (3.5), the transposition of legs $N-1$ and N between $A_{\rm YM}$ and $\tilde{A}_{\rm YM}$ makes sure that these bilinears exhaust all pole channels present in the corresponding field theory amplitude. At the level of the functions $Z_\Pi(1,\ldots,N,N-1)$, the order of legs $N, N-1, 1, \ldots$ (with $N-1$ and 1 adjacent) implies that the Green's function $z_{N-1,1}^{-1}$ in the cyclic denominator cancels the factor $z_{1,N-1}$ from the $SL(2,\mathbb{R})$ Jacobian due to Eq. (3.3). The remaining rational function is then more suitable to the methods of Sect. 5.

The representation (3.8) of the functions $F_\Pi^{\,\sigma}$ in terms of Z_Π casts the open-string amplitude (2.1) into the same form as (3.5)

$$A_{\rm open}(1, \Pi(2,\ldots,N-2), N-1, N) = (-1)^{N-3} \sum_{\sigma \in S_{N-3}} A_{\rm YM}(1, \sigma(2,\ldots,N-2), N-1, N)$$

$$\times \sum_{\rho \in S_{N-3}} S[\,\rho(2,\ldots,N-2)\,|\,\sigma(2,\ldots,N-2)\,]_1 \, Z_\Pi(1, \rho(2,3,\ldots,N-2), N, N-1)\,,$$

$$(3.10)$$

where the replacement $\tilde{A}_{\rm YM}(\rho) \to Z_\Pi(\rho)$ builds up the functions (3.8) in the second line. The result (3.10) does not depend on which YM sector $\tilde{A}_{\rm YM}$ or $A_{\rm YM}$ in (3.5) is replaced by the integral Z_Π. As a consequence, the open-string amplitude (3.10) is symmetric under the exchange of the YM subamplitude $A_{\rm YM}$ and the world-sheet integral Z_Π. Note also that color ordering Π of the string amplitude is a spectator in the S_{N-3} summation over σ and ρ. That is, the open superstring amplitude $A_{\rm open}$ is totally symmetric in all indices as long as the color ordering Π remains unspecified. The formal equivalence between (3.5) and (3.10)

$$\tilde{A}_{\rm YM}(\rho) \; \simeq \; Z_\Pi(\rho) \tag{3.11}$$

makes the exchange symmetry between YM subamplitudes and disk integrals manifest. We shall point out further faces of this correspondence in Subsect. 3.4.

Let us remark, that in the four-dimensional spinor helicity formalism, MHV disk amplitudes [27] allow to establish the correspondence (3.11) even at the level of individual Green's functions z_{ij}^{-1} [28].

3.3 Examples

Let us illustrate the statements above by some examples at multiplicities $N = 4, 5$ and $N = 6$.

3.3.1 Four points

The four-point amplitude

$$A_{\rm open}(\Pi(1,2,3,4)) = -A_{\rm YM}(1,2,3,4)\, s_{12}\, Z_\Pi(1,2,4,3) \tag{3.12}$$

is governed by the integral

$$Z_\Pi(1,2,4,3) = \frac{1}{V_{\rm CKG}} \prod_{i=1}^{4} \int_{D(\Pi)} {\rm d}z_i \, \frac{\prod_{i<j}^{4} |z_{ij}|^{s_{ij}}}{z_{12} z_{24} z_{43} z_{31}} = \int_{D(\Pi)} {\rm d}z_2 \, \frac{|z_{12}|^{s_{12}}\, |z_{23}|^{s_{23}}}{z_{12}}\,. \tag{3.13}$$

In the limit $z_4 \to \infty$, the factors of $z_{24}z_{43}z_{31}$ are cancelled by the Jacobian $z_{13}z_{14}z_{34}$. Given the 1×1 momentum kernel $S[2|2]_1 = s_{12}$, we can rewrite the function $F_\Pi^{(2)}$ as

$$F_\Pi^{(2)} = -S[\,2\,|\,2\,]_1\, Z_\Pi(1,2,4,3) = -\int_{D(\Pi)} \mathrm{d}z_2\, |z_{12}|^{s_{12}}\, |z_{23}|^{s_{23}}\, \frac{s_{12}}{z_{12}}\,, \tag{3.14}$$

which is in agreement with (3.2).

3.3.2   Five points

The five-point integrand involves two permutations $2_\rho = \rho(2)$, $3_\rho = \rho(3)$ of the labels $2, 3$,

$$Z_\Pi(1,2_\rho,3_\rho,5,4) = \frac{1}{\mathcal{V}_{\mathrm{CKG}}} \prod_{i=1}^{5} \int_{D(\Pi)} \mathrm{d}z_i\, \frac{\prod_{i<j}^{5} |z_{ij}|^{s_{ij}}}{z_{12_\rho} z_{2_\rho 3_\rho} z_{3_\rho 5} z_{54} z_{41}} = \int_{D(\Pi)} \mathrm{d}z_2\, \mathrm{d}z_3\, \frac{\prod_{i<j}^{4} |z_{ij}|^{s_{ij}}}{z_{12_\rho} z_{2_\rho 3_\rho}} \tag{3.15}$$

which are tied together by the 2×2 momentum kernel

$$S[\,\rho(2,3)\,|\,\sigma(2,3)\,]_1 = \begin{pmatrix} S[\,23\,|\,23\,]_1 & S[\,32\,|\,23\,]_1 \\ S[\,23\,|\,32\,]_1 & S[\,32\,|\,32\,]_1 \end{pmatrix} = \begin{pmatrix} s_{12}(s_{13}+s_{23}) & s_{12}s_{13} \\ s_{12}s_{13} & s_{13}(s_{12}+s_{23}) \end{pmatrix} \tag{3.16}$$

Inserting these expressions into (3.8) reproduces the form (3.2) of the F_Π^σ after using partial-fraction identities:

$$F_\Pi^{(23)} = \sum_{\rho \in S_2} S[\,\rho(2,3)\,|\,2,3\,]_1\, Z_\Pi(1,2_\rho,3_\rho,5,4)$$

$$= \int_{D(\Pi)} \mathrm{d}z_2\, \mathrm{d}z_3 \prod_{i<j}^{4} |z_{ij}|^{s_{ij}}\, \frac{s_{12}}{z_{12}} \left(\frac{s_{13}}{z_{13}} + \frac{s_{23}}{z_{23}} \right) \tag{3.17}$$

$$F_\Pi^{(32)} = \sum_{\rho \in S_2} S[\,\rho(2,3)\,|\,3,2\,]_1\, Z_\Pi(1,2_\rho,3_\rho,5,4)$$

$$= \int_{D(\Pi)} \mathrm{d}z_2\, \mathrm{d}z_3 \prod_{i<j}^{4} |z_{ij}|^{s_{ij}}\, \frac{s_{13}}{z_{13}} \left(\frac{s_{12}}{z_{12}} + \frac{s_{32}}{z_{32}} \right)$$

3.3.3   Six points

The six-point string corrections are governed by the $\rho \in S_3$ basis of $(6-3)! = 6$ functions

$$Z_\Pi(1,2_\rho,3_\rho,4_\rho,6,5) = \int_{D(\Pi)} \mathrm{d}z_2\, \mathrm{d}z_3\, \mathrm{d}z_4\, \frac{\prod_{i<j}^{5} |z_{ij}|^{s_{ij}}}{z_{12_\rho} z_{2_\rho 3_\rho} z_{3_\rho 4_\rho}} \tag{3.18}$$

and the (6×6)-momentum kernel with the following entries in its first row:

$$\begin{array}{ll}
S[\,234\,|\,234\,]_1 = s_{12}(s_{13}+s_{23})(s_{14}+s_{24}+s_{34})\,, & S[\,342\,|\,234\,]_1 = s_{12}s_{13}(s_{14}+s_{34}) \\
S[\,243\,|\,234\,]_1 = s_{12}(s_{13}+s_{23})(s_{14}+s_{24})\,, & S[\,423\,|\,234\,]_1 = s_{12}(s_{13}+s_{23})s_{14} \\
S[\,324\,|\,234\,]_1 = s_{12}s_{13}(s_{14}+s_{24}+s_{34})\,, & S[\,432\,|\,234\,]_1 = s_{12}s_{13}s_{14}
\end{array}\; \cdot$$

$$(3.19)$$

Repeated use of partial-fraction identities casts the result of (3.8) into the form (3.2)

$$F_\Pi{}^\sigma = -\sum_{\rho \in S_3} S[\, \rho(2,3,4) \,|\, \sigma(2,3,4)\,]_1 \, Z_\Pi(1, 2_\rho, 3_\rho, 4_\rho, 6, 5) \tag{3.20}$$

$$= \int_{D(\Pi)} \mathrm{d}z_2 \, \mathrm{d}z_3 \, \mathrm{d}z_4 \prod_{i<j}^{5} |z_{ij}|^{s_{ij}} \, \frac{s_{12_\sigma}}{z_{2_\sigma 1}} \left(\frac{s_{13_\sigma}}{z_{13_\sigma}} + \frac{s_{2_\sigma 3_\sigma}}{z_{2_\sigma 3_\sigma}} \right) \left(\frac{s_{14_\sigma}}{z_{14_\sigma}} + \frac{s_{2_\sigma 4_\sigma}}{z_{2_\sigma 4_\sigma}} + \frac{s_{3_\sigma 4_\sigma}}{z_{3_\sigma 4_\sigma}} \right) \, .$$

Higher–point analogues of the functions consist of more and more factors $\sum_{m=1}^{k-1} \frac{s_{mk}}{z_{mk}}$ with an increasing number of terms each.

3.4 World-Sheet analogues of KK and BCJ relations

The gravity amplitude (3.5) does not have any notion of color ordering, it is a totally symmetric function w.r.t. its labels $1, 2, \ldots, N$. Although this symmetry is obscured on the right hand side of the KLT formula, it can be verified to hold through the KK relations (discovered in [1] and proven in [25]) and BCJ relations [2] obeyed by the amplitudes $A_{\mathrm{YM}}(\sigma)$ and $\tilde{A}_{\mathrm{YM}}(\rho)$. The connection between gravity amplitudes (3.5) and open-string subamplitudes (3.10) motivates to investigate whether the underlying disk integrals $Z_\Pi(\rho)$ (3.9) taking the role of $\tilde{A}_{\mathrm{YM}}(\rho)$ satisfy the same KK and BCJ relations.

As demonstrated in appendix B, the $Z_\Pi(\rho)$ at fixed color ordering Π share all the algebraic properties of $A_{\mathrm{YM}}(\rho)$ when the permutation ρ determining the integrand is varied. Firstly, they satisfy a world-sheet analogue of KK relations

$$Z_\Pi(1, \alpha, N-1, \beta) = (-1)^{|\beta|} \sum_{\sigma \in \alpha \, \sqcup\!\sqcup \, \beta^t} Z_\Pi(1, \sigma, N-1) \tag{3.21}$$

where α and β denote disjoint ordered subsets of $\{2, \ldots, N-2, N\}$ such that $\alpha \cup \beta = \{2, \ldots, N-2, N\}$, the ^{t} operation reverses the order of the elements in β, and $|\beta|$ is given by the number of elements in β. The summation range $\alpha \sqcup\!\sqcup \beta^t$ includes those permutations of $\alpha \cup \beta$ which preserve the order of elements in α and β^t. Secondly, the BCJ relations among A_{YM} [2] literally translate into

$$s_{1,N-2} \, Z_\Pi(1, N-2, 2, 3, \ldots, N, N-1)$$
$$+ \sum_{j=2}^{N-3} \sum_{k=1}^{j} s_{k,N-2} \, Z_\Pi(1, 2, \ldots, j, N-2, j+1, \ldots, N, N-1)$$
$$- s_{N-2,N-1} \, Z_\Pi(1, 2, \ldots, N-3, N, N-2, N-1) = 0 \, . \tag{3.22}$$

A detailed discussion and derivation of (3.21) and (3.22) based on world-sheet manipulations can be found in appendix B.

These analogues of KK and BCJ relations fulfilled by the integrals $Z_\Pi(\rho)$ (regardless of Π) imply that the open-string CFT correlator enjoys the same total exchange symmetry in $1, 2, \ldots, N$ as the gravity amplitude. Individual disk subamplitudes $A_{\mathrm{open}}(\Pi)$ then stem from integrating a totally symmetric correlator over a domain specified by $z_{\Pi(i)} < z_{\Pi(i+1)}$, that is, the ordering of the vertex operators along the disk boundary. The integration region determined by Π breaks the complete permutation symmetry S_N of the integrand down to a cyclic subgroup $\mathbb{Z}_N$. Hence, establishing an analogue of KK and BCJ relations among the $Z_\Pi(\rho)$ ultimately guarantees cyclic symmetry of $A_{\mathrm{open}}(\Pi)$ given by (3.10).

3.5 Different KLT representations and integration by parts

An important feature to conveniently determine the α'-expansion of the functions $F_\Pi{}^\sigma$ in the string amplitude (2.1) is the choice of a convenient basis of integrals Z_Π's. In this subsection, we will show that there are $N-2$ different S_{N-3}-families of such $Z_\Pi(\ldots)$. One is free to focus on the S_{N-3} slice of $Z_\Pi(\ldots)$ whose low-energy behavior is easiest to access (the criterion will follow in the later Sect. 4). The freedom in constructing the $F_\Pi{}^\sigma$ can be explained both by world-sheet integration by parts and by the analogy of the disk amplitude (3.10) with the KLT formula (3.5) for gravity tree amplitudes.

Contour deformation and the underlying monodromy relations [4,5] allow to rewrite the KLT formula in a variety of ways such that the number of terms in (3.5) or (3.10) can be reduced from $(N-3)! \times (N-3)!$ down to $(N-3)! \times (\nu-1)! \times (N-2-\nu)!$ for some $\nu = 1, 2, \ldots, N-2$ [3,29]. In the language of the momentum kernel [26], this gives rise to equivalent representations[13]

$$
\mathcal{M}(1, 2, \ldots, N) = (-1)^{N-3} \sum_{\sigma \in S_{N-3}} A_{\mathrm{YM}}(1, \sigma(2, 3, \ldots, N-2), N-1, N)
$$

$$
\times \sum_{\rho \in S_{\nu-1}} \sum_{\tau \in S_{N-2-\nu}} S[\,\rho(2_\sigma, \ldots, \nu_\sigma)\,|\,2_\sigma, \ldots, \nu_\sigma\,]_1
$$

$$
\times S[\,\tau((N-2)_\sigma, \ldots, (\nu+1)_\sigma)\,|\,(N-2)_\sigma, \ldots, (\nu+1)_\sigma\,]_{N-1}
$$

$$
\times \tilde{A}_{\mathrm{YM}}(1, \rho(2_\sigma, \ldots, \nu_\sigma), N, \tau^t((N-2)_\sigma, \ldots, (\nu+1)_\sigma), N-1)\,, \qquad (3.23)
$$

for all values $\nu = 1, \ldots, N-2$. Eq. (3.5) corresponds to taking $\nu = N-2$ in (3.23). At odd multiplicity N, the most economic choice $\nu = \frac{1}{2}(N-1)$ w.r.t. the number of terms makes reflection symmetry manifest: At five points, $\nu = 2$ yields a two-term representation

$$
\mathcal{M}(1, 2, \ldots, 5) = \sum_{\sigma \in S_2} A_{\mathrm{YM}}(1, \sigma(2, 3), 4, 5)\, S[\,2_\sigma\,|\,2_\sigma\,]_1\, S[\,3_\sigma\,|\,3_\sigma\,]_4\, \tilde{A}_{\mathrm{YM}}(1, 2_\sigma, 5, 3_\sigma, 4)\,, \quad (3.24)
$$

with $S[2_\sigma|2_\sigma]_1 = s_{12_\sigma}$ and $S[3_\sigma|3_\sigma]_4 = s_{3_\sigma 4}$, and the seven-point amplitude at $\nu = 3$ takes the form

$$
\mathcal{M}(1, 2, \ldots, 7) = \sum_{\sigma \in S_4} A_{\mathrm{YM}}(1, \sigma(2, 3, 4, 5), 6, 7) \sum_{\rho \in S_2} S[\,\rho(2_\sigma, 3_\sigma)\,|\,2_\sigma, 3_\sigma\,]_1
$$

$$
\times \sum_{\tau \in S_2} S[\,\tau(5_\sigma, 4_\sigma)\,|\,5_\sigma, 4_\sigma\,]_6\, \tilde{A}_{\mathrm{YM}}(1, \rho(2_\sigma, 3_\sigma), 7, \tau^t(5_\sigma, 4_\sigma), 6)\,, \qquad (3.25)
$$

with 24×4 terms rather than 24×24. Even multiplicities, on the other hand, leave two values $\nu = \frac{1}{2}N-1, \frac{1}{2}N$ for the minimal number of terms in (3.23).

The dictionary (3.11) between the supergravity amplitude $\mathcal{M}$ and the open superstring amplitude $A_{\mathrm{open}}(\Pi)$ maps the freedom in the choice of the parameter ν in (3.23) to world-sheet integrations by parts. By discarding a total z_{N-2} derivative, we can reduce the number of terms in the representation (3.2) of $F_\Pi{}^\sigma$ by $N-3$

$$
\int_{D(\Pi)} \mathrm{d}z_{N-2} \prod_{i<j}^{N-1} |z_{ij}|^{s_{ij}} \left(\frac{s_{1,N-2}}{z_{1,N-2}} + \ldots + \frac{s_{N-3,N-2}}{z_{N-3,N-2}} \right) = \int_{D(\Pi)} \mathrm{d}z_{N-2} \prod_{i<j}^{N-1} |z_{ij}|^{s_{ij}} \frac{s_{N-2,N-1}}{z_{N-2,N-1}}\,.
$$

$$(3.26)$$

This results in moving from case $\nu = N-2$ to case $\nu = N-3$. Generically, for any $\nu = 1, \ldots, N-2$ we have:

---

[13] Similar to (3.21), the permutation τ^t is related to τ by reversal of its elements.

$$A_{\text{open}}(1, \Pi(2, \ldots, N-2), N-1, N) = (-1)^{N-3} \sum_{\sigma \in S_{N-3}} A_{\text{YM}}(1, \sigma(2, 3, \ldots, N-2), N-1, N)$$

$$\times \sum_{\rho \in S_{\nu-1}} \sum_{\tau \in S_{N-2-\nu}} S[\rho(2_\sigma, \ldots, \nu_\sigma) \,|\, 2_\sigma, \ldots, \nu_\sigma]_1$$

$$\times S[\tau((N-2)_\sigma, \ldots, (\nu+1)_\sigma) \,|\, (N-2)_\sigma, \ldots, (\nu+1)_\sigma]_{N-1}$$

$$\times Z_\Pi(1, \rho(2_\sigma, \ldots, \nu_\sigma), N, \tau^t((N-2)_\sigma, \ldots, (\nu+1)_\sigma), N-1) \,. \tag{3.27}$$

Note, that the Green's functions in $Z_\Pi(1, \rho(2_\sigma, \ldots, \nu_\sigma), N, \tau^t((N-2)_\sigma, \ldots, (\nu+1)_\sigma), N-1)$ factorize into $\rho(z_{12_\sigma} \ldots z_{(\nu-1)_\sigma, \nu_\sigma})^{-1}$ and $\tau(z_{(\nu+1)_\sigma, (\nu+2)_\sigma} \ldots z_{(N-2)_\sigma, N-1})^{-1}$ after $SL(2, \mathbb{R})$ fixing. Carrying out the ρ-sum in (3.27) yields

$$\sum_{\rho \in S_{\nu-1}} \frac{S[\rho(2_\sigma, \ldots, \nu_\sigma) \,|\, 2_\sigma, \ldots, \nu_\sigma]_1}{\rho(z_{12_\sigma} \ldots z_{(\nu-1)_\sigma, \nu_\sigma})} = \sigma \left(\prod_{k=2}^{\nu} \sum_{m=1}^{k-1} \frac{s_{mk}}{z_{mk}} \right) \,. \tag{3.28}$$

Eventually, after repeating the manipulation (3.26) for derivatives in $z_{N-3}, z_{N-4}, \ldots, z_{\lfloor N/2 \rfloor + 1}$ casts the function $F_\Pi{}^\sigma$ into a form with a minimal number of terms at $\nu = \lfloor N/2 \rfloor$:

$$F_\Pi{}^\sigma = (-1)^{N-3} \sum_{\rho \in S_{\lfloor N/2 \rfloor - 1}} S[\rho(2_\sigma, \ldots, \lfloor N/2 \rfloor_\sigma) | 2_\sigma, \ldots, \lfloor N/2 \rfloor_\sigma]_1$$

$$\times \sum_{\tau \in S_{\lceil N/2 \rceil - 2}} S[\tau((N-2)_\sigma, \ldots, (\lfloor N/2 \rfloor + 1)_\sigma) | (N-2)_\sigma, \ldots, (\lfloor N/2 \rfloor + 1)_\sigma]_{N-1}$$

$$\times Z_\Pi(1, \rho(2_\sigma, \ldots, \lfloor N/2 \rfloor_\sigma), N, \tau^t((N-2)_\sigma, \ldots, (\lfloor N/2 \rfloor + 1)_\sigma), N-1) \tag{3.29}$$

$$= (-1)^{N-3} \prod_{i=2}^{N-2} \int_{D(\Pi)} dz_i \prod_{i<j}^{N-1} |z_{ij}|^{s_{ij}} \sigma \left\{ \left(\prod_{k=2}^{\lfloor N/2 \rfloor} \sum_{m=1}^{k-1} \frac{s_{mk}}{z_{mk}} \right) \left(\prod_{k=\lfloor N/2 \rfloor + 1}^{N-2} \sum_{n=k+1}^{N-1} \frac{s_{kn}}{z_{kn}} \right) \right\} \,.$$

At five points, e.g. the representation for $\nu = 2$

$$F_\Pi{}^\sigma = \int_{D(\Pi)} dz_2 \, dz_3 \prod_{i<j}^{4} |z_{ij}|^{s_{ij}} \frac{s_{12_\sigma}}{z_{12_\sigma}} \frac{s_{3_\sigma 4}}{z_{3_\sigma 4}} \quad : \nu = 2 \tag{3.30}$$

involves fewer terms than the representations of the basis functions for $\nu = 1, 3$:

$$F_\Pi{}^\sigma = \begin{cases} \displaystyle\int_{D(\Pi)} dz_2 \, dz_3 \prod_{i<j}^{4} |z_{ij}|^{s_{ij}} \frac{s_{12_\sigma}}{z_{12_\sigma}} \left(\frac{s_{13_\sigma}}{z_{13_\sigma}} + \frac{s_{2_\sigma 3_\sigma}}{z_{2_\sigma 3_\sigma}} \right) \quad : \nu = 3 \,, \\[4ex] \displaystyle\int_{D(\Pi)} dz_2 \, dz_3 \prod_{i<j}^{4} |z_{ij}|^{s_{ij}} \left(\frac{s_{2_\sigma 3_\sigma}}{z_{2_\sigma 3_\sigma}} + \frac{s_{2_\sigma 4}}{z_{2_\sigma 4}} \right) \frac{s_{3_\sigma 4}}{z_{3_\sigma 4}} \quad : \nu = 1 \,. \end{cases} \tag{3.31}$$

As we can see in (3.29), the freedom in rewriting the KLT-like formula for A_{open} enables to reconstruct the $F_\Pi{}^\sigma$ from any set of $(N-3)!$ functions $Z_\Pi(1, 2_\sigma, \ldots, \nu_\sigma, N, (\nu+1)_\sigma, \ldots, (N-2)_\sigma, N-1)$ with $\nu = 1, 2, \ldots, N-2$. The situation is summarized in Fig. 1.

3.6 Basis expansion of individual integrals

Even though the general aim of this article is the construction of basis functions $F_\Pi{}^\sigma$ from disk integrals $Z_\Pi(1, \ldots, N-1)$ as shown in Eq. (3.29), we shall now invert these relations to facilitate the translation

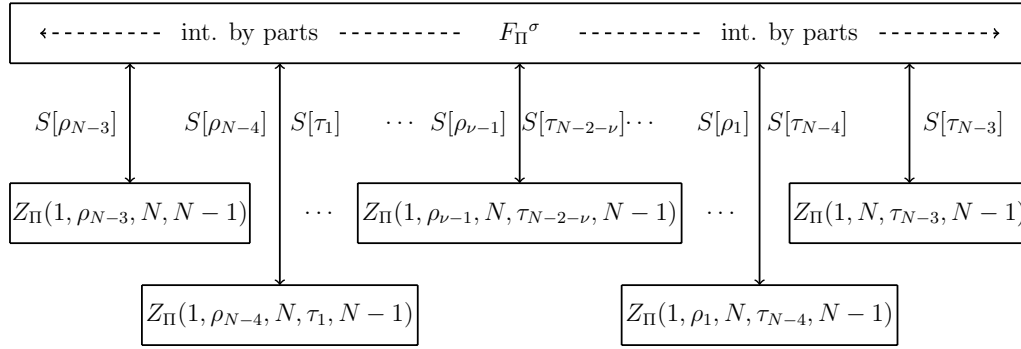**Fig. 1** Each of the $N-2$ integration-by-parts equivalent representations of the basis functions $F_\Pi{}^\sigma$ can be mapped to another $(N-3)!$ basis of KK integrals $Z_\Pi(1,\ldots,N-1)$. Depending on the position $\nu+1$ of leg N in the Z_Π, the transformation matrix is given by a product of $(\nu-1)$- and $(N-2-\nu)$-particle momentum kernels.

between the two types of objects. The expansion of KK basis integrals in terms of $F_\Pi{}^\sigma$ introduces the inverse of the momentum kernel whose entries involve $N-3$ simultaneous poles in Mandelstam invariants. The four-point case $S^{-1}[2|2]_1 = s_{12}^{-1}$ extends as follows to five and six points:

$$
\begin{pmatrix} S^{-1}[\,23\,|\,23\,]_1 & S^{-1}[\,32\,|\,23\,]_1 \\ S^{-1}[\,23\,|\,32\,]_1 & S^{-1}[\,32\,|\,32\,]_1 \end{pmatrix} = \frac{1}{s_{123}} \begin{pmatrix} \frac{1}{s_{12}} + \frac{1}{s_{23}} & -\frac{1}{s_{23}} \\ -\frac{1}{s_{23}} & \frac{1}{s_{13}} + \frac{1}{s_{23}} \end{pmatrix}
\tag{3.32}
$$

$$
S^{-1}[\,234\,|\,234\,]_1 = \frac{1}{s_{1234}} \left(\frac{1}{s_{234}} \left(\frac{1}{s_{23}} + \frac{1}{s_{34}} \right) + \frac{1}{s_{12}s_{34}} + \frac{1}{s_{123}} \left(\frac{1}{s_{12}} + \frac{1}{s_{23}} \right) \right)
$$

$$
S^{-1}[\,243\,|\,234\,]_1 = -\frac{1}{s_{1234}s_{34}} \left(\frac{1}{s_{12}} + \frac{1}{s_{234}} \right), \qquad S^{-1}[\,342\,|\,234\,]_1 = -\frac{1}{s_{1234}s_{34}s_{234}}
$$

$$
S^{-1}[\,324\,|\,234\,]_1 = -\frac{1}{s_{1234}s_{23}} \left(\frac{1}{s_{123}} + \frac{1}{s_{234}} \right), \qquad S^{-1}[\,423\,|\,234\,]_1 = -\frac{1}{s_{1234}s_{23}s_{234}}
$$

$$
S^{-1}[\,432\,|\,234\,]_1 = \frac{1}{s_{1234}s_{234}} \left(\frac{1}{s_{23}} + \frac{1}{s_{34}} \right).
\tag{3.33}
$$

These combinations of $(N-3)$-fold poles govern the inverse of (3.29),

$$
Z_\Pi(1,2_\sigma,\ldots,\nu_\sigma,N,(\nu+1)_\sigma,\ldots,(N-2)_\sigma,N-1) = (-1)^{N-3} \sum_{\rho \in S_{N-3}} F_\Pi{}^{\rho(2_\sigma,3_\sigma,\ldots,(N-2)_\sigma)}
$$

$$
\times\, S^{-1}[2_\sigma,\ldots,\nu_\sigma\,|\,\rho(2_\sigma,\ldots,\nu_\sigma)]_1\, S^{-1}[(N-2)_\sigma,\ldots,(\nu+1)_\sigma\,|\,\rho((N-2)_\sigma,\ldots,(\nu+1)_\sigma)]_{N-1}\,,
\tag{3.34}
$$

where $S^{-1}[\alpha|\beta]_j = 0$ unless α is a permutation of β, that is, the inverse momentum kernel can only take nonzero values if the same set of labels appears in its two slots.

Let us illustrate (3.34) by two examples:

- At $N=5$ and $\nu=2$, we have

$$
Z_\Pi(1,2_\sigma,5,3_\sigma,4) = \sum_{\rho \in S_2} F_\Pi{}^{\rho(2_\sigma,3_\sigma)}\, S^{-1}[2_\sigma\,|\,\rho(2_\sigma)]_1\, S^{-1}[3_\sigma\,|\,\rho(3_\sigma)]_4 = \frac{F_\Pi{}^{\sigma(23)}}{s_{12_\sigma}s_{3_\sigma4}}
\tag{3.35}
$$

since the second term with $\rho(2_\sigma,3_\sigma) = (3_\sigma,2_\sigma)$ yields $S^{-1}[2_\sigma\,|\,3_\sigma]_1 = S^{-1}[3_\sigma\,|\,2_\sigma]_4 = 0$.

- At $N = 6$ and $\nu = 1$, on the other hand,

$$Z_\Pi(1, 6, 2_\sigma, 3_\sigma, 4_\sigma, 5) = -\sum_{\rho \in S_3} F_\Pi^{\rho(2_\sigma, 3_\sigma, 4_\sigma)} \, S^{-1}[4_\sigma 3_\sigma 2_\sigma \,|\, \rho(4_\sigma, 3_\sigma, 2_\sigma)]_5$$

$$= -\frac{F_\Pi^{\sigma(234)}}{s_{2345}} \left(\frac{1}{s_{3_\sigma 4_\sigma} s_{3_\sigma 4_\sigma 5}} + \frac{1}{s_{4_\sigma 5} s_{3_\sigma 4_\sigma 5}} + \frac{1}{s_{2_\sigma 3_\sigma} s_{4_\sigma 5}} + \frac{1}{s_{3_\sigma 4_\sigma} s_{234}} + \frac{1}{s_{2_\sigma 3_\sigma} s_{234}} \right)$$

$$+ \frac{F_\Pi^{\sigma(243)}}{s_{2345} s_{3_\sigma 4_\sigma}} \left(\frac{1}{s_{234}} + \frac{1}{s_{3_\sigma 4_\sigma 5}} \right) + \frac{F_\Pi^{\sigma(324)}}{s_{2345} s_{2_\sigma 3_\sigma}} \left(\frac{1}{s_{234}} + \frac{1}{s_{4_\sigma 5}} \right) + \frac{F_\Pi^{\sigma(342)}}{s_{2345} s_{234} s_{3_\sigma 4_\sigma}}$$

$$+ \frac{F_\Pi^{\sigma(423)}}{s_{2345} s_{234} s_{2_\sigma 3_\sigma}} - \frac{F_\Pi^{\sigma(432)}}{s_{2345} s_{234}} \left(\frac{1}{s_{2_\sigma 3_\sigma}} + \frac{1}{s_{3_\sigma 4_\sigma}} \right). \tag{3.36}$$

We will make use of the relation (3.34) in assembling the final results in Sect. 6 below.

4 Kinematic poles

In the previous section we have seen how the string corrections $F_\Pi^\sigma(\alpha')$ to disk amplitudes are built from generalized Euler integrals $Z_\Pi(\rho)$ with $\rho \in S_N$. Their integrands are characterized by the totally symmetric Koba-Nielsen factor $\prod_{i<j}^N |z_{ij}|^{s_{ij}}$ decorated by a cyclic product $\rho(z_{12} z_{23} \ldots z_{N-1,N} z_{N,1})^{-1}$ of Green's functions (on the sphere). The latter endows the functions $Z_\Pi(\rho)$ with the same algebraic properties as the YM amplitudes $A_{\rm YM}(\rho)$. In particular, the freedom due to world-sheet integration by parts allows to assemble the complete basis of $F_\Pi^\sigma(\alpha')$ from each of the $N - 2$ different sectors of $Z_\Pi(\rho)$, classified by $\nu = 1, 2, \ldots, N - 2$:

$$\left\{ Z_\Pi(1, 2_\rho, \ldots, \nu_\rho, N, (\nu+1)_\rho, \ldots, (N-2)_\rho, N-1), \ \rho \in S_{N-3} \right\}, \quad \nu = 1, 2, \ldots, N - 2. \tag{4.1}$$

The multiple resonance exchange in an N-point scattering process is reflected in up to $N - 3$ simultaneous poles in the Mandelstam variables. They are the source of field theory propagators within disk amplitudes (or, equivalently, non-vanishing field theory limits for some of the F_Π^σ) and pose a major complication in performing an α'-expansion of the Z_Π. The identification and classification of pole channels has been thoroughly explained in [7]. In the following, we will review selected aspects thereof and formulate the resulting prescriptions to directly read off the pole structure from the permutations Π, ρ characterizing $Z_\Pi(\rho)$. Moreover, we will investigate the α'-expansion of pole residues and reveal their recursive structure. Massless poles occur in integration regions where several neighboring vertex operators collide. So far, we have left the disk ordering Π unspecified since it has no impact on the structure and possible manipulations of the integrand. Kinematic poles of the integrated amplitude, however, are sensitive to the color ordering. Hence, we choose the canonical disk ordering $\Pi = (1, 2, \ldots, N)$ in the following (keeping the usual $SL(2, \mathbb{R})$ fixing $z_1 = 0$, $z_{N-1} = 1$, $z_N \to \infty$) and omit the Π subscripts of $Z_\Pi(\rho)$ and F_Π^σ. The integration domain $0 = z_1 \leq z_2 \leq z_3 \leq \ldots \leq z_{N-2} \leq z_{N-1} = 1$ allows for singularities in the multiparticle Mandelstam variables $s_{i,i+1\ldots,j-1,j}$ defined by (2.4) from the regime $z_{i+1}, z_{i+2}, \ldots, z_{j-1}, z_j \to z_i$ for $1 \leq i < j \leq N - 1$. Momentum conservation $s_{j+1,\ldots,N,1,\ldots,i-1} = s_{i,i+1\ldots,j-1,j}$ guarantees that we can avoid explicit reference to the momentum k_N from the vertex operator at $z_N \to \infty$ without omitting any pole channel.

The first Subsect. 4.1 provides a criterion to find the complete set of pole channels present in the individual $Z(\rho)$ of the form (4.1). Subsect. 4.2 then explains how to reduce such $Z(\rho)$ to integrals whose low-energy limit reproduces the propagators of a *single* diagram from the underlying YM field theory comprising cubic vertices only (we shall refer to such diagrams as 'cubic' below). Factorization properties imply that the residues of N-point amplitudes on a k-fold massless pole are given by (off-shell) $(N - k)$-point amplitudes (or products of p lower-point amplitudes with overall multiplicity $N - k + 3(p - 1)$). The manifestation of factorization at the level of integrals and the systematics of the residues' recursive structure is presented in Subsect. 4.3.

4.1 Identifying pole channels

The kinematic poles present in a given $Z(\rho)$ are not immediately obvious from the integrand. Thus we will give a criterion (4.6) below which relates the pole structure of an integral $Z(\rho)$ with the ρ-dependent integrand, see (3.9). We address this problem for the complete KK basis[14], firstly for the sake of generality, and secondly to gather evidence in favor of the $\nu = \lfloor N/2 \rfloor$ slice of (4.1) which leads to a complete F^σ basis in the most economic way. Once we have stripped off the Koba-Nielsen factor, one can refer to the Green's functions in the integrand by a polynomial $R^{\nu,\rho}(z_{ij})$ of degree $(N-3)$:

$$Z(1, 2_\rho, \ldots, \nu_\rho, N, (\nu+1)_\rho, \ldots, (N-2)_\rho, N-1) = \prod_{i=2}^{N-2} \int_0^{z_{i+1}} \mathrm{d}z_i \prod_{i<j}^{N-1} |z_{ij}|^{s_{ij}} \frac{1}{R^{\nu,\rho}(z_{ij})}$$

$$R^{\nu,\rho}(z_{ij}) := (z_{12_\rho} z_{2_\rho 3_\rho} \ldots z_{(\nu-1)_\rho, \nu_\rho}) \times (z_{(\nu+1)_\rho, (\nu+2)_\rho} \ldots z_{(N-2)_\rho, N-1}) . \quad (4.2)$$

Note that the world-sheet parity transformation $(z_i, k_i \mapsto z_{N-i}, k_{N-i})$ for $i = 1, 2, \ldots, N-1$ preserves the form of (4.2) including the $SL(2, \mathbb{R})$ fixing and exchanges S_{N-3} sectors of the KK basis associated with ν and $N-1-\nu$.

The following change of integration variables allows to conveniently probe several pole channels which will turn out to be sufficient for finding the general criterion (4.6):

$$z_2 = x_1 x_2 \ldots x_{N-3} , \qquad z_3 = x_2 x_3 \ldots x_{N-3} , \qquad \ldots \qquad , \qquad z_{N-2} = x_{N-3} . \quad (4.3)$$

This transformation maps the integration region from a simplex $0 \le z_2 \le z_3 \le \ldots \le z_{N-2} \le 1$ to the unit cube $0 \le x_i \le 1$, and the integration measure becomes

$$\prod_{i=2}^{N-2} \int_0^{z_{i+1}} \mathrm{d}z_i \prod_{i<j}^{N-1} |z_{ij}|^{s_{ij}}$$

$$= \prod_{i=1}^{N-3} \int_0^1 \mathrm{d}x_i \prod_{j=1}^{N-3} x_j^{j-1+s_{12\ldots j+1}} (1-x_j)^{s_{j+1,j+2}} \prod_{l=j+1}^{N-3} \left(1 - \prod_{k=j}^{l} x_k\right)^{s_{j+1,l+2}} \quad (4.4)$$

with $\prod_{j=1}^{N-3} x_j^{j-1}$ emerging from the Jacobian $\left|\frac{\partial z_i}{\partial x_j}\right|$. Including the polynomials $R^{\nu,\rho}(z_{ij})$ of (4.2) into (4.4) effectively shifts $s_{ij} \mapsto s_{ij} - 1$ for each z_{ij}-factor present in $R^{\nu,\rho}$, see e.g. [7]. This is crucial for identifying kinematic poles: For functions f, g which are regular at $x = 0$ and $x = 1$, respectively, we find the following behavior[15]

$$\int_0^a \mathrm{d}x\, x^{s-1} f(x) = \frac{f(0)}{s} + \mathcal{O}(s^0) , \qquad \int_{1-b}^1 \mathrm{d}x\, (1-x)^{s-1} g(x) = \frac{g(1)}{s} + \mathcal{O}(s^0) \quad (4.5)$$

due to the limits of the integration range. Hence, the exponents $j-1+s_{12\ldots j+1}$ and $s_{j+1,j+2}$ of the x_j and $(1-x_j)$ factors in the integrand (4.4) trigger propagators $s_{12\ldots j+1}^{-1}$ and $s_{j+1,j+2}^{-1}$, whenever presence of $R^{\nu,\rho}$ shifts them to $x_j^{s_{12\ldots j+1}-1}$ and $(1-x_j)^{s_{j+1,j+2}-1}$, respectively:

- $(j+1)$-particle pole channels $s_{12\ldots j+1}^{-1}$ encompassing legs $1, 2, \ldots, j+1 \le N-2$ emerge from the $x_j \to 0$ regime of the integral (4.4) whenever $R^{\nu,\rho}$ covers j factors of z_{pq} with $1 \le p < q \le j+1$

- two-particle channels $s_{j+1,j+2}^{-1}$ emerge as $x_j \to 1$ whenever $R^{\nu,\rho} \sim z_{j+1,j+2}$

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[14] We want to emphasize that any other integral beyond the KK basis $\{Z(1, \rho(2, 3, \ldots, N-2, N), N), \rho \in S_{N-2}\}$ can be brought into the form Eq. (4.2) by means of KK relations Eq. (3.21). So there is no loss of generality in restricting the discussion to integrals of the form (4.2) with a degree $N-3$ polynomial $R^{\nu,\rho}$ in z_{ij}.

[15] This is a consequence of the δ-function representation $\delta(x) = \lim_{s\to 0} s x^{s-1}$.

The factors of $(1 - x_j x_{j+1} \ldots x_l)^{s_{j+1,l+2}}$ in (4.4) with $l > j$ or potential $z_{j+1,l+2}^{-1}$ admixtures from $R^{\nu,\rho}$ do not affect these poles. They take the role of the functions f, g in (4.5) with regular behavior in the critical range of integration.

This criterion to find the pole channels can neither depend on the particular $j + 1$-particle channel $s_{12\ldots j+1}, s_{23\ldots j+2}, \ldots, s_{N-j-1\ldots N-1}$ in question nor on the parametrization of the disk boundary. Hence, we can lift the above correlation between $s_{12\ldots j+1}$ poles and $R^{\nu,\rho}$ to cyclic images $s_{k,k+1,\ldots j+k}$ with $k \leq N - 1$ and formulate the following general criterion[16] for massless poles:

> *Integrals Z of type (4.2) contribute to the $(j - i + 1)$-particle pole channel $s_{i,i+1\ldots j-1,j}$*
> *with $1 \leq i < j \leq N - 1$ if $R^{\nu,\rho}$ contains $(j - i)$ factors of z_{pq} in the range $i \leq p < q \leq j$.*   (4.6)

The pole channels accessible in Eq. (4.4) additionally point out *incompatible* (or dual) channels[17] [7,30]: The two pole channels $s_{12\ldots j+1}$ and $s_{j+1,j+2}$ are caused by complementary integration regions $x_j \to 0$ and $x_j \to 1$, respectively, this is why a simultaneous pole in $s_{12\ldots j+1}$ and $s_{j+1,j+2}$ cannot appear. Again, this statement must be independent on the label 1 bounding the multi-particle channel, so we conclude that the pairs $(s_{i,i+1}, s_{i+1,i+2\ldots j-1,j})$ and $(s_{i,i+1\ldots j-1,j}, s_{j,j+1})$ are incompatible pole channel for any $1 \leq i < j \leq N - 1$ and always appear in separate denominators (see Eq. (C.5) below for an example).

Examples for the pole criterion (4.6) are given in appendix C.

4.1.1   Momentum kernel representation of field theory limits

We can cast the results of the pole analysis (4.6) into a closed form by demanding a consistent field theory limit for the disk amplitude (2.1): The pole structure of the functions (4.2) must be compatible with

$$F_\Pi{}^\sigma \Big|_{\alpha' \to 0} = \delta_\Pi^\sigma \ . \tag{4.7}$$

The representation (3.34) for $F_\Pi{}^\sigma$ in terms of inverse momentum kernels yields

$$Z_\Pi(1, 2_\sigma, \ldots, \nu_\sigma, N, (\nu+1)_\sigma, \ldots, (N-2)_\sigma, N-1) \Big|_{\alpha' \to 0} = (-1)^{N-3} \tag{4.8}$$

$$\times S^{-1}[2_\sigma, \ldots, \nu_\sigma \,|\, \Pi(2, \ldots, \nu)]_1 \, S^{-1}[(N-2)_\sigma, \ldots, (\nu+1)_\sigma \,|\, \Pi(N-2, \ldots, \nu+1)]_{N-1} \ .$$

Like in Subsect. 3.6, the entry $S^{-1}[\alpha|\beta]$ is bound to vanish unless α is a permutation of β.

4.2   From a KK basis to a pole channel basis

We would like to take advantage of the rule (4.6) identifying pole channels to efficiently construct the α'-expansion of an integral basis. The examples of appendix C show that generic KK integrals contribute to several cubic diagrams of YM field theory. In the later Subsect. 4.3, we aim to identify the residues for less singular string corrections to YM diagrams where only a subset of the $N - 3$ poles remain (the others are said to be collapsed in the following). Since there is a total of 2^{N-3} such subsets per cubic diagram, it is preferable to rearrange the integrals such that they generate only one cubic diagram each, even on the expense of leaving the KK basis (4.2). So we will start with the KK basis and rewrite its elements by means of partial-fraction relations in order to obtain a pole channel basis – another $(N - 2)!$ set of integrals which

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[16] Although the rule (4.6) is formulated for a particular $SL(2, \mathbb{R})$ fixed version of the integrals it does not depend on this choice: in the picture $\frac{1}{V_{CKG}} \prod_{i=1}^N \int dz_i \ldots$ with unspecified $SL(2, \mathbb{R})$ fixing, the S_{N-3} basis of $Z(1, \ldots, N, N-1)$ at $\nu = N-2$ exhibits a pole in $s_{N-1,N}$ from the $(z_{N-1,N})^{s_{N-1,N}-1}$ part of the integrand. Fixing $z_{N-1} = 1$ and $z_N \to \infty$ appears to prevent these two positions from colliding, but the rule (4.6) reveals $s_{N-1,N} = s_{12\ldots N-2}$ as an $(N - 2)$-particle channel caused by the $N - 3$ factors of z_{pq} with $1 \leq p < q \leq N - 2$ in the associated polynomial $R^{\nu=N-2,\rho}$. A similar argument applies to the $s_{N,1} = s_{23\ldots N-1}$ pole present in any $Z(1, N, \ldots, N-1)$.

[17] Compatibility conditions for two channels involving ≥ 3 particles are not directly obvious from Eq. (4.4). One can however show by other means that they are compatible if they can appear as simultaneous propagators in a cubic diagram.

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by construction involve at most one field theory diagram. Subsectors of different $\nu = 1, 2, \ldots, N - 2$ do not mix under these partial-fraction manipulations.

In order to determine the function associated with a particular field theory diagram which is closest to the KK basis (4.2), one is faced with the inverse problem to the pole detection rule (4.6): For the given cubic diagram characterized by $N-3$ compatible pole channels $s_{a_i, a_i+1, \ldots, b_i-1, b_i}$ at $1 \leq a_i < b_i \leq N-1$ for $i = 1, 2, \ldots, N - 3$, we want to construct a polynomial[18] $R[a_i, b_i] \sim (z_{pq})^{N-3}$ such that the integral

$$Z[s_{a_i, a_i+1, \ldots, b_i-1, b_i}] := \prod_{i=2}^{N-2} \int_0^{z_{i+1}} \mathrm{d}z_i \prod_{i<j}^{N-1} |z_{ij}|^{s_{ij}} \frac{1}{R[a_i, b_i]} \tag{4.9}$$

contributes to those $N - 3$ channels exclusively. Indeed, we can verify using (4.6) that the desired polynomial is given by

$$R[a_i, b_i] = \prod_{i=1}^{N-3} z_{b_i, a_i} , \tag{4.10}$$

where each multiparticle Mandelstam variable is associated with a factor z_{pq} according to its bounding momenta k_{a_i} and k_{b_i}, for example $s_{a_i, a_i+1, \ldots, b_i} \mapsto z_{b_i, a_i}$. In other words, the diagram with poles $\prod_{i=1}^{N-3} s_{a_i, a_i+1, \ldots, b_i}$ can be generated by the function

$$Z[s_{a_i, a_i+1, \ldots, b_i-1, b_i}] = \prod_{i=2}^{N-2} \int_0^{z_{i+1}} \mathrm{d}z_i \prod_{i<j}^{N-1} |z_{ij}|^{s_{ij}} \prod_{i=1}^{N-3} \frac{1}{z_{b_i, a_i}}$$

$$\sim \prod_{i=1}^{N-3} \frac{1}{s_{a_i, a_i+1, \ldots, b_i-1, b_i}} . \tag{4.11}$$

In our setup, an integral $Z(\ldots)$ involved in the construction of the F^σ-basis must be written as a linear combinations of functions (4.11). We shall see in the following examples that this can always be achieved through repeated use of partial-fraction identities.

As mentioned in appendix C.3, the KK basis of $Z(1, \ldots, N - 1)$ also incorporates functions with $k \leq N - 5$ simultaneous poles whose α'-expansion starts at transcendentality ζ_{N-3-k}. In this case, the mapping $s_{a_i, a_i+1, \ldots, b_i} \mapsto z_{b_i, a_i}$ for $k \leq N-5$ pole channels leaves $N-3-k$ factors of z_{pq} undetermined. It is desirable to complete the pole basis in a way such that the minimal number of poles need to be considered in performing the α'-expansion. On the other hand, this guideline can conflict with the form of the integrand Eq. (5.1) required by the integration methods of Sect. 5, we will come back to this point later on.

We should point out that higher α'-corrections of the functions constructed from (4.11) do not transform into each other under a shift $i \mapsto i + 1$ subject to cyclic identifications $i := i + 1$. This is a consequence of the fact that neither the set of $R[a_i, b_i]$ nor the KK basis (4.2) is preserved under cyclic shifts.

4.2.1 Five-point examples

The four-point KK basis functions Eqs. (C.1) and (C.2) each generate no more than one cubic diagram, $Z[s_{12}] = -Z(1, 2, 4, 3)$ and $Z[s_{23}] = -Z(1, 4, 2, 3)$, so there is no need to modify the basis. Starting from five points, the map (4.11) from cubic diagrams to disk integrals leads to functions beyond the KK basis elements. According to (C.3) and (C.5), we find

$$Z[s_{23}s_{123}] = \int_0^1 \mathrm{d}z_3 \int_0^{z_3} \mathrm{d}z_2 \frac{\prod_{i<j}^4 |z_{ij}|^{s_{ij}}}{z_{31}z_{32}} = -Z(1, 3, 2, 5, 4)$$

---

[18] We have to demand polynomial form $\sim (z_{pq})^{N-3}$ for the $R[a_i, b_i]$ because rational functions $\sim (z_{pq})^{N-2}/z_{rs}$ of the same degree are not accessible to the integration methods of Sect. 5.

$$Z[s_{23}s_{234}] = \int_0^1 dz_3 \int_0^{z_3} dz_2 \frac{\prod_{i<j}^4 |z_{ij}|^{s_{ij}}}{z_{32}z_{42}} = -Z(1,5,3,2,4) \tag{4.12}$$

whereas the diagrams with poles in $s_{12}s_{123}$ and $s_{23}s_{234}$ require a linear combination of KK functions,

$$Z[s_{12}s_{123}] = \int_0^1 dz_3 \int_0^{z_3} dz_2 \frac{\prod_{i<j}^4 |z_{ij}|^{s_{ij}}}{z_{21}z_{31}} = Z(1,3,2,5,4) + Z(1,2,3,5,4)$$

$$Z[s_{34}s_{234}] = \int_0^1 dz_3 \int_0^{z_3} dz_2 \frac{\prod_{i<j}^4 |z_{ij}|^{s_{ij}}}{z_{42}z_{43}} = Z(1,5,3,2,4) + Z(1,5,2,3,4). \tag{4.13}$$

Partial fraction manipulations mediate between the KK basis and the pole basis containing all the functions generated by (4.11). This change of basis acts block-diagonally on the S_2 sectors labeled by $\nu = 1,2,3$ (with self-inverse 2×2 blocks):

$$\begin{pmatrix} Z(1,2,3,5,4) \\ Z(1,3,2,5,4) \\ Z(1,2,5,3,4) \\ Z(1,3,5,2,4) \\ Z(1,5,2,3,4) \\ Z(1,5,3,2,4) \end{pmatrix} = \begin{pmatrix} 1 & 1 & & & & \\ 0 & -1 & & & & \\ & & 1 & 0 & & \\ & & 0 & 1 & & \\ & & & & 1 & 1 \\ & & & & 0 & -1 \end{pmatrix} \begin{pmatrix} Z[s_{12}s_{123}] \\ Z[s_{23}s_{123}] \\ Z[s_{12}s_{34}] \\ Z[\zeta_2] \\ Z[s_{34}s_{234}] \\ Z[s_{23}s_{234}] \end{pmatrix}. \tag{4.14}$$

The $\nu = 2$ elements of the KK basis are suitable to enter the pole basis,

$$Z[s_{12}s_{34}] = \int_0^1 dz_3 \int_0^{z_3} dz_2 \frac{\prod_{i<j}^4 |z_{ij}|^{s_{ij}}}{z_{21}z_{43}} = Z(1,2,5,3,4)$$

$$Z[\zeta_2] = \int_0^1 dz_3 \int_0^{z_3} dz_2 \frac{\prod_{i<j}^4 |z_{ij}|^{s_{ij}}}{z_{31}z_{42}} = Z(1,3,5,2,4). \tag{4.15}$$

Here we introduced the notation $Z[\zeta_2]$ for the only regular function $Z(1,3,5,2,4)$ in the KK basis. Given the absence of poles in $Z[\zeta_2]$, it is most economic to construct the basis functions $F^{(23)}$ and $F^{(32)}$ from the $\nu = 2$ sector formed by $Z[s_{12}s_{34}]$ and $Z[\zeta_2]$.

4.2.2   Six-point examples

Let us discuss the $\nu = 4, 3$ sectors of the six-point KK separately because the partial-fraction transformation towards a pole channel basis acts block diagonally. As already mentioned in Subsect. 4.1, the $\nu = 1, 2$ sectors follow from world-sheet parity $i \mapsto 6 - i$ and are thus not addressed explicitly. The functions therein complete the set of fourteen field-theory pole-channels (and six subleading pole channels $\zeta_2/s_{i,i+1}$ in integrals with a vanishing field theory limit).

- At $\nu = 4$, a series of partial-fraction operations leads to the transformation matrix

$$\begin{pmatrix} Z(1,2,3,4,6,5) \\ Z(1,2,4,3,6,5) \\ Z(1,3,2,4,6,5) \\ Z(1,3,4,2,6,5) \\ Z(1,4,2,3,6,5) \\ Z(1,4,3,2,6,5) \end{pmatrix} = \begin{pmatrix} -1 & -1 & -1 & 0 & -1 & -1 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & -1 \end{pmatrix} \begin{pmatrix} Z[s_{12}s_{123}s_{1234}] \\ Z[s_{23}s_{123}s_{1234}] \\ Z[s_{12}s_{34}s_{1234}] \\ Z[\zeta_2 s_{1234}] \\ Z[s_{34}s_{234}s_{1234}] \\ Z[s_{23}s_{234}s_{1234}] \end{pmatrix}, \tag{4.16}$$

which is in agreement with equations (C.6), (C.7) and (C.8). In addition to the triple-pole functions defined by (4.11), we have introduced a function

$$Z[\zeta_2 s_{1234}] = \int_0^1 dz_4 \int_0^{z_4} dz_3 \int_0^{z_3} dz_2 \prod_{i<j}^5 |z_{ij}|^{s_{ij}} \frac{1}{z_{31} z_{41} z_{42}} = \frac{\zeta_2}{s_{1234}} + \mathcal{O}(s^0) \qquad (4.17)$$

which generalizes the five-point integral $Z(1,3,5,2,4) = Z[\zeta_2]$ to a six-point setting with a single pole in s_{1234}.

- In the $\nu = 3$ sector of the six-point KK basis, the factorization (3.28) of the F^σ integrands' rational part leads to a 2×2 block-diagonal transformation to the desired pole channel basis:

$$\begin{pmatrix} Z(1,2,3,6,4,5) \\ Z(1,3,2,6,4,5) \\ Z(1,2,4,6,3,5) \\ Z(1,4,2,6,3,5) \\ Z(1,3,4,6,2,5) \\ Z(1,4,3,6,2,5) \end{pmatrix} = \begin{pmatrix} -1 & -1 & & & & \\ 0 & 1 & & & & \\ & & -1 & -1 & & \\ & & 0 & 1 & & \\ & & & & -1 & -1 \\ & & & & 0 & 1 \end{pmatrix} \begin{pmatrix} Z[s_{12}s_{123}s_{45}] \\ Z[s_{23}s_{123}s_{45}] \\ Z[\zeta_2 s_{12}] \\ \tilde{Z}[\zeta_3] \\ Z[\zeta_3] \\ Z[\zeta_2 s_{34}] \end{pmatrix} . \quad (4.18)$$

The α'-expansion for the following pole basis elements starts at subleading order:

$$Z[\zeta_2 s_{12}] = \prod_{i=2}^4 \int_0^{z_{i+1}} dz_i \frac{\prod_{i<j}^5 |z_{ij}|^{s_{ij}}}{z_{21} z_{41} z_{53}} , \qquad \tilde{Z}[\zeta_3] = \prod_{i=2}^4 \int_0^{z_{i+1}} dz_i \frac{\prod_{i<j}^5 |z_{ij}|^{s_{ij}}}{z_{41} z_{42} z_{53}} \qquad (4.19)$$

$$Z[\zeta_2 s_{34}] = \prod_{i=2}^4 \int_0^{z_{i+1}} dz_i \frac{\prod_{i<j}^5 |z_{ij}|^{s_{ij}}}{z_{41} z_{43} z_{52}} , \qquad Z[\zeta_3] = \prod_{i=2}^4 \int_0^{z_{i+1}} dz_i \frac{\prod_{i<j}^5 |z_{ij}|^{s_{ij}}}{z_{31} z_{41} z_{52}} . \qquad (4.20)$$

The functions (4.19) and (4.20) are not the unique choice of completing the pole basis. For the sake of computing the F^σ economically, it is preferable to include as many functions as possible with leading low-energy behavior ζ_3 rather than $\zeta_2/s_{i,i+1}$. It has been already noticed in [14] that a six-point basis requires four linearly independent functions whose α'-expansion starts with ζ_3. Those functions can for instance be taken as $Z[\zeta_3], \tilde{Z}[\zeta_3]$ and parity or cyclicity images thereof.

Comparing (4.16) with (4.18), one can identify the $\nu = 3$ sector to be the most suitable starting point towards an α'-expansion of the $F^{\sigma(234)}$: apart from the block-diagonal partial-fraction transformations, it contains only two instead of five functions with triple poles.

4.2.3 Seven-point examples

In view of the six-point example above, both the pole channel analysis and the number of terms in (3.27) suggests to compute the low-energy expansion of the F^σ from the $\nu = \lfloor N/2 \rfloor$ sector of the KK basis (4.1). At seven points, this amounts to considering the $\nu = 3$ sector $\{Z(1, 2_\rho, 3_\rho, 7, 4_\rho, 5_\rho, 6), \ \rho \in S_4\}$ associated with polynomials $R^{\nu=3,\rho} = (z_{12_\rho} z_{2_\rho 3_\rho}) \times (z_{4_\rho 5_\rho} z_{5_\rho 6})$. A suitable pole channel basis can be picked separately in the six subsectors of $Z(1, 2_\rho, 3_\rho, 7, 4_\rho, 5_\rho, 6)$ according to $\{2_\rho, 3_\rho\} = \{2, 3\}, \{2, 4\}, \{2, 5\}, \{3, 4\}, \{3, 5\}, \{4, 5\}$, with four elements each. This generalizes the block diagonal form of (4.18) for the six-point $\nu = 3$ functions due to the factorization (3.28) of the rational functions in the F^σ integrand.

In Subsect. 6.3 we will construct the seven-point α'-expansion in blocks of four functions each, the $\{2_\rho, 3_\rho\}$ block encompassing $F^{\rho(2345)}, F^{\rho(2354)}, F^{\rho(3245)}$ and $F^{\rho(3254)}$.

4.3 Identifying pole residues

This subsection is devoted to determining the full α'-expansion of kinematic pole residues. The factorization of N-point amplitudes into (products of) lower-point amplitudes on kinematic poles extends to all orders in α'. That is why residues on $N - 3 - p$ simultaneous poles carry the fingerprints of the full $(p + 3)$-point amplitudes including string corrections. In order to make these statements precise, we shall focus on integrals of type (4.11) generating one cubic field-theory diagram and recall the association of each propagator $s_{a_i, a_i+1...b_i}^{-1}$ with the rational factor of z_{b_i, a_i}^{-1} in the integrand. At the residue of the pole, the integration over the associated z_{b_i, a_i} variable collapses to the singular region due to (4.5). The same correspondence $s_{a_i, a_i+1...b_i} \leftrightarrow z_{b_i, a_i}$ holds for functions with less than $N - 3$ simultaneous poles, so their residues can be determined by the same methods.

Singularities in Mandelstam variables pose the obstruction to perform a Taylor expansion of the Koba-Nielsen factor

$$\prod_{i<j}^{N-1} |z_{ij}|^{s_{ij}} = \prod_{i<j}^{N-1} \sum_{n_{ij}=0}^{\infty} \frac{(s_{ij} \ln|z_{ij}|)^{n_{ij}}}{n_{ij}!} \tag{4.21}$$

before carrying out the world-sheet integrals. Putting it differently, integrating the product of the Taylor series (4.21) with the rational function $\prod_{l=1}^{N-3} z_{u_l v_l}^{-1}$ yields a regular object without any poles in $s_{u_i, u_i+1...v_i}$,

$$I_{u_1 v_1, u_2 v_2, ..., u_{N-3} v_{N-3}}^{\text{reg}} := \prod_{i<j}^{N-1} \sum_{n_{ij}=0}^{\infty} \frac{(s_{ij})^{n_{ij}}}{n_{ij}!} \prod_{l=1}^{N-3} \int_0^{z_{l+2}} \frac{\mathrm{d}z_{l+1}}{z_{u_l v_l}} \left(\ln|z_{ij}| \right)^{n_{ij}} . \tag{4.22}$$

These regularized world-sheet integrals are the natural objects in order to describe the residues on kinematic poles. The polylogarithmic integrals therein will be evaluated in Sect. 5. Of course, there are many alternative ways to obtain pole residues of the $Z(\rho)$, examples for five and six points can be found in [14]. In this section we want to cast both the residues and the regular parts of relevant world-sheet integrals into the form (4.22). The integration techniques of Sect. 5 then render the α'-expansion suitable for computer automatization.

In what follows, we promote the regular part $I_{u_1 v_1, u_2 v_2, ..., u_{N-3} v_{N-3}}^{\text{reg}}$ of N-point integrals to functions $I_{u_1 v_1, u_2 v_2, ..., u_{N-3} v_{N-3}}^{\text{reg}}(k_1, k_2, ..., k_{N-1})$ of $N - 1$ on-shell momenta. (By convention, $k_1, ..., k_{N-1}$ are understood to be the arguments for any $I_{...}^{\text{reg}}$ unless we specify them differently.) This allows for a diagrammatic and intuitive method to determine the behavior of the Koba-Nielsen factor at pole residues. The following procedure systematically resolves the singularity structure of an integral with poles in $s_{a_i, a_i+1...b_i}$ due to z_{b_i, a_i} and leaves residues and regular parts of the form (4.22) which can be treated with the methods of Sect. 5:

- Sum over all ways to collapse a subset of the $(s_{a_i, a_i+1...b_i}^{-1})$-propagators in the cubic YM diagram associated with the integral in question. It is convenient to start with the maximally singular term (leaving the cubic YM diagram untouched) and to gradually increase the number of relaxed poles. Draw a diagram for each set of collapsed propagators $s_{a_i, a_i+1...b_i}^{-1}$ and replace them by a contact vertex (drawn as a bubble) associated with the rational factor z_{b_i, a_i}^{-1} in the integrand, see the five-point example in Fig. 2.
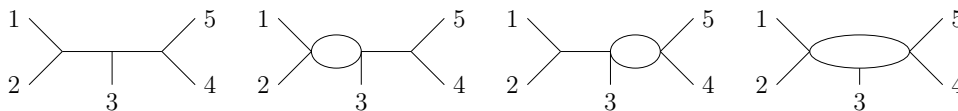

Fig. 2 Propagator collapses contributing to the $Z[s_{12} s_{123}]$ integral.
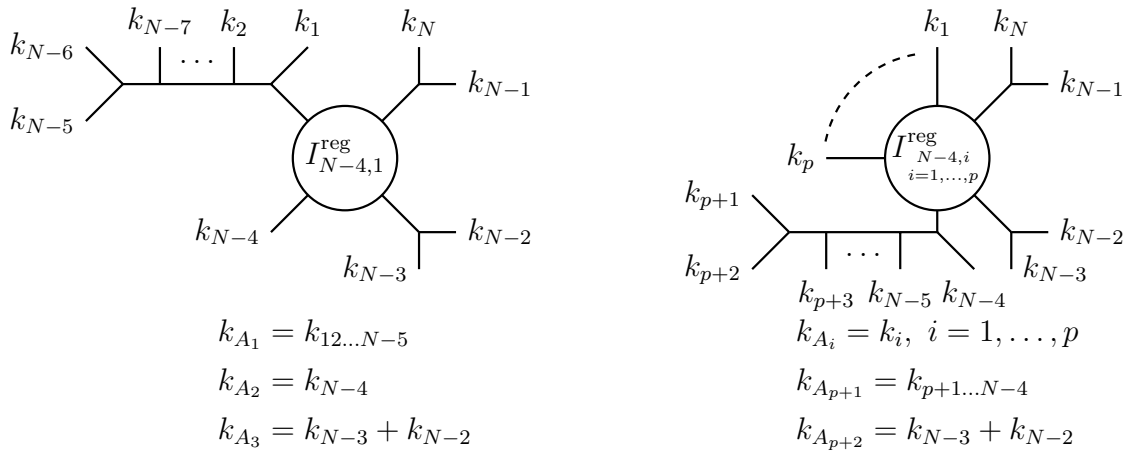
Fig. 3 Examples of a four-point and a $(p+3)$-point contact vertex together with their region momenta k_{A_i}.

- Each connected set of p contact vertices represents a p-fold world-sheet integral as it appears in a $(p+3)$-point amplitude. The propagator structure determines the $p+3$ inflowing composite (or region-) momenta k_A of type

$$k_{12...p} \quad := \quad k_1 + k_2 + \ldots + k_p \tag{4.23}$$

with $A = \{1, 2, \ldots, p\}$ in (4.23) which are generically off-shell with $k_A^2 \neq 0$. The $SL(2, \mathbb{R})$ fixing $z_N \to \infty$ decouples the composite momentum encompassing k_N at infinity, i.e. we don't list it as a separate argument. The $p+2$ region momenta $k_{A_1}, k_{A_2}, \ldots, k_{A_{p+2}}$ remain as independent arguments of the $(p+3)$-point integral, examples are shown in Fig. 3.

- Relabel the z_{b_i,a_i} from the collapsed propagators according to the composite momenta k_A of the associated contact vertices where legs b_i and a_i are attached:

$$a_i \in A_{v_i}, \ b_i \in A_{u_i} \ \Rightarrow \ z_{b_i,a_i} \ \mapsto \ z_{u_i,v_i} \tag{4.24}$$

Fig. 3 gives two examples with a four-point and a $(p+3)$-point contact vertex.

- The momentum expansion of a $(p+3)$-point contact vertex at momenta $k_{A_1}, k_{A_2}, \ldots, k_{A_{p+2}}$ is given by the regular part (4.22) of a $(p+3)$-point integral

$$I^{\text{reg}}_{u_1 v_1, u_2 v_2, \ldots, u_p w_p}(k_{A_1}, k_{A_2}, \ldots, k_{A_{p+2}}) \ = \ \prod_{i<j}^{p+2} \sum_{n_{ij}=0}^{\infty} \frac{(2\alpha' k_{A_i} \cdot k_{A_j})^{n_{ij}}}{n_{ij}!}$$

$$\times \prod_{l=1}^{p} \int_0^{z_{l+2}} \frac{\mathrm{d}z_{l+1}}{z_{u_l,v_l}} \ (\ln|z_{ij}|)^{n_{ij}} \Big|_{z_1=0}^{z_{p+2}=1} \ + \ \mathcal{O}(k_{A_i}^2) \tag{4.25}$$

- In contrast to the regular N-point integral (4.22), generic contact diagrams (4.25) in presence of propagators $s_{a_i,a_i+1...b_i}^{-1} \sim 1/k_{A_i}^2$ depend on off-shell momenta $k_{A_i}^2 \neq 0$. The dependence of on-shell integrals (4.22) on the vanishing k_i^2 is invisible, so for each pair $1 \leq i < j \leq p+2$, it is not a priori clear whether their off-shell generalizations (4.25) depend on $2\alpha' k_{A_i} \cdot k_{A_j}$, on $\alpha'(k_{A_i} + k_{A_j})^2$ or on one of $2\alpha' k_{A_i} \cdot k_{A_j} + \alpha' k_{A_{i,j}}^2$. Each of the $\frac{1}{2}p(p+3)$ Mandelstam invariants governing a $(p+3)$-point contact vertex admits shifts by one of 0, $\alpha' k_{A_i}^2$, $\alpha' k_{A_j}^2$ or $\alpha' k_{A_i}^2 + \alpha' k_{A_j}^2$, and our present method does

not single out a preferred choice. This is a minor drawback of the otherwise constructive method. As we shall see in Sect. 6, there are plenty of consistency conditions on the amplitude (such as cyclicity) which fix the missing $p(p + 3)$ "bit" of information, and it will turn out that the majority of the potential $k_{A_i}^2$-dependencies is absent[19]. Moreover, the $k_{A_i}^2$-dependence is a global property of contact vertices (4.25), so the information on $(2\alpha' k_{A_i} \cdot k_{A_j})$-shifts only needs to be determined once for each set of labels u_i, v_i in $I_{u_1 v_1, u_2 v_2, \ldots, u_p w_p}^{\text{reg}}$. Any dependence of the (partially off-shell) integral (4.25) on the masses $k_{A_i}^2 \neq 0$ vanishes at the residue of the multiple pole in question but contributes to less singular parts and interferes with the associated $I_{b_1 a_1, b_2 a_2, \ldots, b_{N-3} a_{N-3}}^{\text{reg}}$.

- The same reasoning applies to integrals with low-energy behavior $\zeta_{N-3-k} \prod_{i=1}^{k} s_{a_i, a_i+1 \ldots b_i}^{-1}$.

In many cases, the validity of these diagrammatic rules can be verified through the cube parametrization (4.3) and (4.4) introduced in Subsect. 4.1: Sending individual cube variables to $x_i \to 0, 1$ leads to poles in $s_{12 \ldots i+1}, s_{i+1, i+2}$ and fixes the associated z_{b_i, a_i} in the simplex parametrization. The behavior of the Koba-Nielsen factor at the cube boundaries then determines the momentum configuration describing the residue.

A word of caution is appropriate here: The symmetries in the graphical arrangement of contact vertices in general do not apply to the dependence on the external momenta. Firstly, the omnipresent choice to send $z_N \to \infty$ treats the cubic subdiagram containing k_N on special footing. Secondly the integral representations (4.25) are coherently modded out by the conformal Killing group such that also k_{A_1} and $k_{A_{p+2}}$ associated with $z_1 = 0$ and $z_{p+2} = 1$ play a distinguished role. The diagrams serve as a mnemonic to determine the labels u_i, v_i and momenta k_{A_i} in (4.25) and not to reflect any exchange symmetries in $k_{A_i} \leftrightarrow k_{A_j}$.

4.3.1   Four-point examples

Four-point integrals involve a kinematic pole whose residue is determined by the cubic YM vertex without any α'-correction. Hence, the s-channel integral $Z[s_{12}] = Z(1, 2, 4, 3)$ is related to its regularized part as follows:

Fig. 4   Pole structure of the function $Z[s_{12}]$.

The general expression (4.22) for regular parts simplifies to

$$I_{21}^{\text{reg}} := I_{21}^{\text{reg}}(k_1, k_2, k_3) = \sum_{n_{12}=0}^{\infty} \sum_{n_{23}=0}^{\infty} \frac{s_{12}^{n_{12}} s_{23}^{n_{23}}}{n_{12}! \, n_{23}!} \int_0^1 \frac{\mathrm{d}z_2}{z_2} \left(\ln|z_2|\right)^{n_{12}} \left(\ln|1 - z_2|\right)^{n_{23}} \quad (4.26)$$

which can be resummed in closed form: the basis function is given by the Veneziano amplitude

$$F^{(2)} = s_{12} Z[s_{12}] = 1 + s_{12} I_{21}^{\text{reg}} = \exp\left(\sum_{k=2}^{\infty} \frac{\zeta_k}{k} (-1)^k \left(s_{12}^k + s_{23}^k - (s_{12} + s_{23})^k\right), \right) \quad (4.27)$$

---

[19]   In fact, the unique five-point example for a nontrivial "mass" dependence occurs at the s_{23} single pole residue of $Z[s_{23} s_{123}]$, see (4.38) and (4.39): In this example, the associated I_{21}^{reg} function depends on the two combinations of Mandelstam variables given by $2\alpha' k_1 \cdot k_{23} + \alpha' k_{23}^2$ and $2\alpha' k_{23} \cdot k_4$.

which translates into the following α'-expansion for I_{21}^{reg}:

$$I_{21}^{\text{reg}}(k_1, k_2, k_3) = \frac{1}{s_{12}} \exp\left(\sum_{k=2}^{\infty} \frac{\zeta_k}{k} (-1)^k \left(s_{12}^k + s_{23}^k - (s_{12} + s_{23})^k\right) \right) - \frac{1}{s_{12}}$$

$$= -\zeta_2 s_{23} + \zeta_3 s_{23}(s_{12} + s_{23}) - \zeta_4 s_{23}(s_{12}^2 + \tfrac{1}{4} s_{12} s_{23} + s_{23}^2)$$

$$+ \zeta_5 s_{23}(s_{12}^3 + 2s_{12}^2 s_{23} + 2s_{12} s_{23}^2 + s_{23}^3) - \zeta_2 \zeta_3 s_{12} s_{23}^2 (s_{12} + s_{23}) + \dots .$$
$$(4.28)$$

It will prove convenient to also introduce the regular part I_{32}^{reg} of the u-channel function

$$Z[s_{23}] = \frac{1}{s_{23}} + I_{32}^{\text{reg}} \tag{4.29}$$

which is related to its parity image via

$$I_{32}^{\text{reg}}(k_A, k_B, k_C) = I_{21}^{\text{reg}}(k_C, k_B, k_A) = -\zeta_2 s_{12} + \zeta_3 s_{12}(s_{12} + s_{23}) - \zeta_4 s_{12}(s_{12}^2 + \tfrac{1}{4} s_{12} s_{23} + s_{23}^2)$$

$$+ \zeta_5 s_{12}(s_{12}^3 + 2s_{12}^2 s_{23} + 2s_{12} s_{23}^2 + s_{23}^3) - \zeta_2 \zeta_3 s_{12}^2 s_{23}(s_{12} + s_{23}) + \dots . \tag{4.30}$$

In the following subsections, we will not spell out the expansion of the $I_{\dots}^{\text{reg}}$ explicitly. A systematic way to obtain their expansion is explained in Sect. 5.

4.3.2 Five-point examples

At five points, the two basis functions F^σ are proportional to the integrals $Z[s_{12}s_{34}] = Z(1, 2, 5, 3, 4)$ and $Z[\zeta_2] = Z(1, 3, 5, 2, 4)$. The former requires a pole treatment following the procedure of Subsect. 4.3, this time with an infinite tower of α'-corrections at the single pole residues.

The two propagators s_{12}^{-1} and s_{34}^{-1} in the cubic field-theory diagram of $Z[s_{12}s_{34}]$ can collapse separately. This leads to quartic contact vertices whose contributing momenta (in addition to k_5 from $z_5 \to \infty$) can be read off to be (k_1, k_2, k_{34}) and (k_{12}, k_3, k_4) from the second and third diagram in Fig. 5. The $z_{u_i v_i}$ correspondent of the collapsed propagator determines those vertices to be of I_{21}^{reg} and I_{32}^{reg}-type, respectively: the s_{12}^{-1} residue leaves the rational function $z_{43} \leftrightarrow s_{34}$ referring to the second and third region momentum $(k_{12}, k_3, k_4) := (k_{A_1}, k_{A_2}, k_{A_3})$. Applying the relabeling prescription (4.24) to z_{43} with $a_i = 3 \in A_2$ and $b_i = 4 \in A_3$ identifies the s_{12}^{-1} residue to be of I_{32}^{reg}-type. At the s_{34}^{-1} residue, on the other hand, the rational function $z_{21} \leftrightarrow s_{12}$ stays invariant under the relabeling prescription (4.24) leading to a contact vertex of I_{21}^{reg}-type. Hence, the pole structure of $Z[s_{12}s_{34}]$ is determined by:

$$Z[s_{12}s_{34}] = \frac{1}{s_{12}s_{34}} + \frac{1}{s_{34}} I_{21}^{\text{reg}}(k_1, k_2, k_{34}) + \frac{1}{s_{12}} I_{32}^{\text{reg}}(k_{12}, k_3, k_4) + I_{21,43}^{\text{reg}} \tag{4.31}$$

If we view I_{21}^{reg} and I_{32}^{reg} as a function of the four-point Mandelstam invariants s_{12} and s_{23},

$$I_{uv}^{\text{reg}}(k_1, k_2, k_3) = I_{uv}^{\text{reg}}[s_{12}, s_{23}], \tag{4.32}$$

then the composite momenta in $I_{21}^{\text{reg}}(k_1, k_2, k_{34})$ and $I_{32}^{\text{reg}}(k_{12}, k_3, k_4)$ replace the Mandelstam variables $[s_{12}, s_{23}]$ as follows:

$$I_{21}^{\text{reg}}(k_1, k_2, k_{34}) = I_{21}^{\text{reg}}[s_{12}, s_{23} + s_{24}], \qquad I_{32}^{\text{reg}}(k_{12}, k_3, k_4) = I_{32}^{\text{reg}}[s_{13} + s_{23}, s_{34}]. \tag{4.33}$$

The regular part $I_{21,43}^{\text{reg}}$, on the other hand, constitutes an intrinsic five-point vertex whose low-energy behavior is computed in the later Sect. 5.

The second integral basis element $Z[\zeta_2]$ is regular by itself,
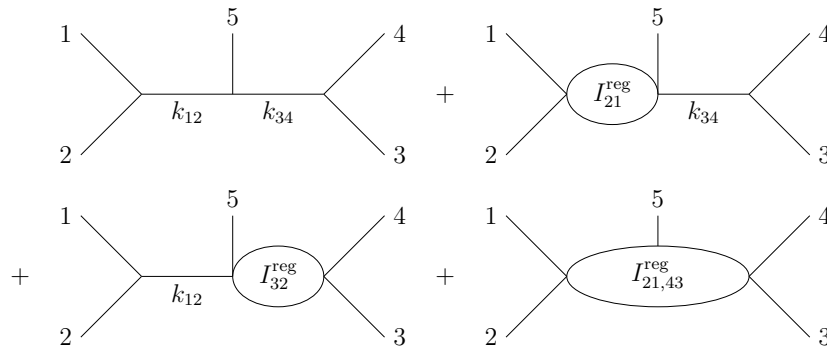
$$Z[\zeta_2] = I_{31,42}^{\text{reg}}. \tag{4.34}$$

Fig. 5   Pole structure of the function $Z[s_{12}s_{34}]$.

This introduces another five-point contact vertex $I^{\mathrm{reg}}_{31,42}$ to be expanded in Sect. 5. Together with (4.31), this completes the F^σ basis, and any other five-point integral follows from linear combinations (3.34) of $F^{(23)}$ and $F^{(32)}$, e.g.

$$Z[s_{12}s_{123}] = \frac{F^{(23)}}{s_{12}s_{123}} + \frac{F^{(32)}}{s_{13}s_{123}} \tag{4.35}$$

$$Z[s_{23}s_{123}] = \frac{F^{(23)}}{s_{23}s_{123}} - \left(\frac{1}{s_{13}} + \frac{1}{s_{23}} \right) \frac{F^{(32)}}{s_{123}} . \tag{4.36}$$

Nevertheless, we carry out the pole analysis for the remaining parity independent[20] field-theory channels to illustrate our method. Repeating the diagrammatic method applied to $Z[s_{12}s_{34}]$ leads to

$$Z[s_{12}s_{123}] = \frac{1}{s_{12}s_{123}} + \frac{I^{\mathrm{reg}}_{21}(k_1, k_2, k_3)}{s_{123}} + \frac{I^{\mathrm{reg}}_{21}(k_{12}, k_3, k_4)}{s_{12}} + I^{\mathrm{reg}}_{21,31} \tag{4.37}$$

$$Z[s_{23}s_{123}] = \frac{1}{s_{23}s_{123}} + \frac{I^{\mathrm{reg}}_{32}(k_1, k_2, k_3)}{s_{123}} + \frac{I^{\mathrm{reg}}_{21}(k_1, k_{23}, k_4)}{s_{23}} + I^{\mathrm{reg}}_{32,31} , \tag{4.38}$$

see Fig. 6 for the former.
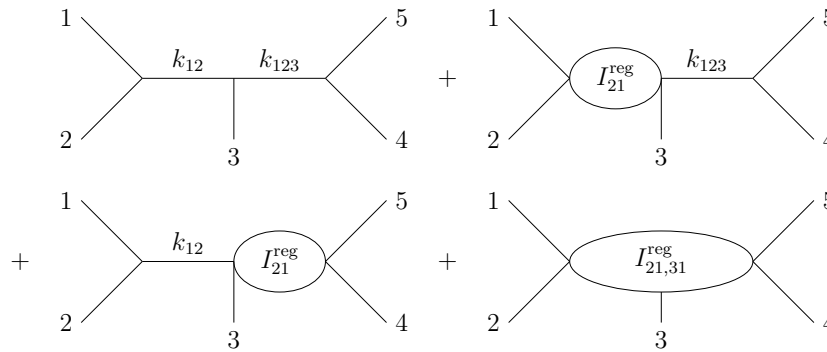

Fig. 6   Pole structure of the function $Z[s_{12}s_{123}]$.

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[20] The functions $Z[s_{34}s_{234}]$ and $Z[s_{23}s_{234}]$ associated with the remaining channels follow from parity $(1, 2, 3, 4, 5) \mapsto (4, 3, 2, 1, 5)$.

The $I_{21,31}^{\text{reg}}$ and $I_{31,32}^{\text{reg}}$ vertices enter various six- and seven-point residues. Combining (4.35) and (4.36) with (4.37) and (4.38) allows to infer their α'-expansion from $F^{(23)}$ and $F^{(32)}$, bypassing the need to compute them from the scratch.

The residues at the pole in s_{123} involve three on-shell momenta because the composite momentum $k_{123} = k_{45}$ associated with $z_5 \to \infty$ is suppressed, see Fig. 6. More importantly, the s_{23}^{-1} residue in (4.38) is the first examples where the norm of an off-shell momentum k_{23} enters:

$$I_{21}^{\text{reg}}(k_{12}, k_3, k_4) = I_{21}^{\text{reg}}[s_{13} + s_{23}, s_{34}] , \qquad I_{21}^{\text{reg}}(k_1, k_{23}, k_4) = I_{21}^{\text{reg}}[s_{123}, s_{24} + s_{34}] . \qquad (4.39)$$

The first Mandelstam argument of the latter is composed of $s_{123} = 2\alpha' k_1 \cdot k_{23} + \alpha' k_{23}^2$. Together with (4.33), this identifies the off-shell completion of the I_{21}^{reg} and I_{32}^{reg} vertices:

$$I_{21}^{\text{reg}}(k_A, k_B, k_C) = I_{21}^{\text{reg}}[2\alpha' k_A \cdot k_B + \alpha' k_B^2, \ 2\alpha' k_B \cdot k_C]$$

$$I_{32}^{\text{reg}}(k_A, k_B, k_C) = I_{32}^{\text{reg}}[2\alpha' k_A \cdot k_B, \ 2\alpha' k_B \cdot k_C + \alpha' k_B^2] . \qquad (4.40)$$

The k_{23}^2 dependence is invisible at the s_{23} residue since $I_{21}^{\text{reg}}[s_{123}, s_{24} + s_{34}] - I_{21}^{\text{reg}}[s_{12} + s_{13}, s_{24} + s_{34}] = \mathcal{O}(s_{23})$. However, it can be detected by comparing the regular part $I_{32,31}^{\text{reg}}$ computed in Sect. 5 with the basis decomposition (4.36) of $Z[s_{23}s_{123}]$.

4.3.3 Six-point examples

According to Subsect. 4.2.2, the six-point basis F^σ of integrals can be most efficiently built from the six elementary integrals in (4.18) corresponding to the $\nu = 3$ sector of the KK basis (4.2). Two of them introduce a field-theory channel with three simultaneous poles $s_{12}s_{123}s_{45}$ and $s_{23}s_{123}s_{45}$, respectively. The diagrammatic method identifies the following residues for the subleading poles:

$$\begin{aligned}
Z[s_{12}s_{123}s_{45}] \ &= \ \frac{1}{s_{12}s_{123}s_{45}} + \frac{I_{21}^{\text{reg}}(k_1, k_2, k_3)}{s_{123}s_{45}} + \frac{I_{21}^{\text{reg}}(k_{12}, k_3, k_{45})}{s_{12}s_{45}} + \frac{I_{32}^{\text{reg}}(k_{123}, k_4, k_5)}{s_{123}s_{12}} \\
&\quad + \frac{I_{21,31}^{\text{reg}}(k_1, k_2, k_3, k_{45})}{s_{45}} + \frac{I_{21}^{\text{reg}}(k_1, k_2, k_3) I_{32}^{\text{reg}}(k_{123}, k_4, k_5)}{s_{123}} \\
&\quad + \frac{I_{21,43}^{\text{reg}}(k_{12}, k_3, k_4, k_5)}{s_{12}} + I_{21,31,54}^{\text{reg}} , \qquad (4.41)
\end{aligned}$$

$$\begin{aligned}
Z[s_{23}s_{123}s_{45}] &= \frac{1}{s_{23}s_{123}s_{45}} + \frac{I_{32}^{\text{reg}}(k_1, k_2, k_3)}{s_{123}s_{45}} + \frac{I_{21}^{\text{reg}}(k_1, k_{23}, k_{45})}{s_{23}s_{45}} + \frac{I_{32}^{\text{reg}}(k_{123}, k_4, k_5)}{s_{23}s_{123}} \\
&\quad + \frac{I_{31,32}^{\text{reg}}(k_1, k_2, k_3, k_{45})}{s_{45}} + \frac{I_{32}^{\text{reg}}(k_1, k_2, k_3) I_{32}^{\text{reg}}(k_{123}, k_4, k_5)}{s_{123}} \\
&\quad + \frac{I_{21,43}^{\text{reg}}(k_1, k_{23}, k_4, k_5)}{s_{23}} + I_{31,32,54}^{\text{reg}} . \qquad (4.42)
\end{aligned}$$

The diagrams contributing to the first case are shown in Fig. 7.

Two simultaneous poles remain whenever one collapsed propagator is bypassed by a quartic vertex I_{21}^{reg} or I_{32}^{reg}, possibly at composite momenta such as k_{123}, k_{12} and k_{45}. The nature of the three single-pole diagrams depends on the relative positions of the collapsed propagators: If they are connected, the resulting contact vertex is quintic and depends on momenta through $I_{21,31}^{\text{reg}}$ and $I_{21,43}^{\text{reg}}$, see (4.31) and (4.37). In the disconnected situation, two independent quartic vertices remain, and the residue factorizes into $I_{21}^{\text{reg}}(k_1, k_2, k_3) I_{32}^{\text{reg}}(k_{123}, k_4, k_5)$ and $I_{32}^{\text{reg}}(k_1, k_2, k_3) I_{32}^{\text{reg}}(k_{123}, k_4, k_5)$, respectively.

Some of the residues in (4.42) depend on the norm of their off-shell momenta, first of all $I_{21}^{\text{reg}}(k_1, k_{23}, k_{45})$ $= I_{21}^{\text{reg}}[s_{123}, s_{24} + s_{25} + s_{34} + s_{35}]$ in agreement with the general off-shell completion (4.40) of the four-point
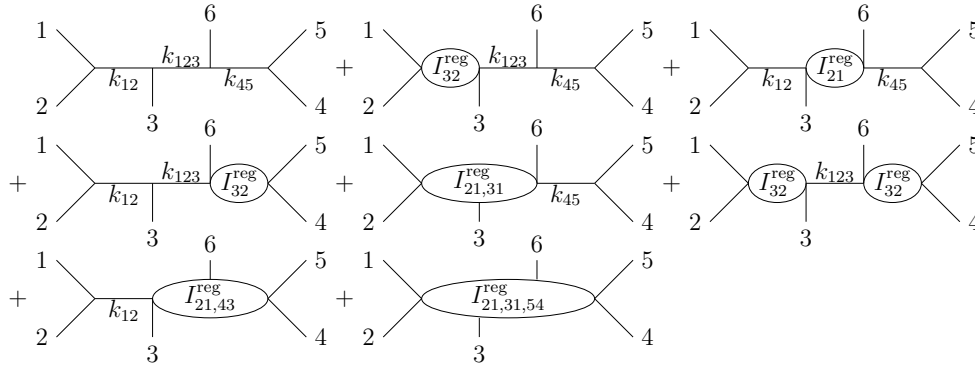
Fig. 7   Pole structure of the function $Z[s_{12}s_{123}s_{45}]$.

contact vertices. In addition, we have the following mass-dependent five-point vertex

$$I_{21,43}^{\text{reg}}(k_1, k_{23}, k_4, k_5) = I_{21,43}^{\text{reg}}[s_{123}, s_{14}, s_{24} + s_{34}, s_{25} + s_{35}, s_{45}] ,\qquad(4.43)$$

see the explanations below (4.25). We observe that the norm $k_{A_1}^2$ and $k_{A_{p+2}}^2$ of the first and last momentum drops out from any contact vertex $I_{\cdots}^{\text{reg}}$ investigated up to seven points.

According to Eq. (4.18) the remaining four integrals (4.19) and (4.20) contributing to F^σ have at most single poles. The residues within $Z[\zeta_2 s_{12}]$ and $Z[\zeta_2 s_{34}]$ are

$$Z[\zeta_2 s_{12}] = \frac{I_{31,42}^{\text{reg}}(k_{12}, k_3, k_4, k_5)}{s_{12}} + I_{21,41,53}^{\text{reg}} \qquad(4.44)$$

$$Z[\zeta_2 s_{34}] = \frac{I_{31,42}^{\text{reg}}(k_1, k_2, k_{34}, k_5)}{s_{34}} + I_{41,43,52}^{\text{reg}} , \qquad(4.45)$$

see Fig. 4. We can identify another explicit dependence on a momentum norm k_{34}^2,

$$I_{31,42}^{\text{reg}}(k_1, k_2, k_{34}, k_5) = I_{31,42}^{\text{reg}}[s_{12}, s_{134}, s_{23} + s_{24}, s_{25}, s_{35} + s_{45}] . \qquad(4.46)$$
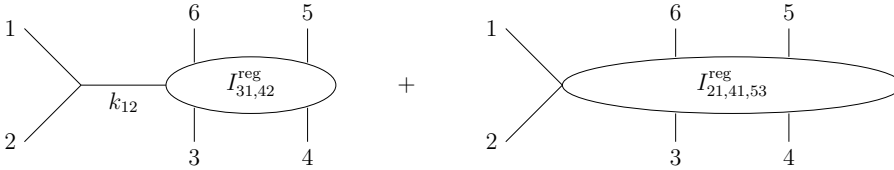
Fig. 8   Pole structure of the function $Z[\zeta_2 s_{12}]$.

Finally, the integrals $\tilde{Z}[\zeta_3]$ and $Z[\zeta_3]$ are regular by themselves,

$$\tilde{Z}[\zeta_3] = I_{41,42,53}^{\text{reg}} , \qquad Z[\zeta_3] = I_{41,43,52}^{\text{reg}} . \qquad(4.47)$$

Once we have expanded the regular parts in (4.41) to (4.47) using the methods of Sect. 5, the information on the six-point basis is complete. In appendix D, we discuss the treatment of the remaining field-theory pole channels and thereby introduce further six-point vertices $I_{ij,kl,mn}^{\text{reg}}$ relevant for residues in $(N \geq 7)$-point integrals.

4.3.4 Seven-point examples

At seven points, integrals which generate one cubic YM diagram in their field theory limit leave fourteen residues associated with subleading poles. This cornucopia of diagrams motivates to introduce an economic setup in Sect. 6 to construct the F^σ basis from integrals with a minimal number of poles. As it will turn out, it is sufficient to address one field-theory diagram, e.g.

$$
\begin{aligned}
Z[s_{12}s_{123}s_{1234}s_{56}] = {} & \frac{1}{s_{12}s_{123}s_{1234}s_{56}} + \frac{I_{21}^{\mathrm{reg}}(k_1, k_2, k_3)}{s_{123}s_{1234}s_{56}} + \frac{I_{21}^{\mathrm{reg}}(k_{12}, k_3, k_4)}{s_{12}s_{1234}s_{56}} \\
& + \frac{I_{21}^{\mathrm{reg}}(k_{123}, k_4, k_{56})}{s_{12}s_{123}s_{56}} + \frac{I_{32}^{\mathrm{reg}}(k_{1234}, k_5, k_6)}{s_{12}s_{123}s_{1234}} \\
& + \frac{I_{21,31}^{\mathrm{reg}}(k_1, k_2, k_3, k_4)}{s_{56}s_{1234}} + \frac{I_{21}^{\mathrm{reg}}(k_1, k_2, k_3)I_{21}^{\mathrm{reg}}(k_{123}, k_4, k_{56})}{s_{123}s_{56}} \\
& + \frac{I_{21}^{\mathrm{reg}}(k_1, k_2, k_3)I_{32}^{\mathrm{reg}}(k_{1234}, k_5, k_6)}{s_{123}s_{1234}} + \frac{I_{21,31}^{\mathrm{reg}}(k_{12}, k_3, k_4, k_{56})}{s_{12}s_{56}} + \frac{I_{21,43}^{\mathrm{reg}}(k_{123}, k_4, k_5, k_6)}{s_{12}s_{123}} \\
& + \frac{I_{21}^{\mathrm{reg}}(k_{12}, k_3, k_4)I_{32}^{\mathrm{reg}}(k_{1234}, k_5, k_6)}{s_{12}s_{1234}} + \frac{I_{21,31,41}^{\mathrm{reg}}(k_1, k_2, k_3, k_4, k_{56})}{s_{56}} \\
& + \frac{I_{21,31,54}^{\mathrm{reg}}(k_{12}, k_3, k_4, k_5, k_6)}{s_{12}} + \frac{I_{21}^{\mathrm{reg}}(k_1, k_2, k_3)I_{21,43}^{\mathrm{reg}}(k_{123}, k_4, k_5, k_6)}{s_{123}} \\
& + \frac{I_{21,31}^{\mathrm{reg}}(k_1, k_2, k_3, k_4)I_{32}^{\mathrm{reg}}(k_{1234}, k_5, k_6)}{s_{1234}} + I_{21,31,41,65}^{\mathrm{reg}}
\end{aligned}
\tag{4.48}
$$

None of the $I_{\cdots}^{\mathrm{reg}}$ shown depends on $k_{A_i}^2$.

According to the discussion in Subsect. 4.2.3, we construct the F^σ from the $\nu = 3$ sector of the KK basis and think of its integrals $Z(1, 2_\rho, 3_\rho, 7, 4_\rho, 5_\rho, 6)$ as forming six blocks labeled by $\{2_\rho, 3_\rho\}$. The $\{3, 5\}$ functions are characterized by polynomials $\in \{z_{31}z_{51}, z_{51}z_{53}\} \times \{z_{42}z_{64}, z_{62}z_{64}\}$ and do not exhibit any poles. Let us display the residues necessary to construct the α'-expansion of the $\{2, 5\}$ block (covering the $\{3, 4\}$ block by parity):

$$
\prod_{i=2}^{5} \int_0^{z_{i+1}} \mathrm{d}z_i \, \frac{\prod_{i<j}^6 |z_{ij}|^{s_{ij}}}{z_{21}z_{51}z_{34}z_{46}} = \frac{I_{31,42}^{\mathrm{reg}}(k_{12}, k_{34}, k_5, k_6)}{s_{12}s_{34}} + \frac{I_{41,32,53}^{\mathrm{reg}}(k_{12}, k_3, k_4, k_5, k_6)}{s_{12}}
$$
$$
+ \frac{I_{21,41,53}^{\mathrm{reg}}(k_1, k_2, k_{34}, k_5, k_6)}{s_{34}} + I_{21,51,43,64}^{\mathrm{reg}}
\tag{4.49}
$$

$$
\prod_{i=2}^{5} \int_0^{z_{i+1}} \mathrm{d}z_i \, \frac{\prod_{i<j}^6 |z_{ij}|^{s_{ij}}}{z_{25}z_{51}z_{34}z_{46}} = - \frac{I_{41,42,53}^{\mathrm{reg}}(k_1, k_2, k_{34}, k_5, k_6)}{s_{34}} - I_{51,52,43,64}^{\mathrm{reg}}
\tag{4.50}
$$

$$
\prod_{i=2}^{5} \int_0^{z_{i+1}} \mathrm{d}z_i \, \frac{\prod_{i<j}^6 |z_{ij}|^{s_{ij}}}{z_{21}z_{51}z_{36}z_{46}} = \frac{I_{41,52,53}^{\mathrm{reg}}(k_{12}, k_3, k_4, k_5, k_6)}{s_{12}} + I_{21,51,63,64}^{\mathrm{reg}}
\tag{4.51}
$$

$$
\prod_{i=2}^{5} \int_0^{z_{i+1}} \mathrm{d}z_i \, \frac{\prod_{i<j}^6 |z_{ij}|^{s_{ij}}}{z_{25}z_{51}z_{36}z_{46}} = - I_{51,52,63,64}^{\mathrm{reg}}
\tag{4.52}
$$

with mass dependence

$$
I_{31,42}^{\mathrm{reg}}(k_{12}, k_{34}, k_5, k_6) = I_{31,42}^{\mathrm{reg}}[s_{1234} - s_{12}, s_{15} + s_{25}, s_{35} + s_{45}, s_{36} + s_{46}, s_{56}]
\tag{4.53}
$$

$$I^{\text{reg}}_{21,41,53}(k_1, k_2, k_{34}, k_5, k_6) = I^{\text{reg}}_{21,41,53}[s_{12}, s_{134}, s_{15}, s_{23} + s_{24}, s_{25}, s_{35} + s_{45}, s_{26}, s_{36} + s_{46}, s_{56}]$$

$$I^{\text{reg}}_{41,42,53}(k_1, k_2, k_{34}, k_5, k_6) = I^{\text{reg}}_{41,42,53}[s_{12}, s_{134}, s_{15}, s_{23} + s_{24}, s_{25}, s_{35} + s_{45}, s_{26}, s_{36} + s_{46}, s_{56}] \,.$$

The Mandelstam arguments of six-point contact vertices are displayed in the order

$$I^{\text{reg}}_{ij,kl,mn}(k_1, k_2, k_3, k_4, k_5) = I^{\text{reg}}_{ij,kl,mn}[s_{12}, s_{13}, s_{14}, s_{23}, s_{24}, s_{34}, s_{25}, s_{35}, s_{45}] \tag{4.54}$$

which is reversed by $i \mapsto 6 - i$ world-sheet parity. The $\{2, 4\}$ and $\{4, 5\}$ blocks are analogously addressed in appendix F.1. Let us now turn to the remaining task of expanding the regular parts $I^{\text{reg}}_{\cdots}$ in α'.

5 Polylogarithms – Calculation of the regulated integrals

This section is devoted to the evaluation of the regular part of N-point disk integrals. As explained in the previous section, their singular parts are recursively determined by lower-point disk amplitudes. Thus, we will assume that the pole residues have already been accounted for by the methods of Subsect. 4.3. In the following, we set the stage for extracting the intrinsic N-point information required for the $(N - 3)!$ basis functions F^σ from a convenient set of regulated parts I^{reg} defined in (4.22). More precisely, we will provide methods to expand integrals

$$
\begin{aligned}
I^{\text{reg}} &= \prod_{k=2}^{N-2} \int_0^{z_{k+1}} \frac{\mathrm{d}z_k}{z_k - a_k} \prod_{i<j}^{N-1} |z_{ij}|^{s_{ij}} \,, \qquad a_k \in \{ 0, z_{k+1}, z_{k+2}, \ldots, z_{N-2}, 1 \} \\
&= \prod_{k=2}^{N-2} \int_0^{z_{k+1}} \frac{\mathrm{d}z_k}{z_k - a_k} \prod_{i<j}^{N-1} \sum_{n_{ij}=0}^{\infty} (s_{ij})^{n_{ij}} \frac{(\ln|z_{ij}|)^{n_{ij}}}{n_{ij}!}
\end{aligned}
\tag{5.1}
$$

where the constraint on the coefficients a_k guarantees that one can successively integrate over $z_2, z_3, \ldots, z_{N-2}$ with only one differential form $\frac{\mathrm{d}z_k}{z_k - a_k}$ depending on the integration variable z_k in each step[21]. This leaves us with $(N - 2)!$ functions[22] in (5.1) accessible to the techniques described below, and one can always apply partial fraction to express a generic integrand in this form.

In the following, expressions of the form (5.1) shall be thought of as disk integrals (4.22) whose poles have already been subtracted. That is why the Koba-Nielsen factor is expanded via (4.21). The various Mandelstam variables s_{ij} which have been pulled out from the integral will contribute to the α'-expansion of the matrix F in Eq. (2.2), while the multiple integration yields the MZVs defined in Sect. 2. While for lower transcendentalities those types of integrals can be solved by standard methods, we will here formalize the way to obtain a solution for any multiplicity and weight. Multiple polylogarithms serve as a suitable language to achieve this formalization and to convert the integrals within (4.22) or (5.1) into MZVs. So before explicitly solving the integral, let us collect a couple of facts about and relations among polylogarithms in the next subsection.

---

[21] There are numerous situations in which techniques for solving iterated integrals have been applied and explored, for example [31, 32]. In particular, handling the criterion in Eq. (5.1) was discussed in [33] and – in a physics context – in [34].

[22] On the one hand, the ability to expand the regular part of the $(N - 2)!$ functions in (5.1) appears to be fully sufficient for the construction of an $(N - 3)!$ basis of F^σ. On the other hand, several of these $(N - 2)!$ functions exhibit incompatible pole channels such that we cannot properly assemble the regular and singular parts with the methods from the previous section. Hence, the applicability of the present techniques to $(N - 2)!$ regular parts is crucial for gathering the complete information on the N-point integral basis.

5.1 Multiple polylogarithms

Multiple polylogarithms – sometimes referred to as Goncharov polylogarithms – are defined as[23]

$$G(a_1, a_2, \ldots, a_n; z) := \int_0^z \frac{dt}{t - a_1} G(a_2, \ldots, a_n; t) \tag{5.2}$$

where $G(z) = G(; z) = 1$ except for $G(\vec{a}; 0) = G(; 0) = 0$. In order to keep the terminology clear, we will refer to $\vec{a} = (a_1, \ldots, a_n)$ as the *label* and to z as *argument* of a polylogarithm G. Multiple polylogarithms constitute a graded Lie algebra with the shuffle product (cf. Subsect. 3.4)

$$G(a_1, \ldots, a_r; z) G(a_{r+1}, \ldots, a_{r+s}; z) = \sum_{\sigma \in \Sigma(r,s)} G(a_{\sigma(1)}, \ldots, a_{\sigma(r+s)}; z) \tag{5.3}$$

where $\Sigma(r, s)$ is the subset of the permutation group S_{r+s} acting on $\{a_1, \ldots, a_{r+s}\}$ which leaves the order of the elements of the individual sets $\{a_1, \ldots, a_r\}$ and $\{a_{r+1}, \ldots, a_{r+s}\}$ unchanged. The unit element for shuffling is $G(; z)=1$.

Multiple polylogarithms of uniform labels are related to powers of ordinary logarithms via

$$G(\underbrace{0, 0, \ldots, 0}_{w}; z) = \frac{1}{w!} (\ln z)^w \qquad \text{and} \qquad G(\underbrace{1, 1 \ldots, 1}_{w}; z) = \frac{1}{w!} \ln^w (1 - z)$$

$$G(\underbrace{a, a, \ldots, a}_{w}; z) = \frac{1}{w!} \ln \left(1 - \frac{z}{a} \right)^w \tag{5.4}$$

and satisfy the scaling property

$$G(k\vec{a}; kz) = G(\vec{a}; z), \qquad k \neq 0. \tag{5.5}$$

The scaling relation does not apply to labels $\vec{a}$ with $a_i = 0 \, \forall \, i$.

MZVs as defined in Eq. (2.5) are special cases of multiple polylogarithms with $a_i \in \{0, 1\}$ evaluated at argument $z = 1$ (where the numbers below the underbraces denote the number of entries.[24]):

$$\zeta_{n_1, \ldots, n_r} = (-1)^r G(\underbrace{0, 0, \ldots, 0, 1}_{n_r}, \ldots, \underbrace{0, 0, \ldots, 0, 1}_{n_1}; 1), \tag{5.6}$$

As an immediate benefit of the scaling property Eq. (5.5), multiple polylogarithms $G(\vec{a}, z)$ with $a_i \in \{0, z\}$ can be rescaled to yield

$$G(\{0, z\}_w; z) = G(\{0, 1\}_w; 1) \tag{5.7}$$

which in turn can be translated into MZVs via Eq. (5.6). In the above equation $\{a, b, \ldots\}_w$ refers to a word of length w built from the letters $a, b, \ldots$.

Multiple polylogarithms are divergent integrals in general. As can be seen from the definition Eq. (5.2), the divergences occur, when either $a_1 = z$ or $a_n = 0$. Regularization of those integrals is discussed in detail in references [35, 40]. The general idea is to slightly move the endpoints of the integration by a small parameter ε and to expand in this parameter afterwards. The regularized value is defined to be the piece

---

[23] Our definitions for polylogarithms agree with Goncharovs paper [35] and references [12] and [36]. Other aspects of mutiple polylogarithms are discussed for example in references [37, 38].

[24] Note, that the order of the $n_1 \ldots n_r$ on the right hand side of the equation is reversed with respect to the left hand side. Our convention for the MZVs agrees with references [12, 35] and [39].

in the expansion, which does not depend on the parameter ε. Extracting the ε-independent piece by using shuffle relations, one can show that for the case where $a_1 = z$ the regularized value can be obtained via

$$G(z, a_2, \ldots, a_n; z) = G(z; z)\, G(a_2, \ldots, a_n; z) - G(a_2, z, a_3, \ldots, a_n; z)$$
$$-G(a_2, a_3, z, a_4, \ldots, a_n; z) - \ldots - G(a_2, \ldots, a_n, z; z) \qquad (5.8)$$

where one defines

$$G(z, \ldots, z; z) = 0 \,. \qquad (5.9)$$

The other case, where $a_n = 0$ can be dealt with in the same way

$$G(a_1, a_2, \ldots, a_{n-1}, 0; z) = G(a_1, a_2, \ldots, a_{n-1}; z)G(0; z) - G(a_1, a_2, \ldots, 0, a_{n-1}; z)$$
$$- G(a_1, a_2, \ldots, 0, a_{n-2}, a_{n-1}; z) - \ldots - G(0, a_2, \ldots, a_{n-1}; z) \,. \qquad (5.10)$$

Here, however, $G(0; z) = \ln(z) \neq 0$. The above rewriting in order to keep the pure logarithms explicit will nevertheless prove convenient below in order to rewrite the polylogarithms in a form where the identity Eq. (5.21) can be readily used.

In the same way as multiple polylogarithms have to be regulated, this is true for the MZVs. In particular, one defines all MZVs with $n_r = 1$ by their shuffled version using Eq. (5.8). From Eq. (5.9) one immediately finds $G(1, \ldots, 1; 1) = 0$ using Eq. (5.5).

Before continuing with the evaluation of the integral in Eq. (5.1), a couple of remarks are appropriate concerning the multiple zeta data mine [41]. This collection of identities between different MZVs for weights up to $w = 22$ allows expressing any MZV in the basis spelled out in table 1 for weights $w \leq 12$. The accompanying articles and the formulae presented in [41,42] make use of harmonic polylogarithms, which are – up to a sign – multiple polylogarithms with entries $a_i \in \{-1, 0, 1\}$:

$$H(\vec{a}; z) = (-1)^k G(\vec{a}; z) \qquad a_i \in \{-1, 0, 1\} \qquad (5.11)$$

where k is the number of elements equal to $(+1)$ in $\vec{a}$ [25].

5.2 Performing the integration using polylogarithm identities

In order to employ the rules from Eq. (5.4), one has to slightly rewrite the terms in the integrand of Eq. (5.1). For example, at $N = 5$ one finds the term

$$I_{\mathrm{Ex}_1}^{\mathrm{reg}} := \frac{1}{2} s_{23}^2 \int_0^1 \frac{\mathrm{d}z_3}{z_3 - 1} \int_0^{z_3} \frac{\mathrm{d}z_2}{z_2} \left(\ln(z_3 - z_2) \right)^2 \qquad (5.12)$$
$$= \frac{1}{2} s_{23}^2 \int_0^1 \frac{\mathrm{d}z_3}{z_3 - 1} \int_0^{z_3} \frac{\mathrm{d}z_2}{z_2} \left[(\ln(z_3))^2 + 2 \ln z_3 \ln\left(1 - \frac{z_2}{z_3}\right) + \left(\ln\left(1 - \frac{z_2}{z_3}\right) \right)^2 \right]$$

as part of the integral $I_{21,34}^{\mathrm{reg}}$, where the absolute value is automatically taken care of by our parametrization $z_i < z_{i+1} \in \mathbb{R}$ of the disk boundary. After having written all terms in the expansion of the integrand in Eq. (5.1) in the above form, one can express them in terms of multiple polylogarithms G using Eq. (5.4), which will remove the combinatorial factors from expanding powers of logarithms.

By rewriting logarithms and employing the rules in Eq. (5.4), one can express the complete Koba-Nielsen factor in terms of multiple polylogarithms [26]:

---

[25] This definition agrees with Remiddi and Vermaseren [42] and Maitre [43].
[26] The $k = N - 1$ term in the double product of (5.13) covers the factors of $\prod_{i=2}^{N-2} \sum_{m_i=0}^{\infty} s_{i,N-1}^{m_i} G(\vec{1}_{m_i}; z_i)$.

$$\prod_{i<j}^{N-1} |z_{ij}|^{s_{ij}} = \left\{ \prod_{i=2}^{N-2} \sum_{n_i=0}^{\infty} \left(\sum_{l=1}^{i-1} s_{il} \right)^{n_i} G(\vec{0}_{n_i}; z_i) \right\}$$

$$\times \left\{ \prod_{2 \le j < k \le N-1} \sum_{n_{jk}=0}^{\infty} s_{jk}^{n_{jk}} G((\vec{z}_k)_{n_{jk}}; z_j) \right\}, \tag{5.13}$$

where $\vec{0}_{n_i}$ and $(\vec{z}_k)_{n_{jk}}$ denotes label vectors with n_i entries 0 and n_{jk} entries z_k, respectively. For the particular example in Eq. (5.12) one will obtain

$$I_{\text{Ex}_1}^{\text{reg}} = s_{23}^2 \int_0^1 \frac{dz_3}{z_3 - 1} \int_0^{z_3} \frac{dz_2}{z_2} \left(G(0,0; z_3) + G(0; z_3) G(z_3; z_2) + G(z_3, z_3; z_2) \right). \tag{5.14}$$

After the integrand has been cast into polylogarithms, one can start with the formal integration. As long as the label $\vec{a}$ does not contain the argument z one can readily apply Eq. (5.2). If, however, the integrand should not contain the integration variable at all, it has to be replaced by $G(; z) = 1$, which is the case for the first term in Eq. (5.14). Performing the integral over z_2 promotes Eq. (5.14) to

$$I_{\text{Ex}_1}^{\text{reg}} = s_{23}^2 \int_0^1 \frac{dz_3}{z_3 - 1} \left(G(0,0; z_3) G(0; z_3) + G(0; z_3) G(0, z_3; z_3) + G(0, z_3, z_3; z_3) \right). \tag{5.15}$$

In the above equation, one can recognize several MZVs in the integrand after rescaling with $k = 1/z_3$ (cf. Eq. (5.7)). However, as already mentioned above, scaling is not allowed for labels consisting of zeros exclusively.

After rewriting $G(0, z_3; z_3) = -\zeta_2$ and $G(0, z_3, z_3; z_3) = \zeta_3$, only the first term contains the product of two multiple polylogarithms with argument z_3. Using the shuffle relation Eq. (5.3), one finds $G(0,0; z_3) G(0; z_3) = 3 G(0,0,0; z_3)$, which allows integration by means of Eq. (5.2) and leads to

$$I_{\text{Ex}_1}^{\text{reg}} = s_{23}^2 \int_0^1 \frac{dz_3}{z_3 - 1} \left(3 G(0,0,0; z_3) + G(0; z_3) G(0,1; 1) + G(0,1,1; 1) \right)$$

$$= s_{23}^2 \left(3 G(1,0,0,0; 1) + G(1,0; 1) G(0,1; 1) + G(1; 1) G(0,1,1; 1) \right)$$

$$= s_{23}^2 \left(-3 \left(G(0,1,0,0; 1) + G(0,0,1,0; 1) + G(0,0,0,1; 1) \right) - G(0,1; 1) G(0,1; 1) \right)$$

$$= s_{23}^2 \left(3 \zeta_4 - \zeta_2^2 \right)$$

$$= \frac{s_{23}^2}{5} \zeta_2^2, \tag{5.16}$$

where we performed the shuffle regulation Eq. (5.8) for $G(1,0; 1)$ and $G(1,0,0,0; 1)$ in the third line as well as recognized $G(1; 1) = 0$.

Unfortunately not all integrals can be solved with the same ease as Eq. (5.12). Consider for example

$$I_{\text{Ex}_2}^{\text{reg}} := \int_0^1 \frac{dz_3}{z_3 - 1} \int_0^{z_3} \frac{dz_2}{z_2} G(z_3; z_2) G(1; z_2) \tag{5.17}$$

which is part of the calculation for the coefficient of $s_{23} s_{24}$ in $I_{21,43}^{\text{reg}}$. After shuffling the product of polylogarithms and performing the integration over z_2 one is faced with the following integral:

$$I_{\text{Ex}_2}^{\text{reg}} = \int_0^1 \frac{dz_3}{z_3 - 1} \left(G(0, z_3, 1; z_3) + G(0, 1, z_3; z_3) \right). \tag{5.18}$$

Obviously, the definition Eq. (5.2) can not be applied here, because the vector of labels, $\vec{a}$, still contains the argument z_3 in both polylogarithms. Furthermore, the disturbing z_3 in $\vec{a}$ can not be removed by means of Eq. (5.7). The general problem is the following: how can a multiple polylogarithm of the form

$$G(\{0, a_1, a_2, \ldots, z, \ldots, a_n\}_w; z) \tag{5.19}$$

be rewritten as a linear combination of objects which do not contain the argument z in their labels any more and can thus be integrated using Eq. (5.2)? Naturally, the desired objects have to be of the same weight as the original polylogarithms Eq. (5.19). A canonical ansatz is of the form

$$
\begin{aligned}
G(\{0, a_1, a_2, \ldots, z, \ldots, a_n\}_w; z) &= \sum c_i \, G(\{0, a_1, a_2, \ldots, a_n\}_w; z)_i \\
&+ \zeta_2 \sum c_j \, G(\{0, a_1, a_2, \ldots, a_n\}_{w-2}; z)_j \; + \zeta_3 \sum c_k \, G(\{0, a_1, a_2, \ldots, a_n\}_{w-3}; z)_k + \\
&\vdots \\
&+ \zeta_5 \sum c_l \, G(\{0, a_1, a_2, \ldots, a_n\}_{w-5}; z)_l \; + \zeta_2 \zeta_3 \sum c_m \, G(\{0, a_1, a_2, \ldots, a_n\}_{w-5}; z)_m + \\
&\vdots \\
&+ c_{max} \zeta_w
\end{aligned}
\tag{5.20}
$$

with rational coefficients c. While the first line consists of the obvious sum, the other lines contain terms in which the weight is partially carried by MZVs. For weights $w \geq 5$ one has to consider all basis elements at weight w spelled out in table 1, and the last term $c_{max} \zeta_w$ is a shorthand for having one coefficient c_{max} for each $\mathbb{Q}$ independent MZV product at weight w, see appendix A.1. The remaining sums run over a basis in the space of polylogarithms of the form $G(\{0, a_1, a_2, \ldots, a_n\}_{w-k}; z)$ with $0 \leq k \leq w$.

The canonical way to derive those relations, that is, to find all coefficients c in Eq. (5.20), is thoroughly described in [44]: one uses the fact that the multiple polylogarithms constitute a Hopf algebra endowed with a coproduct, which iteratively provides – colloquially spoken – the opposite operation of the shuffle product Eq. (5.3). Demanding agreement of the coproduct at every iteration of its calculation on both sides of Eq. (5.20) allows to fix all coefficients but the one(s) in the last line. Considering the ansatz numerically at a special value determines those coefficients as well. The so-called symbol [45, 46], which has been proven a valuable tool to derive identities between (poly)logarithms is equivalent to the maximally iterated coproduct and thus a special case of the method used here.

While we initially used the method described in [44], it turned out that the identities produced in this way can be cast into a closed formula. For simplicity we will write the equality for the case of only one occurrence of the argument z in the label $\vec{a}$:

$$G(a_1, \ldots, a_{i-1}, z, a_{i+1}, \ldots, a_n; z) = G(a_{i-1}, a_1, \ldots, a_{i-1}, \hat{z}, a_{i+1}, \ldots, a_n; z) \tag{5.21a}$$

$$- G(a_{i+1}, a_1, \ldots, a_{i-1}, \hat{z}, a_{i+1}, \ldots, a_n; z) \tag{5.21b}$$

$$- \int_0^z \frac{dt}{t - a_{i-1}} G(a_1, \ldots, \hat{a}_{i-1}, t, a_{i+1}, \ldots, a_n; t) \tag{5.21c}$$

$$+ \int_0^z \frac{dt}{t - a_{i+1}} G(a_1, \ldots, a_{i-1}, t, \hat{a}_{i+1}, \ldots, a_n; t) \tag{5.21d}$$

$$+ \int_0^z \frac{dt}{t - a_1} G(a_2, \ldots, a_{i-1}, t, a_{i+1}, \ldots, a_n; t). \tag{5.21e}$$

In the above equation a hat denotes omission of the corresponding element[27]. Thus, one can rewrite the $G(a_1, \ldots, a_{i-1}, z, a_{i+1}, \ldots, a_n; z)$ in terms of a sum of polylogarithms which are simpler in the following sense:

- terms (5.21a) and (5.21b) do not contain the argument in their labels any more
- terms (5.21c), (5.21d) and (5.21e) still contain the argument z in their labels, but their weight is reduced by one compared to the initial polylogarithm.

Applying Eq. (5.21) recursively allows to completely remove the argument z in the labels of the multiple polylogarithms on the right hand side. A couple of remarks are in order here:

- As mentioned above, Eq. (5.21) deals with the situation, where z appears only once in the label of the polylogarithm. If there are several z's in $\vec{a}$, one has to write down the first four terms on the right hand side for each of the occurrences of z. The cancellations between neighboring terms ensure that the reduction still leads to an expression where the labels of the polylogarithms on the right hand side are independent of z or shorter.
- If the argument z occurs at the last position in the label, that is $a_n = z$, the term (5.21d) in Eq. (5.21) has to be dropped. In addition, the term (5.21b) has to be modified to read

$$-G(0, a_1, \ldots, a_{i-1}, \hat{z}; z). \tag{5.22}$$

- The opposite situation, where the argument z resides at first position in the label of the polylogarithm would require special attention as well. For simplicity we will assume here, that those polylogarithms will be dealt with applying the shuffle regulation rule Eq. (5.8).
- While $a_n = 0$ does not require special attention in Eq. (5.21), it is nevertheless convenient to apply Eq. (5.10) first in order to keep the number of different identities small. Especially for higher weights this shortens the calculation significantly.

Another property of the above identity is that it preserves the shuffle regulation: If the initial polylogarithm to be expanded is shuffle-regulated, so will be all expressions on the right hand side.

The comments above will become clear in proving the identity: let us write formula (5.21) as an integral of its derivative:

$$G(a_1, \ldots, a_{i-1}, z, a_{i+1}, \ldots, a_n; z) = \int_0^z \mathrm{d}t \, \frac{\mathrm{d}}{\mathrm{d}t} G(a_1, \ldots, a_{i-1}, t, a_{i+1}, \ldots, a_n; t). \tag{5.23}$$

The total t derivative in (5.23) splits into partial derivatives acting on the t among the labels

$$\frac{\partial}{\partial a_i} G(\vec{a}; z) = \frac{1}{a_{i-1} - a_i} G(\ldots, \hat{a}_{i-1}, \ldots; z) + \frac{1}{a_i - a_{i+1}} G(\ldots, \hat{a}_{i+1}, \ldots; z)$$
$$- \frac{a_{i-1} - a_{i+1}}{(a_{i-1} - a_i)(a_i - a_{i+1})} G(\ldots, \hat{a}_i, \ldots; z) \tag{5.24}$$

and the argument

$$\frac{\partial}{\partial z} G(\vec{a}; z) = \frac{1}{z - a_1} G(a_2, \ldots, a_n; z). \tag{5.25}$$

---

[27] For illustration, here is an example employing the above formula:

$$G(0, z, 1; z) = G(0, 0, 1; z) - G(1, 0, 1; z) - \int_0^z \frac{\mathrm{d}t}{t - 0} G(t, 1; t) + \int_0^z \frac{\mathrm{d}t}{t - 1} \underbrace{G(0, t; t)}_{-\zeta_2} + \int_0^z \frac{\mathrm{d}t}{t - 0} G(t, 1; t)$$

$$= G(0, 0, 1; z) - G(1, 0, 1; z) - \zeta_2 G(1; z)$$

Applying partial-fraction identities leads straightforwardly to Eq. (5.21).

If there are several occurrences of z in the label of the polylogarithm, several partial derivatives (5.24) contribute to (5.23) which explains the first point in the remarks above. If the argument appears in the last position, the modified derivative rule reads

$$\frac{\partial}{\partial a_n} G(\vec{a}; z) = \frac{1}{a_{n-1} - a_n} G(\ldots, \hat{a}_{n-1}, a_n; z) - \frac{a_{n-1}}{(a_{n-1} - a_n) a_n} G(\ldots, a_{n-1}; z) \qquad (5.26)$$

which explains the necessity to drop the term (5.21d) in Eq. (5.21). Finally, comparing the last term in Eq. (5.26) with the last term in Eq. (5.25) shows that the term (5.21b) in Eq. (5.21) should be modified as mentioned in the remarks above.   □

Applying Eq. (5.21) recursively one finds identities like

$$G(a_1, 0, z; z) = G(0, 0, a_1; z) - G(0, a_1, a_1; z) - G(a_1; z)\zeta_2$$

$$G(a_1, z, a_2; z) = G(a_1, a_1, a_2; z) - G(a_2, 0, a_1; z) + G(a_2, a_1, a_1; z) - G(a_2, a_1, a_2; z)$$

$$(5.27)$$

where in the above formulæ $a_1, a_2 \neq 0$. In addition, a collection of identities for weights two and three can be found in appendix E. With the methods described above one can now solve all regular integrals I^{reg}. Let us now combine all previous information and finally obtain the results for the functions F^σ in the next section.

6 Assembling the basis

The techniques of Sects. 4 and 5 provide access to the α'-expansion of any integral whose rational function takes the form (5.1) provided that its field-theory limit does not involve dual pole channels, that is, more than one cubic YM diagram. In some cases, these two constraints are incompatible. For instance the seven-point world-sheet integrals from the $\{2, 3\}$-block – see Sect. 4.2.3 – cannot be brought into a basis of functions of the form (5.1) with one field-theory diagram each[28]. We will now introduce methods to bypass this subtlety.

In bypassing this subtlety, we pay particular attention to increase the computational efficiency. As we will see, the number of disk integrals whose α'-expansions need to be performed independently can be pushed well below the number $(N-3)!$ of basis functions F^σ, more precisely to 1, 3 and 11 at multiplicities $N = 5, 6$ and $N = 7$, respectively. We will present the basis construction for these multiplicities and investigate possible shortcuts from exploiting world-sheet parity and cyclic transformations along the lines of [14].

The action of world-sheet parity is implemented by $z_{i,j} \mapsto z_{N-i, N-j}$, $k_i \mapsto k_{N-i}$ on the integrand, accompanied by reflection of the integration domain. Let σ^t denote the image of the permutation σ under reversal of its elements, then one finds

$$F^{i_1 i_2 \ldots i_{N-3}} \Big|_{i \mapsto N-i} = F^{N-i_{N-3}, N-i_{N-4}, \ldots, N-i_1} . \qquad (6.1)$$

Cyclic shifts have a more subtle action on the F^σ. As in Sect. 2, $\mathbf{A}_{\text{YM}}$ denotes the vector of BCJ-independent subamplitudes, and $\mathbf{F}$ is the corresponding $(N-3)!$ component vector of functions F^σ entering the canonically ordered disk amplitude (that is, the first row of the matrix F defined in Eq. (2.2)). Then,

---

[28] A factor of $(z_{45} z_{46})^{-1}$ in the integrand violates the criterion (5.1) imposed by our polylogarithm integration techniques whereas $(z_{45} z_{56})^{-1}$ introduces incompatible poles in s_{45}^{-1} and s_{56}^{-1}.

To give a related example for the limitation of our method: It is not possible to directly construct the six-point basis from the $\nu = 4$ sector (4.16) of the pole channel basis: On the one hand, the integrand $\sim (z_{23} z_{24} z_{14})^{-1}$ of the $Z[s_{23} s_{234} s_{1234}]$ function violates the criterion (5.1). On the other hand, partial-fraction elimination of the obstruction $(z_{23} z_{24})^{-1}$ introduces a rational factor $(z_{23} z_{34})^{-1}$ which triggers singularities in incompatible pole channels s_{23} and s_{34}.

the shift $i \mapsto i \pm 1$ (subject to cyclic identifications $i := i + N$) can be implemented by $(N-3)! \times (N-3)!$ matrices

$$\mathbf{A}_{\text{YM}}\Big|_{i \mapsto i \pm 1} = \mathcal{U}_\pm \mathbf{A}_{\text{YM}} \tag{6.2}$$

Their entries are quotients of Mandelstam invariants determined by the BCJ relations[29] [2]. Since the composition of opposite shifts $(i \mapsto i \pm 1)(i \mapsto i \mp 1)$ leaves $\mathbf{A}_{\text{YM}}$ invariant, the matrices $\mathcal{U}_\pm$ satisfy

$$\mathcal{U}_\pm^{-1} = \mathcal{U}_\mp\Big|_{i \mapsto i \pm 1}. \tag{6.3}$$

Cyclic invariance of the open-string subamplitude $A_{\text{open}}(1, 2, \dots, N)$ follows from total symmetry of the CFT correlator[30] together with cyclic invariance of the integration domain defined by $z_i < z_{i+1}$. Hence, the cyclic $i \mapsto i \pm 1$ action on the functions F^σ must be linear

$$\mathbf{F}\Big|_{i \mapsto i \pm 1} = \mathcal{V}_\pm \mathbf{F} \tag{6.4}$$

with $(N-3)! \times (N-3)!$ matrices $\mathcal{V}_\pm$ determined by

$$\mathbf{A}_{\text{YM}}^t \mathbf{F} = \mathbf{A}_{\text{YM}}^t \mathcal{U}_\pm^t \mathcal{V}_\pm \mathbf{F}. \tag{6.5}$$

Together with (6.3), we obtain:

$$\mathcal{V}_\pm = \mathcal{U}_\mp^t\Big|_{i \mapsto i \pm 1}. \tag{6.6}$$

At five points, for instance, the 2×2 matrices $\mathcal{U}_+, \mathcal{V}_+$ implementing a cyclic $i \mapsto i + 1$ shift on $A_{\text{YM}}(1, \sigma(2,3), 4, 5)$ and $F^{\sigma(23)}$ are given by

$$\mathcal{U}_+ = \begin{pmatrix} 1 & 0 \\ -1 - s_{34}/s_{35} & s_{13}/s_{35} \end{pmatrix}, \quad \mathcal{V}_+ = \begin{pmatrix} 1 & -1 - s_{23}/s_{13} \\ 0 & s_{35}/s_{13} \end{pmatrix}. \tag{6.7}$$

They can be checked to obey $\mathcal{U}_+ \mathcal{V}_+^t = 1$ using momentum conservation.

In order to finally assemble the functions F^σ, one has to perform the following steps

- express the functions F^σ in terms of $Z(1, 2_\sigma, \dots, \nu_\sigma, N, (\nu+1)_\sigma, \dots, (N-2)_\sigma, N-1)$ choosing the most convenient representation $\nu = \lfloor N/2 \rfloor$ as in Eq. (3.29).

- convert the functions $Z(\dots)$ into the pole basis $Z[\dots]$ using the partial-fraction manipulations described in Subsect. 4.2

- decompose the elements of the pole basis $Z[\dots]$ into their residues as in Sect. 4.3 and rewrite the residues in terms of lower-point regular integrals as in Eq. (4.25).

- evaluate all regular integrals using the techniques from Sect. 5.

In order to keep the dependence of the regular integrals on squares $k_{A_i}^2$ of off-shell momenta explicit, the arguments are given in terms of Mandelstam invariants (see (4.54) for our conventions for regular six-point integrals). In addition to the direct construction of F^σ following the enumeration above, we demonstrate how parity and cyclic invariance of the open-string amplitude allows to bypass the expansion of some basis functions.

---

[29] The exact form of the matrices $\mathcal{U}_\pm$ follows by expanding the images $A_{\text{YM}}(1, 2, \sigma(3, \dots, N-1), N)$ and $A_{\text{YM}}(\sigma(1, \dots, N-3), N-2, N-1, N)$ under $i \mapsto i+1$ and $i \mapsto i-1$, respectively, in the original basis of $A_{\text{YM}}(1, \sigma(2, \dots, N-2), N-1, N)$ used in (2.1).

[30] We emphasize again that before specifying the integration domain $D(\Pi)$ the open-string amplitude (3.10) is totally symmetric in its labels for the same reasons as the gravity amplitude (3.5), see Subsect. 3.4.

6.1 Five-point

Given the closed form expression (4.27) for the four-point basis function $F^{(2)}$, we shall start our series of examples by assembling the five-point basis integrals $F^{\sigma(23)}$. The α'-expansion of their $\nu = 2$ representation (3.30) follows from the pole structures of the underlying disk integrals $Z(1, \sigma(2), 5, \sigma(3), 4)$ given in (4.31) and (4.34),

$$
\begin{aligned}
F^{(23)} &= s_{12}s_{34}\, Z[s_{12}s_{34}] \\
&= 1 \,+\, s_{12}\, I_{21}^{\mathrm{reg}}[s_{12}, s_{23} + s_{24}] \,+\, s_{34}\, I_{21}^{\mathrm{reg}}[s_{34}, s_{13} + s_{23}] \,+\, s_{12}s_{34}\, I_{21,43}^{\mathrm{reg}} \\
F^{(32)} &= s_{13}s_{24}\, Z[\zeta_2] = s_{13}s_{24}\, I_{31,42}^{\mathrm{reg}} \,.
\end{aligned}
\tag{6.8}
$$

It is instructive to compare with the equivalent $\nu = 3$ representation (3.31)

$$
\begin{aligned}
F^{(23)} &= 1 \,+\, s_{34}\, I_{21}^{\mathrm{reg}}[s_{34}, s_{13} + s_{23}] \,+\, s_{12}\left(I_{21}^{\mathrm{reg}}[s_{123}, s_{24} + s_{34}] \,+\, I_{21}^{\mathrm{reg}}[s_{12}, s_{23}] \right) \\
&\quad +\, s_{12}\left((s_{13} + s_{23})\, I_{21,31}^{\mathrm{reg}} \,+\, s_{23}\, I_{31,32}^{\mathrm{reg}} \right) \\
F^{(32)} &= s_{13}\left(I_{21}^{\mathrm{reg}}[s_{13} + s_{23}, s_{34}] \,-\, I_{21}^{\mathrm{reg}}[s_{123}, s_{24} + s_{34}] \,+\, s_{12}\, I_{21,31}^{\mathrm{reg}} \,-\, s_{23}\, I_{31,32}^{\mathrm{reg}} \right)
\end{aligned}
\tag{6.9}
$$

following from (4.37) and (4.38) with $s_{12}I_{21}^{\mathrm{reg}}[s_{12}, s_{23}] = s_{23}I_{21}^{\mathrm{reg}}[s_{23}, s_{12}]$. Due to the larger number of four-point contact vertices, Eq. (6.9) is less appealing for practical purposes. The two representations (6.8) and (6.9) are equal if the $I_{\cdots}^{\mathrm{reg}}$ satisfy

$$
\begin{aligned}
I_{21}^{\mathrm{reg}}[s_{12}, s_{23} + s_{24}] \,+\, s_{34}\, I_{21,43}^{\mathrm{reg}} &= I_{21}^{\mathrm{reg}}[s_{123}, s_{24} + s_{34}] \,+\, I_{21}^{\mathrm{reg}}[s_{12}, s_{23}] \\
&\quad +\, (s_{13} + s_{23})\, I_{21,31}^{\mathrm{reg}} \,+\, s_{23}\, I_{31,32}^{\mathrm{reg}}
\end{aligned}
\tag{6.10}
$$

as well as

$$
s_{24}\, I_{31,42}^{\mathrm{reg}} = s_{12}\, I_{21,31}^{\mathrm{reg}} \,-\, s_{23}\, I_{31,32}^{\mathrm{reg}} \,+\, I_{21}^{\mathrm{reg}}[s_{13} + s_{23}, s_{34}] \,-\, I_{21}^{\mathrm{reg}}[s_{123}, s_{24} + s_{34}] \,.
\tag{6.11}
$$

Relations among the $I_{\cdots}^{\mathrm{reg}}$ which follow from comparing different F^σ-representations (3.27) mix regular parts at different multiplicity. In other words, integration by parts identities (B.4) and (B.6) only apply to the full integrals including all their singular terms. The naive attempt to lift the vanishing of integrals over $\partial_2 \prod_{i<j}^4 |z_{ij}|^{s_{ij}}/z_{31}$ to regular parts $s_{12}I_{21,31}^{\mathrm{reg}} - s_{23}I_{31,32}^{\mathrm{reg}} - s_{24}I_{31,42}^{\mathrm{reg}}$ fails because of the admixture of I_{21}^{reg} in (6.11).

6.1.1 The cyclicity shortcut

Cyclic properties of the functions $F^{(23)}$ and $F^{(32)}$ allow to bypass the evaluation of $I_{31,42}^{\mathrm{reg}}$ in Eq. (6.8). Using the matrix $\mathcal{V}_+$ given in (6.7) and encoding the cyclic action on $F^{\sigma(23)}$, one can infer the second basis function from

$$
F^{(32)} = \frac{s_{13}}{s_{13} + s_{23}}\left(F^{(23)} \,-\, F^{(23)}\big|_{i \mapsto i+1} \right).
\tag{6.12}
$$

Since I_{21}^{reg} and I_{32}^{reg} are available in closed form (4.28) and (4.30), only $I_{21,43}^{\mathrm{reg}}$ is left to expand through the methods of Sect. 5, i.e. half of the effort at five points is saved by virtue of (6.12).

6.2 Six-point

The six-point basis integrals F^σ can be obtained from the $\nu = 3$ sector of the pole basis. The singularity structure (4.41) and (4.42) of $Z[s_{12}s_{123}s_{45}]$ and $Z[s_{23}s_{123}s_{45}]$ yields the following expressions:

$$
F^{(234)} = s_{12}(s_{13} + s_{23})s_{45}\, Z[s_{12}s_{123}s_{45}] \,+\, s_{12}s_{23}s_{45}\, Z[s_{23}s_{123}s_{45}]
$$

$$=(1 \; + \; s_{12} \, I_{21}^{\text{reg}}[s_{12}, s_{23}]) \, (1 \; + \; s_{45} \, I_{21}^{\text{reg}}[s_{45}, s_{14} + s_{24} + s_{34}])$$

$$+ \; (s_{13} + s_{23}) \, I_{21}^{\text{reg}}[s_{13} + s_{23}, s_{34} + s_{35}] \; + \; s_{12} \, I_{21}^{\text{reg}}[s_{123}, s_{24} + s_{25} + s_{34} + s_{35}]$$

$$+ \; s_{12}(s_{13} + s_{23}) \, I_{21,31}^{\text{reg}}[s_{12}, s_{13}, s_{23}, s_{24} + s_{25}, s_{34} + s_{35}]$$

$$+ \; s_{45}(s_{13} + s_{23}) \, I_{21,43}^{\text{reg}}[s_{13} + s_{23}, s_{14} + s_{24}, s_{34}, s_{35}, s_{45}]$$

$$+ \; s_{12}s_{23} \, I_{31,32}^{\text{reg}}[s_{12}, s_{13}, s_{23}, s_{24} + s_{25}, s_{34} + s_{35}]$$

$$+ \; s_{12}s_{45} \, I_{21,43}^{\text{reg}}[s_{123}, s_{14}, s_{24} + s_{34}, s_{25} + s_{35}, s_{45}]$$

$$+ \; s_{12}(s_{13} + s_{23})s_{45} \, I_{21,31,54}^{\text{reg}} \; + \; s_{12}s_{23}s_{45} \, I_{31,32,54}^{\text{reg}} \tag{6.13}$$

$$F^{(324)} = s_{12}s_{13}s_{45} \, Z[s_{12}s_{123}s_{45}] \; - \; s_{13}s_{23}s_{45} \, Z[s_{23}s_{123}s_{45}]$$

$$= s_{13} \Big(I_{21}^{\text{reg}}[s_{13} + s_{23}, s_{34} + s_{35}] \; - \; I_{21}^{\text{reg}}[s_{123}, s_{24} + s_{25} + s_{34} + s_{35}]$$

$$+ \; s_{12} \, I_{21,31}^{\text{reg}}[s_{12}, s_{13}, s_{23}, s_{24} + s_{25}, s_{34} + s_{35}]$$

$$+ \; s_{45} \, I_{21,43}^{\text{reg}}[s_{13} + s_{23}, s_{14} + s_{24}, s_{34}, s_{35}, s_{45}]$$

$$- \; s_{23} \, I_{31,32}^{\text{reg}}[s_{12}, s_{13}, s_{23}, s_{24} + s_{25}, s_{34} + s_{35}]$$

$$- \; s_{45} \, I_{21,43}^{\text{reg}}[s_{123}, s_{14}, s_{24} + s_{34}, s_{25} + s_{35}, s_{45}]$$

$$+ \; s_{12}s_{45} \, I_{21,31,54}^{\text{reg}} \; - \; s_{23}s_{45} \, I_{31,32,54}^{\text{reg}} \Big) \, . \tag{6.14}$$

We have used $s_{12}I_{21}^{\text{reg}}[s_{12}, s_{23}] = s_{23}I_{21}^{\text{reg}}[s_{23}, s_{12}]$ in order to cast these two basis functions into a manifestly local form. The remaining four functions are built from significantly less $I_{\cdots}^{\text{reg}}$ since the underlying $Z[\ldots]$ have at most single poles as spelled out in Eqs. (4.44) and (4.45):

$$F^{(243)} = s_{12}(s_{14} + s_{24})s_{35} \, Z[\zeta_2 s_{12}] \; + \; s_{12}s_{24}s_{35} \, \tilde{Z}[\zeta_3]$$

$$= s_{12}(s_{14} + s_{24})s_{35} \, I_{21,41,53}^{\text{reg}} \; + \; s_{12}s_{24}s_{35} \, I_{41,42,53}^{\text{reg}}$$

$$+ \; (s_{14} + s_{24})s_{35} \, I_{31,42}^{\text{reg}}[s_{13} + s_{23}, s_{14} + s_{24}, s_{34}, s_{35}, s_{45}] \tag{6.15}$$

$$F^{(423)} = s_{12}s_{14}s_{35} \, Z[\zeta_2 s_{12}] \; - \; s_{14}s_{24}s_{35} \, \tilde{Z}[\zeta_3]$$

$$= s_{12}s_{14}s_{35} \, I_{21,41,53}^{\text{reg}} \; - \; s_{14}s_{24}s_{35} \, I_{41,42,53}^{\text{reg}}$$

$$+ \; s_{14}s_{35} \, I_{31,42}^{\text{reg}}[s_{13} + s_{23}, s_{14} + s_{24}, s_{34}, s_{35}, s_{45}] \tag{6.16}$$

$$F^{(342)} = s_{13}(s_{14} + s_{34})s_{25} \, Z[\zeta_3] \; + \; s_{13}s_{34}s_{25} \, Z[\zeta_2 s_{34}]$$

$$= s_{13}(s_{14} + s_{34})s_{25} \, I_{31,41,52}^{\text{reg}} \; + \; s_{13}s_{34}s_{25} \, I_{41,43,52}^{\text{reg}}$$

$$+ \; s_{13}s_{25} \, I_{31,42}^{\text{reg}}[s_{12}, s_{134}, s_{23} + s_{24}, s_{25}, s_{35} + s_{45}] \tag{6.17}$$

$$F^{(432)} = s_{13}s_{14}s_{25} \, Z[\zeta_3] \; - \; s_{14}s_{34}s_{25} \, Z[\zeta_2 s_{34}]$$

$$= s_{13}s_{14}s_{25} \, I_{31,41,52}^{\text{reg}} \; - \; s_{14}s_{34}s_{25} \, I_{41,43,52}^{\text{reg}}$$

$$- \; s_{14}s_{25} \, I_{31,42}^{\text{reg}}[s_{12}, s_{134}, s_{23} + s_{24}, s_{25}, s_{35} + s_{45}] \tag{6.18}$$

This pedestrian approach to the six-point F^{σ} basis requires the expansion of six integrals. Let us now demonstrate how parity and cyclicity reduce this number.

6.2.1 The parity shortcut

As a first method to improve the efficiency of the basis construction, we employ the parity transformation of the functions F^σ in Eq. (6.1). Under parity, one finds two singlets $F^{(234)}$ and $F^{(432)}$ and two doublets

$$F^{(243)} = F^{(324)}\Big|_{i\mapsto 6-i}, \qquad F^{(423)} = F^{(342)}\Big|_{i\mapsto 6-i}. \tag{6.19}$$

This allows to bypass the evaluation of $I^{\mathrm{reg}}_{21,41,53} = Z[s_{12}\zeta_2]$ and $I^{\mathrm{reg}}_{41,42,53} = \tilde{Z}[\zeta_3]$. Parity has reduced the required number of $I^{\mathrm{reg}}_{ij,kl,mn}$-expansions by one third, and we will next show that cyclicity leads to a further reduction.

6.2.2 The cyclic shortcut

Each of the pole basis integrals $Z[\ldots]$ contains partial information on the F^σ basis such as

$$Z[s_{12}s_{123}s_{45}] = \frac{1}{s_{123}s_{45}}\left(\frac{F^{(234)}}{s_{12}} + \frac{F^{(324)}}{s_{13}}\right) \tag{6.20}$$

$$Z[\zeta_2 s_{12}] = \frac{1}{s_{124}s_{35}}\left(\frac{F^{(243)}}{s_{12}} + \frac{F^{(423)}}{s_{14}}\right) \tag{6.21}$$

$$Z[\zeta_3] = \frac{1}{s_{134}s_{25}}\left(\frac{F^{(342)}}{s_{13}} + \frac{F^{(432)}}{s_{14}}\right), \tag{6.22}$$

see (3.34). In order to obtain the missing information on the F^σ beyond (6.20) to (6.22), we apply cyclic transformations to their right hand sides. At six-point level, the shifts $i \mapsto i \pm 1$ act on the $\mathbf{A}_{\mathrm{YM}}$ and $\mathbf{F}$ through 6×6 matrices $\mathcal{U}_\pm$ and $\mathcal{V}_\pm$. The entries of $\mathcal{U}_\pm$ are again straightforwardly determined by BCJ relations, and the $\mathcal{V}_\pm$ then follow from (6.6). Instead of displaying $\mathcal{V}_+$ in matrix form, let us list the cyclic images of the basis functions relevant for the $i \mapsto i+1$ shift of (6.22):

$$F^{(342)}\Big|_{i\mapsto i+1} = -\frac{F^{(342)}\,(s_{134}+s_{24})\,s_{36}}{s_{13}s_{134}} - \frac{F^{(432)}\,s_{24}s_{36}}{s_{14}s_{134}} + \frac{F^{(324)}\,s_{36}}{s_{13}} \tag{6.23}$$

$$F^{(432)}\Big|_{i\mapsto i+1} = -\frac{F^{(342)}\,s_{36}\,(s_{13}-s_{46})}{s_{13}s_{134}} - \frac{F^{(432)}\,(s_{24}+s_{45})\,s_{36}}{s_{14}s_{134}}. \tag{6.24}$$

We can then obtain the decomposition of $Z[\zeta_3]\big|_{i\mapsto i+1}$, $Z[\zeta_3]\big|_{i\mapsto i-1}$ and $Z[\zeta_3]\big|_{i\mapsto i-2}$ into a basis of F^σ from (6.22) together with $F^{\sigma(234)}$ transformations such as (6.23) and (6.24). The 6×6 matrix of expansion coefficients for $Z[s_{12}s_{123}s_{45}]$, $Z[\zeta_2 s_{12}]$, $Z[\zeta_3]$ and cyclic images thereof has an inverse with local entries:

$$\begin{pmatrix} F^{(234)} \\ F^{(243)} \\ F^{(324)} \\ F^{(342)} \\ F^{(423)} \\ F^{(432)} \end{pmatrix} = \begin{pmatrix} s_{12}s_{123}s_{45} & -s_{12}^2\,(s_{24}+s_{25}) & -s_{12}\,(s_{25}s_{134}+s_{24}(s_{245}+s_{134})) & -s_{12}s_{24}s_{245} & \cdots \\ 0 & s_{12}\,(s_{14}+s_{24})\,s_{35} & 0 & 0 & \cdots \\ 0 & s_{12}s_{13}\,(s_{24}+s_{25}) & s_{13}\,(s_{25}s_{134}+s_{24}(s_{245}+s_{134})) & s_{13}s_{24}s_{245} & \cdots \\ 0 & s_{12}s_{13}s_{25} & s_{13}s_{25}s_{134} & 0 & \cdots \\ 0 & s_{12}s_{14}s_{35} & 0 & 0 & \cdots \\ 0 & -s_{12}s_{14}s_{25} & 0 & 0 & \cdots \end{pmatrix}$$

$$\left.\begin{array}{ccc} \cdots & s_{12}\,(s_{24}+s_{25})\,(s_{23}+s_{24}+s_{25}+s_{35}) & s_{12}s_{24}\,(s_{24}+s_{25}) \\ \cdots & s_{12}s_{24}s_{35} & s_{12}s_{24}s_{35} \\ \cdots & -s_{13}\,(s_{24}+s_{25})\,(s_{23}+s_{24}+s_{25}+s_{35}) & -s_{13}s_{24}\,(s_{24}+s_{25}) \\ \cdots & -s_{13}s_{25}\,(s_{23}+s_{24}+s_{25}+s_{35}) & -s_{13}s_{24}s_{25} \\ \cdots & -s_{14}s_{24}s_{35} & -s_{14}s_{24}s_{35} \\ \cdots & s_{14}s_{25}\,(s_{23}+s_{24}+s_{25}+s_{35}) & s_{14}s_{24}s_{25} \end{array}\right) \left(\begin{array}{c} Z[s_{12}s_{123}s_{45}] \\ Z[\zeta_2 s_{12}] \\ Z[\zeta_3] \\ Z[\zeta_3]\big|_{i\mapsto i+1} \\ Z[\zeta_3]\big|_{i\mapsto i-1} \\ Z[\zeta_3]\big|_{i\mapsto i-2} \end{array}\right)$$

$$(6.25)$$

In this setting, the α'-expansions of $Z[s_{12}s_{123}s_{45}]$, $Z[\zeta_2 s_{12}]$ and $Z[\zeta_3]$ obtained on the basis of (4.41), (4.44) and (4.47), respectively, encode the low-energy behavior of the complete six-point amplitude.

6.3 Seven-point

The pole expansions given in Subsect. 4.3.4 are already adapted to a parity- and cyclicity-inspired construction of the seven-point basis for F^σ. We start by computing the basis functions from the parity independent $\{3,5\}$, $\{2,4\}$, $\{2,5\}$ and $\{4,5\}$ blocks. The $\{3,5\}$ block does not involve any poles and directly translates into $I^{\rm reg}_{ij,kl,mn,pq}$

$$\left(\begin{array}{c} \frac{F^{(3524)}}{s_{13}s_{46}} \\ \frac{F^{(3542)}}{s_{13}s_{26}} \\ \frac{F^{(5324)}}{s_{15}s_{46}} \\ \frac{F^{(5342)}}{s_{15}s_{26}} \end{array}\right) = \left(\begin{array}{cccc} (s_{15}+s_{35})s_{24} & s_{35}s_{24} & (s_{15}+s_{35})s_{26} & s_{35}s_{26} \\ -(s_{15}+s_{35})s_{24} & -s_{35}s_{24} & (s_{15}+s_{35})(s_{24}+s_{46}) & s_{35}(s_{24}+s_{46}) \\ s_{13}s_{24} & -s_{35}s_{24} & s_{13}s_{26} & -s_{35}s_{26} \\ -s_{13}s_{24} & s_{35}s_{24} & s_{13}(s_{24}+s_{46}) & -s_{35}(s_{24}+s_{46}) \end{array}\right)$$

$$\times \left(\begin{array}{c} I^{\rm reg}_{31,51,42,64} \\ I^{\rm reg}_{51,53,42,64} \\ I^{\rm reg}_{31,51,62,64} \\ I^{\rm reg}_{51,53,62,64} \end{array}\right)$$

$$(6.26)$$

As explained in [7], the α'-expansions of $F^{(3524)}$, $F^{(3542)}$, $F^{(5324)}$ and $F^{(5342)}$ start at order $\alpha'^4\zeta_4$ thanks to the absence of poles and the properties of the $I^{\rm reg}_{31,51,62,64}$.

All other blocks incorporate functions with two simultaneous poles. For the $\{2,4\}$ block of the F^σ-basis,

$$\left(\begin{array}{c} (s_{12}s_{56})^{-1}\,F^{(2435)} \\ (s_{12}s_{36})^{-1}\,F^{(2453)} \\ (s_{14}s_{56})^{-1}\,F^{(4235)} \\ (s_{14}s_{36})^{-1}\,F^{(4253)} \end{array}\right)$$

$$= \left(\begin{array}{cccc} (s_{14}+s_{24})s_{35} & -s_{24}s_{35} & (s_{14}+s_{24})s_{36} & -s_{24}s_{36} \\ -(s_{14}+s_{24})s_{35} & s_{24}s_{35} & (s_{14}+s_{24})(s_{35}+s_{56}) & -s_{24}(s_{35}+s_{56}) \\ s_{12}s_{35} & s_{24}s_{35} & s_{12}s_{36} & s_{24}s_{36} \\ -s_{12}s_{35} & -s_{24}s_{35} & s_{12}(s_{35}+s_{56}) & s_{24}(s_{35}+s_{56}) \end{array}\right)$$

$$\times \prod_{i=2}^{5}\int_0^{z_{i+1}} {\rm d}z_i \prod_{i<j}^{6}|z_{ij}|^{s_{ij}} \left(\begin{array}{c} (z_{21}z_{35}z_{41}z_{56})^{-1} \\ (z_{24}z_{35}z_{41}z_{56})^{-1} \\ (z_{21}z_{36}z_{41}z_{56})^{-1} \\ (z_{24}z_{36}z_{41}z_{56})^{-1} \end{array}\right)$$

$$(6.27)$$

the singularities for the integrals on the right hand side are given in (F.1) to (F.4). The Mandelstam variables in the 4×4 matrix then render the resulting expressions for the F^σ manifestly local. Similarly, locality of the following functions from the $\{2,5\}$ and $\{4,5\}$ block can be seen from interplay of the 4×4 matrix entries with the poles in (4.49) to (4.52) and (F.5) to (F.8):

$$
\begin{pmatrix}
(s_{12}s_{46})^{-1} F^{(2534)} \\
(s_{12}s_{36})^{-1} F^{(2543)} \\
(s_{15}s_{46})^{-1} F^{(5234)} \\
(s_{15}s_{36})^{-1} F^{(5243)}
\end{pmatrix}
=
\begin{pmatrix}
(s_{15}+s_{25})s_{34} & -s_{25}s_{34} & (s_{15}+s_{25})s_{36} & -s_{25}s_{36} \\
-(s_{15}+s_{25})s_{34} & s_{25}s_{34} & (s_{15}+s_{25})(s_{34}+s_{46}) & -s_{25}(s_{34}+s_{46}) \\
s_{12}s_{34} & s_{25}s_{34} & s_{12}s_{36} & s_{25}s_{36} \\
-s_{12}s_{34} & -s_{25}s_{34} & s_{12}(s_{34}+s_{46}) & s_{25}(s_{34}+s_{46})
\end{pmatrix}
$$
$$
\times \prod_{i=2}^{5} \int_0^{z_{i+1}} \mathrm{d}z_i \prod_{i<j}^{6} |z_{ij}|^{s_{ij}}
\begin{pmatrix}
(z_{21}z_{51}z_{34}z_{46})^{-1} \\
(z_{25}z_{51}z_{34}z_{46})^{-1} \\
(z_{21}z_{51}z_{36}z_{46})^{-1} \\
(z_{25}z_{51}z_{36}z_{46})^{-1}
\end{pmatrix}
\tag{6.28}
$$

$$
\begin{pmatrix}
(s_{14}s_{36})^{-1} F^{(4523)} \\
(s_{14}s_{26})^{-1} F^{(4532)} \\
(s_{15}s_{36})^{-1} F^{(5423)} \\
(s_{15}s_{26})^{-1} F^{(5432)}
\end{pmatrix}
=
\begin{pmatrix}
(s_{15}+s_{45})s_{23} & s_{45}s_{23} & (s_{15}+s_{45})s_{26} & s_{45}s_{26} \\
-(s_{15}+s_{45})s_{23} & -s_{45}s_{23} & (s_{15}+s_{45})(s_{23}+s_{36}) & s_{45}(s_{23}+s_{36}) \\
s_{14}s_{23} & -s_{45}s_{23} & s_{14}s_{26} & -s_{45}s_{26} \\
-s_{14}s_{23} & s_{45}s_{23} & s_{14}(s_{23}+s_{36}) & -s_{45}(s_{23}+s_{36})
\end{pmatrix}
$$
$$
\times \prod_{i=2}^{5} \int_0^{z_{i+1}} \mathrm{d}z_i \prod_{i<j}^{6} |z_{ij}|^{s_{ij}}
\begin{pmatrix}
(z_{32}z_{63}z_{41}z_{51})^{-1} \\
(z_{32}z_{63}z_{51}z_{54})^{-1} \\
(z_{62}z_{63}z_{41}z_{51})^{-1} \\
(z_{62}z_{63}z_{51}z_{54})^{-1}
\end{pmatrix}
\tag{6.29}
$$

6.3.1 The parity shortcut

Four out of six seven-point blocks of basis functions contain a doublet and two fixed points under worldsheet parity. On the contrary, the functions from the $\{3,4\}$ block transform into those of the $\{2,5\}$ block; one can therefore infer

$$
F^{(3425)} = F^{(2534)}\Big|_{i\mapsto 7-i}, \qquad F^{(3452)} = F^{(5234)}\Big|_{i\mapsto 7-i} \tag{6.30}
$$

$$
F^{(4325)} = F^{(2543)}\Big|_{i\mapsto 7-i}, \qquad F^{(4352)} = F^{(5243)}\Big|_{i\mapsto 7-i}. \tag{6.31}
$$

Within the $\{2,4\}, \{3,5\}$ and $\{4,5\}$ blocks, on the other hand, one can take advantage of parity to bypass one out of the four integral expansions each, e.g.

$$
\prod_{i=2}^{5} \int_0^{z_{i+1}} \mathrm{d}z_i \frac{\prod_{i<j}^{6} |z_{ij}|^{s_{ij}}}{z_{35}z_{51}z_{26}z_{46}} = \prod_{i=2}^{5} \int_0^{z_{i+1}} \mathrm{d}z_i \frac{\prod_{i<j}^{6} |z_{ij}|^{s_{ij}}}{z_{31}z_{51}z_{26}z_{46}}
$$
$$
- \left(\prod_{i=2}^{5} \int_0^{z_{i+1}} \mathrm{d}z_i \frac{\prod_{i<j}^{6} |z_{ij}|^{s_{ij}}}{z_{31}z_{51}z_{24}z_{46}} \right)\Big|_{i\mapsto 7-i}. \tag{6.32}
$$

The analogous relations for the $\{2,4\}$ and $\{4,5\}$ blocks follow by relabelling $(3,5) \leftrightarrow (2,4)$ and $3 \leftrightarrow 4$, respectively. Note that the form (5.1) of the integrand required for applicability of the polylogarithm techniques of Sect. 5 is not invariant under parity, this explains the asymmetry in equation (6.32).

6.3.2 The cyclicity shortcut

So far we have addressed the α'-expansion of 20 basis functions, the results are given in (6.26) to (6.29) together with their parity images (6.30) and (6.31). As explained in the previous subsection, parity can be used to obtain the $\{2, 4\}$, $\{3, 5\}$ and $\{4, 5\}$ blocks from three independent integrals each. In addition, the cyclic transformation $Z(1, 5, 3, 7, 2, 4, 6) = -Z(1, 4, 2, 7, 5, 3, 6)\big|_{i \mapsto i+4}$ allows to infer another $\{3, 5\}$ integral from the $\{2, 4\}$ block:

$$\prod_{i=2}^{5} \int_0^{z_{i+1}} \mathrm{d}z_i \, \frac{\prod_{i<j}^6 |z_{ij}|^{s_{ij}}}{z_{51} z_{53} z_{42} z_{64}} = \prod_{i=2}^{5} \int_0^{z_{i+1}} \mathrm{d}z_i \, \frac{\prod_{i<j}^6 |z_{ij}|^{s_{ij}}}{z_{41} z_{42} z_{53} z_{63}} \bigg|_{i \mapsto i+4}. \tag{6.33}$$

Similarly, the $i \mapsto i+1$ shift of the disk integrals $Z(1, 2, 5, 7, 3, 4, 6)$ and $Z(1, 2, 5, 7, 4, 3, 6)$ from the $\{2, 5\}$ block yields two elements in the $\{4, 5\}$ block:

$$\prod_{i=2}^{5} \int_0^{z_{i+1}} \mathrm{d}z_i \, \frac{\prod_{i<j}^6 |z_{ij}|^{s_{ij}}}{z_{54} z_{41} z_{63} z_{32}} = \prod_{i=2}^{5} \int_0^{z_{i+1}} \mathrm{d}z_i \, \frac{\prod_{i<j}^6 |z_{ij}|^{s_{ij}}}{z_{52} z_{21} z_{64} z_{43}} \bigg|_{i \mapsto i+1} \tag{6.34}$$

$$\prod_{i=2}^{5} \int_0^{z_{i+1}} \mathrm{d}z_i \, \frac{\prod_{i<j}^6 |z_{ij}|^{s_{ij}}}{z_{54} z_{51} z_{63} z_{32}} = \prod_{i=2}^{5} \int_0^{z_{i+1}} \mathrm{d}z_i \, \frac{\prod_{i<j}^6 |z_{ij}|^{s_{ij}}}{z_{52} z_{21} z_{63} z_{43}} \bigg|_{i \mapsto i+1}. \tag{6.35}$$

In summary, parity relations such as (6.32) as well as cyclicity relations (6.33) to (6.35) reduce 20 basis functions outside the $\{2, 3\}$ block to ten cyclicity- and parity-independent computations. Now, the $\{2, 3\}$ block with basis functions $F^{(2345)}$, $F^{(2354)}$, $F^{(3245)}$ and $F^{(3254)}$ is left to determine, and we shall again make use of cyclicity methods.

The cyclic transformation of the seven-point F^σ can be understood along the lines of the six-point integrals, see the discussion around (6.23) and (6.24). We can infer $F^{(3245)}$ and $F^{(3254)}$ by solving the cyclic transformation of two known functions $F^{(3425)}$ and $F^{(4235)}$ shown in (F.10) and (F.11). These identities relate the unknowns $F^{(3245)}$ and $F^{(3254)}$ to the 20 basis functions from above. Knowledge of $F^{(3245)}$ by virtue of (F.10) allows to determine $F^{(2354)}$ via parity,

$$F^{(2354)} = F^{(3245)} \bigg|_{i \mapsto 7-i}. \tag{6.36}$$

The final basis function $F^{(2345)}$ with non-vanishing field-theory limit cannot be obtained from parity and cyclicity (relations of that type would lead to contradictions in the $\alpha' \to 0$ limit). Hence, we make use of the basis expansion (3.34) and the pole decomposition (4.48) for the integral

$$Z[s_{12} s_{123} s_{1234} s_{56}] = \frac{1}{s_{1234} s_{56}} \sum_{\sigma \in S_3} \frac{F^{(\sigma(234)5)}}{s_{1\sigma(2)} s_{1\sigma(23)}} \tag{6.37}$$

and obtain the last basis function by solving (6.37) for $F^{(2345)}$.

7 Conclusions

This paper aims to deepen the conceptual and computational understanding of superstring disk amplitudes and their α'-corrections. On the conceptual side, the world-sheet integrand for the open-string tree amplitude has been presented in close analogy to field-theory trees in supergravity in Subsect. 3.2. The underlying dictionary between YM subamplitudes and disk integrals at various permutations is supported by world-sheet analogues of KK and BCJ relations in Subsect. 3.4.

On the computational side, we provide the tools to calculate the α'-expansion for any multiplicity and to any order in α' in principle. The poles of world-sheet disk integrals are classified by the pole criterion (4.6) and the recursive structure of their residues is accounted for by the diagrammatic methods of Subsect. 4.3. The regular part carrying the intrinsic N-point information on contact terms can be analytically evaluated by the polylogarithm method introduced in Sect. 5. The polylogarithm identity (5.20) renders any world-sheet integration elementary. The limitation is set by calculational power as the number of terms to consider and the number of steps necessary to solve the world-sheet integrals iteratively grows immensely at higher multiplicities and orders in α'. Nevertheless, the methods have allowed us to determine the α'-corrections to the open-string tree-level N-point amplitude up to the order α'^{22} (weight $w = 22$) for $N = 5$, up to α'^9 (weight $w = 9$) for $N = 6$ and α'^7 (weight $w = 7$) for $N = 7$. For these cases we could prove the explicit form (2.7) of the image of the motivic superstring amplitude. We refer the reader to [47] and the data at [17] for more details.

These results provide further testing grounds for the algebraic structure of the open-string amplitude explored in [12]. Further mathematical investigation towards expressing all tree-level amplitudes in open superstring theory using the Drinfeld associator as started for $N = 4$ in [48] will be left for another publication [49].

Appendix

A   Introduction to (motivic) multiple zeta values

This appendix gathers some background information on MZVs, their motivic versions, their isomorphic images $f_2^k f_{i_1} \ldots f_{i_p}$ in (2.7) and the underlying Hopf algebra structure [12, 21]. The choice of material is adapted to the needs of the present article.

A.1   Multiple zeta values

MZVs have been a very active field of research during the last years. While there is a vast amount of articles available on the subject, we will here briefly collect the necessary information. MZVs $\zeta_{n_1,\ldots,n_r}$ of depth r and weight $w = \sum_{i=1}^r n_i$ are defined in (2.5). There are numerous relations between different MZVs such as for example the quasi-shuffle (or stuffle) relation

$$\zeta_m \zeta_n = \zeta_{m,n} + \zeta_{n,m} + \zeta_{m+n}. \tag{A.1}$$

The collection of all relations allows to define the commutative graded algebra $\mathcal{Z}$ as the span of all MZVs over the rational numbers $\mathbb{Q}$, where the algebra is conjectured to be graded by the weights of the MZVs

$$\mathcal{Z} = \bigoplus \mathcal{Z}_w. \tag{A.2}$$

The dimension d_w of $\mathcal{Z}_w$ has been conjectured to be $d_w = d_{w-2} + d_{w-3}$ where $d_0 = 1$, $d_1 = 0$ and $d_2 = 1$ [50]. A possible choice of basis elements at each weight w is given in Table 1, cf. also [41].

Table 1 A possible choice for the basis elements of $\mathcal{Z}_w$ for $2 \leq w \leq 12$.

w	2	3	4	5	6	7	8	9	10	11		12	
$\mathcal{Z}_w$	ζ_2	ζ_3	ζ_2^2	ζ_5	ζ_3^2	ζ_7	$\zeta_{3,5}$	ζ_9	$\zeta_{3,7}$	$\zeta_{3,3,5}$	$\zeta_2\,\zeta_3^3$	$\zeta_{1,1,4,6}$	$\zeta_2\,\zeta_{3,7}$
				$\zeta_2\,\zeta_3$	ζ_2^3	$\zeta_2\,\zeta_5$	$\zeta_3\,\zeta_5$	ζ_3^3	$\zeta_3\,\zeta_7$	$\zeta_{3,5}\,\zeta_3$	$\zeta_2\,\zeta_9$	$\zeta_{3,9}$	$\zeta_2^2\,\zeta_{3,5}$
					$\zeta_2^2\,\zeta_3$		$\zeta_2\,\zeta_3^2$	ζ_5^2	ζ_7^2	ζ_{11}	$\zeta_2^2\,\zeta_7$	$\zeta_3\,\zeta_9$	$\zeta_2\,\zeta_5^2$
							ζ_2^4	$\zeta_2\,\zeta_7$	$\zeta_2\,\zeta_{3,5}$	$\zeta_3^2\,\zeta_5$	$\zeta_2^3\,\zeta_5$	$\zeta_5\,\zeta_7$	$\zeta_2\,\zeta_3\,\zeta_7$
								$\zeta_2^2\,\zeta_5$	$\zeta_2\,\zeta_3\,\zeta_5$		$\zeta_2^4\,\zeta_3$	ζ_3^4	$\zeta_2^2\,\zeta_3\,\zeta_5$
								$\zeta_2^3\,\zeta_3$	$\zeta_2^2\,\zeta_3^2$				$\zeta_2^3\,\zeta_3^2$
									ζ_2^5				ζ_2^6
d_w	1	1	1	2	2	3	4	5	7	9		12	

There is a profound difference between single ζ-functions of even and odd weight. All single zeta values of even weight can be expressed as rational multiples of even powers of π and are thus expressable as powers of ζ_2, e.g. $\zeta_2^2 = \frac{5}{2}\zeta_4$. This immediately renders them transcendental numbers. There is no analogue of this property for the remaining single zeta values: there are no known relations of this type relating two single zeta values of different odd weight. Another difficulty arising for ζ-values of odd weight is their conjectured transcendentality: the only fact which have been proven are the irrationality of ζ_3 and that there is an infinite number of odd irrational ζ's [51, 52].

Once the entries $F_\Pi{}^\sigma$ are expanded in terms of basis MZVs shown in table 1, the α'-expansion of the matrix F in Eq. (2.1) takes the following form

$$
\begin{aligned}
F = {}& 1_{(N-3)! \times (N-3)!} + \zeta_2 P_2 + \zeta_3 M_3 + \zeta_2^2 P_4 + \zeta_2\zeta_3 P_2 M_3 + \zeta_5 M_5 \\
& + \zeta_2^3 P_6 + \frac{1}{2}\zeta_3^2 M_3 M_3 + \zeta_7 M_7 + \zeta_2\zeta_5 P_2 M_5 + \zeta_2^2\zeta_3 P_4 M_3 \\
& + \zeta_2^4 P_8 + \zeta_3\zeta_5 M_5 M_3 + \frac{1}{2}\zeta_2\zeta_3^2 P_2 M_3 M_3 + \frac{1}{5}\zeta_{3,5}[M_5, M_3] \\
& + \ldots + \left(9\zeta_2\zeta_9 + \frac{6}{25}\zeta_2^2\zeta_7 - \frac{4}{35}\zeta_2^3\zeta_5 + \frac{1}{5}\zeta_{3,3,5} \right) [M_3, [M_5, M_3]] \\
& + \ldots + \zeta_{3,5}\zeta_{3,7}\frac{208926}{894845}\,[M_3[M_3[M_7, M_5]]] + \ldots \,,
\end{aligned}
\tag{A.3}
$$

see [12] for the explicit results up to weight $w = 16$. In (A.3), P_w and M_w denote $(N-3)! \times (N-3)!$-matrices of homogeneity degree w in Mandelstam variables (2.4) and are defined in Eq. (2.8). The form of the expansion in Eq. (A.3) is far from obvious. In particular the fact that any product of single ζ's comes with the appropriate product of matrices and in addition the appearance of MZVs as prefactors of commutators containing the corresponding matrices has not been expected at all.

In spite of its beautiful structure, the expansion Eq. (A.3) has three drawbacks: the coefficients in front of higher order terms related to commutators of matrices turn into unwieldy rational numbers soon, as can be seen by the examples at weight 11 and 18 noted in the last two lines of Eq. (A.3). Furthermore, the form of Eq. (A.3) depends on the particular choice of a basis for $\mathcal{Z}_w$. Moreover, the graded algebra spanned by the matrices M_w with the Lie bracket $[\ldots, \ldots]$ depends on N. E.g. for $N = 5$ at weight $w = 18$ the commutator structure becomes ambiguous [47]. So it would be preferable to find a language, in which the rational coefficients disappear, which simultaneously does not depend on a particular choice of basis and is not sensitive to the dependence of the underlying algebra on N.

A.2 Hopf algebra structure of MZVs

The language providing the desired simplification is the graded noncommutative Hopf algebra comodule $\mathcal{U}$ composed from words

$$f_{2i_1+1} \ldots f_{2i_r+1}\, f_2^k \quad , \quad r,k \geq 0, i_1, \ldots, i_r \geq 1 \tag{A.4}$$

of weight $w = 2(i_1 + \ldots + i_r) + r + 2k$. The objects f_{2i+1} alone constitute a noncommutative Hopf algebra. Upon adding powers of f_2, which commute with all f_{2i+1}, the resulting object is a Hopf algebra comodule [39]. It is not difficult to convince oneself that the bases of the gradings $\mathcal{U}_w$ do indeed have the same dimension as $\mathcal{Z}_w$: writing down all noncommutative words of the form in Eq. (A.4) yields the correct number d_w.

The missing piece is the link between the $\mathbb{Q}$-algebra $\mathcal{Z}$ of MZVs Eq. (A.2) and $\mathcal{U}$. Before constructing a map, one first needs to promote the algebra $\mathcal{Z}$ to a Hopf algebra. A Hopf algebra is an algebra which is its own coalgebra and encompasses – besides the usual product of the algebra – a coproduct in a way that product and coproduct are compatible [44]. While there is an obvious candidate for the coproduct of MZVs [39], it is however not easy to promote $\mathcal{Z}$ to a Hopf algebra: a consistent coproduct needs to be $\mathbb{Q}$-linear. This would not pose a problem at all if one could prove, that all MZVs are transcendental. However, as the transcendental nature of many MZVs is undetermined, one needs to a employ a mathematical tool, which ensures correct treatment of this uncertainty: *motives*. An introduction to the theory of motivic zeta values in the context of string theory amplitudes can be found in [12]; the original papers are [21, 39].

Lifting the regular ζ's to their motivic versions ζ^m allows to promote the commutative graded $\mathbb{Q}$-algebra $\mathcal{Z}$ to the commutative graded Hopf algebra of motivic ζ-values $\mathcal{H}$, which is defined over a finite extension of $\mathbb{Q}$. What remains is the construction of an isomorphism ϕ from the commutative, but non-cocommutative Hopf algebra $\mathcal{H}$ to the noncommutative, but co-commutative algebra comodule $\mathcal{U}$:

$$\phi : \mathcal{H} \to \mathcal{U} \,. \tag{A.5}$$

The construction of the map ϕ assigning a linear combination of noncommutative words (A.4) to each motivic MZV is described in [21], where a complete set of examples for $w \leq 16$ can be found in [12]. In order to fix the normalization of the map ϕ, we choose single ζ's to be directly translated into one-letter words

$$\phi(\zeta_w^m) = f_w \,, \qquad f_{2k} \;:=\; \frac{\zeta_{2k}}{(\zeta_2)^k}\, f_2^k \,. \tag{A.6}$$

The map ϕ preserves all the different relations between the MZVs, e.g. (cf. Eq. (A.1)):

$$\phi(\zeta_k^m\, \zeta_l^m) = \phi(\zeta_{k,l}^m) + \phi(\zeta_{l,k}^m) + \phi(\zeta_{k+l}^m) \,. \tag{A.7}$$

In addition one finds for example:

$$\phi(\zeta_{3,9}^m) = -6f_5 f_7 - 15 f_7 f_5 - 27 f_9 f_3$$

$$\phi(\zeta_{3,3,5}^m) = -5f_5 f_3 f_3 + \frac{4}{7} f_5 f_2^3 - \frac{6}{5} f_7 f_2^2 - 45 f_9 f_2 \,. \tag{A.8}$$

Employing the second example simplifies the coefficient of $[M_3, [M_5, M_3]]$ in Eq. (A.3) to $-f_5 f_3 f_3$.

Applying the map ϕ to F^m, the motivic version of the matrix F defined in Eq. (2.2), yields

$$
\begin{aligned}
\phi(F^m) = (\,&1 + f_2 P_2 + f_2^2 P_4 + f_2^3 P_6 + f_2^4 P_8 + f_2^5 P_{10} + f_2^6 P_{12} + \dots\,) \\
\times (\,&1 + f_3\,M_3 + f_5\,M_5 + f_3^2\,M_3^2 + f_7\,M_7 + f_3 f_5\,M_5 M_3 + f_5 f_3\,M_3 M_5 \\
&+ f_9\,M_9 + f_3^3\,M_3^3 + f_5^2\,M_5^2 + f_3 f_7\,M_7 M_3 + f_7 f_3\,M_3 M_7 + f_{11}\,M_{11} \\
&+ f_3^2 f_5\,M_5 M_3^2 + f_3 f_5 f_3\,M_3 M_5 M_3 + f_5 f_3^2\,M_3^2 M_5 + f_3^4\,M_3^4 + f_3 f_9\,M_9 M_3 \\
&+ f_9 f_3\,M_3 M_9 + f_5 f_7\,M_7 M_5 + f_7 f_5\,M_5 M_7 + \dots\,) \, .
\end{aligned}
\tag{A.9}
$$

Thus, by means of the map ϕ all rational coefficients in Eq. (A.3) are converted to 1. The result is a sum over all words in Eq. (A.4). In the odd part of Eq. (A.9), words $f_{i_1} f_{i_2} \dots f_{i_p}$ are accompanied by matrix products $M_{i_p} \dots M_{i_2} M_{i_1}$ with reversed ordering of indices. Writing Eq. (A.9) in a closed form yields the formula (2.7) which completely covers the structure of the open superstring amplitude. All the $s_{i_1 \dots i_p}$ content is expressed in terms of matrices, which already appeared as coefficients of single ζ's in Eq. (A.3). In order to use the same matrices in both expressions, one has to fix the freedom[31] in the construction of the map ϕ [39] such that the only elements of weight w in table 1 whose images contain f_w or $(f_2)^{w/2}$ are ζ_w and $(\zeta_2)^{w/2}$, respectively. However, for any choice of basis for the MZVs the amplitude can be brought into the form Eq. (2.7) by appropriately constructing the map and possibly redefining the matrices P_w and M_w.

B　Additional material on field-theory patterns in world-sheet integrals

This appendix provides supplementing material for Sect. 3. In particular, it supports the correspondence between field-theory amplitudes $A_{\mathrm{YM}}(\sigma)$ and world-sheet disk integrals $Z_\Pi(\sigma)$ by connecting KK and BCJ relations between A_{YM} with partial-fraction and integration by parts manipulations on the world-sheet, see Subsect. 3.4.

B.1　KK relations between disk integrals

KK relations among the $A_{\mathrm{YM}}(\sigma)$ correspond to partial-fraction relations between various permutations of the Green's functions $(z_{12} z_{23} \dots z_{N-1,N} z_{N,1})^{-1}$ in the integrand of $Z_\Pi(\sigma)$. Let us demonstrate this through the four-point example, where the photon decoupling identity

$$
A_{\mathrm{YM}}(1,2,3,4) + A_{\mathrm{YM}}(1,2,4,3) + A_{\mathrm{YM}}(1,3,2,4) = 0
\tag{B.1}
$$

corresponds to a partial-fraction relation at the level of the integrand,

$$
Z_\Pi(1,2,3,4) + Z_\Pi(1,2,4,3) + Z_\Pi(1,3,2,4) \;\sim\; \frac{1}{\mathcal{V}_{\mathrm{CKG}}} \prod_{i=1}^4 \int_{D(\Pi)} \mathrm{d}z_i \prod_{i<j}^4 |z_{ij}|^{s_{ij}}
$$

$$
\times \left(\frac{1}{z_{12} z_{23} z_{34} z_{41}} + \frac{1}{z_{12} z_{24} z_{43} z_{31}} + \frac{1}{z_{13} z_{32} z_{24} z_{41}} \right) = 0 \, .
\tag{B.2}
$$

By iterating the manipulation shown in Eq. (B.2) one can verify the general KK relation (3.21). As mentioned before, the choice of the KK basis $\{ Z_\Pi(1, \rho(2,3,\dots,N), N-1), \ \rho \in S_{N-2} \}$ in (3.21) is convenient in view of the $SL(2,\mathbb{R})$ fixing (3.3). Our methods for constructing the F_Π^σ basis of functions take this $(N-2)!$ family of disk integrals as a starting point.

---

[31] The ϕ action on MZVs $\zeta_{n_1,\dots,n_r}^m$ of depth $r \geq 2$ is fixed by the coproduct, except for the coefficient of $f_{n_1+\dots+n_r}$. In the conventions of [12], the ϕ images of the basis MZVs in table 1 do not involve $f_{n_1+\dots+n_r}$, with the exception of single zeta values.

B.2   BCJ relations between disk integrals

BCJ relations correspond to the vanishing of a total derivative under the world-sheet integral $\int_{D(\Pi)} \mathrm{d}z_i$, regardless of the integration domain $D(\Pi)$ defined by $z_{\Pi(i)} < z_{\Pi(i+1)}$. The linear Mandelstam dependence enters through z_i derivatives acting on the Koba-Nielsen factor:

$$\frac{\partial}{\partial z_k} \prod_{i<j}^{N-1} |z_{ij}|^{s_{ij}} = \prod_{i<j}^{N-1} |z_{ij}|^{s_{ij}} \sum_{l \neq k}^{N-1} \frac{s_{kl}}{z_{kl}} \tag{B.3}$$

The world-sheet analogue of BCJ relations is most conveniently derived after $SL(2,\mathbb{R})$ fixing:

$$0 = \int_{D(\Pi)} \mathrm{d}z_2 \, \mathrm{d}z_3 \, \frac{\partial}{\partial z_3} \frac{\prod_{i<j}^{4} |z_{ij}|^{s_{ij}}}{z_{12}} = \int_{D(\Pi)} \mathrm{d}z_2 \, \mathrm{d}z_3 \, \frac{\prod_{i<j}^{4} |z_{ij}|^{s_{ij}}}{z_{12}} \left(\frac{s_{13}}{z_{31}} + \frac{s_{23}}{z_{32}} + \frac{s_{34}}{z_{34}} \right)$$

$$= - s_{13} \, Z_\Pi(1,3,2,5,4) \, - \, (s_{13}+s_{23}) \, Z_\Pi(1,2,3,5,4) \, + \, s_{34} \, Z_\Pi(1,2,5,3,4) \tag{B.4}$$

Partial fraction $\frac{1}{z_{12}z_{31}} = \frac{1}{z_{12}z_{32}} + \frac{1}{z_{13}z_{23}}$ in the first line of (B.4) leads to a literal copy of the five-point BCJ relation

$$s_{13} \, A_{\mathrm{YM}}(1,3,2,5,4) \, + \, (s_{13}+s_{23}) \, A_{\mathrm{YM}}(1,2,3,5,4) = s_{34} \, A_{\mathrm{YM}}(1,2,5,3,4) \,. \tag{B.5}$$

More generally, we can obtain the Z_Π analogue of the N-point BCJ relation [2,4,5] generalizing (B.5) by means of the following total derivative:

$$0 = - \prod_{i=2}^{N-2} \int_{D(\Pi)} \mathrm{d}z_i \, \frac{\partial}{\partial z_{N-2}} \frac{\prod_{i<j}^{N-1} |z_{ij}|^{s_{ij}}}{z_{12}z_{23} \ldots z_{N-4,N-3}}$$

$$= s_{1,N-2} \, Z_\Pi(1,N-2,2,3,\ldots,N,N-1)$$

$$+ \, (s_{1,N-2}+s_{2,N-2}) \, Z_\Pi(1,2,N-2,3\ldots,N,N-1)$$

$$+ \, \ldots \, + \, (s_{1,N-2}+\ldots+s_{N-3,N-2}) \, Z_\Pi(1,2,\ldots,N-3,N-2,N,N-1)$$

$$- \, s_{N-2,N-1} \, Z_\Pi(1,2,\ldots,N-3,N,N-2,N-1) \tag{B.6}$$

The proof of (B.6) is shown in appendix B.4.

B.3   The momentum-kernel representation of the basis functions

This subsection is devoted to proving equivalence between the momentum-kernel representation $F_\Pi^\sigma = \sum_{\rho \in S_{N-3}} S[\rho|\sigma]_1 Z_\Pi(1, \rho(2,3,\ldots,N-2), N, N-1)$ of the basis functions and their original form (3.2) found in [6,7]. Instead of performing algebraic manipulations between (3.2) and (3.8), we shall start from from the $(N-2)!$ representation

$$A_{\mathrm{open}}(\Pi(1,\ldots,N)) = \sum_{\sigma \in N-2} n[1|\sigma(2,\ldots,N-2,N)|N-1] \, Z_\Pi(1,\sigma(2,3,\ldots,N-2,N),N-1) \tag{B.7}$$

of the disk amplitude [6]. The numerators $n[\ldots]$ therein have been identified with BRST building blocks $T_{12\ldots p}$ [19,53]

$$n[1|2,\ldots,\nu,N,\nu+1,\ldots,N-2|N-1] = \langle \, T_{12\ldots\nu} \, T_{N-1,N-2,\ldots,\nu+1} \, T_N \, \rangle \,, \tag{B.8}$$

and the bracket $\langle \ldots \rangle$ denotes the pure-spinor zero-mode integration to extract superspace components [54].

Without loss of generality, we restrict our discussion to the $(N-3)!$ terms of (B.7) with $\sigma(N) = N$ since the remaining terms where $\sigma(N) \neq N$ simply provide the BRST invariant completion towards A_{YM}, see [19]. The BRST building blocks $T_{1\sigma(2\ldots N-2)}$ are related to their corresponding Berends-Giele currents $M_{1\rho(2\ldots N-2)}$ [19] through the momentum kernel,

$$T_{1\sigma(2\ldots N-2)} = \sum_{\rho \in S_{N-3}} S[\sigma(2,\ldots,N-2)\,|\,\rho(2,\ldots,N-2)]_1 \, M_{1\rho(2\ldots N-2)} \,. \tag{B.9}$$

The latter build up field-theory subamplitudes via $A_{\mathrm{YM}}(1,2,\ldots,N) = \langle M_{12\ldots N-2}T_{N-1}T_N\rangle$+gauge invariant completion [19]. The claim (3.8) then follows by substituting (B.9) into the $(N-2)!$ representation (B.7) of the open-string correlator:

$$
\begin{aligned}
A_{\mathrm{open}}(\Pi(1,\ldots,N)) &= \sum_{\sigma \in S_{N-3}} \Big(\langle\, T_{1\sigma(23\ldots N-2)}T_{N-1}T_N \,\rangle\, Z_{\Pi}(1,\sigma(2,3,\ldots,N-2),N,N-1) \\
&\qquad\qquad + \text{ gauge invariant completion} \Big) \\
&= \sum_{\sigma \in S_{N-3}} \Big(\sum_{\rho \in S_{N-3}} S[\sigma(2,\ldots,N-2)\,|\,\rho(2,\ldots,N-2)]_1 \langle\, M_{1\rho(23\ldots N-2)}T_{N-1}T_N \,\rangle \\
&\qquad\qquad \times\ Z_{\Pi}(1,\sigma(2,3,\ldots,N-2),N,N-1) \ + \text{ gauge invariant completion} \Big) \\
&= \sum_{\sigma \in S_{N-3}} \sum_{\rho \in S_{N-3}} S[\sigma(2,\ldots,N-2)\,|\,\rho(2,\ldots,N-2)]_1 \\
&\qquad\qquad \times\ A_{\mathrm{YM}}(1,\rho(2,\ldots,N-2),N-1,N)\, Z_{\Pi}(1,\sigma(2,3,\ldots,N-2),N,N-1) \tag{B.10}
\end{aligned}
$$

Matching the last line of (B.10) with (2.1) completes the proof of (3.8).

B.4   Deriving the N-point BCJ relation for disk integrals

We shall carry out the intermediate steps of (B.6) here. The claim can be reexpressed as

$$
\begin{aligned}
-\prod_{i=2}^{N-2} \int_{D(\Pi)} \mathrm{d}z_i\, \frac{\partial}{\partial z_{N-2}}\, \frac{\prod_{i<j}^{N-1}|z_{ij}|^{s_{ij}}}{z_{12}z_{23}\ldots z_{N-4,N-3}} &= -\,s_{N-2,N-1}\, Z_{\Pi}(1,2,\ldots,N-3,N,N-2,N-1) \\
&\quad + \sum_{j=1}^{N-3} s_{N-2-j,N-2} \sum_{i=1}^{j} Z_{\Pi}(1,2,\ldots,N-2-i,N-2,N-1-i,\ldots,N-3,N,N-1)\,.
\end{aligned}
\tag{B.11}
$$

The left hand side contains $N-2$ terms from

$$\frac{\partial}{\partial z_{N-2}} \prod_{i<j}^{N-1}|z_{ij}|^{s_{ij}} = \left(\sum_{k=1}^{N-3} \frac{s_{N-2,k}}{z_{N-2,k}} + \frac{s_{N-2,N-1}}{z_{N-2,N-1}} \right) \prod_{i<j}^{N-1}|z_{ij}|^{s_{ij}} \tag{B.12}$$

So we have to check the $s_{N-2,N-1}$ coefficients to match,

$$\prod_{i=2}^{N-2} \int_{D(\Pi)} \mathrm{d}z_i\, \frac{\prod_{i<j}^{N-1}|z_{ij}|^{s_{ij}}}{z_{12}z_{23}\ldots z_{N-4,N-3} \cdot z_{N-2,N-1}} = Z_{\Pi}(1,2,\ldots,N-3,N,N-2,N-1)\,, \tag{B.13}$$

and, after relabeling $k = N - 2 - j$,

$$\prod_{i=2}^{N-2} \int_{D(\Pi)} dz_i \, \frac{\prod_{i<j}^{N-1} |z_{ij}|^{s_{ij}}}{z_{12}z_{23}\ldots z_{N-4,N-3} \cdot z_{N-2,N-2-j}} \tag{B.14}$$

$$= -\sum_{i=1}^{j} Z_\Pi(1, 2, \ldots, N-2-i, N-2, N-1-i, \ldots, N-3, N, N-1)$$

for the range $j = 1, 2, \ldots, N-3$. As usual, we fix $z_1 = 0$, $z_{N-1} = 1$ and $z_N \to \infty$ such that any ratio $\frac{z_{i,N}}{z_{j,N}} \to 1$ for $i, j = 1, 2, \ldots, N-1$. Then, (B.13) easily follows from

$$Z_\Pi(1, 2, \ldots, N-3, N, N-2, N-1) = \prod_{i=2}^{N-2} \int_{D(\Pi)} dz_i$$

$$\times \frac{\prod_{i<j}^{N-1} |z_{ij}|^{s_{ij}}}{z_{12}z_{23}\ldots z_{N-4,N-3}z_{N-2,N-1}} \times \underbrace{\frac{z_{1,N-1}z_{1,N}z_{N-1,N}}{z_{N-3,N}z_{N,N-2}z_{N-1,1}}}_{\to \, 1 \text{ as } z_N \to \infty} \tag{B.15}$$

and (B.14) can be shown inductively: At $j = 1$, it holds by virtue of

$$Z_\Pi(1, 2, \ldots, N-3, N-2, N, N-1)$$

$$= \prod_{i=2}^{N-2} \int_{D(\Pi)} dz_i \, \frac{\prod_{i<j}^{N-1} |z_{ij}|^{s_{ij}}}{z_{12}z_{23}\ldots z_{N-4,N-3}z_{N-3,N-2}} \times \underbrace{\frac{z_{1,N-1}z_{1,N}z_{N-1,N}}{z_{N-2,N}z_{N,N-1}z_{N-1,1}}}_{\to \, 1 \text{ as } z_N \to \infty}$$

$$= -\prod_{i=2}^{N-2} \int_{D(\Pi)} dz_i \, \frac{\prod_{i<j}^{N-1} |z_{ij}|^{s_{ij}}}{z_{12}z_{23}\ldots z_{N-4,N-3} \cdot z_{N-2,N-3}} . \tag{B.16}$$

Assuming validity of (B.14) at $j = l - 1$, we conclude

$$\sum_{i=1}^{l} Z_\Pi(1, 2, \ldots, N-2-i, N-2, N-1-i, \ldots, N-3, N, N-1)$$

$$= Z_\Pi(1, 2, \ldots, N-2-l, N-2, N-1-l, \ldots, N-3, N, N-1)$$

$$- \prod_{i=2}^{N-2} \int_{D(\Pi)} dz_i \, \frac{\prod_{i<j}^{N-1} |z_{ij}|^{s_{ij}}}{z_{12}z_{23}\ldots z_{N-4,N-3} \cdot z_{N-2,N-1-l}}$$

$$= \prod_{i=2}^{N-2} \int_{D(\Pi)} dz_i \, \frac{\prod_{i<j}^{N-1} |z_{ij}|^{s_{ij}}}{z_{12}z_{23}\ldots z_{N-4,N-3} \cdot z_{N-2,N-1-l}} \left(\frac{z_{N-2-l,N-1-l}}{z_{N-2-l,N-2}} - 1 \right)$$

$$= \prod_{i=2}^{N-2} \int_{D(\Pi)} dz_i \, \frac{\prod_{i<j}^{N-1} |z_{ij}|^{s_{ij}}}{z_{12}z_{23}\ldots z_{N-4,N-3} \cdot z_{N-2-l,N-2}} \tag{B.17}$$

Hence, the coefficients of all the $s_{N-2,k}$ match in (B.11).

C   Application of the pole criterion for disk integrals

In this appendix, we gather examples for the rule (4.6) determining the pole content of disk integrals from their KK basis (4.2). All of the results shown are in agreement with the closed formula (4.8) which is equivalent to having a consistent field-theory limit $F_\Pi{}^\sigma|_{\alpha'\to 0} = \delta_\Pi^\sigma$ for the disk amplitude (2.1).

C.1   Four-point examples

The first kinematic poles occur at four points:

$$Z(1,2,4,3) = \int_0^1 \mathrm{d}z_2 \, \frac{|z_{12}|^{s_{12}}|z_{23}|^{s_{23}}}{z_{12}} \quad \Leftrightarrow \quad R^{\nu=2}(z_{ij}) = z_{12} \quad \Leftrightarrow \quad Z(1,2,4,3) \sim -\frac{1}{s_{12}} \qquad \text{(C.1)}$$

$$Z(1,4,2,3) = \int_0^1 \mathrm{d}z_2 \, \frac{|z_{12}|^{s_{12}}|z_{23}|^{s_{23}}}{z_{23}} \quad \Leftrightarrow \quad R^{\nu=1}(z_{ij}) = z_{23} \quad \Leftrightarrow \quad Z(1,4,2,3) \sim -\frac{1}{s_{23}} \qquad \text{(C.2)}$$

In the four-point KK basis $\{Z(1,2,4,3),\, Z(1,4,2,3)\}$, the R^ν are inevitably of the form $z_{i,i+1}$, so they directly translate into two-particle channels $s_{i,i+1}$ according to (4.6). The cubic diagrams arising from the low-energy limits of $Z(1,2,4,3)$ and $Z(1,4,2,3)$ are shown in Fig. 9:
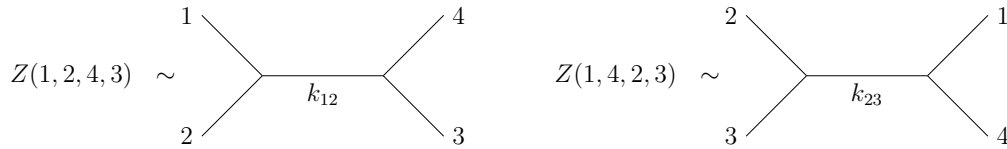

Fig. 9   Cubic diagrams generated by four-point integrals.

C.2   Five-point examples

Five points provide the first cases of incompatible pole channels within the same function. The six KK basis elements $\{Z(1,2_\rho,3_\rho,5_\rho,4),\ \rho \in S_3\}$ exhibit the following poles (cf. 4.6):

Table 2   Pole channels present in the five-point KK basis $Z(1,\rho(2,3,5),4)$.

$Z(\dots)$	$R^{\nu,\rho}$	s_{12}	s_{23}	s_{34}	s_{123}	s_{234}	$Z(\dots)$	$R^{\nu,\rho}$	s_{12}	s_{23}	s_{34}	s_{123}	s_{234}
$Z(1,2,3,5,4)$	$z_{12}z_{23}$	✓	✓	×	✓	×	$Z(1,5,2,3,4)$	$z_{23}z_{34}$	×	✓	✓	×	✓
$Z(1,3,2,5,4)$	$z_{13}z_{32}$	×	✓	×	✓	×	$Z(1,5,3,2,4)$	$z_{32}z_{24}$	×	✓	×	×	✓
$Z(1,2,5,3,4)$	$z_{12}z_{34}$	✓	×	✓	×	×	$Z(1,3,5,2,4)$	$z_{13}z_{24}$	×	×	×	×	×

The disk integrals $Z(1,3,2,5,4),\ Z(1,5,3,2,4)$ have the right number of poles to describe a cubic five-point diagram of YM field theory:

$$Z(1,3,2,5,4) \sim \frac{-1}{s_{23}s_{123}}, \quad Z(1,5,3,2,4) \sim \frac{-1}{s_{23}s_{234}}, \quad Z(1,2,5,3,4) \sim \frac{1}{s_{12}s_{34}} \qquad \text{(C.3)}$$

Two functions $Z(1,2,3,5,4)$ and $Z(1,5,2,3,4)$ are singular in three Mandelstam variables with incompatible pairs therein: Neither s_{12}, s_{23} nor s_{23}, s_{34} can appear simultaneously. Hence, their field-theory

limits give rise to two cubic diagrams each[32], as depicted in Fig. 10:

$$Z(1,2,3,5,4) \sim \frac{1}{s_{123}}\left(\frac{1}{s_{12}} + \frac{1}{s_{23}}\right), \quad Z(1,5,2,3,4) \sim \frac{1}{s_{234}}\left(\frac{1}{s_{23}} + \frac{1}{s_{34}}\right) \quad \text{(C.5)}$$

The sixth function $Z(1,3,5,2,4)$ is regular: As we shall see below, its α'-expansion starts at ζ_2. According to table 2 the five-point KK basis exhausts all the five cubic diagrams compatible with the color ordering $(1,2,3,4,5)$.
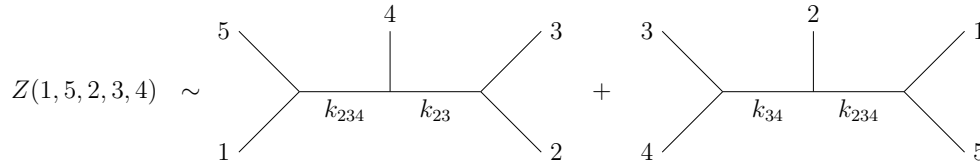

$$Z(1,5,2,3,4) \sim$$

Fig. 10 Cubic diagrams generated by the five-point integral $Z(1,5,2,3,4)$.

C.3   Six-point examples

At six points, the 24 elements $Z(1,2_\rho,3_\rho,4_\rho,6_\rho,5)$ of the KK basis can be split into four S_3 subsets according to $\nu = 4,3,2,1$ in the notation of (4.2). World-Sheet parity $(z_i, k_i) \mapsto (z_{6-i}, k_{6-i})$ connects the $\nu = 4$ integrals $Z(1,2_\rho,3_\rho,4_\rho,6,5)$ with their $\nu = 1$ counterparts $Z(1,6,2_\rho,3_\rho,4_\rho,5)$ and likewise the $Z(1,2_\rho,3_\rho,6,4_\rho,5)$ at $\nu = 3$ with $Z(1,2_\rho,6,3_\rho,4_\rho,5)$ at $\nu = 2$. Hence, it is sufficient to discuss the pole structure of the parity-independent $\nu = 3, 4$ sectors. Let us start with the $\nu = 4$-sector:

Table 3   Pole channels present in the six-point integrals $Z(1,2_\rho,3_\rho,4_\rho,6,5)$ at $\nu = 4$.

$Z(\dots)$	$R^{\nu,\rho}$	s_{12}	s_{23}	s_{34}	s_{45}	s_{1234}	s_{2345}	s_{123}	s_{234}	s_{345}
$Z(1,2,3,4,6,5)$	$z_{12}z_{23}z_{34}$	✓	✓	✓	✗	✓	✗	✓	✓	✗
$Z(1,2,4,3,6,5)$	$z_{12}z_{24}z_{43}$	✓	✗	✓	✗	✓	✗	✗	✓	✗
$Z(1,3,2,4,6,5)$	$z_{13}z_{32}z_{24}$	✗	✓	✗	✗	✓	✗	✓	✓	✗
$Z(1,3,4,2,6,5)$	$z_{13}z_{34}z_{42}$	✗	✗	✓	✗	✓	✗	✗	✓	✗
$Z(1,4,2,3,6,5)$	$z_{14}z_{42}z_{23}$	✗	✓	✗	✗	✓	✗	✗	✓	✗
$Z(1,4,3,2,6,5)$	$z_{14}z_{43}z_{32}$	✗	✓	✓	✗	✓	✗	✗	✓	✗

Two of the six KK integrals $Z(1,\dots,6,5)$ have exactly three pole channels and thus corresponds to a single cubic six-point diagram,

$$Z(1,3,4,2,6,5) \sim \frac{1}{s_{34}s_{234}s_{1234}}, \quad Z(1,4,2,3,6,5) \sim \frac{1}{s_{23}s_{234}s_{1234}}. \quad \text{(C.6)}$$

---

[32] In the parametrization (4.3), the poles of $Z(1,2,3,5,4)$ can be traced back to the $x_2 \to 0$ limit of

$$Z(1,2,3,5,4) = \int_0^1 dz_3 \int_0^{z_3} dz_2 \frac{\prod_{i<j}^4 |z_{ij}|^{s_{ij}}}{z_{12}z_{23}}$$

$$= \int_0^1 dx_1 \int_0^1 dx_2\, x_1^{s_{12}-1}(1-x_1)^{s_{23}-1} x_2^{s_{123}-1}(1-x_2)^{s_{34}}(1-x_1 x_2)^{s_{24}}. \quad \text{(C.4)}$$

The boundaries of the x_1 integration range give rise to the dual two-particle channel in either s_{12} (from $x_1 \to 0$) or s_{23} (from $x_1 \to 1$).

Three functions involve pairs of incompatible pole channels (s_{12}, s_{234}), (s_{123}, s_{234}) and (s_{23}, s_{34}), respectively, so their field-theory limits involve two diagrams

$$
\begin{aligned}
Z(1,2,4,3,6,5) &\sim \frac{1}{s_{1234}s_{34}}\left(\frac{1}{s_{12}} + \frac{1}{s_{234}}\right) \\
Z(1,3,2,4,6,5) &\sim \frac{1}{s_{1234}s_{23}}\left(\frac{1}{s_{123}} + \frac{1}{s_{234}}\right) \\
Z(1,4,3,2,6,5) &\sim -\frac{1}{s_{1234}s_{234}}\left(\frac{1}{s_{23}} + \frac{1}{s_{34}}\right).
\end{aligned} \tag{C.7}
$$

The $Z(1,2,3,4,6,5)$ integral has six pole channels, five of which form a cycle $s_{12}, s_{23}, s_{34}, s_{123}, s_{234}$ with incompatible neighbors. The only way of exhausting all possible singularities is a sum of five diagrams

$$
Z(1,2,3,4,6,5) \sim -\frac{1}{s_{1234}}\left(\frac{1}{s_{12}s_{34}} + \frac{1}{s_{23}s_{123}} + \frac{1}{s_{34}s_{234}} + \frac{1}{s_{123}s_{12}} + \frac{1}{s_{234}s_{23}}\right). \tag{C.8}
$$

Table 4   Pole channels present in the six-point integrals $Z(1, 2_\rho, 3_\rho, 6, 4_\rho, 5)$ at $\nu = 3$.

$Z(\ldots)$	$R^{\nu,\rho}$	s_{12}	s_{23}	s_{34}	s_{45}	s_{1234}	s_{2345}	s_{123}	s_{234}	s_{345}
$Z(1,2,3,6,4,5)$	$z_{12}z_{23}z_{45}$	✓	✓	×	✓	×	×	✓	×	×
$Z(1,3,2,6,4,5)$	$z_{13}z_{32}z_{45}$	×	✓	×	✓	×	×	✓	×	×
$Z(1,2,4,6,3,5)$	$z_{12}z_{24}z_{35}$	✓	×	×	×	×	×	×	×	×
$Z(1,4,2,6,3,5)$	$z_{14}z_{42}z_{35}$	×	×	×	×	×	×	×	×	×
$Z(1,3,4,6,2,5)$	$z_{13}z_{34}z_{25}$	×	×	✓	×	×	×	×	×	×
$Z(1,4,3,6,2,5)$	$z_{14}z_{43}z_{25}$	×	×	✓	×	×	×	×	×	×

The $\nu = 3$ sector incorporates two integrals with a non-vanishing field-theory limit which resembles the five-point $Z(1, 2_\rho, 3_\rho, 5, 4)$ functions (up to an additional s_{45}-propagator):

$$
Z(1,2,3,6,4,5) \sim -\frac{1}{s_{123}s_{45}}\left(\frac{1}{s_{12}} + \frac{1}{s_{23}}\right), \qquad Z(1,3,2,6,4,5) \sim \frac{1}{s_{123}s_{23}s_{45}} \tag{C.9}
$$

Table 4 provides further examples of a general feature of $(N \geq 5)$-point integrals: some of them contribute to fewer pole channels than present in a cubic YM diagrams. Functions with $k = 0, 1, 2, \ldots, N - 5$ pole channels decouple from the field-theory limit, and their leading low-energy contribution occurs at transcendentality ζ_{N-3-k} at the residue of k simultaneous poles[33]. From table 4 one can extract three six-point examples

$$
Z(1,2,4,6,3,5) \sim -\frac{\zeta_2}{s_{12}}, \qquad Z(1,3,4,6,2,5) \sim \frac{\zeta_2}{s_{34}}, \qquad Z(1,4,3,6,2,5) \sim -\frac{\zeta_2}{s_{34}} \tag{C.10}
$$

with a single pole channel $k = 1$ and leading transcendentality ζ_2. The integral $Z(1,4,2,6,3,5)$ is free of poles and furnishes a $k = 0$ example: the leading terms of its α'-expansion is accompanied by ζ_3.

---

[33] Note that the non-existence of (a shuffle-regularized version of) ζ_1 ties in with the absence of $N - 4$ simultaneous poles in the leading low energy behavior of disk integrals.

C.4   Seven-point examples

Instead of showing the lengthy list of 120 seven-point KK integrals $Z(1, 2_\rho, 3_\rho, 4_\rho, 5_\rho, 7_\rho, 6)$ including all permutations $\rho \in S_5$, we shall present some examples here, others can be found in [7]. In the $\nu = 5$ sector $Z(1, 2_\rho, 3_\rho, 4_\rho, 5_\rho, 7, 6)$ of (4.2), the pole prescription (4.6) applies to polynomials $R^{\nu=5,\rho} = z_{12_\rho} z_{2_\rho 3_\rho} z_{3_\rho 4_\rho} z_{4_\rho 5_\rho}$.

- $Z(1, 2, 3, 4, 5, 7, 6)$ has ten pole channels $s_{12}, s_{23}, s_{34}, s_{45}, s_{123}, s_{234}, s_{345}, s_{1234}, s_{2345}$ and s_{12345} where the former nine are subject to various incompatibilities. The complete set of singularities is captured by summing fourteen diagrams resembling a six-point YM subamplitude:

$$Z(1, 2, 3,4,5, 7, 6) \sim \frac{1}{s_{12345}} \left(\frac{1}{s_{12} s_{123} s_{1234}} + \frac{1}{s_{23} s_{123} s_{1234}} + \frac{1}{s_{23} s_{234} s_{1234}} + \frac{1}{s_{34} s_{234} s_{1234}} \right.$$

$$+ \frac{1}{s_{12} s_{34} s_{1234}} + \frac{1}{s_{23} s_{234} s_{2345}} + \frac{1}{s_{34} s_{234} s_{2345}} + \frac{1}{s_{34} s_{345} s_{2345}} + \frac{1}{s_{45} s_{345} s_{2345}}$$

$$\left. + \frac{1}{s_{23} s_{45} s_{2345}} + \frac{1}{s_{12} s_{123} s_{45}} + \frac{1}{s_{23} s_{123} s_{45}} + \frac{1}{s_{12} s_{34} s_{345}} + \frac{1}{s_{12} s_{45} s_{345}} \right) \qquad \text{(C.11)}$$

- $Z(1, 2, 3, 5, 4, 7, 6)$ has seven pole channels, two universally compatible ones s_{12345}, s_{45} and a cycle of five pole channels $s_{12}, s_{23}, s_{345}, s_{123}, s_{2345}$ with incompatibility between neighbors

$$Z(1, 2, 3, 5, 4, 7, 6) \sim -\frac{1}{s_{12345} s_{45}} \left(\frac{1}{s_{12} s_{345}} + \frac{1}{s_{23} s_{123}} + \frac{1}{s_{345} s_{2345}} + \frac{1}{s_{12} s_{123}} + \frac{1}{s_{23} s_{2345}} \right) \text{(C.12)}$$

- $Z(1, 2, 5, 4, 3, 7, 6)$ has six pole channels, two universally compatible ones s_{12345}, s_{345}, and two incompatible pairs s_{12}, s_{2345} as well as s_{34}, s_{45}.

$$Z(1, 2, 5, 4, 3, 7, 6) \sim \frac{1}{s_{12345} s_{345}} \left(\frac{1}{s_{12}} + \frac{1}{s_{2345}} \right) \left(\frac{1}{s_{34}} + \frac{1}{s_{45}} \right) \qquad \text{(C.13)}$$

- $Z(1, 2, 4, 5, 3, 7, 6)$ has five pole channels, $s_{12345}, s_{345}, s_{45}$ being universally compatible and s_{12}, s_{2345} as an incompatible pair:

$$Z(1, 2, 4, 5, 3, 7, 6) \sim -\frac{1}{s_{12345} s_{345} s_{45}} \left(\frac{1}{s_{12}} + \frac{1}{s_{2345}} \right) \qquad \text{(C.14)}$$

- $Z(1, 3, 5, 4, 2, 7, 6)$ has four pole channels $s_{45}, s_{345}, s_{2345}, s_{12345}$ leading to a unique diagram

$$Z(1, 3, 5, 4, 2, 7, 6) \sim \frac{1}{s_{45} s_{345} s_{2345} s_{12345}} \qquad \text{(C.15)}$$

- $Z(1, 3, 5, 2, 4, 7, 6)$ has two pole channels s_{2345}, s_{12345} such that its leading low-energy contribution occurs at ζ_2 order

$$Z(1, 3, 5, 2, 4, 7, 6) \sim \frac{\zeta_2}{s_{2345} s_{12345}} \qquad \text{(C.16)}$$

The $\nu \leq 4$ sectors of the seven-point KK basis additionally involve integrals with leading transcendentality ζ_3 and ζ_4, respectively, e.g.

$$Z(1, 2, 5, 3, 7, 4, 6) \sim \frac{2\zeta_3}{s_{12}}, \qquad Z(1, 5, 3, 7, 2, 4, 6) \sim -\frac{17}{4} \zeta_4. \qquad \text{(C.17)}$$

Further examples of integrals with leading low-energy behavior $\sim \zeta_{N-3-k} \prod_{i=1}^{k} (s_{a_i, a_i+1, \dots, b_i})^{-1}$ can be found in Subsect. 4.2.

D   Singularity structure of further six-point integrals

This appendix supplements the discussion of Subsect. 4.3.3 on residues in six-point integrals. We investigate pole structures present in the $\nu = 4$ sector (4.16) of the pole basis which we can relate to the F^σ via (3.34):

$$Z[s_{12}s_{123}s_{1234}] = \frac{1}{s_{1234}} \left(\frac{F^{(234)}}{s_{12}s_{123}} + \frac{F^{(243)}}{s_{12}s_{124}} + \frac{F^{(324)}}{s_{13}s_{123}} + \frac{F^{(342)}}{s_{13}s_{134}} + \frac{F^{(423)}}{s_{14}s_{124}} + \frac{F^{(432)}}{s_{14}s_{134}} \right)$$

$$Z[s_{12}s_{34}s_{1234}] = \frac{1}{s_{1234}} \left(\frac{F^{(234)}}{s_{12}s_{34}} + \frac{F^{(342)}}{s_{34}s_{134}} - \frac{F^{(432)}}{s_{134}} \left(\frac{1}{s_{14}} + \frac{1}{s_{34}} \right) \right.$$
$$\left. - \frac{F^{(423)}}{s_{14}s_{124}} - \frac{F^{(243)}}{s_{12}} \left(\frac{1}{s_{124}} + \frac{1}{s_{34}} \right) \right)$$

$$Z[s_{23}s_{123}s_{1234}] = \frac{1}{s_{1234}} \left(\frac{F^{(234)}}{s_{23}s_{123}} - \frac{F^{(324)}}{s_{123}} \left(\frac{1}{s_{13}} + \frac{1}{s_{23}} \right) - \frac{F^{(342)}}{s_{13}s_{134}} \right.$$
$$\left. + \frac{F^{(423)}}{s_{14}s_{23}} - \frac{F^{(432)}}{s_{14}} \left(\frac{1}{s_{23}} + \frac{1}{s_{134}} \right) \right)$$

$$Z[s_{34}s_{234}s_{1234}] = \frac{1}{s_{1234}} \left(\frac{F^{(234)}}{s_{34}s_{234}} - F^{(243)} \left(\frac{1}{s_{34}s_{234}} + \frac{1}{s_{24}s_{124}} + \frac{1}{s_{24}s_{234}} \right) + \frac{F^{(324)}}{s_{24}s_{234}} \right. \tag{D.1}$$
$$- F^{(342)} \left(\frac{1}{s_{34}s_{134}} + \frac{1}{s_{24}s_{234}} + \frac{1}{s_{34}s_{234}} \right)$$
$$+ F^{(423)} \left(\frac{1}{s_{14}s_{124}} + \frac{1}{s_{24}s_{124}} + \frac{1}{s_{24}s_{234}} \right)$$
$$\left. + F^{(432)} \left(\frac{1}{s_{14}s_{134}} + \frac{1}{s_{34}s_{134}} + \frac{1}{s_{34}s_{234}} \right) \right)$$

$$Z[s_{23}s_{234}s_{1234}] = \frac{1}{s_{1234}} \left(\frac{F^{(234)}}{s_{23}s_{234}} + \frac{F^{(243)}}{s_{24}} \left(\frac{1}{s_{124}} + \frac{1}{s_{234}} \right) - \frac{F^{(324)}}{s_{234}} \left(\frac{1}{s_{23}} + \frac{1}{s_{24}} \right) + \frac{F^{(342)}}{s_{24}s_{234}} \right.$$
$$- F^{(423)} \left(\frac{1}{s_{14}s_{23}} + \frac{1}{s_{14}s_{124}} + \frac{1}{s_{23}s_{234}} + \frac{1}{s_{124}s_{24}} + \frac{1}{s_{24}s_{234}} \right)$$
$$\left. + \frac{F^{(432)}}{s_{23}} \left(\frac{1}{s_{14}} + \frac{1}{s_{234}} \right) \right)$$

$$Z[\zeta_2 s_{1234}] = \frac{1}{s_{1234}} \left(\frac{F^{(243)}}{s_{24}s_{124}} + \frac{F^{(324)}}{s_{13}s_{24}} - \frac{F^{(342)}}{s_{13}} \left(\frac{1}{s_{134}} + \frac{1}{s_{24}} \right) \right.$$
$$\left. - \frac{F^{(423)}}{s_{124}} \left(\frac{1}{s_{14}} + \frac{1}{s_{24}} \right) - \frac{F^{(432)}}{s_{14}s_{134}} \right)$$

Figures 11 and 12 summarize the diagrams contributing to the singular part of $Z[s_{12}s_{123}s_{1234}]$ and $Z[s_{12}s_{34}s_{1234}]$. They cover the two topologies possible for cubic six point graphs. The methods of Subsect. 4.3 yield the following residues for the six functions in (D.1):

$$Z[s_{12}s_{123}s_{1234}] = \frac{1}{s_{12}s_{123}s_{1234}} + \frac{I_{21}^{\text{reg}}(k_1, k_2, k_3)}{s_{123}s_{1234}} + \frac{I_{21}^{\text{reg}}(k_{12}, k_3, k_4)}{s_{12}s_{1234}} + \frac{I_{21}^{\text{reg}}(k_{123}, k_4, k_5)}{s_{12}s_{123}}$$

$$+ \frac{I_{21,31}^{\mathrm{reg}}(k_1, k_2, k_3, k_4)}{s_{1234}} + \frac{I_{21}^{\mathrm{reg}}(k_1, k_2, k_3) I_{21}^{\mathrm{reg}}(k_{123}, k_4, k_5)}{s_{123}}$$

$$+ \frac{I_{21,31}^{\mathrm{reg}}(k_{12}, k_3, k_4, k_5)}{s_{12}} + I_{21,31,41}^{\mathrm{reg}} \tag{D.2}$$

$$Z[s_{12} s_{34} s_{1234}] = \frac{1}{s_{12} s_{34} s_{1234}} + \frac{I_{21}^{\mathrm{reg}}(k_1, k_2, k_{34})}{s_{34} s_{1234}} + \frac{I_{32}^{\mathrm{reg}}(k_{12}, k_3, k_4)}{s_{12} s_{1234}} + \frac{I_{21}^{\mathrm{reg}}(k_{12}, k_{34}, k_5)}{s_{12} s_{34}}$$

$$+ \frac{I_{21,43}^{\mathrm{reg}}(k_1, k_2, k_3, k_4)}{s_{1234}} + \frac{I_{21,31}^{\mathrm{reg}}(k_1, k_2, k_{34}, k_5)}{s_{34}}$$

$$+ \frac{I_{31,32}^{\mathrm{reg}}(k_{12}, k_3, k_4, k_5)}{s_{12}} + I_{21,43,41}^{\mathrm{reg}} \tag{D.3}$$

$$Z[s_{23} s_{123} s_{1234}] = \frac{1}{s_{23} s_{123} s_{1234}} + \frac{I_{21}^{\mathrm{reg}}(k_{123}, k_4, k_5)}{s_{23} s_{123}} + \frac{I_{32}^{\mathrm{reg}}(k_1, k_2, k_3)}{s_{123} s_{1234}} + \frac{I_{21}^{\mathrm{reg}}(k_1, k_{23}, k_4)}{s_{23} s_{1234}}$$

$$+ \frac{I_{21,31}^{\mathrm{reg}}(k_1, k_{23}, k_4, k_5)}{s_{23}} + \frac{I_{21}^{\mathrm{reg}}(k_{123}, k_4, k_5) I_{32}^{\mathrm{reg}}(k_1, k_2, k_3)}{s_{123}}$$

$$+ \frac{I_{31,32}^{\mathrm{reg}}(k_1, k_2, k_3, k_4)}{s_{1234}} + I_{31,32,41}^{\mathrm{reg}} \tag{D.4}$$

$$Z[s_{23} s_{234} s_{1234}] = \frac{1}{s_{23} s_{234} s_{1234}} + \frac{I_{21}^{\mathrm{reg}}(k_1, k_{234}, k_5)}{s_{23} s_{234}} + \frac{I_{32}^{\mathrm{reg}}(k_1, k_{23}, k_4)}{s_{23} s_{1234}} + \frac{I_{21}^{\mathrm{reg}}(k_2, k_3, k_4)}{s_{234} s_{1234}}$$

$$+ \frac{I_{23,24}^{\mathrm{reg}}(k_1, k_2, k_3, k_4)}{s_{1234}} + \frac{I_{21}^{\mathrm{reg}}(k_1, k_{234}, k_5) I_{21}^{\mathrm{reg}}(k_2, k_3, k_4)}{s_{234}}$$

$$+ \frac{I_{31,32}^{\mathrm{reg}}(k_1, k_{23}, k_4, k_5)}{s_{23}} + I_{32,41,42}^{\mathrm{reg}} \tag{D.5}$$

$$Z[s_{34} s_{234} s_{1234}] = \frac{1}{s_{34} s_{234} s_{1234}} + \frac{I_{21}^{\mathrm{reg}}(k_1, k_{234}, k_5)}{s_{34} s_{234}} + \frac{I_{32}^{\mathrm{reg}}(k_2, k_3, k_4)}{s_{1234} s_{234}} + \frac{I_{32}^{\mathrm{reg}}(k_1, k_2, k_{34})}{s_{34} s_{1234}}$$

$$+ \frac{I_{31,32}^{\mathrm{reg}}(k_1, k_2, k_{34}, k_5)}{s_{34}} + \frac{I_{21}^{\mathrm{reg}}(k_1, k_{234}, k_5) I_{32}^{\mathrm{reg}}(k_2, k_3, k_4)}{s_{234}}$$

$$+ \frac{I_{42,43}^{\mathrm{reg}}(k_1, k_2, k_3, k_4)}{s_{1234}} + I_{41,42,43}^{\mathrm{reg}} \tag{D.6}$$

$$Z[\zeta_2 s_{1234}] = \frac{I_{31,42}^{\mathrm{reg}}(k_1, k_2, k_3, k_4)}{s_{1234}} + I_{31,41,42}^{\mathrm{reg}} \tag{D.7}$$

Following the off-shell completion (4.40) of the quartic contact vertices, the non-vanishing mass contributions to I_{21}^{reg} and I_{23}^{reg} are given by

$$I_{21}^{\mathrm{reg}}(k_1, k_{23}, k_4) = I_{21}^{\mathrm{reg}}[s_{123}, s_{24} + s_{34}], \qquad I_{21}^{\mathrm{reg}}(k_1, k_{234}, k_5) = I_{21}^{\mathrm{reg}}[s_{1234}, s_{25} + s_{35} + s_{45}]$$

$$I_{21}^{\mathrm{reg}}(k_{12}, k_{34}, k_5) = I_{21}^{\mathrm{reg}}[s_{13} + s_{14} + s_{23} + s_{24} + s_{34}, s_{35} + s_{45}] \tag{D.8}$$

$$I_{32}^{\mathrm{reg}}(k_1, k_{23}, k_4) = I_{21}^{\mathrm{reg}}[s_{234}, s_{12} + s_{13}]$$

Also, we discover additional off-shell completions for five-point contact vertices

$$I_{21,31}^{\mathrm{reg}}(k_1, k_2, k_{34}, k_5) = I_{21,31}^{\mathrm{reg}}[s_{12}, s_{134}, s_{23} + s_{24}, s_{25}, s_{35} + s_{45}] \tag{D.9}$$
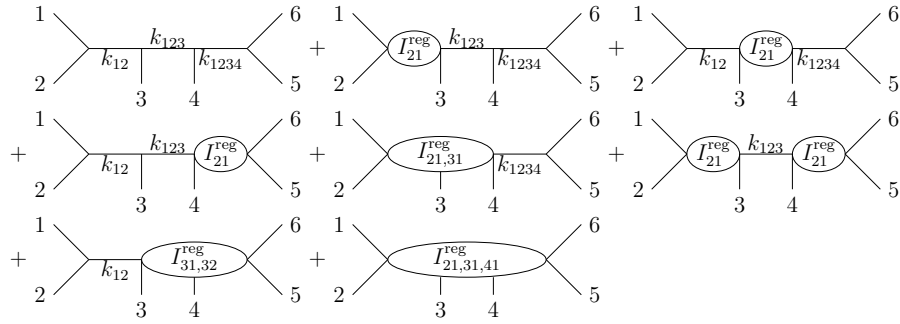
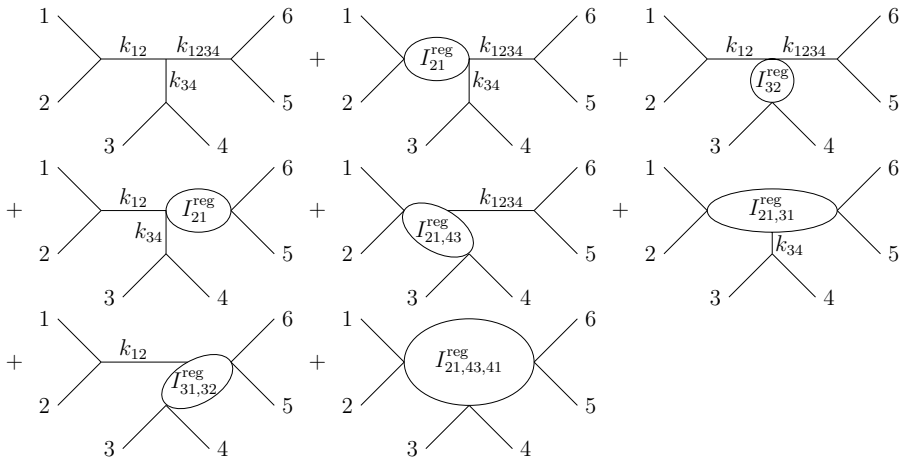**Fig. 11** Pole structure of the function $Z[s_{12}s_{123}s_{1234}]$.

Fig. 12 Pole structure of the function $Z[s_{12}s_{34}s_{1234}]$.

$$I_{21,31}^{\text{reg}}(k_1, k_{23}, k_4, k_5) = I_{21,31}^{\text{reg}}[s_{123}, s_{14}, s_{24} + s_{34}, s_{25} + s_{35}, s_{45}] \tag{D.10}$$

$$I_{31,32}^{\text{reg}}(k_1, k_2, k_{34}, k_5) = I_{31,32}^{\text{reg}}[s_{12}, s_{134}, s_{23} + s_{24}, s_{25}, s_{35} + s_{45}] . \tag{D.11}$$

Unfortunately, the regular part $I_{32,41,42}^{\text{reg}}$ is incompatible with the prerequisite (5.1) for the polylogarithm methods of Sect. 5, hence, we cannot determine the dependence of $I_{31,32}^{\text{reg}}(k_1, k_{23}, k_4, k_5)$ on k_{23}^2. The remaining six-point vertices $I_{21,31,41}^{\text{reg}}, I_{21,43,41}^{\text{reg}}, I_{31,32,41}^{\text{reg}}, I_{41,42,43}^{\text{reg}}$ and $I_{31,41,42}^{\text{reg}}$, on the other hand, can be inferred by equating the pole structures (D.2) to (D.7) with the $F^{\sigma(234)}$ basis expansions (D.1) of the $Z[\ldots]$ functions. The low-energy expansion of these $I_{\ldots}^{\text{reg}}$ is relevant for seven point pole residues, so we can increase the efficiency of our setup by extracting it from the six point integral basis.

E Polylogarithm material

In this part of the appendix, we collect some examples of the identities derivable from Eq. (5.21) at weights two and three. Note that the idenities containing $0's$ are treated separately and thus $a_k \neq 0$ below:

$$G(a_1, z; z) = -G(0, a_1; z) + G(a_1, a_1; z) \tag{E.1}$$

$$G(0, z, a_1; z) = G(0, 0, a_1; z) - G(a_1, 0, a_1; z) - G(a_1; z)\zeta_2$$

$$G(0, a_1, z; z) = -2\,G(0, 0, a_1; z) + G(0, a_1, a_1; z) + G(a_1, 0, a_1; z) + G(a_1; z)\,\zeta_2$$

$$G(a_1, 0, z; z) = G(0, 0, a_1; z) - G(0, a_1, a_1; z) - G(a_1; z)\,\zeta_2$$

$$G(a_1, z, z; z) = G(0, 0, a_1; z) - G(0, a_1, a_1; z) - G(a_1, 0, a_1; z) + G(a_1, a_1, a_1; z)$$

$$G(a_1, z, a_2; z) = G(a_1, a_1, a_2; z) - G(a_2, 0, a_1; z) + G(a_2, a_1, a_1; z) - G(a_2, a_1, a_2; z)$$

$$G(a_1, a_2, z; z) = -G(0, a_1, a_2; z) - G(a_1, 0, a_2; z) + G(a_1, a_2, a_2; z)$$

$$+ G(a_2, 0, a_1; z) - G(a_2, a_1, a_1; z) + G(a_2, a_1, a_2; z)\,. \tag{E.2}$$

These identities are given only for polylogs with the first entry not equal to the argument and, at the same time, nonzero last entry. The remaining polylogs have to be dealt with using the methods of Eq. (5.8) and Eq. (5.10) beforehand.

F Seven-point material

This appendix closes some of the gaps in the presentation of the seven-point α'-expansion.

F.1 The pole structure of integrals from the $\{2, 4\}$ and $\{4, 5\}$ blocks

In this appendix we show the singularity structure of seven point integrals from the $\{2, 4\}$ and $\{4, 5\}$ blocks which were omitted in Subsect. 4.3.4.

The $\{2, 4\}$ functions are characterized by polynomials $\in \{z_{21}z_{41}, z_{41}z_{42}\} \times \{z_{53}z_{65}, z_{63}z_{65}\}$

$$\prod_{i=2}^{5} \int_0^{z_{i+1}} \mathrm{d}z_i \, \frac{\prod_{i<j}^{6} |z_{ij}|^{s_{ij}}}{z_{21}z_{35}z_{41}z_{56}} = \frac{I_{31,42}^{\mathrm{reg}}(k_{12}, k_3, k_4, k_{56})}{s_{12}s_{56}} + \frac{I_{31,42,54}^{\mathrm{reg}}(k_{12}, k_3, k_4, k_5, k_6)}{s_{12}}$$

$$+ \frac{I_{21,41,53}^{\mathrm{reg}}(k_1, k_2, k_3, k_4, k_{56})}{s_{56}} + I_{21,41,53,65}^{\mathrm{reg}} \tag{F.1}$$

$$\prod_{i=2}^{5} \int_0^{z_{i+1}} \mathrm{d}z_i \, \frac{\prod_{i<j}^{6} |z_{ij}|^{s_{ij}}}{z_{24}z_{35}z_{41}z_{56}} = - \frac{I_{41,42,53}^{\mathrm{reg}}(k_1, k_2, k_3, k_4, k_{56})}{s_{56}} - I_{41,42,53,65}^{\mathrm{reg}} \tag{F.2}$$

$$\prod_{i=2}^{5} \int_0^{z_{i+1}} \mathrm{d}z_i \, \frac{\prod_{i<j}^{6} |z_{ij}|^{s_{ij}}}{z_{21}z_{36}z_{41}z_{56}} = \frac{I_{31,42}^{\mathrm{reg}}(k_{12}, k_3, k_4, k_{56})}{s_{12}s_{56}} + \frac{I_{31,52,54}^{\mathrm{reg}}(k_{12}, k_3, k_4, k_5, k_6)}{s_{12}}$$

$$+ \frac{I_{21,41,53}^{\mathrm{reg}}(k_1, k_2, k_3, k_4, k_{56})}{s_{56}} + I_{21,41,63,65}^{\mathrm{reg}} \tag{F.3}$$

$$\prod_{i=2}^{5} \int_0^{z_{i+1}} \mathrm{d}z_i \, \frac{\prod_{i<j}^{6} |z_{ij}|^{s_{ij}}}{z_{24}z_{36}z_{41}z_{56}} = - \frac{I_{41,42,53}^{\mathrm{reg}}(k_1, k_2, k_3, k_4, k_{56})}{s_{56}} - I_{41,42,63,65}^{\mathrm{reg}} \tag{F.4}$$

without any mass dependence.

The $\{4, 5\}$ functions are characterized by polynomials $\in \{z_{41}z_{51}, z_{51}z_{54}\} \times \{z_{32}z_{63}, z_{62}z_{63}\}$

$$\prod_{i=2}^{5} \int_0^{z_{i+1}} \mathrm{d}z_i \, \frac{\prod_{i<j}^{6} |z_{ij}|^{s_{ij}}}{z_{32}z_{63}z_{51}z_{54}} = \frac{I_{31,42}^{\mathrm{reg}}(k_1, k_{23}, k_{45}, k_6)}{s_{23}s_{45}} + \frac{I_{41,43,52}^{\mathrm{reg}}(k_1, k_{23}, k_4, k_5, k_6)}{s_{23}}$$

$$+ \frac{I_{32,41,53}^{\mathrm{reg}}(k_1, k_2, k_3, k_{45}, k_6)}{s_{45}} + I_{32,51,54,63}^{\mathrm{reg}} \tag{F.5}$$

$$\prod_{i=2}^{5}\int_{0}^{z_{i+1}}\mathrm{d}z_{i}\,\frac{\prod_{i<j}^{6}|z_{ij}|^{s_{ij}}}{z_{32}z_{63}z_{41}z_{51}} = \frac{I_{31,41,52}^{\mathrm{reg}}(k_{1},k_{23},k_{4},k_{5},k_{6})}{s_{23}} + I_{32,41,51,63}^{\mathrm{reg}} \tag{F.6}$$

$$\prod_{i=2}^{5}\int_{0}^{z_{i+1}}\mathrm{d}z_{i}\,\frac{\prod_{i<j}^{6}|z_{ij}|^{s_{ij}}}{z_{62}z_{63}z_{51}z_{54}} = \frac{I_{41,52,53}^{\mathrm{reg}}(k_{1},k_{2},k_{3},k_{45},k_{6})}{s_{45}} + I_{51,54,62,63}^{\mathrm{reg}} \tag{F.7}$$

$$\prod_{i=2}^{5}\int_{0}^{z_{i+1}}\mathrm{d}z_{i}\,\frac{\prod_{i<j}^{6}|z_{ij}|^{s_{ij}}}{z_{62}z_{63}z_{41}z_{51}} = I_{41,51,62,63}^{\mathrm{reg}} \tag{F.8}$$

with mass dependence

$$I_{31,42}^{\mathrm{reg}}(k_{1},k_{23},k_{45},k_{6}) = I_{31,42}^{\mathrm{reg}}[s_{123},s_{145},s_{2345}-s_{23}-s_{45},s_{26}+s_{36},s_{46}+s_{56}] \tag{F.9}$$

$$I_{41,43,52}^{\mathrm{reg}}(k_{1},k_{23},k_{4},k_{5},k_{6}) = I_{41,43,52}^{\mathrm{reg}}[s_{123},s_{14},s_{15},s_{24}+s_{34},s_{25}+s_{35},s_{45},s_{26}+s_{36},s_{46},s_{56}]$$

$$I_{32,41,53}^{\mathrm{reg}}(k_{1},k_{2},k_{3},k_{45},k_{6}) = I_{32,41,53}^{\mathrm{reg}}[s_{12},s_{13},s_{145},s_{23},s_{24}+s_{25},s_{34}+s_{35},s_{26},s_{36},s_{46}+s_{56}]$$

$$I_{31,41,52}^{\mathrm{reg}}(k_{1},k_{23},k_{4},k_{5},k_{6}) = I_{31,41,52}^{\mathrm{reg}}[s_{123},s_{14},s_{15},s_{24}+s_{34},s_{25}+s_{35},s_{45},s_{26}+s_{36},s_{46},s_{56}]$$

$$I_{41,52,53}^{\mathrm{reg}}(k_{1},k_{2},k_{3},k_{45},k_{6}) = I_{41,52,53}^{\mathrm{reg}}[s_{12},s_{13},s_{145},s_{23},s_{24}+s_{25},s_{34}+s_{35},s_{26},s_{36},s_{46}+s_{56}]$$

F.2 Cyclic transformations towards functions from the $\{2,3\}$ block

According to Subsect. 6.3.2, the seven point basis functions $F^{(3245)}$ and $F^{(3254)}$ can be inferred from cyclicity. The required identities are

$$\begin{aligned}
F^{(3425)}\Big|_{i\mapsto i+1} =\ & \frac{F^{(3245)}\left(s_{36}+s_{37}\right)}{s_{13}} + F^{(2453)} - \frac{F^{(4325)}s_{24}\left(s_{36}+s_{37}\right)}{s_{14}s_{134}} - \frac{F^{(4253)}\left(s_{14}+s_{24}\right)}{s_{14}} \\
& - \frac{F^{(3425)}\left(s_{134}+s_{24}\right)\left(s_{36}+s_{37}\right)}{s_{13}s_{134}} - \frac{F^{(5324)}\left(s_{36}+s_{37}\right)\left(s_{25}+s_{45}\right)}{s_{15}s_{135}} \\
& + \frac{F^{(5423)}\left(s_{145}+s_{24}\right)\left(s_{1245}-s_{124}\right)}{s_{15}s_{145}} - \frac{F^{(5243)}\left(s_{1245}-s_{124}\right)}{s_{15}} + \frac{F^{(4523)}s_{24}\left(s_{1245}-s_{124}\right)}{s_{14}s_{145}} \\
& - \frac{F^{(3524)}\left(s_{36}+s_{37}\right)\left(s_{135}+s_{25}+s_{45}\right)}{s_{13}s_{135}} + \frac{F^{(5342)}\left(s_{36}+s_{37}\right)\left(s_{25}+s_{45}\right)\left(s_{1345}+s_{24}\right)}{s_{15}s_{135}s_{1345}} \\
& + \frac{F^{(3542)}\left(s_{36}+s_{37}\right)\left(s_{135}+s_{25}+s_{45}\right)\left(s_{1345}+s_{24}\right)}{s_{13}s_{135}s_{1345}} + \frac{F^{(4352)}s_{24}\left(s_{36}+s_{37}\right)\left(s_{1345}+s_{25}\right)}{s_{14}s_{134}s_{1345}} \\
& + \frac{F^{(5432)}s_{24}\left(s_{36}+s_{37}\right)\left(s_{1245}-s_{124}\right)}{s_{15}s_{145}s_{1345}} - \frac{F^{(3452)}s_{24}\left(s_{36}+s_{37}\right)\left(s_{56}+s_{57}\right)}{s_{13}s_{134}s_{1345}} \\
& - \frac{F^{(4532)}s_{24}\left(s_{14}-s_{25}\right)\left(s_{36}+s_{37}\right)}{s_{14}s_{145}s_{1345}}\,. \tag{F.10}
\end{aligned}$$

$$\begin{aligned}
F^{(4235)}\Big|_{i\mapsto i+1} =\ & -\frac{F^{(3254)}\left(s_{13}+s_{23}+s_{35}\right)}{s_{13}} + F^{(2534)} + \frac{F^{(3524)}\left(s_{13}+s_{23}+s_{35}\right)\left(s_{135}+s_{25}\right)}{s_{13}s_{135}} \\
& - \frac{F^{(5234)}\left(s_{15}+s_{25}\right)}{s_{15}} + \frac{F^{(5324)}s_{25}\left(s_{13}+s_{23}+s_{35}\right)}{s_{15}s_{135}} - \frac{F^{(3425)}\left(s_{23}+s_{35}\right)\left(s_{46}+s_{47}\right)}{s_{13}s_{134}} \\
& - \frac{F^{(5423)}s_{25}\left(s_{46}+s_{47}\right)}{s_{15}s_{145}} - \frac{F^{(4523)}\left(s_{145}+s_{25}\right)\left(s_{46}+s_{47}\right)}{s_{14}s_{145}} + \frac{F^{(4253)}\left(s_{46}+s_{47}\right)}{s_{14}}
\end{aligned}$$

$$- \frac{F^{(4325)} \left(s_{134} + s_{23} + s_{35}\right) \left(s_{46} + s_{47}\right)}{s_{14} s_{134}} + \frac{F^{(5432)} s_{25} \left(s_{46} + s_{47}\right) \left(s_{1345} + s_{23}\right)}{s_{15} s_{145} s_{1345}}$$

$$+ \frac{F^{(3452)} \left(s_{23} + s_{35}\right) \left(s_{46} + s_{47}\right) \left(s_{1345} + s_{25}\right)}{s_{13} s_{134} s_{1345}} + \frac{F^{(3542)} s_{25} \left(s_{13} + s_{23} + s_{35}\right) \left(s_{46} + s_{47}\right)}{s_{13} s_{135} s_{1345}}$$

$$+ \frac{F^{(4352)} \left(s_{134} + s_{23} + s_{35}\right) \left(s_{46} + s_{47}\right) \left(s_{1345} + s_{25}\right)}{s_{14} s_{134} s_{1345}} - \frac{F^{(5342)} \left(s_{15} - s_{23}\right) s_{25} \left(s_{46} + s_{47}\right)}{s_{15} s_{135} s_{1345}}$$

$$- \frac{F^{(4532)} s_{25} \left(s_{36} + s_{37}\right) \left(s_{46} + s_{47}\right)}{s_{14} s_{145} s_{1345}} . \tag{F.11}$$

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