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A Decay Estimate for a Wave Equation with Trapping and a Complex Potential

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In this brief note, we consider a wave equation that has both trapping and a complex potential. For this problem, we prove a uniform bound on the energy and a Morawetz (or integrated local energy decay) estimate. The equation is a model problem for certain scalar equations appearing in the Maxwell and linearized Einstein systems on the exterior of a rotating black hole.

1 Introduction

We consider the Cauchy problem:

$$(-\partial_t^2 + \partial_x^2 + V(\Delta_\omega - N) + \mathrm{i}\epsilon W)u = 0, \tag{1}$$

$$u(0, x, \omega) = \psi_0(x, \omega), \quad \partial_t u(0, x, \omega) = \psi_1(x, \omega), \tag{2}$$

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on $(t, x, \omega) \in M = \mathbb{R} \times \mathbb{R} \times S^2$ with smooth, compactly supported initial data. Here, u is a complex function $u = v + \mathrm{i}w$,

$$V = \frac{1}{x^2 + 1},$$

W is a smooth, real-valued, compactly supported function which is nonvanishing at $x = 0$ and uniformly bounded by 1, and $\epsilon > 0$ is a small parameter. Finally, Δ_ω is the Laplacian in the angular variables and N is a number chosen to be sufficiently large to allow us to avoid certain technical problems.

Equation (1) has both trapping, which occurs at $x = 0$, and a complex potential, which does not vanish at the trapped set. The interaction of these creates problems, which appear to frustrate the use of energy and Morawetz estimates at the classical level. By adapting known pseudodifferential methods, we show how to overcome these problems. We now state our main result in terms of the energy

$$E(t) = \frac{1}{2} \int_{\{t\} \times \mathbb{R} \times S^2} |\partial_t u|^2 + |\partial_x u|^2 + V(|\nabla_\omega u|^2 + N|u|^2)\, \mathrm{d}x\, \mathrm{d}^2\omega.$$

Theorem 1. There is a constant C such that, if ψ_0 and ψ_1 are such that $E(0)$ is finite, then

$$\forall t \in \mathbb{R} : E(t) \leq C E(0), \tag{3a}$$

$$\int_M \frac{|\partial_x u|^2}{x^2 + 1} + \frac{x^2}{1 + x^2} \left(\frac{|\nabla_\omega u|^2}{1 + |x|^3} + \frac{|\partial_t u|^2}{x^2 + 1} \right) + \frac{|u|^2}{1 + |x|^3}\, \mathrm{d}t\, \mathrm{d}x\, \mathrm{d}^2\omega \leq C E(0), \tag{3b}$$

$$\int_M \frac{|u||\partial_t u|}{1 + |x|^3}\, \mathrm{d}t\, \mathrm{d}x\, \mathrm{d}^2\omega \leq C E(0). \tag{3c}$$

□

Since Equation (1) has t independent coefficients, one might naively think that Noether's theorem provides a positive conserved energy. However, for the Lagrangian $\mathcal{L}_1[u, \partial u] = -(\partial_t u)^2 + (\partial_x u)^2 + V(\nabla_\omega u \cdot \nabla_\omega u + N u^2) - \mathrm{i}\epsilon W u^2$, which has the wave Equation (1) as its Euler–Lagrange equation, the conserved quantity associated to the time translation symmetry is indefinite, being approximately the energy of the real component of u minus the energy of the imaginary component (plus ϵ times a term involving Wvw). On the other hand, a Lagrangian of the form $\mathcal{L}_2[u, \partial u] = -|\partial_t u|^2 + |\partial_x u|^2 + V(|\nabla_\omega u|^2 + N|u|^2)$, which corresponds to the energy expression considered above, does not yield Equation (1) as its Euler–Lagrange equation.

In our proof, it is crucial that the three estimates (3a)–(3c) are proved simultaneously, since the decay estimates (3b)–(3c) are required to prove the energy bound (3a), and the decay estimates require a uniform bound on $E(t)$. To establish these, we combine a Fourier-transform-in-time technique (as in [7, 26]) with a "modulation" (or Fourier-rescaling) technique (from [5]).

The wave Equation (1) is a model for equations arising in the study of the Maxwell and linearized Einstein equations outside a Kerr black hole. The geometry of the Kerr black hole has trapping. Certain components of the Maxwell and linearized Einstein equations can be shown to satisfy wave equations with complex potentials. The imaginary part of these potentials vanishes linearly in the parameter a, which is explained below.

The Kerr black holes are a family of Lorentzian manifolds arising in general relativity, and they are characterized by a mass parameter M and an angular momentum parameter a. Black holes are believed to be the enormously massive objects at the center of most galaxies. The case $|a| \leq M$ is the physically relevant one. The case $a = 0$ is the Schwarzschild class of black holes.

It is expected that every uncharged black hole will asymptotically approach a Kerr solution under the dynamics generated by the Einstein equations of general relativity. The wave, Maxwell, and linearized Einstein equations on a fixed Kerr geometry are a sequence of increasingly accurate models for these dynamics. By projecting on a null tetrad, the Maxwell and linearized Einstein fields can be decomposed into sets of complex scalars, the Newman–Penrose (NP) scalars [14, 20, 21]. It is well known that the NP scalars with extreme spin weights satisfy decoupled wave equations, known as the Teukolsky equations, and that the solutions to these reduced equations can be used to reconstruct the full system [27].

For the Maxwell field on the Kerr background, the spin weight 0 NP scalar can be treated in the same way, and the resulting equation is known as the Fackerell–Ipser equation [11]. For linearized gravity on the Schwarzschild background, it is also well known that the imaginary part of the spin weight 0 NP scalar is governed by a wave equation, the Regge–Wheeler equation [22, 24]. The corresponding equation for the real part is more complicated, cf. [19, 32], see also [1].

It was recently shown [1] that in the general ($|a| < M$) Kerr case, by imposing a gauge condition related to the wave coordinates gauge, the equation for both the real and imaginary parts of the spin weight 0 NP scalar of the linearized gravitational field may be put in a form analogous to the Regge–Wheeler and Fackerell–Ipser equations. Explicitly (in the Kerr space-time with signature $-+++$, working in Boyer–Lindquist coordinates) these take the form

$$\left(\nabla^{\alpha}\nabla_{\alpha} + 2s^2 \frac{M}{(r - \mathrm{i}a\cos\theta)^3}\right) u = 0, \tag{4}$$

where $s=0$ corresponds to the free scalar wave equation, $s=1$ corresponds to the Maxwell (Fackerell–Ipser) case, while $s=2$ corresponds to the linearized gravity (generalized Regge–Wheeler) case. In particular, for the $a \neq 0$ cases, the analogues of the Regge–Wheeler equations have complex potentials, with the imaginary part depending continuously on a.

For the wave equation in the Schwarzschild case, the use of the energy estimate [30], like (3a) with $C=1$, and Morawetz estimates (which are also called integrated local energy estimates) are well established [3, 4, 6, 8, 16]. In the Morawetz estimate (3b), there is a loss of control of time and angular derivatives near $x=0$, in the sense that the integrand cannot control $|x|^p(|u|^{q_t}|\partial_t u|^{2-q_t} + |u|^{q_\omega}|\nabla_\omega u|^{2-q_\omega})$ with both $p=0$ and either $q_t=0$ or $q_\omega=0$. The presence of trapping makes some loss unavoidable [23]. By applying "angular modulation" and "phase space analysis", the range for the angular parameter q_ω can be refined to $p=0$ and $q>0$ [5]. This type of refinement is crucial in the current paper, since estimate (3b) is insufficient to establish the energy bound (3a). Alternatively, certain pseudodifferential operators have been used to obtain refinements near $x=0$, to $p>0$ and $q_t=q_\omega=0$ [18].

For the wave equation in the general ($|a|<M$) Kerr case, it is possible to apply Fourier transforms first in the ϕ and t variables (Here ϕ is the azimuthal angle, which would be one component of ω in the notation of this paper.) and then the remaining variables. The individual ϕ modes decay pointwise [12]. Although the problem has a time-translation symmetry, because the generator of time translations fails to be a time-like vector with respect to the Lorentzian inner product of the Kerr geometry, there is no positive, conserved energy. A major advance was the proof that, in the slowly rotating case $|a| \ll M$, there is a uniform energy bound, like estimate (3a). The first proof used an estimate similar to (3b), but with additional restrictions on the support of the Fourier transform [7]. Independent work [26] established estimates similar to (3a) and (3b), but with no restriction on the Fourier support, and there were subsequent pseudodifferential refinements [28]. Also, the first two authors have proved similar results using methods which require two additional levels of regularity but which completely avoid the use of Fourier transforms. Morawetz estimates and refinements are a crucial step in proving pointwise decay estimates [5, 6, 8, 9, 17] and Strichartz estimates [18, 28], including the long-conjectured, inverse-cubic, Price law [25, 29].

The study of the Maxwell and linearized Einstein systems in the Kerr geometry is still in its infancy. For the general Kerr case, a certain transformed, separated version of the Teukolsky system has no exponentially growing modes [31]. In the Schwarzschild case, the ϕ modes of the Teukolsky equation decay pointwise [13]. Recently, improved decay estimates for the Regge–Wheeler-type Equation (4) on the Schwarzschild background, giving decay rates of t^{-3}, t^{-4}, and t^{-6}, respectively, for $s = 0, 1, 2$, have been proved [10].

Weighted energy estimates have been used to prove decay estimates for the Maxwell field in the Schwarzschild case [2] and for the full (not merely linearized) Einstein equation on asymptotically Schwarzschildean space-times [15]. The estimates for the Maxwell equation used a strategy based on the observation that the spin weight 0 NP scalar is the only one needing to be controlled in weighted energy estimates in order to gain control over the full system. More precisely, it is possible to first prove energy and Morawetz estimates for the spin-weight zero component and then to use these to establish decay for the full Maxwell system. This process of studying the (spin 1) Maxwell system by first studying an equation similar to the (spin 0) wave equation is known as spin reduction. A similar process of spin reduction, involving Maxwell-like and wave-like equations, was used in [15].

The Maxwell field outside a Kerr black hole fails to have a positive, conserved energy. In seeking to prove an energy bound and a Morawetz decay estimate simultaneously for it and the Fackerell–Ipser equation, we have considered two model problems, two causes for the absence of a positive, conserved energy for a time-independent wave equation, and two obstacles to proving a sufficiently strong Morawetz estimate in the presence of trapping. First, for the wave equation in the Kerr geometry, the generator of the time-translation symmetry fails to be time-like everywhere, and the orbiting null geodesics (analogous to trapped rays) fill an open set in the manifold, although their lift to the tangent bundle does not. Second, for a wave equation with a complex potential, there is no positive, conserved energy because no variational argument can provide an energy-momentum tensor that satisfies the dominant energy condition. Furthermore, if there is trapping when the potential is nonzero, a classical Morawetz estimate, of the form (3b), is insufficient to control the growth of the energy, $E(t)$. Any attempt to treat the Maxwell or linearized Einstein equations in the Kerr geometry via spin reduction will have to treat all four of these problems, as well as others, such as the existence of stationary solutions. To treat all four of these problems simultaneously, the variable x would have to be replaced by some pseudodifferential measure of the distance from the trapped geodesics.

As is common, C will be used to denote a constant which may vary from line to line, but which is independent of the choice of u or T. The notation $A \lesssim B$ is used to denote that there is some C such that $A < CB$, with C independent of u and T, and similarly for $\gtrsim$.

2 A Preliminary Energy Estimate

We derive an estimate for an energy for the wave Equation (1) by integrating by parts against $\partial_t \bar{u}$ and following the standard procedure for getting an energy estimate:

$$\begin{aligned}0 &= \mathrm{Re}((\partial_t \bar{u})(-\partial_t^2 + \partial_x^2 + V(\Delta_\omega - N) + \mathrm{i}\epsilon W)u)\\ &= -\tfrac{1}{2}\partial_t |\partial_t u|^2 + \partial_x \mathrm{Re}((\partial_t \bar{u})\partial_x u) - \tfrac{1}{2}\partial_t |\partial_x u|^2 + \nabla_\omega \cdot \mathrm{Re}((\partial_t \bar{u})\nabla_\omega u)\\ &\quad - \tfrac{1}{2}\partial_t (V(|\nabla_\omega u|^2 + N|u|^2)) - \epsilon W \mathrm{Im}((\partial_t \bar{u})u).\end{aligned}$$

Introducing an energy which we denote by

$$E(t) = \frac{1}{2}\int_{\{t\}\times\mathbb{R}\times S^2} |\partial_t u|^2 + |\partial_x u|^2 + V(|\nabla_\omega u|^2 + N|u|^2)\,\mathrm{d}x\,\mathrm{d}^2\omega,$$

assuming that u decays sufficiently rapidly as $|x| \to \infty$, and integrating the previous formula over a region $[t_1, t_2] \times \mathbb{R} \times S^2$, we find

$$E(t_2) - E(t_1) = \int_{[t_1,t_2]\times\mathbb{R}\times S^2} -\epsilon W \mathrm{Im}((\partial_t \bar{u})u)\,\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}^2\omega. \tag{5}$$

In particular, note that the energy fails to be conserved and that an estimate of the form (3b) would be insufficient to control the right-hand side. There is, however, a trivial exponential bound:

$$E(t_2) \le \mathrm{e}^{\epsilon(t_2 - t_1)} E(t_1).$$

3 The Morawetz Estimate

Following the standard procedure for investigating the wave equation, we derive a Morawetz estimate by multiplying the wave equation by $(f(x)\partial_x \bar{u} + q(x)\bar{u})$, where f and q are real-valued functions.

In performing this calculation, it is useful to observe that

$$q'(x)\mathrm{Re}(\bar{u}\,\partial_x u) = \partial_x\left(\frac{q'}{2}\bar{u}u\right) - \frac{1}{2}q''\bar{u}u.$$

Using this and applying the product rule term-by-term, one finds

$$\begin{aligned}&\mathrm{Re}((f\partial_x\bar{u}+q\bar{u})(-\partial_t^2u+\partial_x^2u+V(\Delta_\omega-N)u+\mathrm{i}\epsilon Wu))\\&=\partial_t p_t+\partial_x p_x+\nabla_\omega\cdot p_\omega+(-\tfrac{1}{2}f'+q)|\partial_t u|^2-(\tfrac{1}{2}f'+q)|\partial_x u|^2\\&\quad+((\tfrac{1}{2}f'-q)V+\tfrac{1}{2}f(\partial_x V))|\nabla_\omega u|^2\\&\quad+(N((\tfrac{1}{2}f'-q)V+\tfrac{1}{2}f(\partial_x V))+\tfrac{1}{2}q'')|u|^2-\epsilon fW\,\mathrm{Im}((\partial_x\bar{u})u),\end{aligned}\tag{6}$$

where

$$p_t=p_t(f,q;u)=-\mathrm{Re}((f(\partial_x\bar{u})+q\bar{u})(\partial_t u)),$$

$$p_x=p_x(f,q;u)=\tfrac{1}{2}f|\partial_t u|^2+\tfrac{1}{2}f|\partial_x u|^2-\tfrac{1}{2}fV|\nabla_\omega u|^2+q\mathrm{Re}(\bar{u}\partial_x u)-\tfrac{1}{2}(NfV+q')|u|^2,$$

$$p_w=p_w(f,q;u)=fV\,\mathrm{Re}((\partial_x\bar{u})(\nabla_\omega u))+qV\,\mathrm{Re}(\bar{u}\nabla_\omega u).$$

We take $f=-\arctan(x)$, for which $f'=-(x^2+1)^{-1}=-V$, $f''=2x(x^2+1)^{-2}$, and $f'''=-2(3x^2-1)(x^2+1)^{-3}$. We take $q=f'/2+\delta(1+x^2)^{-1}\arctan(x)^2$ for some sufficiently small δ.

We use the notation

$$E_{f\partial_x+q}(t)=\int_{\{t\}\times\mathbb{R}\times S^2}\mathrm{Re}(f(\partial_x\bar{u})\partial_t u)+\mathrm{Re}(q\bar{u}(\partial_t u))\,\mathrm{d}x\,\mathrm{d}^2\omega,$$

and observe that, by a simple Cauchy–Schwarz argument, there is the estimate $|E_{f\partial_x+q}|\le CE$.

Observing that the left-hand side of (6) vanishes, we have

$$\begin{aligned}0&=\partial_t p_t+\partial_x p_x+\nabla_\omega\cdot p_\omega+\delta\frac{\arctan(x)^2}{1+x^2}|\partial_t u|^2+\frac{1}{1+x^2}(1-\delta\arctan(x)^2)|\partial_x u|^2\\&\quad+\left(\frac{x\arctan(x)-\delta\arctan(x)^2}{(1+x^2)^2}\right)|\nabla_\omega u|^2+\left(N\left(\frac{x\arctan(x)-\delta\arctan(x)^2}{(1+x^2)^2}\right)+\frac{1}{2}q''\right)|u|^2\\&\quad-\epsilon fW\mathrm{Im}((\partial_x\bar{u})u).\end{aligned}$$

Taking ϵ sufficiently small, N sufficiently large, and δ sufficiently small, the factors in front of $|\partial_x u|^2$ and $|u|^2$ are nonnegative and one can dominate the term involving W using these two terms. (These estimates are uniform, in the sense that, if the estimate holds for choices of ϵ_0, N_0, and δ_0, then it remains valid for $\epsilon < \epsilon_0$, $N = N_0$, and $\delta = \delta_0$.) Thus, by integrating over a time-space slab $M_{[t_1,t_2]} = [t_1, t_2] \times \mathbb{R} \times S^2$, one can conclude that there is a constant C such that

$$E(t_2) + E(t_1) \gtrsim \int_{M_{[t_1,t_2]}} \frac{|\partial_x u|^2}{x^2+1} + |\arctan(x)|^2 \left(\frac{|\nabla_\omega u|^2}{1+|x|^3} + \frac{|\partial_t u|^2}{x^2+1} \right) + \frac{|u|^2}{1+|x|^3} \,\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}^2\omega. \quad (7)$$

4 Pseudodifferential Refinements

4.1 The wave equation for an approximate solution

We define a smooth characteristic function of an interval $[a, b]$ to be a function which is identically 1 on $[a, b]$, which is supported on $[a-1, b+1]$, and which is monotonic on each of the intervals $[a-1, a]$ and $[b, b+1]$. A smooth characteristic function of a collection of intervals, each of which are separated by distance at least two, is defined to be the sum of the smooth characteristic functions of each interval.

Let $T > 0$ be a large constant. (Here, large means larger than $-\log|\epsilon|$ and 2.) Let χ_1 be a smooth characteristic function on $[0, T]$, and let χ_2 be a smooth characteristic function of $[-1, 0] \cup [T, T+1]$. Let $\chi_{|x|\le 2}$ be a smooth characteristic function of $[-1, 1]$. We use χ_1, χ_2, and $\chi_{|x|\le 2}$ to denote $\chi_1(t)$, $\chi_2(t)$, and $\chi_{|x|\le 2}(x)$, respectively.

Since χ_1 is smooth, there is a uniform bound on its derivative and second derivative, each of which are supported on $[0, 1] \cup [T, T+1]$, so that there is a constant C such that $|\partial_t \chi_1| + |\partial_t^2 \chi_1| \le C\chi_2$.

The functions

$$u_1 = \chi_1 \chi_{|x|\le 2} u,$$

$$u_2 = \chi_2 \chi_{|x|\le 2} u,$$

$$u_3 = \chi_1 u$$

satisfy the equation

$$(-\partial_t^2 + \partial_x^2 + V(\Delta_\omega - N) + \mathrm{i}\epsilon W)u_1 = F(u_2, \nabla u_2, t, x) + G(u_3, \nabla u_3, t, x), \quad (8)$$

where

$$F(u_2, \nabla u_2, t, x) = -2(\partial_t \chi_1)(\partial_t u_2) - (\partial_t^2 \chi_1)u_2,$$

$$G(u_3, \nabla u_3, t, x) = 2(\partial_x \chi_{|x|\le 2})(\partial_x u_3) + (\partial_x^2 \chi_{|x|\le 2})u_3.$$

Since all functions of t in this equation are smooth and supported in $t \in [-2, T+2]$, they are Schwartz class in t, so we may take the Fourier transform in t and remain in the Schwartz class. We will use $\hat{}$ to denote the Fourier transform in t, and τ for the argument of such functions. We will typically use the word "functions" to describe u, u_1, u_2, and u_3 and the words "Fourier transforms" to describe their Fourier transforms. We will use L^2 to denote $L^2(\mathrm{d}\omega\,\mathrm{d}x\,\mathrm{d}t)$ for functions and to denote $L^2(\mathrm{d}\omega\,\mathrm{d}x\,\mathrm{d}\tau)$ for Fourier transforms. We will use $\|\cdot\|$ for $\|\cdot\|_{L^2}$ unless otherwise specified.

We introduce the following space-time integrals

$$I(T) = \int_{-2}^{T+2}\int_{-2}^{2}\int_{S^2} x^2|\partial_t u|^2 + |\partial_x u|^2 + |u|^2 \mathrm{d}\omega\,\mathrm{d}x\,\mathrm{d}t,$$

$$J(T) = \int_{\mathbb{R}\times\mathbb{R}\times S^2} |\tau|^{6/5}|\hat{u}_1|^2\,\mathrm{d}\omega\,\mathrm{d}x\,\mathrm{d}\tau.$$

The dependence of J upon T is through the smooth cutoff χ_1 in u_1. Typically, the argument T will be clear from context and will be omitted. From the Morawetz estimate (7) and the exponential bound on the energy, it follows that $I \lesssim E(T) + E(0)$.

We now aim to prove a Morawetz estimate using the Fourier transform. We take

$$f = -\arctan(|\tau|^{\alpha}x),$$

$$q = \frac{f'}{2} = \frac{1}{2}\frac{|\tau|^{\alpha}}{1+|\tau|^{2\alpha}x^2}$$

with $\alpha \in [0, \frac{1}{2}]$. We multiply the Fourier transform of Equation (8) by $(f\partial_x + q)\bar{\hat{u}}_1$, and integrate the real part over $\mathbb{R}\times\mathbb{R}\times S^2$. This integral is convergent because all the functions are compactly supported in time, so the Fourier transforms are Schwartz class.

4.2 Controlling the terms arising from the cutoff

We consider first the integral arising from the right-hand side of (8). This is

$$\int_{\mathbb{R}\times\mathbb{R}\times S^2} \mathrm{Re}(((f\partial_x + q)\bar{\hat{u}}_1)(\hat{F} + \hat{G}))\,\mathrm{d}\omega\,\mathrm{d}x\,\mathrm{d}\tau \le \|(f\partial_x + q)\bar{\hat{u}}_1\|\|\hat{F} + \hat{G}\|.$$

The terms on the right can be estimated by

$$\|(f\partial_x + q)\bar{\hat{u}}_1\| \le \|f\partial_x\hat{u}_1\| + \|q\hat{u}_1\|,$$

$$\|f\partial_x\hat{u}_1\| \lesssim \|\partial_x\hat{u}_1\| \lesssim I^{1/2},$$

$$\|q\hat{u}_1\| \lesssim \||\tau|^{\alpha}\hat{u}_1\| \lesssim \|\hat{u}_1\| + \||\tau|^{1/2}\hat{u}_1\| \lesssim I^{1/2} + J^{1/2},$$

and

$$\|\hat{F} + \hat{G}\| \leq \|\hat{F}\| + \|\hat{G}\|.$$

Because G is supported only for $t \in [-1, T+1]$ and $x \in [-2, 2]$, we have

$$\|\hat{G}\| \lesssim I^{1/2}.$$

Similarly, because F is supported only for $t \in [-1, 0] \cup [T, T+1]$ and $x \in [-2, 2]$, we have that at each instant in t, the function F is bounded in $L^2(\mathrm{d}x\,\mathrm{d}\omega)$ by either $CE(0)^{1/2}$ or $CE(T)^{1/2}$. Since we are considering two intervals in t of length 1, we have

$$\|\hat{F}\| = \|F\| \leq C(E(0)^{1/2} + E(T)^{1/2}).$$

Thus, the terms on the right-hand side of the Fourier transform of (8) are bounded by

$$\int_{\mathbb{R}\times\mathbb{R}\times S^2} \mathrm{Re}((f\partial_x + q)\bar{\hat{u}}_1)(\hat{F} + \hat{G})\,\mathrm{d}\omega\,\mathrm{d}x\,\mathrm{d}\tau$$
$$\leq C(E(0)^{1/2} + E(T)^{1/2} + J^{1/2})(E(0)^{1/2} + E(T)^{1/2}). \quad (9)$$

4.3 The Morawetz estimate for the approximate solution

If we multiply the left-hand side of the Fourier transform of the wave Equation (8) by $(f\partial_x + q)\bar{\hat{u}}_1$ and take the real part, then we have the analog of (6)

$$\begin{aligned}
&\mathrm{Re}((f\partial_x\bar{\hat{u}}_1 + q\bar{\hat{u}}_1)(\tau^2\hat{u}_1 + \partial_x\hat{u}_1 + V(\Delta_\omega - N)\hat{u}_1 + \mathrm{i}\epsilon W\hat{u}_1))\\
&\quad = \partial_x p_x + \nabla_\omega \cdot p_\omega + (-\tfrac{1}{2}f' + q)|\tau\hat{u}_1|^2 - (\tfrac{1}{2}f' + q)|\partial_x\hat{u}_1|^2\\
&\qquad + ((\tfrac{1}{2}f' - q)V + \tfrac{1}{2}f(\partial_x V))|\nabla_\omega\hat{u}_1|^2 + (N((\tfrac{1}{2}f' - q)V + \tfrac{1}{2}f(\partial_x V)) + \tfrac{1}{2}q'')|\hat{u}_1|^2\\
&\qquad - \epsilon f W\,\mathrm{Im}((\partial_x\bar{\hat{u}}_1)\hat{u}_1), \qquad (10)
\end{aligned}$$

where

$$p_x = p_x(f, q; u) = \tfrac{1}{2}f|\partial_t\hat{u}_1|^2 + \tfrac{1}{2}f|\partial_x\hat{u}_1|^2 - \tfrac{1}{2}fV|\nabla_\omega\hat{u}_1|^2 + q\,\mathrm{Re}(\bar{\hat{u}}_1\partial_x\hat{u}_1) - \tfrac{1}{2}(NfV + q')|\hat{u}_1|^2,$$

$$p_\omega = p_\omega(f, q; \hat{u}_1) = fV\,\mathrm{Re}((\partial_x\bar{\hat{u}}_1)(\nabla_\omega\hat{u}_1)) + qV\,\mathrm{Re}(\bar{\hat{u}}_1\nabla_\omega\hat{u}_1).$$

Note that there is no p_t term because, for Fourier transforms, the analog of the product rule is simply $-\overline{\mathrm{i}\tau\hat{u}_1}\hat{u}_1 = \bar{\hat{u}}_1\mathrm{i}\tau\hat{u}_1$.

When this equality is integrated over a space-time slab, the p_x and p_ω terms integrate to zero, and the remaining terms are all nonnegative except for those arising from q'' and from W. The integral of the term involving W is bounded by I.

We now consider the term involving q'':

$$\frac{1}{2}q''|\hat{u}_1|^2 = |\tau|^{3\alpha}\frac{1-3|\tau|^{2\alpha}x^2}{(1+|\tau|^{2\alpha}x^2)^3}|\hat{u}_1|^2.$$

From the positivity of the remaining terms and the bound (9) of the terms coming from the right-hand side of the wave Equation (8) for u_1, we have that

$$\int_{\mathbb{R}\times\mathbb{R}\times S^2}|\tau|^{3\alpha}\frac{1-3|\tau|^{2\alpha}x^2}{(1+|\tau|^{2\alpha}x^2)^3}|\hat{u}_1|^2\,\mathrm{d}\omega\,\mathrm{d}x\,\mathrm{d}\tau \le C(E(0)^{1/2}+E(T)^{1/2}+J^{1/2})(E(0)^{1/2}+E(T)^{1/2}).$$

This can be combined with an additional factor of MI, where M is a large constant (700 is sufficient). The integral I dominates the integral of $(|\tau|^2+1)x^2|\hat{u}_1|^2$ and is bounded by $C(E(T)+E(0))$. Thus, we have

$$\begin{aligned}&\int_{\mathbb{R}\times\mathbb{R}\times S^2}\left(|\tau|^{3\alpha}\frac{1-3|\tau|^{2\alpha}x^2}{(1+|\tau|^{2\alpha}x^2)^3}+M(|\tau|^2+1)x^2\right)|\hat{u}_1|^2\mathrm{d}\omega\,\mathrm{d}x\,\mathrm{d}\tau\\ &\quad\le C(E(0)^{1/2}+E(T)^{1/2}+J^{1/2})(E(0)^{1/2}+E(T)^{1/2}).\end{aligned}$$

By considering the two cases $|\tau|^\alpha|x| < M^{-1/2}$ and $|\tau|^\alpha|x| \ge M^{-1/2}$, one can see that if $2-2\alpha = 3\alpha$ (i.e., $\alpha = 2/5$), then

$$\left(|\tau|^{3\alpha}\frac{1-3|\tau|^{2\alpha}x^2}{(1+|\tau|^{2\alpha}x^2)^3}+M(|\tau|^2+1)x^2\right)\chi_{|x|\le 2} \ge C|\tau|^{3\alpha}\chi_{|x|\le 2}$$

and, therefore, we find

$$\begin{aligned}&C(E(0)^{1/2}+E(T)^{1/2}+J^{1/2})(E(0)^{1/2}+E(T)^{1/2})\\ &\quad\ge\int_{\mathbb{R}\times\mathbb{R}\times S^2}\left(|\tau|^{3\alpha}\frac{1-3|\tau|^{2\alpha}x^2}{(1+|\tau|^{2\alpha}x^2)^3}+M(|\tau|^2+1)x^2\right)|\hat{u}_1|^2\,\mathrm{d}\omega\,\mathrm{d}x\,\mathrm{d}\tau \ge J,\end{aligned}$$

which implies

$$J \le C(E(T)+E(0)). \tag{11}$$

4.4 Closing the energy estimate

It is now possible to estimate the integral on the right-hand side of the energy estimate (5). For $|x| \geq 1$, the right-hand side (using the compact support of W and the Cauchy-Schwarz estimate) can be dominated by $I \leq C(E(T) + E(0))$. For $|x| \leq 1$, we would like to dominate the integral over $\mathbb{R} \times \mathbb{R} \times S^2$ of $|W\mathrm{Im}(\bar{u}_1 \partial_t u_1)|$ by the integral J. However, this is not entirely correct, because in J there is a contribution arising from the support of u in the region $t \in [-1, 0] \cup [T, T+1]$. The error in this approximation is bounded by $C(E(T) + E(0))$. Thus, we have

$$E(T) - E(0) \leq C\epsilon \left(E(T) + E(0) + \left| \iint_{\mathbb{R}\times\mathbb{R}\times S^2} W\mathrm{Im}(\bar{u}_1 \partial_t u_1)\, \mathrm{d}\omega\, \mathrm{d}x\, \mathrm{d}t \right| \right).$$

We can also take the Fourier transform to obtain an estimate by

$$\begin{aligned} &C\epsilon \left(E(T) + E(0) + \left| \iint_{\mathbb{R}\times\mathbb{R}\times S^2} W\,\mathrm{Im}(\bar{\hat{u}}_1 \widehat{\partial_t u_1})\, \mathrm{d}\omega\, \mathrm{d}x\, \mathrm{d}t \right| \right) \\ &\leq C\epsilon \left(E(T) + E(0) + \int_{\mathbb{R}\times\mathbb{R}\times S^2} W|\tau||\hat{u}_1|^2\, \mathrm{d}\omega\, \mathrm{d}x\, \mathrm{d}t \right). \end{aligned}$$

The integrand is now controlled by $I + J$, which, from estimates (7) and (11), we know can be estimated also by the sum of the initial and final energies. This leaves the estimate

$$E(T) - E(0) \leq C\epsilon(E(T) + E(0)).$$

By taking ϵ sufficiently small relative to the constant, we obtain a uniform bound on the energy

$$E(T) \leq C\,E(0).$$

We note that, since all the constants were independent of T, the estimate holds uniformly in T. This proves the first statement (3a), in Theorem 1. Combining this with estimate (7) (and estimating $x^2(1+x^2) \lesssim \arctan(x)^2$) gives the second, (3b). Finally, the arguments of this section and the bound on $I + J$, from estimates (7) and (11), give the third result (3c).

Remark 2. Using the method given in [5], the stronger Morawetz estimate (11) can be improved to control the integral of $|\tau|^{2-\varepsilon}|\hat{u}|^2$ for any $\varepsilon > 0$. Because of the presence of trapping, it is not possible to improve this to $|\tau|^2|\hat{u}|^2$. □

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