

# DYNAMICAL ELASTIC BODIES IN NEWTONIAN GRAVITY

LARS ANDERSSON, TODD A. OLIYNYK<sup>†</sup>, AND BERND G. SCHMIDT

ABSTRACT. Well-posedness for the initial value problem for a self-gravitating elastic body with free boundary in Newtonian gravity is proved. In the material frame, the Euler-Lagrange equation becomes, assuming suitable constitutive properties for the elastic material, a fully non-linear elliptic-hyperbolic system with boundary conditions of Neumann type. For systems of this type, the initial data must satisfy compatibility conditions in order to achieve regular solutions. Given a relaxed reference configuration and a sufficiently small Newton's constant, a neighborhood of initial data satisfying the compatibility conditions is constructed.

## 1. INTRODUCTION

In Newtonian physics, the two-body problem is solvable for point particles moving around their common center of gravity on Kepler ellipses. However, if one considers extended bodies, the situations changes drastically. Assuming that a solution exists for extended bodies, one can show that the centers of mass of the bodies move as point particles, but existence could not be established for a long time.

The first successful attack on the problem was made by Leon Lichtenstein [21] who considered self-gravitating fluid bodies moving on circles about their center of gravity. In this case the Euler equations for a self-gravitating system become time independent in a coordinate system co-moving with the bodies. For the case of small bodies or widely separated large bodies, Lichtenstein showed the existence of such solutions.

Well-posedness for the Cauchy problem for fluid bodies with free boundary was proved only recently. Lindblad [22] proved well-posedness for a non-relativistic compressible liquid body (i.e. positive boundary density) with free boundary. In this paper one can also find references to earlier work. A different proof of the result of Lindblad that is valid for both the relativistic and non-relativistic cases was given by Trakhinin [33].

The more singular case of fluids with vanishing boundary density is discussed in [12, 17]. Unfortunately, the problem of proving well-posedness for self-gravitating compressible fluid bodies with free boundary is still open in general relativity and even in Newtonian gravity. However, see [23] for the case of self-gravitating incompressible fluids. See also [29, 10, 18] for results dealing with various restricted versions of the Cauchy problem for fluid bodies in general relativity.

One can argue that the Cauchy problem should be simpler to handle if the bodies consist of elastic material, since such a body can, at least in the case of small bodies, be said to 'have a shape of its own'. In this paper we solve the boundary initial value problem for self-gravitating deformations of a relaxed body. In particular, consider

---

<sup>†</sup> Partially supported by the ARC grant DP1094582 and an MRA grant.

a relaxed (i.e. with vanishing stress) elastic body in the absence of gravity. Then we show existence of a self-gravitating solution for initial data close to those of the static relaxed body and for a small gravitational constant. Thus, the motion of the body will consist of small nonlinear oscillations around the equilibrium solution.

The Cauchy problem for a Newtonian elastic body with free boundary is, under suitable constitutive assumptions on the elastic material, a fully non-linear elliptic-hyperbolic initial-boundary value problem with boundary values of Neumann type. The case of purely hyperbolic problems of this type is covered by a theorem of Koch [19]. The method of proof of this theorem can be adapted to include the non-local elliptic terms in the which appear in the system of differential equations considered in this paper via the Newtonian gravitational field.

In order to achieve a regular evolution for the initial-boundary value problem, the initial data must satisfy certain compatibility conditions. The problem of constructing an open neighborhood of initial data satisfying the compatibility conditions given one such data set is discussed in the work of Koch. The work of Lindblad contains a related discussion for the fluid case. For the case of a Newtonian elastic body, we construct, for small values of Newton's constant, initial data satisfying suitable compatibility conditions near data for a relaxed elastic body in the absence of gravity. It should be noted that due to the presence of the Newtonian gravitational potential this problem is non-local.

The situation considered in this paper has several interesting generalizations. If we consider a large static, self-gravitating body, a relaxed state may not exist. Restricting to the spherically symmetric case, solutions to the field equations can be found by ODE techniques [28]. If we perturb the data slightly, there should exist solutions with small oscillations around the equilibrium configuration. Our approach allows one to consider also such more complicated problems; the essential difficulty is the construction of the initial data satisfying the compatibility conditions and not the time evolution. However, the results in this paper do not immediately cover these more general situations. In particular, a proof of a suitable version of the well-posedness result for the Cauchy problem, in these situations, requires a more general treatment of potential theory in Riemannian manifolds. This problem will be considered in a separate paper.

Given a solution  $u$  to the Cauchy problem for the Newtonian elastic body which exists for times  $t \in [0, T]$ , one may conclude from the main result of this paper, see theorem 4.3, that by choosing initial data sufficiently close to that of  $u$ , the time of existence of the resulting solution can be arbitrarily close to  $T$ . In particular, given a static solution, there exist nearby data such that the corresponding solution to the Cauchy problem has a time of existence not less than any given time  $T$ . The proof of existence of static self-gravitating bodies in Newtonian theory [6] could be used together with a generalization of theorem 4.3 that allows for more general initial data, as alluded to above, as well as a generalization of the construction of initial data to show that nearby data define solutions which exist up to some time  $T$ .

The treatment of two fluid balls in steady rotation due to Lichtenstein, which was mentioned above, has been generalized to the case of elastic bodies, see [8]. Using these solutions in the manner just described, it is possible to obtain classes of solutions of the two body problem in any prescribed finite time interval. We leave these problems for later investigations.

Global existence for small data is an important question. For unbounded fluids in 3-dimensions, it is now known that shocks form for a large class of initial data [31, 11], and it is widely believed that for most equations of state shocks will always form for arbitrary small perturbation of constant density state. On the other hand, there do exist small data global existence results for unbounded elastic bodies provided the material satisfies certain additional conditions, cf. [32, 14]. For elastic bodies of finite extent, it appears to be an open question whether there exist solutions that are global in time or under what conditions shocks form.

**Overview of this paper.** The paper is organized as follows. Section 2 describes Newtonian elasticity, sets up the basic equations and gives the conditions we impose on the material. We derive the equations in the material frame and in spacetime from a variational principle. Section 3 deals with the problem of finding solutions to the compatibility conditions needed for the proof of local existence. Due to the non-local terms in the equations, these conditions imply conditions on the Cauchy data on the whole initial surface. To find initial data satisfying these conditions, we make use of some results from potential theory. These are developed in section 3.1. Further, the Poisson equation must be studied in the material frame, see section 3.2. The results concerning the linearized elasticity operator which are needed can be found in section 3.3. Section 4 generalizes Koch's theorem and proves our main theorem. Appendix A contains some background material for the function spaces used in this paper.

## 2. NEWTONIAN ELASTICITY

**2.1. Kinematics.** The *body*  $\mathcal{B}$  is an open, connected, and bounded set with a  $C^\infty$  boundary in Euclidean space  $\mathbb{R}_\mathcal{B}^3$ . We refer to  $\mathbb{R}_\mathcal{B}^3$  as the *extended body*. We consider *configurations*, i.e. maps  $f : \mathbb{R}_\mathcal{S}^3 \rightarrow \mathcal{B}$ , and deformations  $\phi : \mathcal{B} \rightarrow \mathbb{R}_\mathcal{S}^3$  with

$$f \circ \phi = \mathbf{id}_\mathcal{B}. \quad (2.1)$$

Thus, the physical body is the domain in *space*  $\mathbb{R}_\mathcal{S}^3$  given by  $f^{-1}(\mathcal{B}) = \phi(\mathcal{B})$ .

We shall make use of the extension  $\tilde{\phi}$  of  $\phi$  to a map  $\mathbb{R}_\mathcal{B}^3 \rightarrow \mathbb{R}_\mathcal{S}^3$  and let  $(X^A)_{A=1,2,3}$  and  $(x^i)_{i=1,2,3}$  denote global Cartesian coordinates on the body  $\mathbb{R}_\mathcal{B}^3$  and the configuration  $\mathbb{R}_\mathcal{S}^3$  spaces, respectively. Since we shall consider the Newtonian dynamics of a body, we let  $f, \phi$  depend on time, denoted by  $t$ . Equation (2.1) gives

$$f^A(t, \phi(t, X)) = X^A \quad \text{in } \mathcal{B} \quad \text{and} \quad \phi^i(t, f(t, x)) = x^i \quad \text{in } f^{-1}(\mathcal{B}). \quad (2.2)$$

Writing  $x^\mu = (t, x^i)$ , we introduce  $f^A{}_\mu = \partial_\mu f^A$  and  $\phi^k{}_A = \partial_A \phi^k$ . In particular,  $f^A{}_0 = \partial_t f^A$ . We have

$$\phi^k{}_A f^A{}_\ell = \delta^k{}_\ell \quad \text{and} \quad f^B{}_k \phi^k{}_A = \delta^B{}_A \quad (2.3)$$

where these expressions are defined. This implies

$$\frac{\partial \phi^i{}_A}{\partial f^B{}_k} = -\phi^i{}_B \phi^k{}_A, \quad \text{and} \quad \frac{\partial f^B{}_k}{\partial \phi^i{}_A} = -f^B{}_i f^A{}_k. \quad (2.4)$$

Let

$$\chi_{f^{-1}(\mathcal{B})} = \begin{cases} 1 & \text{in } f^{-1}(\mathcal{B}) \\ 0 & \text{in } \mathbb{R}_\mathcal{S}^3 \setminus f^{-1}(\mathcal{B}) \end{cases} \quad (2.5)$$

be the indicator function of the support of the physical body. Using the above identities, one may calculate the variation of  $\chi_{f^{-1}(\mathcal{B})}$  with respect to  $f^A$ ,

$$\frac{\partial \chi_{f^{-1}(\mathcal{B})}}{\partial (f^A)} = \phi^i{}_A \partial_i \chi_{f^{-1}(\mathcal{B})} \quad (2.6)$$

Let  $H^{AB} = f^A{}_{,i} f^B{}_{,j} \delta^{ij}$  and define  $H_{AB}$  by  $H_{AB} H^{BC} = \delta_A{}^C$ . Then  $H_{AB} = \phi^i{}_A \phi^j{}_B \delta_{ij}$ . We have

$$\frac{\partial H^{AB}}{\partial (f^C{}_k)} = 2 \delta^{kn} \delta^{(A}{}_C f^{B)}{}_n. \quad (2.7)$$

Differentiating (2.3) gives the usual formulas for the derivative of the inverse,

$$\partial_\mu \phi^i{}_A = -\phi^i{}_B \partial_\mu f^B{}_k \phi^k{}_A, \quad (2.8)$$

$$\partial_B f^A{}_i = -f^A{}_j \partial_B \phi^j{}_C f^C{}_i. \quad (2.9)$$

We let  $v^\mu \partial_\mu = \partial_t + v^i \partial_i$  be defined by  $v^\mu f^A{}_\mu = 0$ . This determines the vector field  $v^i(x)$  on  $\mathbb{R}_S^3$  uniquely in terms of  $f$ . The velocity field  $v^\mu \partial_\mu$  describes the trajectories of material particles. From the relation  $v^\mu \partial_\mu f^A = 0$ , we get

$$v^i = -\phi^i{}_A f^A{}_0.$$

On the other hand, time differentiating (2.2) gives

$$v^i(t, x) = (\partial_t \phi^i)(t, f(t, x)).$$

Thus, we have

$$\begin{aligned} \partial_t^2 \phi^k(t, X) &= \partial_t (v^k(t, \phi(t, X))) \\ &= (\partial_t v^k)(t, \phi(t, X)) + \partial_\ell v^k(t, \phi(t, X)) \partial_t \phi^\ell(t, X) \\ &= (v^\mu \partial_\mu v^k)(t, \phi(t, X)). \end{aligned}$$

Further,

$$v^\mu v^\nu \partial_\mu \partial_\nu f^A = -v^\mu \partial_\mu v^k f^A{}_k, \quad (2.10)$$

which shows that

$$H_{AB} v^\mu v^\nu \partial_\mu \partial_\nu f^B = -v^\mu \partial_\mu v^m \delta_{mn} \phi^n{}_A. \quad (2.11)$$

The body  $\mathcal{B}$  carries a reference volume element defined by a 3-form  $V_{ABC}$  on  $\mathcal{B}$ . The number density  $n$  is defined, cf. [3, eq. (3.2)], by

$$f^A{}_i f^B{}_j f^C{}_k V_{ABC}(f(x)) = n(x) \epsilon_{ijk}(x)$$

where  $\epsilon_{ijk}$  is the volume element of the Euclidean metric  $\delta_{ij}$  on  $\mathbb{R}_S^3$ . Since we are considering the Newtonian case with Euclidean geometry, we have simply

$$n = \det Df.$$

The mass density is

$$\rho = nm \quad (2.12)$$

where  $m$  is the specific mass of the particles, i.e.  $mV$  is the mass density<sup>1</sup> for the material in its natural state, see [3]. See also [7] where the specific mass is denoted  $\rho_0$ . We have the relation

$$\frac{\partial n}{\partial f^A{}_i} = n \phi^i{}_A. \quad (2.13)$$

---

<sup>1</sup>Our techniques allow for  $m$  to be a function on the body,  $m = m(X)$ , but for simplicity, we will assume that  $m$  is constant.

The following equations for  $n$  follow from the above definitions, using (2.13),

$$\partial_\mu(nv^\mu) = 0, \quad (2.14)$$

and

$$\partial_k(n\phi^k_A) = 0. \quad (2.15)$$

The elastic material is described by the stored energy function

$$\epsilon = \epsilon(f^A, H^{AB}).$$

The various forms of the stress tensor<sup>2</sup> are defined from the stored energy function via

$$\begin{aligned} \tau_{AB} &= +2\frac{\partial\epsilon}{\partial H^{AB}}, & \tau_{ij} &= nf^A_i f^B_j \tau_{AB}, \\ \tau_i^A &= f^B_i \tau_{BC} H^{CA}, & \tau_A^i &= n\tau_{AB} f^B_j \delta^{ji}. \end{aligned}$$

We have

$$\begin{aligned} \frac{\partial\epsilon}{\partial(f^A_i)} &= n^{-1}\tau_A^i, \\ \frac{\partial\epsilon}{\partial(\phi^i_A)} &= -\tau_i^A. \end{aligned}$$

The elasticity tensor  $L_i^A k^B$  is defined by

$$L_i^A k^B = \frac{\partial\tau_i^A}{\partial(\phi^k_B)}. \quad (2.16)$$

It follows from this definition and the assumptions above that the elasticity tensor has the symmetries

$$L^{iAjB} = L^{jBiA} = L^{iABj} = L^{AijB} \quad (2.17)$$

where

$$L^{iAjB} = \delta^{il}\delta^{jk}L_l^A k^B.$$

Clearly, we also have

$$\partial_A\tau_i^A = L_i^A k^B \partial_A\partial_B\phi^k.$$

**2.2. Variational formulation.** We derive the field equations for a self-gravitating elastic body from the elastic action, supplemented by a term giving Newton's force law and Newton's law of gravitation. The Lagrange density is of the form  $\Lambda\epsilon_{0123}$  where  $\epsilon_{0123}$  is the 4-volume element on spacetime  $\mathbb{R} \times \mathbb{R}_S^3$  in Cartesian coordinates, and

$$\Lambda = \Lambda^{grav} + \Lambda^{pot} + \Lambda^{kin} + \Lambda^{elast}$$

where

$$\begin{aligned} \Lambda^{grav} &= \frac{|\nabla U|^2}{8\pi G}, \\ \Lambda^{pot} &= \rho U \chi_{f^{-1}(\mathcal{B})}, \\ \Lambda^{kin} &= \frac{1}{2}\rho v^2 \chi_{f^{-1}(\mathcal{B})}, \\ \Lambda^{elast} &= -n\epsilon \chi_{f^{-1}(\mathcal{B})}, \end{aligned}$$

---

<sup>2</sup>The sign of the stress tensor here is the opposite of that of  $\sigma_{AB}$  defined in [3, eq. (3.5)], but agrees with the usage in [7, equation (4.2)].

with  $\chi_{f^{-1}(\mathcal{B})}$  given by (2.5),

$$|\nabla U|^2 = \partial_i U \partial_j U \delta^{ij},$$

and

$$v^2 = v^i v^j \delta_{ij}.$$

2.2.1. *Eulerian picture.* The action in the Eulerian picture then takes the form

$$\mathcal{L} = \int \Lambda \epsilon_{0123} dx^0 dx^1 dx^2 dx^3$$

with  $\Lambda = \Lambda(U, \partial_i U, f^A, f^A_0, f^A_i)$ . The Euler-Lagrange equations are of the form  $\mathcal{E}_A = 0$ ,  $\mathcal{E}_U = 0$  with

$$\begin{aligned} -\mathcal{E}_A &= \partial_\mu \frac{\partial \Lambda}{\partial (\partial f^A_\mu)} - \frac{\partial \Lambda}{\partial (f^A)}, \\ -\mathcal{E}_U &= \partial_i \frac{\partial \Lambda}{\partial (\partial_i U)} - \frac{\partial \Lambda}{\partial U}. \end{aligned}$$

We have

$$-\mathcal{E}_U = \frac{\Delta U}{4\pi G} - \rho \chi_{f^{-1}(\mathcal{B})}.$$

Next, we consider the Euler-Lagrange terms generated by variations with respect to the configuration  $f^A$ . A calculation using (2.15) shows that the factor  $\rho \chi_{f^{-1}(\mathcal{B})}$  gives no contribution to the Euler-Lagrange equations, and hence the kinetic term in the action gives

$$\begin{aligned} -\mathcal{E}_A^{kin} &= \partial_\mu \frac{\partial \Lambda^{kin}}{\partial (f^A_\mu)} - \frac{\partial \Lambda^{kin}}{\partial (f^A)} \\ &= [\partial_k (\frac{1}{2} \rho \phi^k_A v^2) - \partial_t (\rho \delta_{mn} v^m \phi^n_A) - \partial_k (\rho \delta_{mn} v^m v^k \phi^n_B)] \chi_{f^{-1}(\mathcal{B})}, \end{aligned}$$

which after some calculations, using (2.14) and (2.15), gives

$$-\mathcal{E}_A^{kin} = -\rho v^\mu \partial_\mu v^m \delta_{mn} \phi^n_A \chi_{f^{-1}(\mathcal{B})}$$

use (2.11)

$$= \rho H_{AB} v^\mu v^\nu f^A \chi_{f^{-1}(\mathcal{B})}.$$

The elastic term gives, using (2.15) and (2.6),

$$\begin{aligned} -\mathcal{E}_A^{elast} &= \partial_i \frac{\Lambda^{elast}}{\partial (f^A_i)} - \frac{\partial \Lambda^{elast}}{\partial (f^A)} \\ &= -\partial_i (\tau_A^i \chi_{f^{-1}(\mathcal{B})}). \end{aligned}$$

In view of [3, lemma 2.2], we have that the divergence  $\partial_i (\tau_A^i \chi_{f^{-1}(\mathcal{B})})$  is integrable only if the zero traction boundary condition

$$\tau_A^j n_j |_{\partial f^{-1}(\mathcal{B})} = 0$$

holds, in which case the identity

$$\partial_i (\tau_A^i \chi_{f^{-1}(\mathcal{B})}) = \partial_i \tau_A^i \chi_{f^{-1}(\mathcal{B})}$$

is valid.

Finally, the potential term gives, using (2.15) and (2.6),

$$\begin{aligned} -\mathcal{E}_A^{pot} &= \partial_i \frac{\Lambda^{pot}}{\partial(f^A_i)} - \frac{\partial \Lambda^{pot}}{\partial f^A} \\ &= \partial_i (\rho \phi^i_A U \chi_{f^{-1}(\mathcal{B})}) - \phi^i_A \partial_i \chi_{f^{-1}(\mathcal{B})} \\ &= \rho \phi^i_A \partial_i U \chi_{f^{-1}(\mathcal{B})}. \end{aligned}$$

Adding the terms, we have

$$-\mathcal{E}_A = \rho H_{AB} v^\mu v^\nu \partial_\mu \partial_\nu f^A - \partial_i \tau_A^i + \rho \phi^i_A \partial_i U \quad \text{in } f^{-1}(\mathcal{B}), \quad (2.18)$$

subject to the boundary condition

$$\tau_A^j n_j \big|_{f^{-1}(\mathcal{B})} = 0.$$

Hence, we find that the Euler-Lagrange equations  $\mathcal{E}_A = 0$ ,  $\mathcal{E}_U = 0$  are equivalent to the system

$$-\rho H_{AB} v^\mu v^\nu \partial_\mu \partial_\nu f^B + \partial_i \tau_A^i = \rho \phi^i_A \partial_i U, \quad \text{in } f^{-1}(\mathcal{B}) \quad (2.19a)$$

$$\Delta U = 4\pi G \rho \chi_{f^{-1}(\mathcal{B})}, \quad (2.19b)$$

$$\tau_A^i n_i \big|_{\partial f^{-1}(\mathcal{B})} = 0 \quad (2.19c)$$

where  $n_i$  is the unit outward pointing normal to  $\partial f^{-1}(\mathcal{B})$ . These are the field equations for a dynamical elastic body in Newtonian gravity displayed in the Eulerian frame. We note that, by using (2.11) and multiplying by  $f^A_i$ , equations (2.19a) and (2.19c) take the form

$$\rho v^\mu \partial_\mu v_i + \partial_j \tau_i^j = \rho \partial_i U \quad \text{in } f^{-1}(\mathcal{B}) \quad \text{and} \quad \tau_i^j n_j \big|_{\partial f^{-1}(\mathcal{B})} = 0. \quad (2.20)$$

2.2.2. *Material frame.* Similarly, the action in the material frame is given by

$$\begin{aligned} \mathcal{L}^{mat} &= \int \phi^* (\Lambda \epsilon_{0123}) dX^0 dX^1 dX^2 dX^3 \\ &= \int J \Lambda^{mat} dX^0 dV_{ABCD} dX^A dX^B dX^C \end{aligned}$$

where

$$J = (\phi^* n)^{-1} = \det(\partial_A \phi^i)$$

is the Jacobian of  $\phi$ , and  $\Lambda^{mat} = \Lambda^{mat}(U, \phi^i, \phi_0^i, \phi^i_A)$  is given by the relation

$$J \Lambda^{mat} = J \frac{|\nabla \bar{U}|_H^2}{8\pi G} + (m\bar{U} + \frac{1}{2}mv^2 - \bar{\epsilon}) \chi_{\mathcal{B}} \quad (2.21)$$

where  $\chi_{\mathcal{B}}$  is the indicator function of the support of the body and  $\bar{U}, \bar{\epsilon}$  are defined along the lines of [3]. In particular,  $\bar{U} = U \circ \phi$ , and

$$|\nabla \bar{U}|_H^2 = H^{AB} \partial_A \bar{U} \partial_B \bar{U}$$

is the pullback to  $\mathbb{R}_g^3$  of  $|\nabla U|^2$  where we recall that

$$H^{AB} = f^A_i f^B_j \delta^{ij} \quad \text{and} \quad f_i^A = (\partial_A \phi^i)^{-1}.$$

Similarly,  $\bar{\epsilon} = \epsilon \circ \phi$  so that  $\bar{\epsilon} = \bar{\epsilon}(f^A_i, H^{AB})$ . We note also that

$$\det(H^{AB}) = \det(f^A_i)^2 = J^2.$$

The Euler-Lagrange equations in material frame are  $\mathcal{E}_i = 0, \mathcal{E}_{\bar{U}} = 0$  where

$$\begin{aligned} -\mathcal{E}_i &= \partial_A \frac{\partial \mathcal{L}^{mat}}{\partial(\phi^i_A)} - \frac{\partial \mathcal{L}^{mat}}{\partial(\phi^i)}, \\ -\mathcal{E}_{\bar{U}} &= \partial_A \frac{\partial \mathcal{L}^{mat}}{\partial(\partial_A \bar{U})} - \frac{\partial \mathcal{L}^{mat}}{\partial(\bar{U})}. \end{aligned}$$

This gives the system of equations

$$-m\partial_t^2 \phi^i + \partial_A(\bar{\tau}^{iA}) = m\delta^{ij} f_j^A \partial_A \bar{U} \quad \text{in } \mathcal{B}, \quad (2.22a)$$

$$\Delta_H \bar{U} = 4\pi G J^{-1} m \chi_{\mathcal{B}} \quad \text{in } \mathbb{R}_{\mathcal{B}}^3, \quad (2.22b)$$

$$\nu_A \bar{\tau}^{iA} |_{\partial \mathcal{B}} = 0 \quad (2.22c)$$

where  $\nu_A$  is the unit outward pointing normal to  $\partial \mathcal{B}$ ,

$$\bar{\tau}^{iA} = \delta^{ik} J(f^A_j \tau_k^j) \circ \phi$$

is the Piola transform of  $\tau_i^j$ , and

$$H = H_{AB} dX^A dX^B = \phi^*(\delta_{ij} dx^i dx^j)$$

is the pull back of the Euclidean metric  $\delta_{ij} dx^i dx^j$  under the map  $\phi$ . Observe that in (2.22b), we could also have used the notation  $\bar{\rho} = J^{-1}m$  since  $J^{-1} = n \circ \phi$ .

Rescaling time and the Newtonian potential, we can write the evolution equations (2.22a)-(2.22c) in the form

$$-\partial_t^2 \phi^i + \partial_A \bar{\tau}^{Ai} - \lambda^2 \delta^{ij} f_j^A \partial_A \bar{U} = 0 \quad \text{in } \mathcal{B}, \quad (2.23a)$$

$$\Delta_H \bar{U} = J^{-1} m \chi_{\mathcal{B}} \quad \text{in } \mathbb{R}_{\mathcal{B}}^3, \quad (2.23b)$$

$$\nu_A \bar{\tau}^{Ai} |_{\partial \mathcal{B}} = 0 \quad (2.23c)$$

where

$$\lambda^2 = 4\pi m G.$$

We note that the Poisson equation (2.23b) can be written out more explicitly as

$$\partial_A (J H^{AB} \partial_B \bar{U}) = m \chi_{\mathcal{B}}. \quad (2.24)$$

It follows from the symmetry properties (2.17) of the elasticity tensor that

$$\bar{L}^{iA}{}_j{}^B = \frac{\partial \bar{\tau}^{iA}}{\partial \partial_B \phi^j} \quad (2.25)$$

satisfies

$$\bar{L}^{iA}{}_j{}^B = \bar{L}^{jD}{}_{iB} = \bar{L}^{iAB}{}_j = \bar{L}^{Aij}{}_B \quad (2.26)$$

where

$$\bar{L}^{iA}{}_j{}^B = \delta^{jk} \bar{L}^{iA}{}_j{}^B.$$

Since we will be working in the material representation for the remainder of the article, we will drop the  $\mathcal{B}$  from the body space  $\mathbb{R}_{\mathcal{B}}$  and denote it simply by  $\mathbb{R}^3$ . For latter use, we define the following nonlinear functionals:

$$E^i(\phi) = \partial_A(\bar{\tau}^{iA}(\partial\phi)), \quad \text{and} \quad E^i_{\delta}(\phi) = \text{tr}_{\partial \mathcal{B}} \nu_A \bar{\tau}^{iA}(\partial\phi) \quad (2.27)$$

where  $\partial\phi = (\partial_A \phi^i)$ .

**2.3. Constitutive conditions.** We make the following assumptions on the elastic material:

- (1)  $\bar{\tau}^{iA}$  is a smooth function of its arguments  $(\partial\phi)$  in the neighborhood of the identity map

$$\psi_0^i : \mathbb{R}^3 \longrightarrow \mathbb{R}^3 : (X^i) \longmapsto (\psi_0^i(X) = X^i),$$

- (2) the identity map is an equilibrium solution to (2.23a)-(2.23c) for  $\lambda = 0$ , i.e.

$$(E^i(\psi_0), E_{\partial}^i(\psi_0)) = (0, 0),$$

and

- (3)

$$a^{iA}{}_{jB} = \bar{L}^{iA}{}_{jB} \Big|_{\partial\phi = \partial\psi_0}$$

satisfies the following properties:

- (a) there exists an  $\omega > 0$  such that

$$a^{iA}{}_{jB}(X)\xi_i\xi_A\eta_j\eta_B \geq \omega|\xi|^2|\eta|^2$$

for all  $\xi, \eta \in \mathbb{R}^3$ , and

- (b) there exists a  $\gamma > 0$  such that

$$\frac{1}{2}a^{iA}{}_{jB}\sigma_{iA}\sigma_{jB} \geq \gamma|\sigma|^2$$

for all  $\sigma = (\sigma_{iB}) \in \mathbb{M}_{3 \times 3}$  with  $\sigma_{iB} = \sigma_{Bi}$ .

We remark that condition (2) above is satisfied for a stored energy function which has a minimum at some reference configuration. See [3, equations (3.20)-(3.22)].

### 3. CONSTRUCTION OF INITIAL DATA

In order to prove the existence of dynamical solutions to the evolution equations (2.23a)-(2.23c), we first need to construct initial data that satisfy the *compatibility conditions* to a sufficiently high order. The compatibility conditions are defined as follows.

**Definition 3.1.** Fixing  $s > 3/2 + 1$ , we say that the initial data

$$(\phi^i|_{t=0}, \partial_t|_{t=0}\phi^i) = (\phi_0^i, \phi_1^i) \in W^{s+1,2}(\mathcal{B}, \mathbb{R}^3) \times W^{s,2}(\mathcal{B}, \mathbb{R}^3)$$

satisfies the *compatibility conditions to order  $r$*  ( $0 \leq r \leq s$ ) if there exists maps

$$\phi_\ell^i \in W^{s+1-\ell,2}(\mathcal{B}, \mathbb{R}^3) \quad \ell = 2, 3, \dots, r$$

that satisfy

$$\begin{aligned} \partial_t^{\ell-2}(-\partial_t^2\phi^i + \partial_A\bar{\tau}^{Ai} - \lambda^2\delta^{ij}f_j^A\partial_A\bar{U})|_{t=0} &= 0 \quad \text{in } \mathcal{B}, \\ \partial_t^\ell(\Delta_H\bar{U} - J^{-1}m\chi_{\mathcal{B}})|_{t=0} &= 0 \quad \text{in } \mathbb{R}^3, \\ \partial_t^\ell((\nu_A\bar{\tau}^{Ai})|_{\partial\mathcal{B}})|_{t=0} &= 0, \end{aligned}$$

for  $\ell = 0, 1, 2, \dots, r$  where, after formally differentiating, we set  $\partial_t^\ell|_{t=0}\phi^i = \phi_\ell^i$ .

**3.1. Potential theory.** Before we can solve the problem of existence of initial data satisfying the compatibility conditions and the time evolution of this data, we first need to develop some potential theory. To begin, we set  $\mathcal{B}_+ = \mathcal{B}$ ,  $\mathcal{B}_- = \mathbb{R}^3 \setminus \overline{\mathcal{B}}$ , and let

$$E(X, Y) = -\frac{1}{4\pi|X - Y|}$$

denote the Newton potential so that  $\Delta E = \delta$ . We also let  $\mathcal{S}$  and  $\mathcal{D}$  denote the single and double layer potentials

$$\mathcal{S}[f](X) = \int_{\partial\mathcal{B}} E(X, Y)f(Y) d\sigma(Y) \quad X \notin \partial\mathcal{B}$$

and

$$\mathcal{D}[f](X) = \int_{\partial\mathcal{B}} \frac{\partial}{\partial\nu_Y} E(X, Y)f(Y) d\sigma(Y) \quad X \notin \partial\mathcal{B},$$

respectively, where  $d\sigma$  is the induced surface measure on  $\partial\mathcal{B}$ , and  $\nu$  is the outward pointing normal. Restricting to  $\partial\mathcal{B}$ , we have for  $X \in \partial\mathcal{B}$  that

$$\mathcal{S}[f]|_{\partial\mathcal{B}}(X) = \int_{\partial\mathcal{B}} E(X, Y)f(Y) d\sigma(Y) \quad \text{and} \quad \mathcal{D}[f]|_{\partial\mathcal{B}_\pm}(X) = (\pm\frac{1}{2}I + K)[f](X)$$

where

$$K[f](X) = P.V. \int_{\partial\mathcal{B}} \frac{\partial}{\partial\nu_Y} E(X, Y)f(Y) d\sigma(Y).$$

Further,

$$\frac{\partial}{\partial\nu} \mathcal{S}[f]|_{\partial\mathcal{B}_\pm} = (\pm I + K^*)[f]$$

where  $K^*$  is the adjoint of  $K$ . We recall the following well known relations between the boundary value problems for  $\Delta$  in  $\mathcal{B}_\pm$  and these potentials:

- (i) The solution to the Dirichlet problem

$$\Delta u = 0, \quad \text{tr}_{\partial\mathcal{B}} u = \psi$$

on  $\mathcal{B}_\pm$  is given by

$$u = \mathcal{D}[f]$$

where  $f$  solves

$$(\pm\frac{1}{2}I + K)[f] = \psi.$$

- (ii) The solution to the Neumann problem

$$\Delta u = 0, \quad \text{tr}_{\partial\mathcal{B}} \frac{\partial}{\partial\nu} u = \psi$$

on  $\mathcal{B}_\pm$  is given by

$$u = \mathcal{S}[f] + C$$

where  $f$  solves

$$(\mp\frac{1}{2}I + K^*)[f] = \psi,$$

and  $C$  is an arbitrary constant. The solution exists if and only if  $\int_{\partial\mathcal{B}} \psi d\sigma = 0$ .

For the moment, we consider the interior Dirichlet and Neumann problems on  $\mathcal{B}_+ = \mathcal{B}$ . The solution for the Dirichlet problem has the property that

$$u \in W^{s,p}(\mathcal{B}) \quad \text{if } \text{tr}_{\partial\mathcal{B}} u \in B^{s-1/p,p}(\partial\mathcal{B}), \quad (3.1)$$

while for the Neumann problem, we have

$$u \in W^{s,p}(\mathcal{B}) \quad \text{if } \text{tr}_{\partial\mathcal{B}} \frac{\partial}{\partial\nu} u \in B^{s-1-1/p,p}(\partial\mathcal{B}).$$

These results are classical, see [2]. Defining the volume potential of a density  $f$  in  $\mathcal{B}$  to be

$$\mathcal{V}[f](X) = \Delta^{-1}(f\chi_{\mathcal{B}})(X) = \int_{\mathcal{B}} E(X, Y) f(Y) d^3 Y,$$

we abuse notation and say that  $\mathcal{V}[f] \in W^{k,p}(\mathcal{B})$  if the restriction to  $\mathcal{B}$  has this property, and similarly for the other potentials. Using the fact that  $\partial_{X^A} E(X, Y) = -\partial_{Y^A} E(X, Y)$ , we have

$$\partial_A \mathcal{V}[f](X) = - \int_{\mathcal{B}} \partial_{Y^A} E(X, Y) f(Y) d^3 Y.$$

This gives, after a partial integration,

$$\partial_A \mathcal{V}[f] = \mathcal{V}[\partial_A f] - \mathcal{S}[\text{tr}_{\partial\mathcal{B}} f \nu^A]. \quad (3.2)$$

Let  $u$  be the solution of the Dirichlet problem

$$\Delta u = 0, \quad \text{tr}_{\partial\mathcal{B}} u = \psi.$$

By Green's theorem, we have

$$\begin{aligned} \int_{\mathcal{B}} (\Delta_Y u(Y) E(X, Y) - u(Y) \Delta_Y E(X, Y)) d^3 Y = \\ \int_{\partial\mathcal{B}} \left( \frac{\partial}{\partial\nu_Y} u(Y) E(X, Y) - u(Y) \frac{\partial}{\partial\nu_Y} E(X, Y) \right) d\sigma(Y). \end{aligned}$$

Since  $u$  solves the Dirichlet problem with boundary data  $\psi$ , and

$$\Delta_Y E(X, Y) = \delta(X - Y), \quad (3.3)$$

this gives

$$\mathcal{D}[u]|_{\mathcal{B}} = \mathcal{S}[\text{tr}_{\partial\mathcal{B}} \frac{\partial}{\partial\nu} u] + u. \quad (3.4)$$

Upon taking the limit from the interior at  $\partial\mathcal{B}$ , we have

$$\left(\frac{1}{2}I + K\right)[u]|_{\partial\mathcal{B}} = \mathcal{S}[\text{tr}_{\partial\Omega} \frac{\partial}{\partial\nu} u] + u.$$

Consider a metric  $g_{AB}$  on  $\mathbb{R}^3$  with covariant derivative  $\nabla_A$ . Let  $\{e_a\}_{a=1,2}$  be a tangential frame on  $\partial\mathcal{B}$ , and let  $h_{ab}$  denote the induced metric on  $\partial\mathcal{B}$  with covariant derivative  $D$ . Let a vector field  $\xi$  be given. Decompose  $\xi$  into tangential and normal component at  $\partial\mathcal{B}$ ,

$$\xi = P^a e_a + \langle \xi, \nu \rangle \nu.$$

Introduce a Gaussian foliation near  $\partial\mathcal{B}$ . Then  $g$  takes the form

$$g_{AB} dX^A dX^B = dr^2 + h_{ab}(y, r) dy^a dy^b$$

where  $y^a$  are coordinates on  $\partial\mathcal{B}$ . Extending  $\nu$  in a neighborhood of  $\partial\mathcal{B}$  using a Gaussian foliation, we have

$$\nabla_A \xi^A = h^{ab} \nabla_a \xi_b + (\nabla \xi)(\nu, \nu).$$

The last term vanishes due to  $\nabla_\nu \nu = 0$ , which is valid in a Gauss foliation. Then we have

$$\nabla_A \xi^A = h^{ab} \nabla_a \xi_b = h^{ab} (D_a P_b - \lambda_{ab} \langle \xi, \nu \rangle)$$

where  $\lambda_{ab} = \langle \nabla_a \nu, e_b \rangle = \frac{1}{2} \mathcal{L}_\nu h_{ab}$  is the second fundamental form. Let  $H = h^{ab} \lambda_{ab}$  denote the mean curvature of  $\partial \mathcal{B}$ . Then we have

$$D_a P^a = H \langle \xi, \nu \rangle + \nabla_A \xi^A.$$

Now specialize to the Euclidean case; let  $g_{AB} = \delta_{AB}$  be the Euclidean metric and let  $\xi = \partial_A$  for some fixed  $A$ . Then  $\nabla_A \xi^A = 0$ , and we have

$$D_a P^a = H \nu^A. \quad (3.5)$$

For  $X \in \mathcal{B}$ , we calculate using (3.3) and the divergence theorem that

$$\begin{aligned} \partial_A \mathcal{S}[f](X) &= - \int_{\partial \mathcal{B}} f(Y) P^a D_a E(X, Y) d\sigma(Y) \\ &\quad - \int_{\partial \mathcal{B}} f(Y) \langle \nu, \partial_{Y^A} \rangle \frac{\partial}{\partial \nu_Y} E(X, Y) d\sigma(Y) \\ &= \int_{\partial \mathcal{B}} (D_a P^a f(Y) + P^a D_a f) E(X, Y) d\sigma(Y) \\ &\quad - \int_{\partial \mathcal{B}} f(Y) \langle \nu, \partial_{y^A} \rangle \frac{\partial}{\partial \nu_Y} E(x, y) d\sigma(Y). \end{aligned}$$

Thus we have by the above result and (3.5) that

$$\partial_A \mathcal{S}[f] = \mathcal{S}[f H \nu^A + \partial_A^\parallel f] - \mathcal{D}[f \nu^A]. \quad (3.6)$$

**Proposition 3.2.** *The operators  $S, \frac{1}{2}I + K$  have the mapping properties*

$$\frac{1}{2}I + K : B^{k-1/p, p}(\partial \Omega) \rightarrow B^{k-1/p, p}(\partial \Omega), \quad (3.7a)$$

$$S : B^{k-1-1/p, p}(\partial \Omega) \rightarrow B^{k-1/p, p}(\partial \Omega), \quad (3.7b)$$

for  $s = k - 1/p$ ,  $k \geq 1$ ,  $k$  an integer. The corresponding statements for the single and double layer potentials are

$$\mathcal{D} : B^{k-1/p, p}(\partial \Omega) \rightarrow W^{k, p}(\Omega) \quad (3.8a)$$

$$\mathcal{S} : B^{k-1-1/p, p}(\partial \Omega) \rightarrow W^{k, p}(\Omega) \quad (3.8b)$$

for  $k \geq 1$ ,  $k$  integer.

*Proof.* We use induction to reduce the statement to the case  $k = 1$ . This case follows from [15], see also [26]. Suppose then that we have proved the statement for  $k - 1$ . To do the induction, assume  $f \in B^{k-1/p, p}(\partial \Omega)$ . Let  $u$  be the solution to the Dirichlet problem with boundary data  $f$ . Then  $u \in W^{k, p}(\Omega)$ , and setting  $\psi = \text{tr}_{\partial \Omega} \frac{\partial}{\partial \nu} u$ , we have  $\psi \in B^{k-1-1/p, p}$ . Equations (3.4) and (3.6) give

$$\begin{aligned} \partial_{x^i} \mathcal{D}[f] &= \partial_{x^i} \mathcal{S}[\psi] + \partial_{x^i} u \\ &= \mathcal{S}[\psi H \nu^i + \partial_{x^i}^\parallel \psi] - \mathcal{D}[\psi \nu^i] + \partial_{x^i} u \end{aligned}$$

which by the induction assumption is in  $W^{k-1, p}$ . It follows that  $\mathcal{D}[f] \in W^{k, p}(\Omega)$  and hence  $(\frac{1}{2}I + K)f \in B^{k-1/p, p}(\partial \Omega)$ . This proves the statement for  $\mathcal{D}$  and  $(\frac{1}{2}I + K)$  at regularity  $k$ .

Next, for  $\mathcal{S}$ , we use the equation (3.6) for  $f \in B^{k-1-1/p,p}(\partial\Omega)$ . Using the induction assumption and the statement for  $\mathcal{D}$  and  $(\frac{1}{2}I + K)$  just proved, we have

$$\partial_{x^i}\mathcal{S}[f] \in W^{k,p},$$

which gives  $\mathcal{S}[f] \in W^{k,p}(\partial\Omega)$ , and  $\mathcal{S}[f] \in B^{k-1/p,p}(\partial\Omega)$ . This completes the induction and the result follows.  $\square$

*Example 3.3.* Let  $f \in W^{1,p}(\Omega)$ . Then  $\text{tr}_{\partial\Omega}f \in B^{1-1/p,p}$  and  $\mathcal{V}[\partial_{x^i}f] \in W^{2,p}(\Omega)$ . Further,  $\mathcal{S}[\text{tr}_{\partial\Omega}f\nu^i]$  solves a Dirichlet problem with boundary data  $\mathcal{S}[\text{tr}_{\partial\Omega}f\nu^i] \in B^{2-1/p,p}(\partial\Omega)$ , and hence  $\mathcal{S}[\text{tr}_{\partial\Omega}f\nu^i] \in W^{2,p}$ . It follows, in view of (3.2), that  $(\mathcal{V}f)\chi_\Omega \in W^{3,p}(\Omega)$ .

*Example 3.4.* Let  $f \in W^{2,p}(\mathcal{B})$ . We have

$$\partial_{x^i}\mathcal{V}[f] = \mathcal{V}[\partial_{x^i}f] + \mathcal{S}[\text{tr}_{\partial\Omega}f\nu^i].$$

Since  $\partial_{x^i}f \in W^{1,p}(\mathcal{B})$ , we have from Example 3.3 that  $\mathcal{V}[\partial_{x^i}f] \in W^{3,p}(\mathcal{B})$ . Further,  $\text{tr}_{\partial\Omega}f \in B^{2-1/p,p}(\partial\Omega)$  and hence  $\mathcal{S}[\text{tr}_{\partial\Omega}f\nu^i] \in W^{3,p}(\mathcal{B})$ . Therefore  $\mathcal{V}[f] \in W^{4,p}(\mathcal{B})$ .

**Proposition 3.5.** *Let  $k \geq 1$ , and assume  $f \in W^{k,p}(\mathcal{B})$ . Then  $\mathcal{V}[f] \in W^{k+2,p}(\mathcal{B})$ .*

*Proof.* The proof proceeds by induction, with base case  $k = 1$ . For this case, the statement follows by the argument in Example 3.3. Suppose we have proved the statement for  $k - 1$ . We will make use of the identity (3.2). By induction,  $\mathcal{V}[\partial_{x^i}f] \in W^{k+1,p}(\Omega)$ . Further,  $\text{tr}_{\partial\Omega}f \in B^{k-1/p}$  and hence by (3.7),  $\mathcal{S}[\text{tr}_{\partial\Omega}f\nu^i] \in B^{k+1-1/p}(\partial\Omega)$ . It follows by (3.1) that  $\mathcal{S}[\text{tr}_{\partial\Omega}f\nu^i] \in W^{k+1,p}(\mathcal{B})$ . This shows that  $\partial_{x^i}\mathcal{V}[f] \in W^{k+1,p}(\mathcal{B})$  and hence  $\mathcal{V}[f] \in W^{k+2,p}(\mathcal{B})$ .  $\square$

Similar arguments combined with the mapping properties (A.7)-(A.8) of the Laplacian on the weighted Sobolev spaces can be used to establish the following proposition for the volume potential on  $\mathcal{B}_-$ .

**Proposition 3.6.** *Let  $k \geq 1$ ,  $-1 < \delta < 0$ , and assume  $f \in W_{\delta-2}^{k,p}(\mathcal{B}_-)$ . Then  $\mathcal{V}[f] \in W_{\delta}^{k+2,p}(\mathcal{B}_-)$ .*

**3.2. The Poisson equation in the material frame.** The next step in solving the problems of the existence of initial data satisfying the compatibility conditions and the time evolution of this data is to establish a number of smoothness properties for solutions to the Poisson equations in the material frame. We begin by defining the spaces

$$\mathcal{W}_{\delta}^{k,s,p}(\mathbb{R}^3) = \{ u \in W_{\delta}^{k,p}(\mathbb{R}^3, V) \mid u|_{\mathcal{B}} \in W^{k+s,p}(\mathcal{B}) \text{ and } u|_{\mathcal{B}_-} \in W_{\delta}^{k+s,p}(\mathcal{B}_-) \}$$

for  $1 < p < \infty$ ,  $k \in \mathbb{Z}$  and  $\delta \in \mathbb{R}$ . It is not difficult to verify that these spaces are complete with respect to the norm

$$\|u\|_{\mathcal{W}_{\delta}^{k,s,p}(\mathbb{R}^3)} = \|u|_{\mathcal{B}}\|_{W^{k+s,p}(\mathcal{B})} + \|u|_{\mathcal{B}_-}\|_{W_{\delta}^{k+s,p}(\mathcal{B}_-)} + \|u\|_{W_{\delta}^{k,p}(\mathbb{R}^3)}, \quad (3.9)$$

and hence Banach spaces.

**Theorem 3.7.** *Suppose  $1 < p < \infty$ ,  $s \in \mathbb{Z}_{\geq 0}$ ,  $s + 1 > 3/p$ , and  $-1 < \delta < 0$ . Then there exist an open neighborhood  $\tilde{\mathcal{O}}^{s+2,p} \subset W^{s+2,p}(\mathcal{B}, \mathbb{R}^3)$  of  $\psi_0$ , and an analytic map*

$$\bar{U} : \tilde{\mathcal{O}}^{s+2,p} \longrightarrow \mathcal{W}_{\delta}^{2,s,p}(\mathbb{R}^3) : \phi \longmapsto \bar{U}(\phi)$$

*such that*

- (i)  $\bar{U}(\phi)$  satisfies the Poisson equation (2.23b) on  $\mathbb{R}^3$ ,
- (ii) for each  $\phi \in \tilde{\mathcal{O}}^{s+2,p}$ , the map  $\tilde{\phi} = \psi_0 + \mathbb{E}_{\mathcal{B}}(\phi - \psi_0|_{\mathcal{B}})$  is a  $C^1$  diffeomorphism on  $\mathbb{R}^3$  that satisfies  $\tilde{\phi} - \psi_0 \in W_{-10}^{2+s,p}(\mathbb{R}^3, \mathbb{R}^3) \subset C_{-10}^1(\mathbb{R}^3, \mathbb{R}^3)$  and  $\tilde{\phi}^{-1} - \psi_0 \in W_{-10}^{2+s,p}(\mathbb{R}^3, \mathbb{R}^3)$ ,
- (iii)  $U = \bar{U}(\phi) \circ \tilde{\phi}^{-1} \in W_{\delta}^{2,p}(\mathbb{R}^3)$  satisfies the Poisson equation  $\Delta U = m\rho\chi_{\phi(\mathcal{B})}$  on  $\mathbb{R}^3$  and  $U(x) = o(|x|^{\delta})$  as  $|x| \rightarrow \infty$ , and
- (iv) for  $k \in \mathbb{Z}_{\geq 1}$  and  $k \leq s+1$ , the derivative of  $\bar{U}$  can be extended to act on  $W^{k,p}(\mathcal{B})$ , and moreover, the map

$$\tilde{\mathcal{O}}^{s+2,p} \ni \phi \longmapsto D\bar{U}(\phi) \in L(W^{k,p}(\mathcal{B}))$$

is well defined and analytic<sup>3</sup>.

*Proof.* (i) Fix  $1 < p < \infty$ ,  $s+1 > 3/p$ , and  $-1 < \delta < 0$ . Given  $\psi \in W^{2+s,p}(\mathcal{B})$ , we define

$$\tilde{\phi} = \psi_0 + \mathbb{E}_{\mathcal{B}}(\psi), \quad \phi_A^i = \partial_A \tilde{\phi}^i, \quad \text{and} \quad (f_i^A) = (\phi_A^i)^{-1}.$$

Since matrix inversion and the determinant both define analytic maps in a neighborhood of the identity, it follows from (A.6), proposition 3.6 of [16], the continuity of extension and differentiation, the analyticity of continuous linear maps, and the property that the composition of analytic maps are again analytic that there exists a  $R > 0$  such that the maps

$$B_R(W^{2+s,p}(\mathcal{B}, \mathbb{R}^3)) \ni \psi \longmapsto \det(\phi_A^i) - 1 \in W_{-11}^{1+s,p}(\mathbb{R}^3) \quad (3.10)$$

and

$$B_R(W^{2+s,p}(\mathcal{B}, \mathbb{R}^3)) \ni \psi \longmapsto (f_i^A - \delta_i^A) \in W_{-11}^{1+s,p}(\mathbb{R}^3, \mathbb{M}_{3 \times 3}) \quad (3.11)$$

are well defined and analytic. Recalling that  $H^{AB} = f_i^A \delta^{ij} f_j^B$ , we see from the same arguments that the map

$$B_R(W^{2+s,p}(\mathcal{B}, \mathbb{R}^3)) \ni \psi \longmapsto JH^{AB} - \delta^{AB} \in W_{-11}^{1+s,p}(\mathbb{R}^3, \mathbb{M}_{3 \times 3}) \quad (3.12)$$

is analytic where  $J = \det(\phi_A^i)$ . Using the multiplication inequalities (A.2) and (A.6), we find that

$$\|\partial_A (JH^{AB} \partial_B \bar{U})|_{\mathcal{B}}\|_{W^{s,p}(\mathcal{B})} \lesssim \|\psi\|_{W^{2+s,p}(\mathcal{B}, \mathbb{R}^3)} \|\bar{U}|_{\mathcal{B}}\|_{W^{2+s,p}(\mathcal{B})}, \quad (3.13)$$

$$\|\partial_A (JH^{AB} \partial_B \bar{U})|_{\mathcal{B}_-}\|_{W_{\delta}^{s,p}(\mathcal{B}_-)} \lesssim \|\psi\|_{W^{2+s,p}(\mathcal{B}, \mathbb{R}^3)} \|\bar{U}|_{\mathcal{B}_-}\|_{W_{\delta}^{2+s,p}(\mathcal{B}_-)} \quad (3.14)$$

and

$$\|\partial_A (JH^{AB} \partial_B \bar{U})\|_{W_{\delta-2}^{0,p}(\mathbb{R}^3)} \lesssim \|\psi\|_{W^{2+s,p}(\mathcal{B}, \mathbb{R}^3)} \|\bar{U}\|_{W_{\delta}^{2,p}(\mathbb{R}^3)}. \quad (3.15)$$

From the analyticity of the map (3.12), the bilinear estimates (3.13)-(3.15), the analyticity of continuous bilinear maps, and the property that the composition of analytic maps are again analytic, it follows that the

$$B_R(W^{2+s,p}(\mathcal{B}, \mathbb{R}^3)) \times \mathcal{W}_{\delta}^{2,s,p}(\mathbb{R}^3) \ni (\psi, \bar{U}) \longmapsto \partial_A (JH^{AB} \partial_B \bar{U}) \in \mathcal{W}_{\delta-2}^{0,s,p}(\mathbb{R}^3) \quad (3.16)$$

is well defined and analytic. Together, (A.7), and Propositions 3.5 and 3.6 imply that

$$\Delta^{-1}(\chi_{\mathcal{B}}) \in \mathcal{W}_{\delta}^{2,s,p}(\mathbb{R}^3), \quad (3.17)$$

and the Laplacian

$$\Delta : \mathcal{W}_{\delta}^{2,s,p}(\mathbb{R}^3) \longrightarrow \mathcal{W}_{\delta-2}^{0,s,p}(\mathbb{R}^3) \quad (3.18)$$

<sup>3</sup>For a Banach space  $X$ ,  $L(X)$  denotes the set of continuous linear operators on  $X$

is an isomorphism with inverse given by (A.8).

From (3.16), (3.17), and (3.18), we see that

$$F : B_R(W^{2+s,p}(\mathcal{B}, \mathbb{R}^3)) \times \mathcal{W}_\delta^{2,s,p}(\mathbb{R}^3) \longrightarrow W_\delta^{2,s,p}(\mathbb{R}^3),$$

$$(\psi, \bar{U}) \longmapsto \Delta^{-1} \left( \partial_A (JH^{AB} \partial_B \bar{U}) - m\chi_{\mathcal{B}} \right)$$

is well defined and analytic. Evaluating  $F$  at  $\psi = 0$  gives

$$F(0, \bar{U}) = \Delta^{-1} \left( \Delta \bar{U} - m\chi_{\mathcal{B}} \right),$$

which shows that

$$\bar{U}_0 = m\Delta^{-1}(\chi_{\mathcal{B}}) \in W_\delta^{2,s,p}(\mathbb{R}^3) \quad (3.19)$$

satisfies

$$F(0, \bar{U}_0) = 0. \quad (3.20)$$

Also, by the linearity of  $F$  in its second argument and the invertibility of the Laplacian, it is clear that

$$D_2 F(0, \bar{U}) \cdot \delta \bar{U} = \Delta^{-1} \left( \Delta \delta \bar{U} \right) = \delta \bar{U}. \quad (3.21)$$

Results (3.20) and (3.21) allow us to apply an analytic version of the implicit function theorem (see [13], theorem 15.3) to conclude the existence of a unique analytic map, shrinking  $R$  if necessary,

$$\bar{U} : B_R(W^{2+s,p}(\mathcal{B}, \mathbb{R}^3)) \longrightarrow W_\delta^{2,s,p}(\mathbb{R}^3) \quad (3.22)$$

that satisfies

$$\bar{U}(0) = \bar{U}_0,$$

and

$$F(\psi, \bar{U}(\psi)) = 0 \quad \forall \psi \in B_R(W^{2+s,p}(\mathcal{B}, \mathbb{R}^3)).$$

From the definition of  $F$  and the invertibility of  $\Delta^{-1}$ , it then follows that  $\bar{U}(\psi)$  satisfies

$$\partial_A (JH^{AB} \partial_B \bar{U}(\psi)) = m\chi_{\mathcal{B}}. \quad (3.23)$$

**(ii) & (iii)** Following Cantor [9], we consider the following group of diffeomorphisms on  $\mathbb{R}^3$

$$\mathcal{D}_\delta^{s,q}(\mathbb{R}^3) := \{ \phi : \mathbb{R}^3 \rightarrow \mathbb{R}^3 \mid \phi - \psi_0 \in W_\eta^{s,q}(\mathbb{R}^3, \mathbb{R}^3), \text{ and } \phi^{-1} - \psi_0 \in W_\eta^{s,q}(\mathbb{R}^3, \mathbb{R}^3) \}$$

where  $s > 3/q + 1$  and  $\eta \leq 0$ . Fixing  $\psi \in B_R(W^{2+s,p}(\mathcal{B}, \mathbb{R}^3))$ , we get from (3.10) and (3.11) that

$$\tilde{\phi} = \phi(\psi) \in \mathcal{D}_{-10}^{2+s,p}(\mathbb{R}^3).$$

Defining,

$$U := U(\psi) \circ \tilde{\phi}^{-1}, \quad (3.24)$$

we can apply Corollary 1.6 of [9] to get<sup>4</sup>

$$U \in W_\delta^{2,p}(\mathbb{R}^3). \quad (3.25)$$

A straightforward calculation using the chain rule and (3.23), (3.24), and (3.25) then shows that

$$\Delta U = m \det(D(\tilde{\phi}^{-1})) \chi_{\tilde{\phi}(\mathcal{B})},$$

---

<sup>4</sup>In [9], Cantor required that  $\delta \leq -3/2$  because that was what he needed to prove the weighted multiplication inequality (A.6). It is clear that his proofs are valid whenever the multiplication inequality holds and  $W_\delta^{k,p} \subset C_b^1$ . Consequently, the only restriction on  $\delta$  is that  $\delta \leq 0$ .

while the fall off condition  $U(x) = o(|x|^\delta)$  as  $|x| \rightarrow \infty$  follows from the weighted Sobolev inequality (A.5).

(iv) To begin, we assume that  $k = 1$  and observe that for  $\theta^i \in W^{1,p}(\mathcal{B})$  and  $\bar{U} \in \mathcal{W}^{2,s,p}(\mathbb{R}^3)$

$$\begin{aligned} \|\mathbb{E}_{\mathcal{B}}(\partial_C \theta^i) \partial_B \bar{U}\|_{L^p_{\delta^{-1}}(\mathbb{R}^3)} &\leq \|\partial_C \theta^i \partial_B \bar{U}\|_{L^p(\mathcal{B})} + \|\mathbb{E}_{\mathcal{B}}(\partial_C \theta^i) \partial_B \bar{U}\|_{L^p_{\delta^{-1}}(\mathcal{B}_-)} \\ &\lesssim \|\theta\|_{W^{1,p}(\mathcal{B})} \|\bar{U}\|_{\mathcal{B}} \|_{W_\delta^{s+2,p}(\mathcal{B})} + \|\theta\|_{W^{1,p}(\mathcal{B})} \|\bar{U}\|_{\mathcal{B}_-} \|_{W_\delta^{s+2,p}(\mathcal{B}_-)} \\ &\lesssim \|\theta\|_{W^{1,p}(\mathcal{B})} \|\bar{U}\|_{\mathcal{W}^{2,s,p}(\mathbb{R}^3)} \end{aligned} \quad (3.26)$$

where in deriving the result we have used property (A.9) of the extension operator  $\mathbb{E}_{\mathcal{B}}$ , the multiplication inequalities (A.2) and (A.6), and the assumption  $1+s > 3/p$ .

Letting

$$H_i^{ABC} = \frac{\partial JH^{AB}}{\partial \phi_C^i},$$

we get, using the estimate (3.26) and the same arguments as above, that for  $R$  small enough the map

$$\begin{aligned} G_A : B_R(W^{2+s,p}(\mathcal{B}, \mathbb{R}^3)) \times W^{1,p}(\mathcal{B}, \mathbb{R}^3) \times \mathcal{W}_\delta^{2,s,p}(\mathbb{R}^3) \times W_\delta^{1,p}(\mathbb{R}^3) &\longrightarrow W_{\delta^{-1}}^{-1,p}(\mathbb{R}^3) \\ : (\psi^i, \theta^i, \bar{U}, \bar{V}) &\longmapsto JH^{AB} \partial_B \bar{V} + H_i^{ABC} \mathbb{E}_{\mathcal{B}}(\partial_C \theta^i) \partial_B(\bar{U}) \end{aligned}$$

is analytic. From the continuity of differentiation and the trace map, we then have that the map

$$G : B_R(W^{2+s,p}(\mathcal{B}, \mathbb{R}^3)) \times W^{1,p}(\mathcal{B}, \mathbb{R}^3) \times \mathcal{W}_\delta^{2,s,p}(\mathbb{R}^3) \times W_\delta^{1,p}(\mathbb{R}^3) \longrightarrow W_{\delta^{-2}}^{-1,p}(\mathbb{R}^3)$$

defined by

$$G(\psi^i, \theta^i, \bar{U}, \bar{V}) = \partial_A G^A(\psi^i, \theta^i, \bar{U}, \bar{V})$$

is analytic. Taking  $\bar{U}_0$  as defined by (3.19), a straightforward calculation and the invertibility of the Laplacian show that

$$G(0, 0, \bar{U}_0, 0) = 0 \quad \text{and} \quad D_4 G(0, 0, \bar{U}_0, 0) \cdot \delta \bar{V} = \delta \bar{V}.$$

Therefore, we can again apply the analytic version of the implicit function theorem (see [13], theorem 15.3) to conclude the existence of a unique analytic map, shrinking  $R$  if necessary,

$$\bar{V} : B_R(W^{2+s,p}(\mathcal{B}, \mathbb{R}^3)) \times W^{1,p}(\mathcal{B}, \mathbb{R}^3) \times (\bar{U}_0 + B_R(\mathcal{W}_\delta^{2,s,p})) \longrightarrow W_\delta^{1,p}(\mathbb{R}^3) \quad (3.27)$$

that satisfies

$$\bar{V}(0, 0, \bar{U}_0, 0) = 0,$$

and

$$G(\psi^i, \theta^i, \bar{U}, \bar{V}(\psi^i, \theta^i, \bar{U})) = 0$$

for all  $(\psi^i, \theta^i, \bar{U}) \in B_R(W^{2+s,p}(\mathcal{B}, \mathbb{R}^3)) \times W^{1,p}(\mathcal{B}, \mathbb{R}^3) \times (\bar{U}_0 + B_R(\mathcal{W}_\delta^{2,s,p}(\mathbb{R}^3)))$ .

From the construction of  $G$ , and the uniqueness of the maps (3.22) and (3.27), it is not difficult to verify that

$$\bar{V}(\psi^i, \partial_A \delta \psi^i, \bar{U}(\psi)) = D\bar{U}(\psi) \cdot \delta \psi \quad \forall (\psi, \delta \psi) \in B_R(W^{2+s,p}(\mathcal{B}, \mathbb{R}^3)) \times W^{2+s,p}(\mathcal{B}, \mathbb{R}^3).$$

From this and the density of  $W^{s+1,p}(\mathcal{B})$  in  $W^{1,p}(\mathcal{B})$ , it follows that the derivative of  $\bar{U}$  can be extended act on  $W^{1,p}(\mathcal{B})$ , and moreover, that the map

$$B_R(W^{2+s,p}(\mathcal{B}, \mathbb{R}^3)) \ni \psi \longmapsto D\bar{U}(\psi) \in L(W^{1,p}(\mathcal{B})) \quad (3.28)$$

is well defined and analytic. By (3.22) above, we also have that the map

$$B_R(W^{2+s,p}(\mathcal{B}, \mathbb{R}^3)) \ni \psi \longmapsto D\bar{U}(\psi) \in L(W^{2+s,p}(\mathcal{B})) \quad (3.29)$$

is well defined and analytic. Together, the maps (3.28)-(3.29) and interpolation imply that the map

$$B_R(W^{2+s,p}(\mathcal{B}, \mathbb{R}^3)) \ni \psi \longmapsto D\bar{U}(\psi) \in L(W^{k,p}(\mathcal{B}))$$

is well defined and analytic for  $k \in \mathbb{Z}$  and  $1 < k < s + 2$ .  $\square$

**Corollary 3.8.** *The map*

$$\Lambda : \tilde{\mathcal{O}}^{s+2,p} \longrightarrow W^{s+1,p}(\mathcal{B}, \mathbb{R}^3)$$

defined by

$$\Lambda^i(\phi) = -\delta^{ij} f_i^A \partial_A \bar{U}(\phi) \quad ((f_i^A) = (\partial_A \phi^i)^{-1})$$

is analytic. Moreover, for  $k \in \mathbb{Z}_{\geq 1}$  and  $k \leq s + 1$ , the derivative of  $\Lambda$  can be extended act on  $W^{k,p}(\mathcal{B})$ , and the map

$$\tilde{\mathcal{O}}^{s+2,p} \ni \phi \longmapsto D\Lambda(\phi) \in L(W^{k,p}(\mathcal{B}), W^{k-1,p}(\mathcal{B}))$$

is well defined and analytic.

*Proof.* This follows directly from theorem 3.7, the multiplication inequality (A.6), proposition 3.6 of [16], and the fact that compositions of analytic maps are again analytic.  $\square$

**3.3. The linearized elasticity operator.** We define the operator linearized elasticity and boundary operators by

$$A(\phi)^i := \partial_B (a^{iB}{}_j{}^D \partial_D \phi^j) \quad (3.30)$$

and

$$A_\partial(\phi)^i := \nu_B a^{iB}{}_j{}^D \partial_D \phi^j, \quad (3.31)$$

respectively. We also define

$$Y^{s,p} = W^{s,p}(\mathcal{B}, \mathbb{R}^3) \times B^{s+1-1/p,p}(\partial\mathcal{B}, \mathbb{R}^3),$$

and for each  $\phi \in W^{s+2,p}(\mathcal{B}, \mathbb{R}^3)$ ,

$$Y_\phi^{s,p} = \{ (b, t) \in Y^{s,p} \mid \mathcal{C}_1(b, t) = 0, \mathcal{C}_2(\phi, b, t) = 0 \}$$

where

$$\mathcal{C}_1(b, t) = \int_{\mathcal{B}} b + \int_{\partial\mathcal{B}} t, \quad (3.32)$$

and

$$\mathcal{C}_2(\phi, b, t) = \int_{\mathcal{B}} b \times \phi + \int_{\partial\mathcal{B}} t \times \phi. \quad (3.33)$$

Here, we are using the notation

$$(b \times \phi)^i = \epsilon^i{}_{jk} b^j \phi^k, \quad (3.34)$$

$$\int_{\mathcal{B}} b = \int_{\mathcal{B}} b d^3 X, \quad (3.35)$$

and

$$\int_{\partial\mathcal{B}} t = \int_{\partial\mathcal{B}} t \, d\sigma. \quad (3.36)$$

For later use, we recall the following theorem from [24] concerning the surjectivity of the linearized elasticity operator.

**Theorem 3.9.** [[24], theorem 1.11, Section 6.1] *Suppose  $1 < p < \infty$ ,  $s \geq 0$ , and  $\mathbb{P} : Y^{s,p} \rightarrow Y_{\psi_0}^{s,p}$  is any projection map. Then the map*

$$\mathbb{A} : W^{s+2,p}(\mathcal{B}, \mathbb{R}^3) \rightarrow Y_{\psi_0}^{s,p} : \phi \mapsto \mathbb{P}(A(\phi), A_{\partial}(\phi))$$

is surjective and

$$\ker \mathbb{A} = \{a + b \times X \mid a = (a^i), b = (b^i) \in \mathbb{R}^3\}$$

where  $(b \times X)^i = \epsilon^i_{jA} b^j X^A$ .

*Remark 3.10.* Letting

$$\tilde{X}^j = \frac{1}{\text{Vol}(\mathcal{B})} \int_{\mathcal{B}} X^j \, d^3 X$$

denote the center of  $\mathcal{B}$ , a short calculation using the change of coordinates  $\bar{X}^j = X^j - \tilde{X}^j$  and  $\bar{\mathcal{B}} = \mathcal{B} - \tilde{X}$  shows that

$$\int_{\bar{\mathcal{B}}} \bar{X}^j \, d^3 \bar{X} = \int_{\mathcal{B}} X^j - \tilde{X}^j \, d^3 X.$$

Therefore, we can always arrange that

$$\int_{\mathcal{B}} X^j \, d^3 X = 0 \quad (3.37)$$

by translating the domain  $\mathcal{B}$ . For the remainder of this article, we will always assume that the condition (3.37) holds.

Next, for  $p > 3/2$ , we define the spaces

$$X^{s,p} = \left\{ \phi \in W^{s,p}(\mathcal{B}, \mathbb{R}^3) \mid \int_{\mathcal{B}} \phi = 0 \right\},$$

and observe, using Sobolev's and Hölder's inequalities, that

$$\begin{aligned} \left| \int_{\mathcal{B}} \phi \times \psi \right| &\leq \|\phi \times \psi\|_{L^1(\mathcal{B}, \mathbb{R}^3)} \lesssim \|\phi\|_{L^\infty(\mathcal{B}, \mathbb{R}^3)} \|\psi\|_{L^1(\mathcal{B}, \mathbb{R}^3)} \\ &\lesssim \|\phi\|_{W^{2,p}(\mathcal{B}, \mathbb{R}^3)} \|\psi\|_{L^p(\mathcal{B}, \mathbb{R}^3)} \\ &\lesssim \|\phi\|_{W^{2+s,p}(\mathcal{B}, \mathbb{R}^3)} \|\psi\|_{W^{s,p}(\mathcal{B}, \mathbb{R}^3)} \end{aligned}$$

from which the continuity of the bilinear map

$$B : X^{s+2,p} \times X^s \rightarrow \mathbb{R}^3 : (\phi, \psi) \mapsto \int_{\mathcal{B}} \phi \times \psi$$

follows. Setting

$$B_\psi(\phi) := B(\phi, \psi),$$

we define for  $p > 3/2$  the following spaces

$$\mathcal{U}^{s,p} = \{ \psi \in X^{s,p} \mid B_\psi : \ker A \cap X^{s+2,p} \rightarrow \mathbb{R}^3 \text{ is an isomorphism} \}.$$

**Lemma 3.11.**  $\psi_0 \in \mathcal{U}^{s,p}$  for all  $s \geq 0$  and  $3/2 < p < \infty$ .

*Proof.* By (3.37), we have that

$$\int_{Bc} \psi_0^i d^3 X = \int_{\mathcal{B}} X^i d^3 X = 0,$$

and

$$\int_{\mathcal{B}} a + b \times X = \int_{\mathcal{B}} a + b \times \int_{\mathcal{B}} X = \text{Vol}(\mathcal{B})$$

for all  $a, b \in \mathbb{R}^3$ . Consequently,

$$\psi_0 \in X^{s,p}, \quad (3.38)$$

and

$$\ker \mathbb{A} \cap X^{s+2,p} = \{ b \times X \mid b \in \mathbb{R}^3 \} \quad (3.39)$$

by theorem 3.9. Next,

$$\begin{aligned} B_{\psi_0}(b \times X) &= \int_{\mathcal{B}} (b \times X) \times \psi_0 = \int_{\mathcal{B}} (b \times X) \times X \\ &= \int_{\mathcal{B}} (X \cdot b)X - b|X|^2, \end{aligned}$$

which, after taking the innerproduct with  $a \in \mathbb{R}^3$ , yields

$$b \cdot B_{\psi_0}(b \times X) = \int_{\mathcal{B}} (X \cdot b)(X \cdot a) - a \cdot b|X|^2. \quad (3.40)$$

The Cauchy-Schwartz inequality shows that

$$(X \cdot b)^2 - |b|^2|X|^2 \leq 0 \quad (3.41)$$

and

$$(X \cdot b)^2 - |b|^2|X|^2 = 0 \quad \forall X \in \mathcal{B} \iff b = 0. \quad (3.42)$$

Combining (3.40)-(3.42), we arrive at

$$b \cdot B_{\psi_0}(b \times X) = 0 \iff b = 0. \quad (3.43)$$

By way of contradiction, suppose that the map

$$\ker \mathbb{A} \cap X^{s+2,p} \longrightarrow \mathbb{R}^3 \quad (3.44)$$

is not surjective. Then the image  $B_{\psi_0}(\ker \mathbb{A} \cap X^{s+2,p})$  is contained in a two dimensional subspace, and therefore, there exists a non zero  $a \in \mathbb{R}^3$  such that

$$a \cdot B_{\psi_0}(b \times X) = 0 \quad \forall b \in \mathbb{R}^3.$$

But this is impossible by (3.43), and hence the map (3.44) is surjective. Since  $\dim \ker \mathbb{A} \cap X^{s+2,p} = 3$ , the map (3.44) must, in fact, be an isomorphism.  $\square$

#### 3.4. Existence of initial data satisfying the compatibility conditions.

**Lemma 3.12.** *Suppose that  $\mathcal{Y}_0, \mathcal{Y}_1, \dots, \mathcal{Y}_r$ , and  $\mathcal{Z}$  are Banach spaces with continuous (linear) embeddings*

$$\iota_{i,j} : \mathcal{Y}_i \longrightarrow \mathcal{Y}_j \quad i, j \in \{0, 1, \dots, r\}, \quad i < j,$$

$\mathcal{U} \subset \mathcal{Y}_r$  is open, and  $F \in C^{r+1}(\mathcal{U}, \mathcal{Z})$ . Then the map defined by

$$F_r(y_0, y_1, \dots, y_r) := \left. \frac{d^r}{dt^r} \right|_{t=0} F(c(t)) \quad \text{where} \quad c(t) = \sum_{j=0}^r t^j \iota_{j,r}(y_j)$$

is in  $C^1(t_{0,r}^{-1}(\mathcal{U}) \times \prod_{j=1}^r \mathcal{Y}_j, \mathcal{Z})$ .

*Proof.* Since  $\mathcal{U} \in Y_r$  is open, it follows from the continuity of the map  $\iota_{0,r}$  that  $\iota_{0,r}^{-1}(\mathcal{U}) \subset Y_0$  is open. Next, fix  $y_0 \in \iota_{0,r}^{-1}(\mathcal{U})$  and  $y_j \in Y_j$  for  $j = 1, 2, \dots, r$ . Then the continuity of the maps  $\iota_{j,r} : Y_j \rightarrow Y_r$  and  $\iota_{0,r}(y_0) \in \mathcal{U}$  guarantees the existence of a  $\delta > 0$  such that

$$c(t) = \sum_{j=0}^r t^j \iota_{j,r}(y_j) \in \mathcal{U} \quad \forall t \in (-\delta, \delta).$$

Clearly, this implies that  $c \in C^\infty((-\delta, \delta), \mathcal{U})$ , and hence, that the map

$$\iota_{0,r}^{-1}(\mathcal{U}) \times \prod_{j=1}^r \mathcal{Y}_j \ni (y_0, \dots, y_r) \mapsto \left. \frac{d^r}{dt^r} \right|_{t=0} F(c(t)) \in \mathcal{Z}$$

is well defined and continuously differentiable.  $\square$

**Proposition 3.13.** *Suppose  $3 < p < \infty$ ,  $s \in \mathbb{Z}_{\geq 0}$ , and  $\tilde{\mathcal{O}}^{s+2,p} \subset W^{s+2,p}(\mathcal{B}, \mathbb{R}^3)$  is the open neighborhood of  $\psi_0$  from theorem 3.7. Then the maps (see (2.27))*

$$\begin{aligned} E &: \tilde{\mathcal{O}}^{s+2,p} \longrightarrow W^{s,p}(\mathcal{B}, \mathbb{R}^3), \\ E_\partial &: \tilde{\mathcal{O}}^{s+2,p} \longrightarrow B^{s+1-1/p,p}(\partial\mathcal{B}) \end{aligned}$$

are  $C^\infty$ .

*Proof.* First, we recall that, by assumption  $\bar{\tau}^{iA}$ , is a smooth function of its arguments  $\partial_A \phi^i$  in the neighborhood of the identity map  $\psi_0^i$ . Since  $p > 3$  and  $s \geq 0$ , we have that  $s+1 > 3/p$ , and it follows from theorem 1, Section 5.5.2, of [30], and the continuity of differentiation (cf. (A.1)) that the map  $\tilde{\mathcal{O}}^{s+2,p} \ni \phi \mapsto \tau^{iA} \in W^{s+1,p}(\mathcal{B}, \mathbb{R}^6)$  is  $C^\infty$ . The proof then follows directly from the continuity of the trace map (A.3).  $\square$

Defining

$$\mathcal{O}^{s+2,p} = \tilde{\mathcal{O}}^{s+2,p} \cap X^{s+2,p},$$

we have that

$$\psi_0 \in \mathcal{O}^{s+2,p}$$

by (3.38). We also define<sup>5</sup>

$$F : \mathcal{O}^{s+2,p} \times X^{s,p} \times \mathbb{R} \longrightarrow Y^{s,p} \times \mathbb{R}^3 : (\phi_0, \phi_2, \lambda) \mapsto (F(\phi_0, \phi_2, \lambda), B(\phi_0, \psi_0))$$

where

$$\mathcal{F}(\phi_0, \phi_2, \lambda) = (E(\phi_0) + \lambda^2 \Lambda(\phi_0) - \phi_2, E_\partial(\phi_0)),$$

and we let

$$\mathbb{P} : \mathcal{O}^{s+2,p} \rightarrow L(Y^{s,p}, Y^{s,p}) \tag{3.45}$$

denote any  $C^\infty$  map for which  $\mathbb{P}_{\psi_0}$  coincides with the projection operator from theorem 3.9. Furthermore, we assume that for each  $\phi_0 \in \mathcal{O}^{s+2,p}$ , the linear operator

$$\mathbb{P}(\phi_0)|_{Y_{\phi_0}^{s,p}} : Y_{\phi_0}^{s,p} \longrightarrow Y_{\psi_0}^{s,p} \tag{3.46}$$

is an isomorphism and

$$Y^{s,p} = Y_{\phi_0}^{s,p} \oplus \ker \mathbb{P}(\phi_0). \tag{3.47}$$

The existence of a map (3.45) satisfying (3.46) and (3.47) can be found in [20].

<sup>5</sup>For two Banach spaces  $X$  and  $Y$ ,  $L(X, Y)$  denotes the set of continuous linear maps from  $X$  to  $Y$ .

For  $r \in \mathbb{Z}_{\geq 0}$ , we define

$$\begin{aligned}\phi_r &= (\phi_0, \phi_1, \dots, \phi_r)^T, \\ \phi_r(t) &= \phi_0 + \sum_{j=1}^r \lambda \frac{t^j}{j!} \phi_j, \\ \phi_{r,2}(t) &= \sum_{j=0}^r \lambda \frac{t^j}{j!} \phi_{j+2},\end{aligned}$$

and

$$\begin{aligned}F_0(\phi_0, \phi_2, \lambda) &= F(\phi_0, \phi_2, \lambda), \\ F_1(\phi_1, \phi_3, \lambda) &= \frac{1}{\lambda} (\mathcal{F}_1(\phi_1, \phi_3, \lambda), \phi_{1,2}(t), \lambda), \lambda B(\psi_0, \phi_1), \\ F_{r+2}(\phi_r, \phi_{r+2}, \lambda) &= \frac{1}{\lambda} \left( \mathcal{F}_{r+2}(\phi_{r+2}, \phi_{r+4}, \lambda), \frac{d^r}{dt^r} \Big|_{t=0} B(\phi_0(t), \phi_{0,2}(t)) \right) \quad (r \geq 0)\end{aligned}$$

where

$$\mathcal{F}_r(\phi_r, \phi_{r+2}, \lambda) = \frac{d^r}{dt^r} \Big|_{t=0} \mathcal{F}(\phi_r(t), \phi_{r,2}(t), \lambda).$$

*Remark 3.14.* Under the identification

$$\phi_r = (\partial_t^r \phi) \Big|_{t=0},$$

it follows from the definition of  $\phi_r(t)$  and  $\phi_{r,2}(t)$  above that

$$\frac{d^r}{dt^r} \Big|_{t=0} \phi_r(t) = \lambda (\partial_t^r \phi) \Big|_{t=0} \quad \text{and} \quad \frac{d^r}{dt^r} \Big|_{t=0} \phi_{r,2}(t) = \lambda (\partial_t^{r+2} \phi) \Big|_{t=0}.$$

Moreover, under this identification, we have that

$$\mathcal{F}(\phi_0, \phi_2, \lambda) = 0$$

if and only if

$$(\partial_A \bar{\tau}^{iA}(\partial\phi) - \lambda^2 \delta^{ij} f_j^A \partial_A \bar{U}(\phi) - \partial_t^2 \phi^i) \Big|_{t=0} = 0 \quad \text{and} \quad (\nu_A \bar{\tau}^{iA}(\partial\phi) \Big|_{\partial\mathcal{B}}) \Big|_{t=0} = 0.$$

Also, by repeatedly differentiating the equations of motion (2.23a)-(2.23c), it is not difficult to see that

$$\mathcal{F}_r(\phi_r, \phi_{r+2}, \lambda) = 0 \tag{3.48}$$

if and only if

$$\partial_t^r (\partial_A \bar{\tau}^{iA}(\partial\phi) - \lambda^2 \delta^{ij} f_j^A \partial_A \bar{U}(\phi) - \partial_t^2 \phi^i) \Big|_{t=0} = 0 \quad \text{and} \quad \partial_t^r (\nu_A \bar{\tau}^{iA}(\partial\phi) \Big|_{\partial\mathcal{B}}) \Big|_{t=0} = 0.$$

This shows that solving (3.48) for  $r = 0, 1, \dots, \ell$  will produce initial data that satisfies the compatibility conditions to order  $\ell$ .

In order to use the implicit function theorem to solve the equations (3.48), we need to introduce the following maps which are closely related to the  $\mathcal{F}_r$  and  $F_r$

maps introduced above:

$$\begin{aligned} \mathcal{G}(\phi_0, \phi_2, \lambda) &= \mathbb{P}(\phi_0) \mathcal{F}(\phi_0, \phi_2, \lambda), \\ G_0(\phi_0, \phi_2, \lambda) &= (\mathcal{G}(\phi_0, \phi_2, \lambda), B(\psi_0, \phi_0)), \\ G_1(\phi_1, \phi_3, \lambda) &= \frac{1}{\lambda} (\mathcal{G}_1(\phi_1, \phi_3, \lambda), \lambda B(\psi_0, \phi_1)), \\ G_{r+2}(\phi_{r+2}, \phi_{r+4}, \lambda) \\ &= \frac{1}{\lambda} \left( \frac{d^{r+2}}{dt^{r+2}} \Big|_{t=0} \mathcal{G}_{r+2}(\phi_{r+2}(t), \phi_{r+4}(t), \lambda), \frac{d^r}{dt^r} \Big|_{t=0} B(\phi_0(t), \phi_{0,2}(t)) \right), \quad (r \geq 0) \end{aligned}$$

and

$$\mathbf{G}_r(\phi_r, \phi_{r+1}, \phi_{r+2}, \lambda) = \begin{pmatrix} G_0(\phi_0, \phi_2, \lambda) \\ G_1(\phi_1, \phi_3, \lambda) \\ \vdots \\ G_{r-1}(\phi_{r-1}, \phi_{r+1}, \lambda) \\ G_r(\phi_r, \phi_{r+2}, \lambda) \end{pmatrix}$$

where

$$\mathcal{G}_r(\phi_r, \phi_{r+2}, \lambda) = \frac{d^r}{dt^r} \Big|_{t=0} \mathcal{G}(\phi_r(t), \phi_{r,2}(t), \lambda).$$

**Proposition 3.15.** *Suppose  $s \in \mathbb{Z}_{\geq r}$  and  $3 < p < \infty$ . Then the maps*

$$F_r : \tilde{\mathcal{O}}^{s+2,p} \times \left( \prod_{j=1}^{r+2} W^{s+2-j,p}(\mathcal{B}, \mathbb{R}^3) \right) \times \mathbb{R}^3 \longmapsto Y^{s-r,p} \times \mathbb{R}^3$$

and

$$G_r : \tilde{\mathcal{O}}^{s+2,p} \times \left( \prod_{j=1}^{r+2} W^{s+2-j,p}(\mathcal{B}, \mathbb{R}^3) \right) \times \mathbb{R}^3 \longmapsto Y^{s-r,p} \times \mathbb{R}^3$$

are  $C^1$ .

*Proof.* First we note that the maps  $F_0$  and  $G_0$  are  $C^\infty$  which follows from Corollary 3.8, proposition 3.13, and the smoothness of the map (3.45). The proof then follows immediately from lemma 3.12 and the definition of  $F_r$  and  $G_r$ .  $\square$

Introducing

$$\mathcal{E}(\phi, \lambda) = (E(\phi) + \lambda^2 \Lambda(\phi), E_\partial(\phi)),$$

we get that

$$F_0(\phi_0, \phi_2, \lambda) = (\mathcal{E}(\phi_0, \lambda) - (\phi_2, 0), B(\psi_0, \phi_0)),$$

and it follows easily from the definition of the  $F_r$  maps above that

$$F_1(\phi_1, \phi_3, \lambda) = (D_\phi \mathcal{E}(\phi_0, \lambda) \cdot \phi_1 - (\phi_3, 0), B(\psi_0, \phi_1)),$$

and

$$F_2(\phi_2, \phi_4, \lambda) = (D_\phi \mathcal{E}(\phi_0, \lambda) \cdot \phi_2 - (\phi_4, 0) + \lambda D_\phi^2 \mathcal{E}(\phi_0, \lambda) \cdot (\phi_1, \phi_1), B(\phi_0, \phi_2)).$$

Proceeding inductively, we obtain for  $r \geq 1$ ,

$$F_r(\phi_r, \phi_{r+2}, \lambda) = (D_\phi \mathcal{E}(\phi_0, \lambda) \cdot \phi_r - (\phi_{r+2}, 0), B(\phi_0, \phi_r)) + \lambda \tilde{F}_r(\phi_{r+1}, \lambda) \quad (3.49)$$

where the map  $\tilde{F}_r$  is  $C^1$ . Setting,

$$\mathbb{P}_r(\phi_r) = \frac{d^r}{dt^r} \Big|_{t=0} \mathbb{P}(\phi_r(t)),$$

the product rule shows that

$$\mathcal{G}_r(\phi_r, \phi_{r+2}, \lambda) = \sum_{j=0}^r \binom{r}{s} \mathbb{P}_{r-j}(\phi_{r-j}) \mathcal{F}_r(\phi_j, \phi_{j+2}, \lambda). \quad (3.50)$$

This and (3.49), in turn, give

$$G_r(\phi_r, \phi_{r+2}, \lambda) = (\mathbb{P}(\phi_0) [D_\phi \mathcal{E}(\phi_0, \lambda) \cdot \phi_r - (\phi_{r+2}, 0)], B(\phi_0, \phi_r)) + \lambda \tilde{G}_r(\phi_{r+1}, \lambda) \quad r \geq 1 \quad (3.51)$$

where  $\tilde{G}_r$  is  $C^1$ .

Letting

$$\psi_r = (\psi_0, 0, \dots, 0),$$

it follows directly from (3.51) that the derivative of  $\mathbf{G}_r$  evaluated at  $(\phi_r, \phi_{r+1}, \phi_{r+2}, \lambda) = (\psi_r, 0, 0, 0)$  is

$$D_{\phi_r} \mathbf{G}_r(\psi_r, 0, 0, 0) \cdot \delta \phi_r = A \cdot \delta \phi_r \quad (3.52)$$

where

$$A = \begin{pmatrix} \mathcal{L}(\psi_0) & 0 & \mathcal{M}(\psi_0) & 0 & \cdots & 0 \\ 0 & \mathcal{L}(\psi_0) & 0 & \mathcal{M}(\psi_0) & & \vdots \\ 0 & 0 & \mathcal{L}(\psi_0) & 0 & \ddots & 0 \\ \vdots & \vdots & 0 & \mathcal{L}(\psi_0) & & \mathcal{M}(\psi_0) \\ & & & & \ddots & 0 \\ 0 & 0 & 0 & \cdots & 0 & \mathcal{L}(\psi_0) \end{pmatrix} \quad (3.53)$$

$$\mathcal{L}(\psi_0) \cdot \delta \psi = (\mathbb{P}(\psi_0) \mathbb{A} \delta \psi, B(\psi_0, \delta \psi)), \quad (3.54)$$

and

$$\mathcal{M}(\psi_0) \cdot \delta \psi = (-\mathbb{P}(\psi_0)(\delta \psi, 0), 0).$$

We note that in deriving (3.52)-(3.53), we have used

$$\mathcal{E}(\psi_0, 0) = 0 \quad \text{and} \quad D_\phi \mathcal{E}(\psi_0, 0) \cdot \delta \psi = \mathbb{A} \delta \psi.$$

Although our main objective is to solve the equations (3.48), we first solve  $\mathbf{G}_r = 0$  and later show that this implies that compatibility conditions are satisfied to order  $r$ . To solve  $\mathbf{G}_r = 0$ , we use the implicit function theorem. The proof we present is modeled on the existence proof for static, self-gravitating elastic bodies presented in [7].

**Proposition 3.16.** *There exists an  $\lambda_0 > 0$ , open neighborhoods  $\mathcal{N}_{r+1} \subset X^{s+1-r,p}$  and  $\mathcal{N}_{r+2} \subset X^{s-r,p}$  both containing 0, and  $C^1$  maps*

$$\Phi_j : \mathcal{N}_{r+1} \times \mathcal{N}_{r+2} \times (-\lambda_0, \lambda_0) \longrightarrow X^{s+2-j,p} : (\phi_{r+1}, \phi_{r+2}, \lambda) \longmapsto \Phi_j(\phi_{r+1}, \phi_{r+2}, \lambda),$$

for  $j = 0, 1, \dots, r$ , such that

$$\Phi_0(0, 0, 0) = \psi_0, \quad \Phi_j(0, 0, 0) = 0, \quad j = 1, 2, \dots, r$$

and

$$\mathbf{G}_r(\Phi_r(\phi_{r+1}, \phi_{r+2}), \phi_{r+1}, \phi_{r+2}, \lambda) = 0 \quad \forall (\phi_{r+1}, \phi_{r+2}, \lambda) \in \mathcal{N}_{r+1} \times \mathcal{N}_{r+2} \times (-\lambda_0, \lambda_0)$$

where  $\Phi_r = (\Phi_0, \Phi_1, \dots, \Phi_r)$ .

*Proof.* We begin by verifying the the linear operator (3.53) is an isomorphism.

**Lemma 3.17.** *For  $s \in \mathbb{Z}_{\geq r}$  and  $3 < p < \infty$ , the map*

$$A : \prod_{j=0}^r X^{s+2-j,p} \longrightarrow \prod_{j=0}^r (Y^{s-j,p} \times \mathbb{R}^3)$$

*is a linear isomorphism.*

*Proof.* By lemma 3.11,  $\psi_0 \in \mathcal{U}^{s,p}$  and so it follows from the definition of  $\mathcal{L}(\psi_0)$  (see (3.54)) and theorem 3.9 that  $\mathcal{L}(\psi_0) : X^{s+2-j,p} \longrightarrow Y^{s-j,p} \times \mathbb{R}^3$  is an isomorphism for  $j = 0, 1, \dots, r$ . The proof now follows immediately from upper triangular structure of the map  $A$  (see (3.53)).  $\square$

Recalling that  $\mathcal{E}(\psi_0) = 0$ , it is clear from the definition of  $\mathcal{F}$  that

$$\mathcal{F}(\psi_0, 0, 0) = 0,$$

and hence, by the antisymmetry of the map  $B$ , that

$$G_0(\psi_0, 0, 0) = 0.$$

Furthermore, it clear from (3.51) that

$$G_j(\psi_j, 0, 0) = 0 \quad j = 1, 2, \dots, r,$$

which, in turn, shows that

$$\mathbf{G}_r(\psi_r, 0, 0, 0) = 0. \quad (3.55)$$

The proof of the proposition now follows proposition 3.15, lemma 3.17, and the implicit function theorem.  $\square$

**Theorem 3.18.** *The maps  $\Phi_j$   $j = 0, 1, \dots, r$  from proposition 3.16 satisfy*

$$\mathcal{F}_j(\Phi_j(\phi_{r+1}, \phi_{r+2}), \Phi_{j+2}(\phi_{r+1}, \phi_{r+2}), \lambda) = 0 \quad j = 0, 1, \dots, r-2,$$

$$\mathcal{F}_{r-1}(\Phi_{r-1}(\phi_{r+1}, \phi_{r+2}), \phi_{r+1}, \lambda) = 0 \quad \text{and} \quad \mathcal{F}_r(\Phi_r(\phi_{r+1}, \phi_{r+2}), \phi_{r+2}, \lambda) = 0$$

*for all  $(\phi_{r+1}, \phi_{r+2}, \lambda) \in \mathcal{N}_{r+1} \times \mathcal{N}_{r+2} \times (-\lambda_0, \lambda_0)$ .*

*Proof.* It is shown in [7] that

$$\mathcal{C}_1(\mathcal{E}(\phi, \lambda)) = 0 \quad \text{and} \quad \mathcal{C}_2(\phi, \mathcal{E}(\phi, \lambda)) = 0 \quad (3.56)$$

are automatically satisfied for all  $\phi \in \mathcal{O}^{s+2,p}$  and  $-\lambda_0 < \lambda < \lambda_0$ . Setting

$$\mathcal{E}_j(\phi_j, \lambda) = \left. \frac{d^j}{dt^j} \right|_{t=0} \mathcal{E}(\phi_r(t), \lambda) \quad j = 0, 1, \dots, r,$$

the formulas

$$\mathcal{C}_1(\mathcal{E}_j(\phi_j, \lambda)) = 0 \quad \text{and} \quad \sum_{k=0}^j \binom{j}{k} \mathcal{C}_2(\phi_k, \mathcal{E}_{j-k}(\phi_{j-k}, \lambda)) = 0 \quad j = 0, 1, \dots, r$$

follow from differentiating (3.56). In particular, this implies that the maps  $\Phi_j$  from theorem 3.16 satisfy

$$\mathcal{C}_1(\mathcal{E}_j(\Phi_j(\phi_{r+1}, \phi_{r+2}, \lambda)), \lambda) = 0 \quad \text{for } j = 0, 1, \dots, r \quad (3.57)$$

and for  $j = 0, 1, \dots, r-2$ ,

$$\sum_{k=0}^j \binom{j}{k} \mathcal{C}_2(\Phi_k(\phi_{r+1}, \phi_{r+2}, \lambda), \mathcal{E}_{j-k}(\Phi_{j-k}(\phi_{r+1}, \phi_{r+2}, \lambda), \lambda)) = 0, \quad (3.58)$$

for all  $(\phi_{r+1}, \phi_{r+2}, \lambda) \in \mathcal{N}_{r+1} \times \mathcal{N}_{r+2} \times (-\lambda_0, \lambda_0)$

Next, we note that

$$\mathcal{C}_1((\phi_j, 0)) = 0, \quad \forall \phi_r \in X^{s+2-j,p}, \quad (3.59)$$

and

$$\begin{aligned} \frac{d^j}{dt^j} \Big|_{t=0} B(\phi_j(t), \phi_{2,j}(t)) = 0 &\iff \frac{d^j}{dt^j} \Big|_{t=0} \mathcal{C}_2(\phi_j(t), (\phi_{2,r}(t), 0)) = 0 \\ &\iff \sum_{k=0}^j \binom{j}{k} \mathcal{C}_2(\phi_k, (\phi_{j+2-k}, 0)) = 0. \end{aligned} \quad (3.60)$$

If we define

$$\Phi_{r+1}(\phi_{r+1}, \phi_{r+2}, \lambda) = \phi_{r+1} \quad \text{and} \quad \Phi_{r+2}(\phi_{r+1}, \phi_{r+2}, \lambda) = \phi_{r+2},$$

then it follows from (3.59), (3.60), and the definition of  $\mathbf{G}_r$  that the maps  $\Phi_j$  satisfy

$$\mathcal{C}_1(\Phi_j(\phi_{r+1}, \phi_{r+2}, \lambda), 0) = 0 \quad j = 0, 1, \dots, r+2, \quad (3.61)$$

and

$$\sum_{k=0}^j \binom{j}{k} \mathcal{C}_2(\Phi_k(\phi_{r+1}, \phi_{r+2}, \lambda), (\Phi_{j+2-k}(\phi_{r+1}, \phi_{r+2}, \lambda), 0)) = 0 \quad j = 0, 1, \dots, r \quad (3.62)$$

for all  $(\phi_{r+1}, \phi_{r+2}, \lambda) \in \mathcal{N}_{r+1} \times \mathcal{N}_{r+2} \times (-\lambda_0, \lambda_0)$ .

Fixing  $(\phi_{r+1}, \phi_{r+2}, \lambda) \in \mathcal{N}_{r+1} \times \mathcal{N}_{r+2} \times (-\lambda_0, \lambda_0)$ , the identities (3.57), (3.58), (3.61), (3.62) and the definition of the maps  $\mathcal{F}_j$  show that

$$\mathcal{C}_1(\mathcal{F}_j(\Phi_j(\phi_{r+1}, \phi_{r+2}, \lambda), \Phi_{j+2}(\phi_{r+1}, \phi_{r+2}, \lambda), \lambda)) = 0, \quad (3.63)$$

and

$$\sum_{k=0}^j \binom{j}{k} \mathcal{C}_2(\Phi_k(\phi_{r+1}, \phi_{r+2}, \lambda), \mathcal{F}_{j-k}(\Phi_{j-k}(\phi_{r+1}, \phi_{r+2}, \lambda), \Phi_{j+2-k}(\phi_{r+1}, \phi_{r+2}, \lambda), \lambda)) = 0 \quad (3.64)$$

for  $0 \leq j \leq r$ , while

$$\mathcal{G}_0(\Phi_0(\phi_{r+1}, \phi_{r+2}, \lambda), \Phi_2(\phi_{r+1}, \phi_{r+2}, \lambda), \lambda) = 0$$

follows from theorem 3.16, or equivalently, by the definition of  $\mathcal{G}_0$ , and

$$\mathbb{P}(\Phi_0(\phi_{k+1}, \phi_{r+2}, \lambda)) \mathcal{F}_0(\Phi_0(\phi_{r+1}, \phi_{r+2}, \lambda), \Phi_2(\phi_{r+1}, \phi_{r+2}, \lambda), \lambda) = 0. \quad (3.65)$$

But, we also have that

$$\mathcal{F}_0(\Phi_0(\phi_{r+1}, \phi_{r+2}, \lambda), \Phi_2(\phi_{r+1}, \phi_{r+2}, \lambda), \lambda) \in Y_{\Phi_0(\phi_{k+1}, \phi_{r+2}, \lambda)}^{s,p}$$

by (3.63)-(3.64), and thus,

$$\mathcal{F}_0(\Phi_0(\phi_{r+1}, \phi_{r+2}, \lambda), \Phi_2(\phi_{r+1}, \phi_{r+2}, \lambda), \lambda) = 0$$

by (3.47) and (3.65).

To finish the proof, we proceed by induction. So, we assume that

$$\mathcal{F}_0(\Phi_j(\phi_{r+1}, \phi_{r+2}, \lambda), \Phi_{j+2}(\phi_{r+1}, \phi_{r+2}, \lambda), \lambda) = 0 \quad (3.66)$$

for  $0 \leq k \leq j < r$ . Then

$$\mathcal{G}_k(\Phi_k(\phi_{r+1}, \phi_{r+2}, \lambda), \Phi_{k+2}(\phi_{r+1}, \phi_{r+2}, \lambda)) = 0 \quad (3.67)$$

for  $0 \leq k \leq j$  by theorem 3.16. Clearly, (3.50), (3.66), and (3.67) imply that

$$\mathbb{P}(\Phi_0(\phi_{k+1}, \phi_{r+2}, \lambda)) \mathcal{F}_{j+1}(\Phi_{j+1}(\phi_{r+1}, \phi_{r+2}, \lambda), \Phi_{j+2}(\phi_{r+1}, \phi_{r+2}, \lambda), \lambda) = 0. \quad (3.68)$$

But, since

$$\mathcal{F}_{j+1}(\Phi_{j+1}(\phi_{r+1}, \phi_{r+2}, \lambda), \Phi_{j+3}(\phi_{r+1}, \phi_{r+2}, \lambda), \lambda) \in Y_{\Phi_0(\phi_{k+1}, \phi_{r+2}, \lambda)}^{s,p}$$

by (3.63)-(3.64), we must, in fact, have

$$\mathcal{F}_{j+1}(\Phi_{j+1}(\phi_{r+1}, \phi_{r+2}, \lambda), \Phi_{j+3}(\phi_{r+1}, \phi_{r+2}, \lambda), \lambda) = 0$$

by (3.47) and (3.68), and the proof is complete.  $\square$

In light of remark 3.14, the following corollary is a direct consequence of the above theorem.

**Corollary 3.19.** *Suppose  $s \in \mathbb{Z}_{\geq 0}$ ,  $3 < p < \infty$ , and the maps  $\Phi_j$ , the neighborhoods  $\mathcal{N}_{s+1} \subset X^{1,p} \subset W^{1,p}(\mathcal{B}, \mathbb{R}^3)$  and  $\mathcal{N}_{s+2} \subset X^{0,p} \subset W^{0,p}(\mathcal{B}, \mathbb{R}^3)$ , and  $\lambda_0$  are as in proposition 3.16. Then for each  $(\phi_{s+1}, \phi_{s+2}, \lambda) \in \mathcal{N}_{s+1} \times \mathcal{N}_{s+2} \times (-\lambda_0, \lambda_0)$ , the initial data*

$$(\phi|_{t=0}, \partial_t|_{t=0}\phi) = (\Phi_0(\phi_{r+1}, \phi_{r+2}, \lambda), \Phi_1(\phi_{r+1}, \phi_{r+2}, \lambda)) \in \tilde{\mathcal{O}}^{s+2,p} \times \tilde{\mathcal{O}}^{s+1,p}$$

*satisfy the compatibility conditions to order  $s$ .*

#### 4. LOCAL WELL-POSEDNESS

In this section, we prove local well-posedness for the system (2.23a)-(2.23c) using the approach of [19]. The system under consideration is an initial-boundary value problem of elliptic-hyperbolic type, due to the presence of the equation (2.23b) in the system, and hence the results of [19] do not apply directly. However, the techniques of [19] are readily adapted to include non-local terms, and we will present an outline of the proof of this fact below. To conclude this section, we will apply the resulting existence theorem to establish the existence of dynamical solutions to (2.23a)-(2.23c).

**4.1. Setup and notation.** For ease of reference, we adopt the index and other notational conventions of [19], with some exceptions, as pointed out below. We are interested only in the case  $n = 3$ , but it is convenient to treat the case of general  $n \geq 3$ . The number of components of the system of equations is in the case of elasticity equal to  $N = 3$ , but the treatment below applies to general  $N$ .

Let  $0 \leq i, k \leq n$ ,  $1 \leq \alpha, \beta \leq n$ ,  $1 \leq j, l \leq N$ . We work in a coordinate system  $(x_i)$  and let  $t = x_0$ . The summation convention is used. Further, we shall denote the unknown field by  $u$  rather than  $\phi$ . Let  $Du = (\partial_t u, \partial_{x_1} u, \dots, \partial_{x_n} u)$ , and  $D_x u = (\partial_{x_1} u, \dots, \partial_{x_n} u)$ . Fix some  $s > n/2 + 1$ ,  $s$  an integer, and consider the system in the domain  $\Omega$  with boundary  $\Gamma$  of regularity class  $C^{s+2}$ , and denote  $\Omega_T = [0, T] \times \Omega$  and  $\Gamma_T = [0, T] \times \Gamma$  where  $0 < T \leq \infty$ .

Following [19], we consider equations of the form

$$\partial_{x_i} F_j^i(t, x, u, Du) = w_j[t, x, u, Du] \quad \text{in } \Omega_T, \quad (4.1a)$$

$$v_j F_j^i(t, x, u, Du) = g_j(t, x, u, Du) \quad \text{in } \Gamma_T, \quad (4.1b)$$

$$u = u_0, \quad \partial_t u = u_1 \quad \text{in } \{0\} \times \Omega \quad (4.1c)$$

where, in contrast to [19],  $w_j[t, x, u, Du]$  is a functional of  $(u, Du)$  that we allow to be non-local. The properties required of  $w$  are specified below in Assumption  $w$ .

*Assumption 1.* We assume  $u_0 \in W^{s+1}(\Omega)$ ,  $u_1 \in W^s(\Omega)$ . We assume  $F, g \in C^{s+1}(U)$  where  $U$  is a neighborhood of the graph of  $u_0, u_1, D_x u_0$  as in [19, §1], and

$$a_{jl}^{ik} = \frac{\partial F_j^i}{\partial(\partial_{x_k} u^l)}, \quad h_{jl}^k = \frac{\partial g_j}{\partial(\partial_{x_k} u^l)}.$$

We decompose  $h_{jl}^k = h_{jl}^{sk} + h_{jl}^{uk}$  where  $h_{jl}^{sk} = h_{(jl)}^{sk}$  is the symmetric part and  $h_{jl}^{uk} = h_{[jl]}^{uk}$  is the antisymmetric part. We assume that the symmetric part  $h_{(jl)}^{sk}$  is of the form  $h_{(jl)}^{sk} = \theta^k h_{jl}$  where  $\theta(t, x, u, Du)$  is a vector field which is tangential to  $\Gamma_T$  and satisfies  $\theta^0 = 1$ , and  $h_{jl}$  is symmetric.

For the elastic body, we have

$$h_{jl}^k = 0 \tag{4.2}$$

while  $a_{jl}^{ik}$  can be calculated in terms of the elasticity tensor  $L$ , cf. (2.16).

*Assumptions 2-5.* The structural relations of the elastic body imply the hyperbolicity of the system. In particular, for the self-gravitating elastic body, the symmetry and coerciveness assumptions, Assumptions 2 and 3 of [19], hold. Further, Assumption 4 of [19] on the time components  $a_{jl}^{00}$  follow directly from the structure of the elastic system. For a discussion of the compatibility conditions on the initial data, see Assumption 5 in [19].

Following [19, p. 25], we introduce the spaces  $E^s$ ,  $G^s$ , and  $F^s$  with norms

$$\|u\|_{E^s}(t) = \left( \sum_{i=0}^s \|\partial_t^i u(t)\|_{W^{s+1-i}}^2 \right)^{1/2},$$

$$\|u\|_{G_{t_1, t_2}^s} = \sup_{t_1 \leq t \leq t_2} \|u\|_{E^s}(t),$$

and

$$\|u\|_{F_{t_1, t_2}^s} = \int_{t_1}^{t_2} \|u\|_{E^s}(t) dt,$$

respectively.

*Assumption w.* For non-local  $w$ , we make the following further assumptions. We assume  $w$  to be well defined if the graph of  $(u, Du)$  lies in a suitable subset of  $U$ , where  $U$  as in Assumption 1 above. There, the following conditions are imposed.

(1) If

$$u \in \cap_{0 \leq j \leq s} C^j([0, T], W^{s+1-j}(\Omega)),$$

we have

$$w[u, Du] \in \cap_{0 \leq j \leq s} C^j([0, T], W^{s-j}(\Omega)).$$

(2) We require the map  $u \mapsto w[u, Du]$  to be Lipschitz in the above topology.

In particular, see section 4.3.1 below, we shall make use of the estimate

$$\|\partial_t^2(w(v_1) - w(v_0))\|_{L^2} \leq c \|(\partial_t(v_1 - v_0))\|_{E^2}. \tag{4.3}$$

(3) Finally, we require the following uniform estimate

$$\|w\|_{E^s} \leq c(1 + \|D^2 u\|_{L^\infty})(1 + \|u\|_{E^{s+1}})$$

where  $c$  is a constant depending on  $\|Du\|_{L^\infty}$  as well as the coercivity constants  $\kappa, \mu$  for the system.

We have the following result, which is the analog of [19, theorem 1.1].

**Theorem 4.1.**

- (1) *Existence, regularity:* There exists a unique  $0 < t_0 \leq T$ , and a unique classical solution  $u \in C^2(\Omega_{t_0} \cup \Gamma_{t_0})$  of (4.1) with  $D^\sigma u(t) \in L^2(\Omega)$  if  $0 \leq \sigma \leq s+1$ . Here  $D^\sigma u$  denotes all derivatives of order  $\sigma$ .
- (2) *Continuous dependence on initial data.*
- (3) *Blow up:*  $t_0$  is characterized by the two alternatives: either the graph of  $(u, Du)$  is not precompact in  $U$  or

$$\int_0^t \|D^2 u(\tau)\|_{L^\infty(\Omega)} d\tau \rightarrow \infty, \quad \text{as } t \rightarrow t_0.$$

**4.2. Linear systems.** For a solution to the system (analogous to [19, (2.10)])

$$\partial_{x_i}(a^{ik}\partial_{x_k}u) = w + \partial_{x_i}f^i \quad \text{in } \Omega_T, \quad (4.4a)$$

$$v_\alpha(a^{\alpha k}\partial_{x_k}u - f^\alpha) = h^{sk}\partial_{x_k}u \quad \text{in } \Gamma_T \quad (4.4b)$$

with the coercivity and structure conditions as in [19, §2], and in particular, the regularity assumptions (see [19, Assumption 2s, p. 26])

$$a, h \in G^s \cap C^1(\Omega_T) \quad \partial_t a, \partial_t h \in F^s, \quad (4.5a)$$

$$w \in G^{s-1} \cap W^{s,1}([0, T], L^2(\Omega)) \cap W^{s-1,1}([0, T], W^1(\Omega)), \quad (4.5b)$$

$$f^i \in G^s, Df^i \in W^{s,1}([0, T], L^2(\Omega)), \quad (4.5c)$$

$$u_0 \in W^{s+1}(\Omega) \quad u_1 \in W^s(\Omega), \quad (4.5d)$$

we have the estimate

$$\begin{aligned} \|u\|_{E^{s+1}}(t_2) \leq & \tilde{c} \left( \|u(t_1)\|_{E^{s+1}} + \|w\|_{G^{s-1}} \right. \\ & \left. + \|f\|_{G^s} + \int_{t_1}^{t_2} \|\partial_t^s w(t)\|_{L^2(\Omega)} + \|\partial_t^{s+1} f^i(t)\|_{L^2(\Omega)} dt \right) \end{aligned} \quad (4.6)$$

where

$$\tilde{c} = \tilde{c}(\kappa, \mu, \|a\|_{G^s \cap C^1(\Omega_T)}, \|\partial_t a\|_{F^s}, \|h^s\|_{G^s \cap C^1(\Omega_T)}, \|\partial_t h^s\|_{F^s}).$$

Note that the system given in (4.4) is of a restricted form with  $g = 0$ ,  $h^u = 0$ . These terms can be absorbed into the others, cf. the discussion in [19, §2]. This is achieved by introducing the modified coefficients  $\bar{a}^{ik}$ ,  $\bar{f}^i$ ,  $\bar{w}$  as in [19, p. 31], see also (4.12) below.

Applying the above estimate to a system of the form

$$\partial_{x_i}(a^{ik}\partial_{x_k}u) = w + \partial_{x_i}f^i \quad \text{in } \Omega_T \quad (4.7a)$$

$$v_\alpha(a^{\alpha k}\partial_{x_k}u - f^\alpha) = h^k\partial_{x_k}u + g \quad \text{in } \Gamma_T, \quad (4.7b)$$

we get, in view of the above discussion, the inequality

$$\begin{aligned} \|u\|_{E^{s+1}}(t_2) \leq & \tilde{c} \left( \|u(t_1)\|_{E^{s+1}} + \|w\|_{G^{s-1}} + \|f\|_{G^s} + \|g\|_{G^s} \right. \\ & + \int_{t_1}^{t_2} \|\partial_t^s w(t)\|_{L^2(\Omega)} + \|\partial_t^{s+1} f^i(t)\|_{L^2(\Omega)} dt \\ & \left. + \int_{t_1}^{t_2} \|\partial_t^s g\|_{W^1(\Omega)} + \|\partial_t^{s+1} g\|_{L^2(\Omega)} ds \right) \end{aligned} \quad (4.8)$$

**4.3. Proof of theorem 4.1.** In this section, we discuss the main steps in the proof of theorem 4.1. In the following calculations, we suppress the indices  $u_j$  on  $u$  and the corresponding indices on  $a_{jt}^{ik}, w_j$ , etc. First, we apply a time derivative to (4.1) which gives<sup>6</sup>

$$\partial_{x_i}(a^{ik}\partial_{x_k,t}^2 u) = \partial_t w + \partial_{x_i} f^i \quad \text{in } \Omega_T, \quad (4.9a)$$

$$v_\alpha(a^{\alpha k}\partial_{x_k,t}^2 u - f^\alpha) = h^k\partial_{x_k,t}^2 u + \bar{g} \quad \text{in } \Gamma_T, \quad (4.9b)$$

where

$$f^i(t, x, u, Du) = a^{ik}\partial_{x_k,t}^2 u - \partial_t F^i, \quad \bar{g} = \partial_t g - h^k\partial_{x_k,t}^2 u.$$

For the elastic system, the coefficients have no explicit time dependence, and the boundary condition is homogenous. So we have that

$$f^i = 0, \quad \bar{g} = 0. \quad (4.10)$$

Next, the system is rewritten in the form

$$\partial_{x_i}(\bar{a}^{ik}\partial_{x_k,t}^2 u) = \bar{w} + \partial_{x_i} \bar{f}^i \quad \text{in } \Omega_T, \quad (4.11a)$$

$$v_\alpha(\bar{a}^{\alpha k}\partial_{x_k,t}^2 u - \bar{f}^\alpha) = h^{sk}\partial_{x_k,t}^2 u \quad \text{in } \Gamma_T, \quad (4.11b)$$

where

$$\bar{f}^i = f + v^i \bar{g}, \quad \bar{a}^{ik} = a^{ik} - v^i h^{uk} + v^k h^{ui} \quad (4.12a)$$

and

$$\bar{w} = \partial_t w - ((\partial_v + \text{div } v)h^{uk})\partial_{x_k,t}^2 u + (\partial_{x_i} v^\beta h^{ui})\partial_{x_\beta,t}^2 u - \partial_v \bar{g} - (\text{div } v)\bar{g}. \quad (4.12b)$$

*Remark 4.2.* By introducing the modifications  $\bar{a}^{ik}, \bar{f}^i, \bar{w}$  of  $a^{ik}, f^i, w$  as in (4.12), the resulting system (4.11) has no term  $g$  and also  $h^u = 0$ . Thus it is of the form of the system (4.4) considered in [19, theorem 2.4].

For the elastic system, we have  $f = g = h = 0$  and hence

$$\bar{a}^{ik} = a^{ik}, \quad \bar{w} = \partial_t w. \quad (4.13)$$

We have that  $\bar{a}, \bar{f}$  depend on  $(t, x, u, Du)$  and  $\bar{w}$  depends linearly on  $D^2 u$ .

For technical reasons, we assume  $s > n/2 + 2$  for the differentiability index  $s$ . This is one more degree of smoothness than one would normally expect to require for a solution of a non-linear wave equation using energy estimates. However, this assumption reflects the use of  $v = \partial_t u$  as the main variable in Koch's approach [19], which has one less degree of differentiability compared to  $u$ . The stated result for initial data with  $s > n/2 + 1$  is recovered by a smoothing and a limit argument, see the discussion in [19, p. 33].

Next, we let  $Y_{\tau,R}$  be the subset of

$$H_\tau = \cap_{1 \leq i \leq s+1} W^{i,\infty}([0, \tau], W^{s+1-i}(\Omega))$$

of functions  $v$  that satisfy  $\partial_t^i v(0) = u_i$ , where  $u_i$  are the formal time derivatives of  $u$  for  $0 \leq i \leq s$  at  $t = 0$ , and  $\|v\|_{H_\tau} \leq R$ . By choosing  $R$  sufficiently large, we can make sure this set is non-empty.

The construction of solutions for the system (4.1) makes use of a standard fixed point argument, where one proves boundedness in a high norm and contraction in

<sup>6</sup>We write  $\partial_{x_k,t}^2$  where  $\partial_{x_k t}$  is used in [19]

a low norm. The high norm in this case is  $W^{s+1}$ , which for the linearized (time-differentiated) system corresponds to  $W^s$ . The low norm for the time-differentiated system is  $W^2$ .

This type of argument has been carried out for a quasi-linear elliptic-hyperbolic system with no boundary conditions in [4]. The difference between that system and the present situation is that we have neumann-type boundary conditions, and the system has symbol depending on  $Du$ , i.e. it is fully non-linear.

In the rest of this section, we consider the details of the contraction estimate and the continuation property. The proof of the continuous dependence given in [19] can be readily adapted to the present case with the details given below.

4.3.1. *Contraction estimate.* Define  $J \in C(Y_{\tau,R}, H_{\tau})$  as the map  $v \mapsto u$ , where  $u$  solves the linear system<sup>7</sup>

$$\partial_{x_i}(\bar{a}^{ik}(v)\partial_{x_k,t}^2 u) = \bar{w}(v) + \partial_{x_i}\bar{f}^i \quad \text{in } \Omega_T, \quad (4.14a)$$

$$v_{\alpha}(\bar{a}^{\alpha k}(v)\partial_{x_k,t}^2 u - \bar{f}^{\alpha}(v)) = h^k(v)\partial_{x_k,t}^2 u \text{ in } \Gamma_T, \quad (4.14b)$$

$$\partial_t u = u_1, \quad \partial_t^2 u = u_2, \text{ in } \{0\} \times \Omega \quad (4.14c)$$

where we have denoted  $a(v) = a(t, x, v, Dv)$  etc.

For the case of the elastic system, we again have

$$\bar{a}^{ik} = a^{ik}, \quad \bar{w} = \partial_t w, \quad \bar{f} = 0, \quad h = 0,$$

and in particular,

$$\bar{w} = D_u w \cdot \partial_t u + D_{D_x u} w \cdot D_x \partial_t u$$

where we have used  $D$  to denote the Frechet derivative. There is no explicit  $(t, x)$  dependence for this case.

Next, we observe that

$$\partial_t w \in G^{s-1}, \quad \partial_t^{s+1} w \in F^0$$

by Assumption  $w$ . From this, we see that for large  $R$ , the image of  $J$  lies in  $Y$  provided  $\tau$  is chosen small enough.

We have

$$Y \subset C^3([0, \tau], L^2) \cap C^2([0, \tau], W^1) \cap C^1([0, \tau], W^2)$$

One shows that the set  $Y$  is compact in the topology of the space defined by the right hand side of this expression.

Let  $u_0 = J(v_0)$ ,  $u_1 = J(v_1)$ . In order to derive the system solved by  $u_1 - u_0$ , we let  $Z(u) = \partial_{x_i}\bar{a}^{ik}(t, x, u, Du)\partial_{x_k,t}^2 u$ , and note that  $Z(u_1) - Z(u_0) = \int_0^1 \frac{d}{d\lambda} Z(u_0 + \lambda(u_1 - u_0))$ . Expanding this out gives

$$\begin{aligned} Z(u_1) - Z(u_0) &= \partial_{x_i} \left( \int_0^1 \bar{a}_{\lambda}^{ik} d\lambda \right) \partial_{x_k,t}^2 (u_1 - u_0) \\ &\quad + \partial_{x_i} \left( \int_0^1 [\bar{a}_{\lambda,u}^{ik} \cdot (u_1 - u_0)] \partial_{x_k,t}^2 (u_0 + \lambda(u_1 - u_0)) \right) d\lambda \\ &\quad + \partial_{x_i} \left( \int_0^1 [\bar{a}_{\lambda,Du}^{ik} \cdot (Du_1 - Du_0)] \partial_{x_k,t}^2 (u_0 + \lambda(u_1 - u_0)) \right) d\lambda \end{aligned}$$

<sup>7</sup>A typo in the boundary condition in [19, eq. (3.2)] is corrected here

where<sup>8</sup>

$$\begin{aligned}\bar{a}_\lambda^{ik} &= \bar{a}^{ik}(t, x, u_0 + \lambda(u_1 - u_0), Du_0 + \lambda(Du_1 - Du_0)), \\ \bar{a}_{\lambda,u}^{ik} &= \frac{\partial \bar{a}}{\partial u}(t, x, u_0 + \lambda(u_1 - u_0), Du_0 + \lambda(Du_1 - Du_0)), \\ \bar{a}_{\lambda,Du}^{ik} &= \frac{\partial \bar{a}}{\partial Du}(t, x, u_0 + \lambda(u_1 - u_0), Du_0 + \lambda(Du_1 - Du_0)).\end{aligned}$$

Now, let

$$\begin{aligned}\tilde{a}^{ik} &= \int_0^1 \bar{a}_\lambda^{ik} d\lambda \\ \tilde{w} &= \bar{w}(v_1) - \bar{w}(v_0) \\ &\quad - \left[ \partial_{x_i} \left( \int_0^1 [\bar{a}_{\lambda,u}^{ik} \cdot (u_1 - u_0)] \partial_{x_k,t}^2 (u_0 + \lambda(u_1 - u_0)) d\lambda \right. \right. \\ &\quad \left. \left. + \partial_{x_i} \left( \int_0^1 [\bar{a}_{\lambda,Du}^{ik} \cdot (Du_1 - Du_0)] \partial_{x_k,t}^2 (u_0 + \lambda(u_1 - u_0)) d\lambda \right) \right] \\ \tilde{f} &= \bar{f}(v_1) - \bar{f}(v_0).\end{aligned}$$

Then  $u_1 - u_0$  solves<sup>9</sup>

$$\partial_{x_i} (\tilde{a}^{ik} \partial_{x_k,t}^2 (u_1 - u_0)) = \tilde{w} + \partial_{x_i} \tilde{f}^i, \quad (4.15a)$$

$$v_\alpha (\tilde{a}^{\alpha k} \partial_{x_k,t}^2 (u_1 - u_0) - \tilde{f}^\alpha) = \tilde{h}^k \partial_{x_k,t}^2 (u_1 - u_0) + \tilde{g} \quad (4.15b)$$

where  $\tilde{h}, \tilde{g}$  can be calculated along the same lines as above.

In particular, for the elastic system, we may calculate  $\tilde{w}$  using  $\bar{w} = \partial_t w$ ,  $\bar{h} = \bar{f} = \bar{g} = 0$ . Note that  $\tilde{g}$  is non-vanishing in general due to contributions from  $\bar{a}$ .

We have the estimate<sup>10</sup>

$$\|\partial_t \tilde{w}(t)\|_{L^2} + \|\partial_t \tilde{f}\|_{E^1}(t) + \|\partial_t \tilde{g}\|_{E^1}(t) \leq c \|\partial_t(v_1 - v_0)\|_{E^2} \quad (4.16)$$

where we made use of

$$\tilde{w} = \bar{w}(v_1) - \bar{w}(v_0) + \text{terms involving } \bar{a}.$$

As discussed above,  $\bar{w}$  can be estimated given estimates for  $\partial_t w$ , and hence  $\partial_t \tilde{w}$  can be estimate in terms of  $\partial_t^2(w(v_1) - w(v_0))$ . The terms involving  $\bar{a}$  can be estimated in terms of  $u_1 - u_0$  and hence can be absorbed when applying Gronwall.

From Assumption  $w$ , (4.3), we have

$$\|\partial_t^2(w(v_1) - w(v_0))\|_{L^2} \leq c \|\partial_t(v_1 - v_0)\|_{E^2}.$$

The required contraction estimate is obtained by applying the estimate for the linear system given in [19, theorem 2.4], as discussed in section 4.2, to the system (4.15). One checks that after suitable modifications, cf. section 4.2, the assumptions of [19, theorem 2.4] holds for this system, and this provides the needed contraction estimates.

<sup>8</sup>A typo in [19, p. 32] is corrected here.

<sup>9</sup>This corrects a typo in [19, p. 32].

<sup>10</sup>This corrects a typo in [19, p. 32], which leads to an incorrect estimate for  $\partial_t(u_1 - u_0)$ .

Applying the inequality (4.8) with  $s = 1$  to the system (4.15), we get the inequality

$$\begin{aligned} \|\partial_t(u_1 - u_0)\|_{E^2} &\leq c \left( \|u_1(0) - u_0(0)\|_{E^2} + \|\tilde{w}\|_{G^0} + \|\tilde{f}\|_{G^1} + \|\tilde{g}\|_{G^1} \right. \\ &\quad \left. + \int_0^t \|\partial_t \tilde{w}(\sigma)\|_{L^2} + \|\partial_t \tilde{f}^i\|_{E^1} + \|\partial_t \tilde{g}(\sigma)\|_{E^1} d\sigma \right) \end{aligned}$$

use that  $u_1, u_0$  have the same initial data, and that  $\tilde{w}, \tilde{f}, \tilde{g}$  vanish at  $t = 0$

$$\leq c \left( \int_0^t \|\partial_t \tilde{w}(\sigma)\|_{L^2} + \|\partial_t \tilde{f}^i\|_{E^1} + \|\partial_t \tilde{g}(\sigma)\|_{E^1} d\sigma \right)$$

use (4.16)

$$\leq c \|\partial_t(v_1 - v_0)\|_{E^2}$$

where we made use of the fact that  $\tilde{w}, \tilde{f}, \tilde{g}$  all vanish at  $t = 0$ .

**4.3.2. Continuation principle.** We next consider the proof of the continuation principle. Suppose the graph of  $(u, Du)$  lies in a compact set  $U$ . We need an estimate of the following form, cf. [19, eq. (3.3)]

$$\|u(t)\|_{E^{s+1}} \leq \tilde{c} \left( U, \int_0^t \|D^2 u(\tau)\|_{L^\infty} d\tau \right) (1 + \|u_0\|_{W^{s+1}} + \|u_1\|_{W^s}). \quad (4.17)$$

This estimate is proved by applying operators  $D_P^s$  of order  $s$ , tangent to  $\Gamma_\tau$ , to both sides of the equation (4.1), and applying the estimate for the linear system. Let  $s > n/2 + 1$ , and let  $u \in G^{s+1}$  be a solution of the system (4.1). We then have  $u \in \cap_{0 \leq j \leq s+1} C^j([0, T], W^{s+1-j})$ . One finds, cf. [19, p. 34], that  $D_P^s u$  solves an equation of the form

$$\partial_{x_i}(\tilde{a}^{ik} \partial_{x_k} D_P^s u) = \hat{w} + \partial_{x_i} \hat{f}^i \quad \text{in } \Omega_T, \quad (4.18a)$$

$$v_i(\tilde{a}^{ik} \partial_{x_k} D_P^s u - \hat{f}^i) = h^{sk} \partial_{x_k} D_P^s u \quad \text{in } \Gamma_T, \quad (4.18b)$$

$$D_P^s u(0) = u_1^s, \quad \partial_t D_P^s u = u_2^s \quad (4.18c)$$

where  $u_1^s, u_2^s$  are obtained by formal calculations from the initial data  $u_0, u_1$ . Here a transformation which absorbs the terms  $g$  and  $h^{uk}$  has been applied, along the lines discussed in section 4.2, see [19, p.34] for details.

The basic energy estimate for systems of this type, see [19, theorem 2.2], gives an estimate of the form<sup>11</sup>

$$\|u\|_{E^{s+1}} \leq c \left( \|u\|_{E^{s+1}}(0) + \int_0^t \|\hat{w}\|_{L^2}(\tau) + \|\partial_t \hat{f}^i\|_{L^2}(\tau) d\tau \right) \quad (4.19)$$

As shown in [19], the  $L^2$  norms in the right hand side of (4.19) that involve local expressions can be estimated at a fixed time in terms of

$$(1 + \|D^2 u\|_{L^\infty})(1 + \|u(t)\|_{E^{s+1}})$$

<sup>11</sup>The norms  $\|\cdot\|_2$  in [19, p. 35] should be  $\|\cdot\|_{L^2}$ .

with a constant depending on  $\|Du\|_{L^\infty}$  as well as the coercivity constants  $\kappa, \mu$ . The term  $\widehat{w}$  contains  $D_P^s w$ , and thus we need the nonlocal term  $w$  to satisfy at a fixed time an estimate of precisely this form, namely

$$\|D_P^s w\|_{L^2} \leq c(1 + \|D^2 u\|_{L^\infty})(1 + \|u\|_{E^{s+1}})$$

which we can state as

$$\|w\|_{E^s} \leq c(1 + \|D^2 u\|_{L^\infty})(1 + \|u\|_{E^{s+1}}).$$

This estimate holds by Assumption  $w$ .

**4.4. Application to the elastic system.** We are now ready to apply the local existence theorem 4.1 to our system (2.23a)-(2.23c).

**Theorem 4.3.** *Suppose  $s \in \mathbb{Z}_{\geq 3}$ ,  $1 < p < \infty$ , and let  $(\phi^i|_{t=0}, \partial_t|_{t=0}\phi^i) = (\phi_0^i, \phi_1^i) \in \widetilde{\mathcal{O}}^{s+2,p} \times \widetilde{\mathcal{O}}^{s+1,p} \subset W^{s+2}(\mathcal{B}) \times W^{s+1}(\mathcal{B})$  be the initial data from Corollary 3.19. Then there exists a  $t_0 > 0$  and a unique classical solution  $\phi^i \in C^2(\mathcal{B}_{t_0} \cup \partial\mathcal{B}_{t_0})$  of (2.23a)-(2.23c) with  $D^\sigma \phi^i(t) \in L^2(\mathcal{B})$  for all  $0 \leq |\sigma| \leq s+1$  and  $0 \leq t < t_0$ .*

*Proof.* First, we observe by Corollary 3.8 that non-local function  $\Lambda^i(\phi) = -\delta^{ij} f_i^A \partial_A \bar{U}(\phi)$  satisfies Assumption  $w$  of section 4.1 for  $\phi^i$  in the open set  $\widetilde{\mathcal{O}}^{s+1} \subset W^{s+1}(\mathcal{B})$ . Since the initial data

$$(\phi^i|_{t=0}, \partial_t|_{t=0}\phi^i) = (\phi_0^i, \phi_1^i) \in \widetilde{\mathcal{O}}^{s+2,p} \times \widetilde{\mathcal{O}}^{s+1,p} \subset \widetilde{\mathcal{O}}^{s+2,2} \times \widetilde{\mathcal{O}}^{s+1,2} \subset W^{s+2}(\mathcal{B}) \times W^{s+1}(\mathcal{B})$$

from Corollary 3.19 satisfies the compatibility conditions to order  $s$ , the proof then follows directly from theorem 4.1.  $\square$

**Acknowledgements.** Part of this work was completed during visits of the authors T.A.O. and B.G.S. to the Albert Einstein Institute. We are grateful to the Institute its support and hospitality during these visits.

## APPENDIX A. FUNCTION SPACES

**A.1.  $W^{k,p}$  spaces.** Give a finite dimensional vector space  $V$ , an open subset  $\Omega \subset \mathbb{R}^3$  with a  $C^\infty$  boundary,  $k \in \mathbb{Z}$ , and  $1 \leq p \leq \infty$ , we let  $W^{k,p}(\Omega, V)$  denote the standard Sobolev space for maps  $u : \Omega \rightarrow V$ . If  $V = \mathbb{R}$ , then we will just write  $W^{k,p}(\mathcal{B})$ , while if  $p = 2$  we set  $W^k(\Omega, V) = W^{k,2}(\Omega, V)$ .

For these spaces, we recall the following results:

(i) Differentiation

$$\partial_A = \frac{\partial}{\partial X^A} : W^{k,p}(\Omega) \longrightarrow W^{k-1,p}(\Omega) \quad A = 1, 2, 3 \quad (\text{A.1})$$

defines a continuous linear map.

(ii) (Multiplication Inequality) If  $1 \leq p < \infty$ ,  $k_1 + k_2 > 0$ ,  $k_1, k_2 \geq k_3$ ,  $k_3 < k_1 + k_2 - 3/p$ , then there exists a  $C > 0$  such that

$$\|uv\|_{W^{k_3,p}(\Omega)} \leq C \|u\|_{W^{k_1,p}(\Omega)} \|v\|_{W^{k_2,p}(\Omega)} \quad (\text{A.2})$$

for all  $u \in W^{k_1,p}(\Omega)$  and  $v \in W^{k_2,p}(\Omega)$ .

(iii) Letting  $B^{k,p}(\partial\Omega)$  to denote the Besov spaces on the boundary  $\partial\Omega$ , the trace map

$$W^{k,p}(\Omega) \ni u \longmapsto \text{tr}_{\partial\Omega} u \in B^{k-1/p,p}(\partial\Omega) \quad (\text{A.3})$$

is well defined and continuous for  $1 < p < \infty$ , and  $k - 1/p > 0$  (cf. theorem 7.70 in [1]).

A.2.  $W_\delta^{k,p}$  **spaces.** For  $k \in \mathbb{Z}_{\geq 0}$ ,  $1 \leq p \leq \infty$ , and  $\delta \in \mathbb{R}$ , we use  $W_\delta^{k,p}(\mathbb{R}^3, V)$  to denote the weighted Sobolev spaces for maps  $u : \mathbb{R}^3 \rightarrow V$  as defined in [5]. For open sets  $\Omega \subset \mathbb{R}^3$  for which  $\mathbb{R}^3 \setminus \Omega$  is bounded, we denote the restriction of the weighted spaces to these subsets by  $W_\delta^{k,p}(\Omega, V)$ . Following [5], the negative index spaces  $W_\delta^{-k,p}(\mathbb{R}^3)$  for  $k \in \mathbb{Z}_{> 0}$ ,  $1 < p < \infty$ , and  $\delta \in \mathbb{R}$  are defined by duality.

We recall the following facts about the weighted Sobolev spaces:

(i) Differentiation

$$\partial_A : W_\delta^{k,p}(\Omega) \longrightarrow W_{\delta-1}^{k-1,p}(\Omega) \quad (\text{A.4})$$

defines a continuous linear map.

(ii) (Sobolev Inequality) If  $k > 3/p$ , then there exists a  $C > 0$  such that

$$\|u\|_{L_\delta^\infty(\Omega, V)} \leq C \|u\|_{W_\delta^{k,p}(\Omega, V)} \quad (\text{A.5})$$

for all  $u \in W_\delta^{k,p}(\mathbb{R}^3, V)$ . Moreover,  $u \in C_\delta^0(\mathbb{R}^3, V)$  and  $u(X) = o(|X|^\delta)$  as<sup>12</sup>  $|X| \rightarrow \infty$ . (See lemma A.7 in [27].)

(iii) (Multiplication Inequality) If  $1 \leq p < \infty$ ,  $k_1 + k_2 > 0$ ,  $k_1, k_2 \geq k_3$ ,  $k_3 < k_1 + k_2 - 3/p$ , and  $\delta_1 + \delta_2 \leq \delta_3$ , then there exists a constant  $C > 0$  such that

$$\|uv\|_{W_{\delta_3}^{k_3,p}(\Omega)} \leq C \|u\|_{W_{\delta_1}^{k_1,p}(\Omega)} \|v\|_{W_{\delta_2}^{k_2,p}(\Omega)} \quad (\text{A.6})$$

for all  $u \in W_{\delta_1}^{k_1,p}(\Omega)$  and  $v \in W_{\delta_2}^{k_2,p}(\Omega)$ . (See lemma A.8 in [27].)

(iv) For  $1 < p < \infty$ ,  $-1 < \delta < 0$ , and  $k \in \mathbb{R}$ , the Laplacian

$$\Delta = \delta^{AB} \partial_A \partial_B : W_\delta^{k+2,p}(\mathbb{R}^3) \longrightarrow W_{\delta-2}^{k,p}(\mathbb{R}^3) \quad (\text{A.7})$$

is a linear isomorphism with inverse given by the formula

$$[\Delta^{-1}(u)](X) = -\frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{u(Y)}{|X-Y|} d^3Y. \quad (\text{A.8})$$

(The proof of this statement follows from using the fact that  $\Delta : W_\delta^{k+2,p}(\mathbb{R}^3) \rightarrow W_{\delta-2}^{k,p}(\mathbb{R}^3)$  is an isomorphism for  $k \in \mathbb{Z}_{\geq 0}$  and  $-1 < \delta < 0$  (cf. [5, proposition 2.2]) together with duality and interpolation<sup>13</sup>.)

For bounded  $\Omega \subset \mathbb{R}^3$ , we let  $\mathbb{E}_\Omega$  denote an extension operator that satisfies

$$\|\mathbb{E}_\Omega(u)\|_{W_{-10}^{k,p}(\mathbb{R}^3, V)} \leq C_{k,p} \|u\|_{W_\delta^{k,p}(\Omega, V)} \quad \forall u \in W_\delta^{k,p}(\Omega, V), \quad (\text{A.9})$$

and

$$[\partial_A^\alpha \mathbb{E}_\Omega(u)]|_\Omega = \partial_A^\alpha u \quad \forall u \in W_\delta^{k,p}(\Omega, V), |\alpha| \leq k. \quad (\text{A.10})$$

<sup>12</sup>Here  $|X| = \sqrt{\delta_{AB} X^A X^B}$ .

<sup>13</sup>As noted by Maxwell [25], the weighted spaces  $W_\delta^{k,p}(\mathbb{R}^3)$  and their fractional extensions correspond to the spaces  $h_{p,p(k-\delta)-3}^k$  in [34, 35] (cf. remark 2 and theorem 2 in [35]). The following duality and interpolation results follow from remark 2, and theorems 2 and 3 in [35]:

(a) For  $1 < p < \infty$  and  $q = p/(p-1)$ ,  $W_{-3-\delta}^{-k,q}(\mathbb{R}^3)$  is the dual of  $W_\delta^{k,p}(\mathbb{R}^3)$ .  
(b) If  $1 < p_1 < \infty$ ,  $1 < p_2 < \infty$ ,  $0 < \theta < 1$ ,  $k = (1-\theta)k_1 + \theta k_2$ ,  $\delta = (1-\theta)\delta_1 + \theta\delta_2$ , and  $1/p = (1-\theta)/p_1 + \theta/p_2$ , then  $W_\delta^{k,p}(\mathbb{R}^3)$  is the interpolation space  $[W_{\delta_1}^{k_1,p}(\mathbb{R}^3), W_{\delta_2}^{k_2,p}(\mathbb{R}^3)]_\theta$ .

## REFERENCES

1. Robert A. Adams, *Sobolev spaces*, Academic Press, New York-London, 1975, Pure and Applied Mathematics, Vol. 65.
2. S. Agmon, A. Douglis, and L. Nirenberg, *Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions. I*, Comm. Pure Appl. Math. **12** (1959), 623–727.
3. Lars Andersson, Robert Beig, and Bernd G. Schmidt, *Static self-gravitating elastic bodies in Einstein gravity*, Comm. Pure Appl. Math. **61** (2008), no. 7, 988–1023.
4. Lars Andersson and Vincent Moncrief, *Elliptic-hyperbolic systems and the Einstein equations*, Ann. Henri Poincaré **4** (2003), no. 1, 1–34.
5. Robert Bartnik, *The mass of an asymptotically flat manifold*, Comm. Pure Appl. Math. **39** (1986), no. 5, 661–693.
6. Robert Beig and Bernd G. Schmidt, *Static, self-gravitating elastic bodies*, R. Soc. Lond. Proc. Ser. A Math. Phys. Eng. Sci. **459** (2003), no. 2029, 109–115.
7. ———, *Static, Self-Gravitating Elastic Bodies*, Proc. Roy. Soc. Lond. **A459** (2003), 109–115.
8. ———, *Celestial mechanics of elastic bodies*, Math. Z. **258** (2008), no. 2, 381–394.
9. M. Cantor, *Perfect fluid flows over  $\mathbf{R}^n$  with asymptotic conditions*, J. Funct. Anal. **18** (1975), 73–84. MR 0380872 (52 #1769)
10. Yvonne Choquet-Bruhat and Helmut Friedrich, *Motion of Isolated bodies*, Class. Quant. Grav. **23** (2006), 5941–5950.
11. Demetrios Christodoulou, *The formation of shocks in 3-dimensional fluids*, EMS Monographs in Mathematics, European Mathematical Society, Zürich, 2007.
12. D. Coutand and S. Shkoller, *Well-posedness in smooth function spaces for the moving-boundary 3-D compressible Euler equations in physical vacuum*, ArXiv e-prints (2010).
13. Klaus Deimling, *Nonlinear functional analysis*, Springer-Verlag, Berlin, 1985.
14. W. Domański and R. W. Ogden, *On the null condition for nonlinearly elastic solids*, Arch. Mech. (Arch. Mech. Stos.) **58** (2006), no. 4-5, 339–361.
15. Eugene Fabes, Osvaldo Mendez, and Marius Mitrea, *Boundary layers on Sobolev-Besov spaces and Poisson's equation for the Laplacian in Lipschitz domains*, J. Funct. Anal. **159** (1998), no. 2, 323–368.
16. Uwe Heilig, *On the existence of rotating stars in general relativity*, Comm. Math. Phys. **166** (1995), no. 3, 457–493.
17. J. Jang and N. Masmoudi, *Well-posedness of compressible Euler equations in a physical vacuum*, ArXiv e-prints (2010).
18. Saskia Kind and Jürgen Ehlers, *Initial-boundary value problem for the spherically symmetric Einstein equations for a perfect fluid*, Classical Quantum Gravity **10** (1993), no. 10, 2123–2136. MR MR1242402 (95c:83009)
19. Herbert Koch, *Mixed problems for fully nonlinear hyperbolic equations*, Math. Z. **214** (1993), no. 1, 9–42.
20. Hervé Le Dret, *Structure of the set of equilibrated loads in nonlinear elasticity and applications to existence and nonexistence*, J. Elasticity **17** (1987), no. 2, 123–141.
21. Leon Lichtenstein, *Gleichgewichtfiguren rotierende flüssigkeiten*, Springer, Berlin, 1933.
22. Hans Lindblad, *Well posedness for the motion of a compressible liquid with free surface boundary*, Comm. Math. Phys. **260** (2005), no. 2, 319–392.
23. Hans Lindblad and Karl Håkan Nordgren, *A priori estimates for the motion of a self-gravitating incompressible liquid with free surface boundary*, J. Hyperbolic Differ. Equ. **6** (2009), no. 2, 407–432.
24. Jerrold E. Marsden and Thomas J. R. Hughes, *Mathematical foundations of elasticity*, Dover Publications Inc., New York, 1994, Corrected reprint of the 1983 original.
25. David Maxwell, *Rough solutions of the Einstein constraint equations*, J. Reine Angew. Math. **590** (2006), 1–29. MR 2208126 (2006j:58044)
26. Marius Mitrea and Michael Taylor, *Potential theory on Lipschitz domains in Riemannian manifolds: Sobolev-Besov space results and the Poisson problem*, J. Funct. Anal. **176** (2000), no. 1, 1–79.
27. Todd A. Oliynyk, *The Newtonian limit for perfect fluids*, Comm. Math. Phys. **276** (2007), no. 1, 131–188.

28. Jiseong Park, *Spherically symmetric static solutions of the Einstein equations with elastic matter source*, Gen. Relativity Gravitation **32** (2000), no. 2, 235–252.
29. A. D. Rendall, *The initial value problem for a class of general relativistic fluid bodies*, Journal of Mathematical Physics **33** (1992), 1047–1053.
30. Thomas Runst and Winfried Sickel, *Sobolev spaces of fractional order, Nemytskij operators, and nonlinear partial differential equations*, de Gruyter Series in Nonlinear Analysis and Applications, vol. 3, Walter de Gruyter & Co., Berlin, 1996.
31. Thomas C. Sideris, *Formation of singularities in three-dimensional compressible fluids*, Commun. Math. Phys. **101** (1985), 475–485.
32. ———, *The null condition and global existence of nonlinear elastic waves*, Invent. Math. **123** (1996), no. 2, 323–342.
33. Yuri Trakhinin, *Local existence for the free boundary problem for nonrelativistic and relativistic compressible Euler equations with a vacuum boundary condition*, Comm. Pure Appl. Math. **62** (2009), no. 11, 1551–1594.
34. Hans Triebel, *Spaces of Kudrjavcev type. I. Interpolation, embedding, and structure*, J. Math. Anal. Appl. **56** (1976), no. 2, 253–277.
35. ———, *Spaces of Kudrjavcev type. II. Spaces of distributions: duality, interpolation*, J. Math. Anal. Appl. **56** (1976), no. 2, 278–287.

*E-mail address:* `lars.andersson@aei.mpg.de`

ALBERT EINSTEIN INSTITUTE, AM MÜHLENBERG 1, D-14476 POTSDAM, GERMANY

SCHOOL OF MATHEMATICAL SCIENCES, MONASH UNIVERSITY, VIC 3800, AUSTRALIA

*E-mail address:* `todd.oliynyk@sci.monash.edu.au`

*E-mail address:* `bernd@aei.mpg.de`

ALBERT EINSTEIN INSTITUTE, AM MÜHLENBERG 1, D-14476 POTSDAM, GERMANY