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RICCI FLOW AND THE DETERMINANT OF THE LAPLACIAN ON NON-COMPACT SURFACES

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Abstract. On compact surfaces with or without boundary, Osgood, Phillips and Sarnak proved that the maximum of the determinant of the Laplacian within a conformal class of metrics with fixed area occurs at a metric of constant curvature and, for negative Euler characteristic, exhibited a flow from a given metric to a constant curvature metric along which the determinant increases. The aim of this paper is to perform a similar analysis for the determinant of the Laplacian on a non-compact surface whose ends are asymptotic to hyperbolic funnels or cusps. In that context, we show that the Ricci flow converges to a metric of constant curvature and that the determinant increases along this flow.

Contents

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Introduction

A non-compact surface of finite topology M has a natural compactification $\overline{M}$ to a surface with boundary obtained by attaching a circle 'at infinity' to each end. To discuss asymptotic expansions of functions or sections of a vector bundle on M, it is very convenient to pass to the compactification and make use of a **boundary defining function** or **bdf**. A boundary defining function for Y, an open and closed subset of the boundary of M, is a smooth non-negative function on $\overline{M}$ that is equal to zero precisely on Y, and has non-vanishing differential on Y. A bdf for ∂M is known as a **total boundary defining function**.

Definition 1. *Let $\overline{M}$ be a compact surface with boundary. Assume that the connected components of the boundary of M have been partitioned into 'funnel ends' and 'cusp ends'*

$$\partial_{\mathrm{F}}\overline{M} = \{Y_1, \ldots, Y_{n_{\mathrm{F}}}\}, \quad \partial_{\mathrm{hc}}\overline{M} = \{Y_{n_{\mathrm{F}}+1}, \ldots, Y_{n_{\mathrm{F}}+n_{\mathrm{hc}}}\}.$$

A metric g on the interior of M is a **funnel, cusp metric** *or* F,hc**-metric** *if, for each $Y_i \in \partial_{\mathrm{F}} M$ there is a bdf x_i and a collar neighborhood $(0,\epsilon)_{x_i} \times \mathbb{S}^1$ of Y in M on which the pull-back of g is equal to*

$$e^{\varphi}\left(\frac{dx_i^2 + d\theta_i^2}{x_i^2}\right), \tag{1}$$

and similarly for each $Y_i \in \partial_{\mathrm{hc}} M$ there is a bdf x_i and a collar neighborhood $(0,\epsilon)_{x_i} \times \mathbb{S}^1$ of Y in M on which the pull-back of g is equal to

$$e^{\varphi}\left(\frac{dx_i^2}{x_i^2} + x_i^2 d\theta_i^2\right). \tag{2}$$

In both cases, $d\theta_i^2$ is the round metric on the circle of length one, and φ is required to be a smooth function on $\overline{M}$ equal to a constant at $x_i = 0$.

A simple example of F,hc metric is the **horn**,

$$\mathscr{F} = \mathbb{S}^1 \times (0,\infty)_s, \text{ with the metric } g_{\mathscr{F}} = \frac{ds^2 + d\theta^2}{s^2}. \tag{3}$$

It has a funnel end for $\{s < 1\}$ and a cusp end for $\{s > 1\}$. In fact, the hyperbolic metric on the quotient of $\mathbb{H}^2$ by a geometrically finite discrete group of hyperbolic isometries is a F,hc metric [29, Example 2.1].

The interior of any manifold with boundary can be endowed with a F,hc metric. A result of Mazzeo and Taylor [21, §2], and also a consequence of the present manuscript, is that any such metric can be conformally transformed to a hyperbolic metric. Note that funnel ends are also referred to as conformally compact or asymptotically hyperbolic [20].

For each i, we will assume that x_i is equal to 1 outside a collar neighborhood of Y_i. As a global boundary defining function, we can therefore consider

$$x = \prod_{i=1}^{n_{\mathrm{hc}}+n_{\mathrm{F}}} x_i. \tag{4}$$

We can also take

$$x_{\mathrm{F}} = \prod_{i=1}^{n_{\mathrm{F}}} x_i \quad \text{and} \quad x_{\mathrm{hc}} = \prod_{i=n_{\mathrm{F}}+1}^{n_{\mathrm{F}}+n_{\mathrm{hc}}} x_i \tag{5}$$

as boundary defining functions for $\partial_{\mathrm{F}}\overline{M}$ and $\partial_{\mathrm{hc}}\overline{M}$ respectively. Any non-compact hyperbolic metric is an example of a F, hc-metric, as is any compact perturbation of a hyperbolic metric.

In analogy with the case of a manifold with boundary, we say that the funnel ends of a F, hc metric (M, g) are **totally geodesic** if, in the description (1), we have

$$\varphi - \varphi\big|_{x_i=0} = o(x_i) \tag{6}$$

and, if this condition holds at the cusp ends, we say that they are totally geodesic. Unless otherwise stated, we will usually assume that the metrics considered are totally geodesic.

In this paper, we propose to define and study the determinant of the Laplacian on such surfaces. On compact manifolds, the determinant of the Laplacian is a global spectral invariant originally defined by Ray and Singer [30] using the zeta function of the Laplacian to regularize the product of its non-zero eigenvalues. For a non-compact manifold, the Laplacian generally has not only eigenvalues but also continuous spectrum and further regularization is necessary to define its determinant. One approach, due to Müller [24] and extended to our situation by the second author [6] and by Borthwick, Judge, and Perry [8], is to define a relative determinant by comparing the Laplacian to a model operator along the ends of the manifold. For non-compact surfaces with constant curvature, other approaches involve techniques from hyperbolic geometry, e.g., [12],[13] [33], [7]. We follow a method originally due to Richard Melrose [22] that allows us to use refined information about the Schwartz kernel of the heat kernel (from [1] and [35]) to define its *renormalized* trace and extend the definition of the zeta function and determinant. This method has the advantage of being very flexible and systematic. It has been used by Andrew Hassell in his proof of the Cheeger-Müller theorem [16], and in recent work of two of the authors [3, 4, 5].

As in the compact case, our determinant admits a Polyakov formula describing the variation of the determinant under conformal deformations of the metric. If all of the ends of (M, g) are asymptotically cusps, then the area of M is finite and the formula is very similar to the one for the compact case. This was done in terms of relative determinants by the second author in [6] and by Borthwick, Judge and Perry [8] (using the Mazzeo-Taylor uniformization [21] as a starting point). In general, if there are any funnel ends, then the area of M is infinite and the behavior of the determinant of the Laplacian is more complicated. The same renormalization process used to extend the determinant of the Laplacian can be used to define renormalized integrals and, in particular, a 'renormalized area', ${}^R\mathrm{Area}$. Complications arise because this renormalized integral is not a positive functional. Indeed, we show below that compactly supported conformal changes can be used to make the renormalized area equal to any given real number, positive or negative!

Explicitly, our Polyakov formula (Theorem 2.9 below) stipulates that if $g_\tau = e^{\omega(\tau)} g_0$ is a family of F, hc metrics on a non-compact surface M with totally geodesic ends, then the determinant of the Laplacian satisfies

$$\partial_\tau \log \det \Delta_\tau = -\frac{1}{24\pi} \int_M \omega'(\tau) R_\tau \, \mathrm{dA}_\tau + \partial_\tau \log \mathrm{Area}_\tau(M) \tag{7}$$

in the finite area case and

$$\partial_\tau \log \det \Delta_\tau = -\frac{1}{24\pi} {}^R\!\!\int_M \omega'(\tau) R_\tau \, \mathrm{dA}_\tau \tag{8}$$

in the infinite area case, where R_τ is the scalar curvature of g_τ.

By analogy with the compact case and the result of [27], one would expect that among all metrics in a given conformal class and with fixed (renormalized) area, the determinant should be maximal on the one with constant scalar curvature. For the relative determinant on Riemann surfaces with cusps ends, such a result was recently obtained by the second author [6]. For F, hc metrics with only funnel ends and of constant curvature outside a compact set, the corresponding result was obtained by Borthwick, Judge, and Perry [8], again for the relative determinant.

For the determinant considered here, our strategy to establish the analog of the result of [27] is to use the Ricci flow. The relevance of Ricci flow for the determinant of the Laplacian on closed surfaces was pointed out already in [27] by Osgood, Phillips, and Sarnak. Later Müller and Wendland [26] (see also the work of Kokotov and Korotkin [19]) verified that the determinant on closed surfaces increases along this flow.

On compact Riemann surfaces, Ricci flow was considered by Hamilton (see also the work of Cao [9] for the Kähler-Ricci flow) who showed in [15] that on surfaces of negative Euler characteristic, the normalized Ricci flow exists for all time and converges to a metric of constant curvature. The proof of this result elegantly follows from the study of the evolution equation of an accessory potential function. Following the same strategy, this result was recently generalized by Ji, Mazzeo, and Šešum [18] to non-compact surfaces with asymptotically cusp ends, the key new difficulty in this case being the construction of the potential function, which turns out to be much more delicate due to the presence of cusps. See also the work of Chau [10] for the generalization of the result of Cao [9] to non-compact Kähler manifolds.

In section 3, we generalize further the result of [18] to include also funnels (see Theorem 3.11). Along the way, we also carefully study how the asymptotic behavior of the metric evolves along the flow, an important point for the study of the determinant. A new feature in our case is that, as long as there is at least one end asymptotic to a funnel, the Euler characteristic, beside being negative, can also be equal to 0 or 1. This is consistent with the fact that there exists a metric with negative constant scalar curvature in these cases, e.g. the horn and the hyperbolic plane $\mathbb{H}^2$. Again, the key step is the construction of the potential function. Essentially by a doubling construction along the funnels, we can reduce to a situation where there are only cusps and obtain our potential function from the one of [18]. Since the area is infinite in this setting, we have to proceed differently to define a normalized Ricci flow. Instead of using the average scalar curvature, we can use any fixed negative real number $\mathscr{C}$ and consider the normalized Ricci flow

$$\frac{\partial g}{\partial t}(t) = (\mathscr{C} - R_{g(t)})g(t). \tag{9}$$

A particularly interesting choice is to take this number to be a renormalized average curvature, in which case the renormalized area is preserved along the flow.

Finally, in the last section, we combine our Polyakov formula with the convergence result for the Ricci flow to get our main result, which says (see Theorem 4.1 for the precise statement) that among all $\mathrm{F,hc}$ metrics g in a given conformal class with totally geodesics ends satisfying

$$^{R}\operatorname{Area}(g) = -2\pi\chi(M)$$

with scalar curvature asymptotically equal to -2 in each funnel ends, the determinant of the Laplacian is greatest at the hyperbolic metric in this conformal class.

Indeed, if g_0 is such a metric and $g(t) = e^{\omega(t)}g_0$ is the solution to the normalized Ricci flow with $\mathscr{C}$ given by the (renormalized) average curvature, then the variation of the determinant along the flow is nonnegative and given by

$$\begin{aligned}
\partial_t \det(\Delta_{g(t)}) &= -\frac{1}{24\pi}\, {}^{R}\!\!\int \omega'(t) R_t \,\mathrm{dA}_t \\
&= \frac{1}{24\pi}\, {}^{R}\!\!\int (R_t - \mathscr{C})^2 \,\mathrm{dA}_t + \frac{\mathscr{C}}{12\pi}\, {}^{R}\!\!\int (R_t - \mathscr{C}) \,\mathrm{dA}_t \\
&= \frac{1}{24\pi}\, {}^{R}\!\!\int (R_t - \mathscr{C})^2 \,\mathrm{dA}_t \geq 0,
\end{aligned} \tag{10}$$

the last integral being nonnegative since it does not need to be renormalized because $R_t - \mathscr{C} = \mathcal{O}(x_{\mathrm{F}})$ along the flow (See section 4 for all the details).

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1. Renormalization on non-compact Riemann surfaces

Our approach to extend the definition of the determinant of the Laplacian to non-compact surfaces is through renormalized integrals. In this section we briefly review how to renormalize the integral of a density on a manifold with boundary provided it has asymptotic expansions at each boundary face. We refer the reader to [22] for more details, where renormalized integrals are called 'b-integrals', as well as [17], [2], [5, Appendix].

1.1. **The choice of the boundary defining function.** The choice of coordinates (x_i, θ_i) in (1) or (2) and in particular the choice of boundary defining function (4) is not fixed by the conformal structure of the metric. A local conformal change of coordinates in (1) or (2) would induce a different boundary defining function. However, the boundary compactification $\overline{M}$, which can be seen as being obtained from (M, g) via our choice of boundary defining function (4), does only depend on the conformal class of g.

Proposition 1.1. *The boundary compactification $\overline{M}$ does not depend on the choice of conformal coordinates in* (1) *and* (2)*. If $(\widehat{x}_i, \widehat{\theta}_i)$ corresponds to a different choice of conformal coordinates, then $\widehat{x}_i = x_i h(x_i, \theta_i)$ for some smooth function h with $h(0, \theta_i)$ nowhere zero on the boundary component Y_i.*

Proof. The horn $\mathscr{F}$ of (3) is conformal to the punctured unit disk using the complex coordinate $\zeta = e^{2\pi i z}$ with $z = \theta + is$, the cusp end corresponding to the puncture and the funnel end corresponding to the boundary of the unit disk. In this coordinate, the boundary defining function of the cusp end is given by $\rho_{\mathrm{hc}} = \frac{-2\pi}{\log|\zeta|}$ and by $\rho_{\mathrm{F}} = -\frac{\log|\zeta|}{2\pi}$ for the funnel end.

Using this coordinate near a cusp end, we see that a local change of conformal coordinates near the cusp is given by a holomorphic function $f(\zeta) = \zeta g(\zeta)$ with $g(0) \neq 0$ and induces the new boundary defining function

$$\widehat{\rho}_{\mathrm{hc}} = \frac{-2\pi}{\log|f(\zeta)|}. \tag{1.1}$$

A quick inspection shows that f extends to give a smooth function on the boundary compactification at the cusp (defined by ρ_{hc}) and that $\widehat{\rho}_{\mathrm{hc}} = \rho_{\mathrm{hc}} h(\rho_{\mathrm{hc}}, \theta)$ with h a smooth function with $h(0, \theta)$ nowhere zero.

Similarly, a local change of conformal coordinates near a funnel end is given by a holomorphic function f defined for $1 - \epsilon < |\zeta| < 1$ for some $\epsilon > 0$ and which extends to a continuous function on the unit circle in such a way that $|f(\zeta)| = 1$ whenever $|\zeta| = 1$. It defines a new boundary defining function $\widehat{\rho}_{\mathrm{F}} = -\frac{\log|f(\zeta)|}{2\pi}$. By the Schwarz reflection principle, f extends to be smooth on the boundary compactification (defined by ρ_{F}) and $\widehat{\rho}_{\mathrm{F}} = \rho_{\mathrm{F}} h(\rho_{\mathrm{F}}, \theta)$ for some smooth function h with $h(0, \theta)$ nowhere zero.

□

In this paper, we will assume that the boundary defining function (4) and the conformal coordinates (x_i, θ_i) in (1) and (2) are given and fixed. Notice that whether or not an end is *totally geodesic* (6) depends on this choice. In the funnel case, this notion is related with the important rôle played by 'special' or 'geodesic' bdf's (see [14], [2]). These are bdf's x such that $|\frac{dx}{x}|_g$ is constant in a neighborhood of the funnel boundary (the value *at* the funnel boundary is independent of the choice of bdf). In order to make use of these results, we point out that a bdf satisfying (6) is 'close to' geodesic for the associated F,hc metric. By [14, Lemma 2.1] we know that there exists a geodesic bdf $\widehat{x}$ for g, unique in a neighborhood of $\partial_{\mathrm{F}} M$, such that $\widehat{x}^2 g\big|_{T\partial_{\mathrm{F}} M}$ is the metric $d\theta_i^2$ on each component Y_i of $\partial_{\mathrm{F}} \overline{M}$.

Lemma 1.2. *Let g be a F,hc metric, x be a bdf satisfying the assumptions above, and let $\varphi_0 = \varphi\big|_{x=0}$. Let $\widehat{x}$ be a geodesic bdf as described above that coincides with x along the cusp ends. If we write $\widehat{x} = e^{\phi} x$ then $\phi = -\frac{1}{2}\varphi_0 + \mathcal{O}(x)$ as $x \to 0$. If the funnel ends are totally geodesic, then $\phi = -\frac{1}{2}\varphi_0 + o(x)$.*

Proof. Assume without loss of generality that we only have funnel ends. We need to have $\widehat{x}^2 g\big|_{T\partial M} = e^{-\varphi_0} x^2 g\big|_{T\partial M}$, but since

$$\widehat{x}^2 g = e^{2\phi} x^2 g$$

this means we must have $2\phi + \varphi_0 = \mathcal{O}(x)$. We are also asking that $|\frac{d\widehat{x}}{\widehat{x}}|_g^2$ be constant near $\widehat{x} = 0$. The constant is necessarily equal to

$$\left|\frac{dx}{x}\right|_g^2 \Bigg|_{x=0} = e^{-\varphi_0}$$

so we need to have

$$\begin{aligned} e^{-\varphi_0} = g\left(\frac{d\widehat{x}}{\widehat{x}}, \frac{d\widehat{x}}{\widehat{x}}\right) &= g\left(\frac{dx}{x} + d\phi, \frac{dx}{x} + d\phi\right) \\ &= \left(e^{-\varphi} + 2g\left(\frac{dx}{x}, d\phi\right) + g\left(d\phi, d\phi\right)\right). \end{aligned}$$

Since $g(d\phi, d\phi) = \mathcal{O}(x^2)$, this implies $\partial_x \phi = o(1)$ if $\varphi - \varphi_0 = o(x)$. □

Since the volume form of a F,hc metric blows-up at the funnel ends to second order, this lemma implies as we will see that, for metrics totally geodesic along the funnel ends, renormalization results that require a geodesic bdf also hold for a bdf satisfying the assumptions above.

1.2. **Renormalized integrals.**

Let M be a manifold with boundary. A function f on M is **polyhomogeneous** if it is smooth in the interior of M and, at each connected component N of ∂M, f has an asymptotic expansion in terms of a bdf x for N and $\log x$.

To be more specific, we say that a set $E \subset \mathbb{C} \times (\mathbb{N} \cup \{0\})$ is a (smooth) index set if:

i) $(s,p) \in E \implies (s+1,p), (s,p-1) \in E$
ii) If $\{(s_j,p_j)\} \subseteq E$ is any infinite sequence of distinct elements, then $\operatorname{Re} s_j \to \infty$.

In particular $\inf E = \inf\{\operatorname{Re} s : (s,p) \in E\}$ is a finite number. The function f is polyhomogeneous if, at each boundary face N_i, there is an index set E_i such that f satisfies

$$f \sim \sum_{(s,p)\in E_i} a_{s,p} x_i^s (\log x_i)^p$$

for some bdf x_i for N_i and smooth functions $a_{s,p}$ on N_i. This means that, for any $\mathcal{N} \in \mathbb{N} \cup \{0\}$,

$$\left(f - \sum_{\substack{(s,p)\in E_i \\ \operatorname{Re} s \leq \mathcal{N} + \inf E}} a_{s,p} x_i^s (\log x_i)^p \right) \in x_i^{\mathcal{N}+\inf E} \mathcal{C}^{\mathcal{N}}(\overline{M}). \tag{1.2}$$

Let x be a total boundary defining function, μ a smooth density on $\overline{M}$, f a polyhomogeneous function, and consider the function

$$\zeta_f(z) = \int_M x^z f \mu.$$

If $f\mu$ is integrable then $\zeta_f(0) = \int_M f\mu$. If $f\mu$ is not integrable (i.e., if f blows-up at the boundary) then it is easy to see that ζ_f is holomorphic on a half-plane of the form $\{\operatorname{Re} z > C\}$ for some $C > 0$. The polyhomogeneity of f allows us to extend $\zeta_f(z)$ meromorphically to $\mathbb{C}$ with the location (and strength) of the potential poles determined by the elements of the index sets of f. In terms of the meromorphic continuation of ζ_f, we define the **renormalized integral** of $f\mu$ by

$$ {}^R\!\!\int_M f\mu = \operatorname*{FP}_{z=0} \zeta_f(z). \tag{1.3}$$

If f is not polyhomogeneous but satisfies (1.2) for some $\mathcal{N} \in \mathbb{N} \cup \{0\}$ such that $x^{\mathcal{N}}\mu$ is integrable, then notice that $\zeta_f(z)$ is still well-defined and meromorphic near $z = 0$ so that (1.3) can still be used to define the renormalized integral. This will be useful in section 3 when we consider metrics with less regularity.

Alternatively, we point out that for small enough $\varepsilon > 0$ the truncated integral

$$\int_{x\geq\varepsilon} f\mu$$

has an asymptotic expansion in ε, and we define

$$^{H}\!\!\int_M f\mu = \underset{\varepsilon=0}{\mathrm{FP}} \int_{x\geq\varepsilon} f\mu.$$

This method of renormalizing is often known as *Hadamard renormalization* and is used in [22] while the previous method is known as *Riesz renormalization* and is used in [23]. In [2, §2.3] it is shown that, under certain natural conditions which will always hold in this manuscript, these two renormalizations coincide.

1.3. **The renormalized area.**

Let (M, g) be a surface of infinite area. Given a total boundary defining function on M, we can define the renormalized area of M by taking the renormalized integral of the volume form of g. In sharp contrast to the usual area, the renormalized area *need not be positive.* Also, the renormalized area generally depends on the choice of bdf used to define it. As we are working with a fixed bdf, the renormalized area of a F, hc-metric is unambiguously defined. Nevertheless it is a somewhat mysterious quantity, and we now point out that if one is free to change the metric g conformally, one can arrange for the renormalized area to be any given real number without changing g outside a compact set.

Lemma 1.3. *Suppose that g is a* F, hc *metric on M of infinite area. Let x be a total bdf on M. For any $L \in \mathbb{R}$ there is a metric g_L in the conformal class of g that coincides with g outside a compact set, whose renormalized area computed with respect to x is equal to L.*

Proof. Since $\mathrm{Area}(x \geq \varepsilon) \to \infty$ as ε goes to zero, we can choose ε small enough so that

$$^{R}\mathrm{Area}(x < \varepsilon/2) < -2|L|.$$

Let $\alpha \mapsto \chi_\alpha(t)$ be a smooth family of smooth functions on $\mathbb{R}^+$ satisfying

$$\chi_\alpha(t) = \begin{cases} 1 & t < \varepsilon/2 \\ \alpha & t > \varepsilon \end{cases}$$

and taking values in between α and 1 for t between $\varepsilon/2$ and ε. Thus $\chi_\alpha(x)$ extends to all of M, and for $\alpha > 0$, $\chi_\alpha(x)g$ is a metric on M, conformally related to g, that coincides with g outside a compact set.

Suppose $L < {}^{R}\mathrm{Area}(g)$. Note that

$$^{R}\mathrm{Area}(\chi_\alpha(x)g) = {}^{R}\mathrm{Area}(x < \varepsilon/2) + \int_{x\in[\varepsilon/2,\varepsilon]} \chi_\alpha(x)\ \mathrm{dA} + \alpha\,\mathrm{Area}(x \geq \varepsilon)$$

so ${}^R\,\mathrm{Area}$ varies continuously with α and satisfies, for $\alpha < 1$,

$$ {}^R\,\mathrm{Area}(\chi_\alpha(x)g) < -2|L| + \alpha\,\mathrm{Area}(x \geq \varepsilon/2). $$

Hence ${}^R\,\mathrm{Area}(\chi_\alpha(x)g)$ is equal to ${}^R\,\mathrm{Area}(g)$ for $\alpha = 1$, and is less than L for small enough α, so by continuity is equal to L for some α.

If $L > {}^R\,\mathrm{Area}(M)$ then since, for $\alpha > 1$,

$$ {}^R\,\mathrm{Area}(\chi_\alpha(x)g) > {}^R\,\mathrm{Area}(x < \varepsilon) + \alpha\,\mathrm{Area}(x > \varepsilon), $$

another appeal to continuity shows that we can find an α for which ${}^R\,\mathrm{Area}(\chi_\alpha(x)g) = L$. □

1.4. **The renormalized Gauss-Bonnet theorem.**

In this section we describe the extension of the Gauss-Bonnet theorem to F,hc-metrics.

Recall that the Gauss-Bonnet theorem for a surface with boundary $(M, \overline{g})$ says that

$$ \int_M R(\overline{g})\ \mathrm{dA}_{\overline{g}} + \int_{\partial M} \mathrm{I\!I} = 4\pi\chi(M) $$

where $\mathrm{I\!I}$ is a density on ∂M built up from the second fundamental form of ∂M in M. In particular, if the boundary of M is totally geodesic, $\int_M R(\overline{g})\ \mathrm{dA}_{\overline{g}} = 4\pi\chi(M)$.

If instead g is a F,hc metric and x is any boundary defining function then applying the Gauss-Bonnet theorem to the surface with boundary $\{x \geq \varepsilon\}$ yields, for ε small enough,

$$ \int_{x \geq \varepsilon} R(g)\ \mathrm{dA}_g + \int_{x = \varepsilon} \mathrm{I\!I} = 4\pi\chi(M_\varepsilon) = 4\pi\chi(M) $$

and taking the finite part as $\varepsilon \to 0$ yields

$$ {}^R\!\!\int R(g)\ \mathrm{dA}_g + \underset{\varepsilon=0}{\mathrm{FP}} \int_{x=\varepsilon} \mathrm{I\!I} = 4\pi\chi(M). \tag{1.4} $$

A straight-forward computation using Lemma 1.2 (see [2, Theorem 4.5]) shows that, if all of the funnel ends of g are totally geodesic (6), and x is as in (4), then this boundary term vanishes.

Theorem 1.4 (Gauss-Bonnet for F,hc metrics)**.** *Let (M, g) be a non-compact surface with a F,hc metric. If $\mathrm{Area}(M)$ is finite, then*

$$ \int_M R(g)\ \mathrm{dA}_g = 4\pi\chi(M). $$

If $\mathrm{Area}(M)$ is infinite and the infinite volume ends of M are totally geodesic, then

$$ {}^R\!\!\int R(g)\ \mathrm{dA}_g = 4\pi\chi(M). $$

Remark 1.5. *For exactly hyperbolic metrics this appears in* [7, §2] *for surfaces and in* [28] *for higher dimensional hyperbolic manifolds without cusps; see also* [2] *for higher dimensional asymptotically hyperbolic metrics that are asymptotically Einstein.*

If the funnel ends of g are not totally geodesic, one can check that the boundary contribution in (1.4) *is essentially the same as the boundary contribution to the Gauss-Bonnet theorem for the incomplete metric $x_{\mathrm{F}}^2 g$ (where x_{F} is a bdf for $\partial_{\mathrm{F}} M$).*

2. A Polyakov formula for the renormalized determinant

In this section we will explain how the renormalized integrals described above can be used to define renormalized traces which in turn can be used to define the determinant of the Laplacian on non-compact surfaces.

2.1. **The renormalized determinant.**

Since the work of Ray and Singer on analytic torsion [30], zeta regularization has been used to define the determinant of the Laplacian. Starting with the identity, for $\{\lambda_i\} \subseteq \mathbb{R}^+$,

$$\partial_s\big|_{s=0}\left(\sum_{i=1}^N \lambda_i^{-s}\right) = -\sum_{i=1}^N \log\lambda_i = -\log\prod_{i=1}^N \lambda_i,$$

Ray and Singer proposed to define the determinant of the Laplacian of (M, g) by first setting

$$\zeta(s) = \sum_{\lambda\in\mathrm{Spec}(\Delta)\setminus\{0\}} \lambda^{-s} \tag{2.1}$$

and then formally defining

$$\det\Delta = e^{-\partial_s\big|_{s=0}\zeta}. \tag{2.2}$$

To make sense of this formula we note that if M is closed then, by Weyl's law, the sum defining $\zeta(s)$ converges if $\mathrm{Re}(s) > \dim M/2$ and defines a holomorphic function on this half-plane. It is possible to extend this function meromorphically to the whole plane and then (2.2) involves taking the derivative of this meromorphically extended function at the origin, which turns out to be a regular point.

One way to justify the meromorphic extension is to rewrite $\zeta(s)$ using the heat kernel of Δ. It is easy to check that, for any $\lambda > 0$,

$$\lambda^{-s} = \frac{1}{\Gamma(s)}\int_0^\infty t^s e^{-t\lambda}\frac{dt}{t}$$

and hence for $\mathrm{Re}(s) > \dim M/2$,

$$\zeta(s) = \frac{1}{\Gamma(s)}\int_0^\infty t^s\,\mathrm{Tr}(e^{-t\Delta} - \mathcal{P})\frac{dt}{t} \tag{2.3}$$

where $\mathcal{P}$ is the projection onto the null space of Δ, i.e., the constant functions. As $t \to 0$ the trace of the heat kernel has an asymptotic expansion

$$\mathrm{Tr}(e^{-t\Delta}) \sim t^{-\dim M/2} \sum_{k \geq 0} a_k t^k$$

which implies that $\zeta(s)$ has a meromorphic continuation to the complex plane with potential poles at

$$s \in \left\{ \frac{\dim M}{2}, \frac{\dim M}{2} - 1, \frac{\dim M}{2} - 2, \dots \right\}.$$

It can then be explicitly checked that zero is a regular point so that the right hand side of (2.2) is well-defined.

In terms of the renormalized integrals of the previous section, since

$$\frac{1}{\Gamma(s)} \sim s + \mathcal{O}(s^2), \text{ as } s \to 0,$$

we have

$$\zeta'(0) = {}^R\!\!\int_0^\infty \mathrm{Tr}(e^{-t\Delta}) \, \frac{dt}{t}$$

where t is used to renormalize the integral at $t = 0$ and t^{-1} is used to renormalize as $t \to \infty$. The right hand side serves as a definition of the (logarithm of the) determinant of the Laplacian that does not require that the origin be a regular point for the meromorphically continued zeta function.

On a non-compact surface (M, g), the spectrum of the Laplacian consists of eigenvalues and a continuous spectrum, so one cannot define $\zeta(s)$ by (2.1). There is also a problem with extending definition (2.3) because the heat kernel of the Laplacian is not of trace class. However this problem can be overcome by means of a *renormalized* trace.

To motivate the renormalized trace recall Lidskii's theorem which says that, if A is an operator that acts via a continuous kernel $\mathcal{K}_A$,

$$Af(\zeta) = \int \mathcal{K}_A(\zeta, \zeta') f(\zeta') \, d\zeta',$$

and A is of trace class, then

$$\mathrm{Tr}(A) = \int \mathcal{K}_A(\zeta, \zeta) \, d\zeta.$$

Since we already know how to renormalize integrals, we define **the renormalized trace** of an operator A to be

$${}^R\,\mathrm{Tr}\,(A) = {}^R\!\!\int \mathcal{K}_A(\zeta, \zeta) \, d\zeta,$$

whenever the right-hand side makes sense.

Fortunately, the heat kernel of a F, hc metric is well-enough understood to define its renormalized trace. The heat kernel for a metric with ends asymptotic to hyperbolic cusps is described in [35] and for a metric with ends asymptotic to hyperbolic funnels in [1], and it is straightforward to patch together a heat kernel for a general F, hc metric from these. In either case it is shown that the distributional kernel of $e^{-t\Delta}\big|_{\mathrm{diag}}$ is a smooth function in the interior of M with polyhomogeneous expansions at the boundary of M and also as $t \to 0$. From the analysis of the Laplacian in [20] and [35], we know that Δ has closed range and hence $e^{-t\Delta}$ converges exponentially to the projection onto the null space of Δ as $t \to \infty$. These properties allow us to make the following definition.

Definition 2.1. *Let M be a non-compact surface with a* F, hc *metric g, and let x be a total boundary defining function. The* **renormalized trace of the heat kernel** *of g is defined to be*

$$ {}^R\operatorname{Tr}\left(e^{-t\Delta}\right) = {}^R\!\!\int \mathcal{K}_{e^{-t\Delta}}\big|_{\mathrm{diag}}\ \mathrm{dA}_g $$

where the renormalization is carried out using x. From [1] *and* [35] *(see* [5, Appendix]*) we know that, as $t \to 0$,*

$$ {}^R\operatorname{Tr}\left(e^{-t\Delta}\right) \sim \sum_{k\geq -2} a_k t^{-k/2} + \sum_{k\geq -1} \widetilde{a}_k t^{k/2}\log t. \tag{2.4} $$

The **determinant of the Laplacian** *is defined to be*

$$ \det\Delta = \exp\left(- {}^R\!\!\int_0^\infty {}^R\operatorname{Tr}(e^{-t\Delta})\ \frac{dt}{t} \right). $$

As mentioned above, this definition directly extends the definition from operators with trace-class heat kernel. We will show in §2.3 that it also extends the definition of the determinant via the Selberg zeta function from hyperbolic metrics (cf. [7], [33]). Another point in its favor is the main result of this paper: it singles out constant curvature metrics as its critical metrics.

In (2.4), the logarithmic terms come from the cusp ends. In fact, as the next lemma shows, many of the coefficients $\widetilde{a}_k$ automatically vanish.

Lemma 2.2. *For any* F, hc *metric, the terms $\widetilde{a}_k$ in the short-time asymptotics of the renormalized trace of its heat kernel* (2.4) *vanish if k is even. In particular, $\widetilde{a}_0 = 0$.*

Proof. Assume without loss of generality that there is only one cusp end. Recall from [5, Appendix] that the asymptotic expansion (2.4) is derived by analyzing the restriction of the integral kernel $\mathcal{K}$ of the heat kernel to the manifold with corners

$$ \mathrm{diag}_H = [M \times \mathbb{R}^+_{\sqrt{t}}; \partial M \times \{0\}] $$

where it is polyhomogeneous. This space has three boundary hypersurfaces: B_{10} and B_{01} coming from the 'old' hypersurfaces $\{x = 0\}$ and $\{t = 0\}$ in $M \times \mathbb{R}^+_{\sqrt{t}}$, respectively, and B_{11} the 'front face' resulting from blowing-up $\partial M \times \{0\}$. The logarithmic terms in the expansion come from the corner $\mathfrak{B}_{11} \cap \mathfrak{B}_{01}$ and, since we are renormalizing, also from the corner $\mathfrak{B}_{10} \cap \mathfrak{B}_{11}$.

At the latter corner, the log terms can be shown to arise from the short-time expansion of the coefficient of x^{-1} in the expansion of the heat kernel at the face $\mathfrak{B}_{10}$. This can be identified following Vaillant [35, Chapter 4] with the heat kernel of a model operator (the 'horizontal family') which in this case is multiplication by a constant, namely $-\frac{r}{8}$ where r is the asymptotic value of the scalar curvature at the cusp. Thus this corner contributes $\frac{e^{\frac{r}{8}t}}{\sqrt{t}}\log t$ to the short-time expansion of the renormalized heat trace, and so does not contribute to $\widetilde{a}_k$ for k even.

To handle the former corner, $\mathfrak{B}_{11}\cap\mathfrak{B}_{01}$, recall the well-known fact that for closed manifolds of dimension n, the short-time asymptotics of the heat trace are of the form $t^{-n/2}$ times an expansion in t (as opposed to $\sqrt{t}$). The same 'even-ness' is true in the expansion of the heat kernel of a $\mathrm{F,hc}$ metric at the boundary face $\mathfrak{B}_{01}$. It follows (see [5, Appendix]) that the asymptotics at the corner $\mathfrak{B}_{11}\cap\mathfrak{B}_{01}$ do not contribute to $\widetilde{a}_k$ for k even.

□

We can also define the ζ function of the Laplacian of g, for $\operatorname{Re} s \gg 0$, by

$$\zeta(s) = \frac{1}{\Gamma(s)}\int_0^\infty t^s \, {}^R\mathrm{Tr}\left(e^{-t\Delta} - \mathcal{P}\right)\frac{dt}{t} \tag{2.5}$$

where $\mathcal{P}$ is the projection onto the L^2 null space of Δ (i.e., the constant functions if g has finite area and the zero function otherwise). Using (2.4) this function has a meromorphic continuation to the complex plane, still denoted ζ. Since $\widetilde{a}_0$ vanishes, zero is a regular point of $\zeta(s)$ and the definition above coincides with

$$\det\Delta = e^{-\zeta'(0)}.$$

Remark 2.3. *The renormalized trace was first defined by Melrose in his proof of the Atiyah-Patodi-Singer index theorem* [22] *for asymptotically cylindrical metrics. Melrose also pointed out that this definition allows one to extend the definition of the zeta function and the determinant of the Laplacian as described above. See* [16] *for an analysis of this extension to asymptotically cylindrical metrics including a proof of the Cheeger-Müller theorem for the corresponding analytic torsion.*

2.2. **Relation with the relative determinant.**

The relative determinant was introduced by Werner Müller [24] as a way to overcome the problem of the heat kernel of the Laplacian not being of trace class. Instead of extending the trace functional to operators that are not of trace class, a model operator is introduced, so that the difference of both operators is of trace class. One can consider an operator that is naturally related to the surface. In the case of finite area, i.e., only cusps ends, the natural model operator is the following: Decompose M as a compact part, M_0, and the cusps ends, Z_{hc}. The cusps ends are each considered with one boundary that joins the end to the compact part M_0. The model operator is the direct sum of the self-adjoint extension of the hyperbolic Laplacian on the cusp ends with respect to Dirichlet boundary conditions at the boundaries, and the operator zero on M_0. We denote this operator as Δ_0. The definition of the relative determinant depends on the following facts:

(1) $e^{-t\Delta_g} - e^{-t\Delta_0}$ is an operator of trace class for all $t > 0$.
(2) As $t \to 0^+$, there is an asymptotic expansion of the form:

$$\mathrm{Tr}(e^{-t\Delta_g} - e^{-t\Delta_0}) \sim \sum_{j=0}^{\infty}\sum_{k=0}^{k(j)} a_{jk} t^{\alpha_j} \log^k t,$$

with $-\infty < \alpha_0 < \alpha_1 < \cdots$ and $\alpha_k \to \infty$. If $\alpha_j = 0$, $a_{jk} = 0$ for $k > 0$.
(3) Since there is a spectral gap at zero we have that:

$$\mathrm{Tr}(e^{-t\Delta_g} - e^{-t\Delta_0}) = 1 + O(e^{-ct}),$$

as $t \to \infty$, where $1 = \dim\ker\Delta_g - \dim\ker\Delta_0$.

Then we can define the relative zeta function as:

$$\zeta(s; \Delta_g, \Delta_0) = \frac{1}{\Gamma(s)} \int_0^{\infty} t^s (\mathrm{Tr}(e^{-t\Delta_g} - e^{-t\Delta_0}) - 1) \frac{dt}{t}.$$

In the same way as for the regularized determinant, the asymptotic expansions guarantee that the relative zeta function has a meromorphic continuation to the complex plane that is regular at zero, and we can define:

$$\det(\Delta_g, \Delta_0) = e^{-\zeta'(0;\Delta_g,\Delta_0)}.$$

The relative determinant has been considered on non-compact surfaces where the metric is exactly hyperbolic outside a compact set by Müller [25], [24] and by Borthwick, Judge, and Perry [8]. In a more recent work [6], the second author has extended the definition of the relative determinant to surfaces whose ends are merely asymptotic to hyperbolic cusps.

In either case, working with renormalized traces allows us to write

$$\mathrm{Tr}(e^{-t\Delta_g} - e^{-t\Delta_0}) = {}^R\mathrm{Tr}\left(e^{-t\Delta_g}\right) - {}^R\mathrm{Tr}\left(e^{-t\Delta_0}\right),$$

$$\zeta(s; \Delta_g, \Delta_0) = \zeta(s; \Delta_g) - \zeta(s; \Delta_0), \text{ and } \det(\Delta_g, \Delta_0) = \frac{\det(\Delta_g)}{\det(\Delta_0)}.$$

Hence, when the relative determinant is defined, it is essentially equivalent to the renormalized determinant.

2.3. **Relation with Selberg zeta function.**

For compact hyperbolic Riemann surfaces, an interesting relationship between the determinant of the Laplacian and the Selberg Zeta function was discovered by D'Hoker and Phong [11] and further refined by Sarnak [31]. If M is a non-compact surface with a F, hc-metric g of *constant curvature*, then this relationship still holds. We can write M as $\mathbb{H}/\Gamma$ and define the Selberg zeta function $Z(s)$, for $\operatorname{Re} s > 1$, by the absolutely convergent product

$$Z(s) = \prod_{\{\gamma\}} \prod_{k=0}^{\infty} (1 - e^{-(s+k)\ell(\gamma)})$$

where the outer product goes over conjugacy classes of primitive hyperbolic elements of Γ, and $\ell(\gamma)$ is the length of the corresponding closed geodesic. The function $Z(s)$ admits a meromorphic continuation to the whole complex plane.

Borthwick, Judge, and Perry showed [7, (5.2)] that if

$$R_g(s) = (\Delta + s(s-1))^{-1}$$

and $\mathcal{L}(s)$ is any function satisfying

$$\left(\frac{1}{2s-1}\frac{\partial}{\partial s}\right)^2 \log \mathcal{L}(s) = -^R \operatorname{Tr}\left(R_g(s)^2\right) \tag{2.6}$$

then there are constants E and F such that

$$\mathcal{L}(s) = Z(s) e^{E+Fs(1-s)} \left(\frac{\Gamma(s)}{(2\pi)^s \Gamma_2^2(s)}\right)^{\chi(M)} \left(2^s \sqrt{\pi(s-\tfrac{1}{2})}\Gamma(s-\tfrac{1}{2})\right)^{-n_C} \tag{2.7}$$

(here $\Gamma_2(s)$ is Barnes' double Gamma function). We will show that, with the determinant defined above, $\det(\Delta + s(s-1))$ satisfies (2.6) and hence (2.7). As in [31], by examining the asymptotics as $s \to \infty$ we will determine the values of E and F in this case.

Theorem 2.4. *Let M be a non-compact surface and g a F, hc-metric on M of constant curvature. The zeta regularized determinant $\det(\Delta + s(s-1))$ satisfies (2.6) and hence (2.7). The constants E and F in this case are equal to*

$$E = \chi(M)\left(\frac{1}{2}\log 2\pi - 2\zeta_R'(-1) + \frac{1}{4}\right), \quad F = -\chi(M)$$

where ζ_R is the Riemann zeta function. It follows that

$$\det(\Delta) = \begin{cases} C_{\mathrm{F,hc}}\, Z'(1) & \text{if } Area(g) < \infty \\ C_{\mathrm{F,hc}}\, Z(1) & \text{otherwise.} \end{cases}$$

with

$$C_{\mathrm{F,hc}} = e^E (2\pi)^{-\chi(M)} (\sqrt{2}\pi)^{-n_C}.$$

Remark 2.5. *The reason a derivative is taken in the case of a finite area surface is that to compute $\det(\Delta)$ one needs to exclude the zero eigenvalue of Δ.*

Proof. In [4, (7.9)] it is shown that (2.4) implies

$$
\begin{aligned}
-\log\det(\Delta + w) = \int_0^\infty & \left({}^R\operatorname{Tr}\left(e^{-t\Delta}\right) - f_0(t)\right) e^{-tw}\,\frac{dt}{t} \\
& - a_0 \log w - 2\sqrt{\pi}a_{-\frac{1}{2}}\sqrt{w} + a_{-1}w\,(-1+\log w) \\
& + \widetilde{a}_{-\frac{1}{2}}\sqrt{w}\left(\Gamma_{\log}(-\frac{1}{2}) - \log w\,\Gamma(-\frac{1}{2})\right)
\end{aligned}
\tag{2.8}
$$

where

$$
\begin{gathered}
f_0(t) = a_{-1}t^{-1} + \widetilde{a}_{-1/2}t^{-1/2}\log t + a_{-1/2}t^{-1/2} + a_0, \\
\text{and } \Gamma_{\log}(z) := \int_0^\infty t^z e^{-t}\log t\,\frac{dt}{t}.
\end{gathered}
\tag{2.9}
$$

It follows that ([4, (7.21)])

$$
\begin{aligned}
&\left(\frac{1}{2s-1}\frac{\partial}{\partial s}\right)\log\det(\Delta + s(s-1)) \\
&\qquad = \int_0^\infty \left({}^R\operatorname{Tr}\left(e^{-t\Delta}\right) - a_{-1}t^{-1}\right) e^{-ts(s-1)}\,dt - a_{-1}\log\left(s(s-1)\right)
\end{aligned}
$$

and hence

$$
\begin{aligned}
\left(\frac{1}{2s-1}\frac{\partial}{\partial s}\right)^2 \log\det(\Delta + s(s-1)) &= -\int_0^\infty t\left({}^R\operatorname{Tr}\left(e^{-t\Delta}\right)\right) e^{-ts(s-1)}\,dt \\
= -{}^R\operatorname{Tr}\left(\int_0^\infty t e^{-t(\Delta+s(s-1))}\,dt\right) &= -{}^R\operatorname{Tr}\left((\Delta + s(s-1))^{-2}\right).
\end{aligned}
$$

which establishes (2.6) and (2.7).

To determine E and F, we can then proceed exactly as in the proof of [4, Theorem 2]. Finally, because zero is in the spectrum of the Laplacian on a hyperbolic surface precisely when its area is finite,

$$
\det(\Delta) = \begin{cases} \lim_{s\to 1}\frac{\det(\Delta+s(s-1))}{s(s-1)} & \text{if } \operatorname{Area}(g) < \infty \\ \det(\Delta + s(s-1))\big|_{s=1} & \text{otherwise} \end{cases}
$$

□

In fact, the proof of [4, Theorem 2] also give us the values of the first few coefficients in the short time asymptotic (2.4) of the trace of the heat kernel

$$
\begin{gathered}
a_{-1} = \frac{{}^R\operatorname{Area}(M)}{4\pi}, \quad \widetilde{a}_{-\frac{1}{2}} = \frac{n_C}{4\sqrt{\pi}}, \quad a_0 = \frac{\chi(M)}{6}, \\
a_{-\frac{1}{2}} = \frac{n_C}{2\sqrt{\pi}}\left(\frac{\Gamma_{\log}(-\frac{1}{2})}{4\sqrt{\pi}} + 1 - \log 2\right).
\end{gathered}
\tag{2.10}
$$

When the scalar curvature is not constant but the metric has totally geodesic ends, we get the same coefficients as the next lemma shows.

Lemma 2.6. *Let g be a* $\mathrm{F,hc}$ *metric on* M.

1) *If the ends of g are totally geodesic with scalar curvature asymptotically equal to -2 in each end, then in the expansion* (2.4) *the first coefficients are given by* (2.10).
2) *If $\varphi = \mathcal{O}(x_{\mathrm{F}}^2)$ and $\varphi = \mathcal{O}(x_{\mathrm{hc}})$, then $\int_0^t e^{-(t-s)\Delta}\varphi\Delta e^{-s\Delta}\, ds$ is of trace class.*

Remark 2.7. *If the asymptotic value of the scalar curvature is different from -2, the coefficients $a_{-\frac{1}{2}}$ and $\widetilde{a}_{-\frac{1}{2}}$ can easily be obtained by an appropriate rescaling argument.*

Remark 2.8. *If there are only cusps, see also* [6] *for a different proof of (2).*

Proof. For the first part of the lemma, notice that the 'interior contributions' are given by the same integration of local terms, but with integrals replaced with renormalized integrals whenever the volume is infinite. Thus, we need to check the contributions coming from the cusp and funnel ends are the same for a metric with totally geodesic ends and a metric which is hyperbolic in a collar neighborhood of each end.

First let us focus on what happens near a cusp. Without loss of generality, we can assume that we have only cusp ends, in fact only one cusp. By assumption, we know that (2) is satisfied with $\varphi|_{\partial\overline{M}} = 0$. For these metrics the heat kernel is an element of Vaillant's heat calculus [35, §4],

$$e^{-t\Delta_g} \in \Psi_H^{2,2,0}(M).$$

The superindices in $\Psi_H^{2,2,0}(M)$ refer to the behavior of $e^{-t\Delta}$ as $t \to 0$ in the interior of M, as $t \to 0$ and $x \to 0$, and as $x \to 0$ for $t > 0$. Let g_{hc} be a metric which is hyperbolic in a neighborhood of the cusp end. We will see, using Duhamel's formula and Vaillant's composition formula, that the difference between $e^{-t\Delta_g}$ and the heat kernel of g_{hc} vanishes at the cusp as $t \to 0$. Suppose that in a neighborhood E of $\partial_{\mathrm{hc}}M$ we can write the metric as

$$g = e^{\varphi}\left(\frac{dx^2}{x^2} + x^2 d\theta^2\right) = e^{\varphi} g_{\mathrm{hc}}$$

for some smooth $\varphi = \mathcal{O}(x^k)$ with $k > 0$. By introducing the interpolating family

$$g_\tau = e^{\tau\varphi} g_{\mathrm{hc}},$$

the heat kernels of g and g_{hc} in this neighborhood are related by

$$\begin{aligned} e^{-t\Delta_g} &= e^{-t\Delta_{\mathrm{hc}}} + \int_0^1 \partial_\tau e^{-t\Delta_\tau}\, d\tau \\ &= e^{-t\Delta_{\mathrm{hc}}} - \int_0^1 \int_0^t e^{-(t-s)\Delta_\tau}\varphi\Delta_\tau e^{-s\Delta_\tau}\, ds\, d\tau. \end{aligned}$$

From the proof of [35, Theorem 4.9], we know that $\Delta_\tau e^{-s\Delta_\tau}$ is an element of $\Psi_H^{0,0,0}(M)$ in Vaillant's heat calculus. For φ in $\mathcal{O}(x^k)$, $\varphi\Delta_\tau e^{-s\Delta_\tau}$ is an element of $\Psi_H^{0,k,k}(M)$ and hence using Vaillant's composition formula [35, Theorem 4.6]

$$\int_0^t e^{-(t-s)\Delta_\tau}\varphi\Delta_\tau e^{-s\Delta_\tau}\, ds \in \Psi_H^{2,2+k,\mathcal{H}}(M)$$

for some index set $\mathcal{H}$ bounded below by k. This implies that the a_j and $\widetilde{a}_j$ in the short-time asymptotics (2.4) are given by the same formula as those in the short-time asymptotics of $e^{-t\Delta_{\mathrm{hc}}}$ for $j < (k-1)/2$. Elements in $\Psi_H^{2,2+k,\mathcal{H}}(M)$ are trace-class at positive time for $k > 0$, so the discussion above also establishes (2) along the cusp ends. A similar argument using [1] instead of [35] (and the fact that $\varphi = \mathcal{O}(x_{\mathrm{F}}^2)$) establishes the lemma along the funnel ends.

□

2.4. **Polyakov formula.**

In this section we extend Polyakov's formula for the change in the determinant of the Laplacian upon a conformal change of metric. For the relative determinant, this formula is due to Borthwick, Judge, and Perry [8] (for metrics with no cusps and exactly hyperbolic outside a compact set) and to the second author [6] (for metrics with no funnels).

We will assume that the metrics involved are F,hc metrics with totally geodesic ends. As explained in the previous section, the latter assumption simplifies the behavior of the heat kernel and includes the metrics of constant scalar curvature. Thus we will analyze the behavior of the determinant of the Laplacian for a family of metrics $g(\tau) = e^{\omega(\tau)}g_0$ where g_0 is a smooth F,hc metric and

$$\omega(\tau) = \widetilde{\omega}(\tau) + \sum_{i=1}^{n_{\mathrm{F}}+n_{\mathrm{hc}}} \omega_i(\tau)\chi(x_i), \text{ with } \widetilde{\omega} \in x^2\mathcal{C}^\infty(\overline{M}\times[0,T]_\tau), \tag{2.11}$$

where $\omega_i \in \mathcal{C}^\infty(\mathbb{R})$ and $\chi \in \mathcal{C}_c^\infty([0,+\infty)_u)$ is a cut-off function equal to 1 for $u < \frac{\epsilon}{2}$ and to 0 for $u > \frac{3\epsilon}{4}$.

Theorem 2.9 (Polyakov formula)**.** *Let (M, g_0) be a non-compact surface with a smooth F,hc metric. Let $\omega(\tau) \in \mathcal{C}^\infty(\overline{M})$ be a smooth family of functions satisfying (2.11) and $g(\tau) = e^{\omega(\tau)}g_0$. Then the determinant of the Laplacian satisfies*

$$\partial_\tau \log\det\Delta_\tau = -\frac{1}{24\pi}\,{}^{R}\!\!\int_M \omega'(\tau)R_\tau\, \mathrm{dA}_\tau \tag{2.12}$$

if the area of M is infinite, and

$$\partial_\tau \log\det\Delta_\tau = -\frac{1}{24\pi}\int_M \omega'(\tau)R_\tau\, \mathrm{dA}_\tau + \partial_\tau \log \mathrm{Area}_\tau(M) \tag{2.13}$$

otherwise.

Proof. Let Δ_τ denote the Laplacian of $g(\tau)$, so that $\Delta_\tau = e^{-\omega(\tau)}\Delta_0$. Then

$$\begin{aligned}
\partial_\tau {}^R\operatorname{Tr}(e^{-t\Delta_\tau}) &= \operatorname*{FP}_{z=0} \partial_\tau \operatorname{Tr}(x^z e^{-t\Delta_\tau}) \\
&= -\operatorname*{FP}_{z=0} \operatorname{Tr}\left[x^z \int_0^t e^{-(t-s)\Delta_\tau}(-\omega'(\tau))\Delta_\tau e^{-s\Delta_\tau}\, ds\right] \\
&= \sum_{i=1}^{n_{\mathrm{F}}+n_{\mathrm{hc}}} \omega_i'(\tau) \operatorname*{FP}_{z=0} \operatorname{Tr}\left[x^z t\chi(x_i)\Delta_\tau e^{-t\Delta_\tau}\right] + \operatorname{Tr}\left[\widetilde{\omega}'(\tau)t\Delta_\tau e^{-t\Delta_\tau}\right] \\
&+ \sum_{i=1}^{n_{\mathrm{F}}+n_{\mathrm{hc}}} \omega_i'(\tau) \int_0^t \operatorname*{FP}_{z=0} \operatorname{Tr}\left[x^z[e^{-(t-s)\Delta_\tau}, \chi(x_i)\Delta_\tau e^{-s\Delta_\tau}]\right]\, ds \\
&= {}^R\operatorname{Tr}\left(\omega'(\tau)t\Delta_\tau e^{-t\Delta_\tau}\right) = -t\partial_t {}^R\operatorname{Tr}\left(\omega'(\tau)e^{-t\Delta_\tau}\right)
\end{aligned}$$

where in the third equality we have used that $\int_0^t e^{-(t-s)\Delta_\tau}\widetilde{\omega}'(\tau)\Delta_\tau e^{-s\Delta_\tau}$ is of trace class and that

$$ {}^R\operatorname{Tr}\left([e^{-(t-s)\Delta_\tau}, \chi(x_i)\Delta_\tau e^{-s\Delta_\tau}]\right) = 0. $$

Indeed, since this term is the regularized trace of a commutator, its value depends on the asymptotic expansion of the various operators at Y_i. Since $\chi(x_i) \equiv 1$ near Y_i, this means

$$ {}^R\operatorname{Tr}\left([e^{-(t-s)\Delta_\tau}, \chi(x_i)\Delta_\tau e^{-s\Delta_\tau}]\right) = {}^R\operatorname{Tr}\left([e^{-(t-s)\Delta_\tau}, \Delta_\tau e^{-s\Delta_\tau}]\right) = 0, $$

the latter commutator vanishing identically.

Hence, with $\mathcal{P}$ equal to the L^2 projection onto constants in the case of finite volume and equal to zero otherwise, we can proceed as follows

$$\begin{aligned}
\frac{\partial}{\partial\tau}\frac{\partial}{\partial s}\Big|_{s=0} \zeta_{e^{\omega(\tau)}g}(s) &= \frac{\partial}{\partial s}\Big|_{s=0}\left[\frac{1}{\Gamma(s)}\int_0^\infty t^s \frac{\partial}{\partial\tau} {}^R\operatorname{Tr}(e^{-t\Delta_\tau} - \mathcal{P})\frac{dt}{t}\right] \\
&= \frac{\partial}{\partial s}\Big|_{s=0}\left[-\frac{1}{\Gamma(s)}\int_0^\infty t^s \partial_t {}^R\operatorname{Tr}(\omega'(\tau)(e^{-t\Delta_\tau} - \mathcal{P}))\, dt\right] \\
&= \frac{\partial}{\partial s}\Big|_{s=0}\left[\frac{s}{\Gamma(s)}\int_0^\infty t^{s-1}\, {}^R\operatorname{Tr}(\omega'(\tau)(e^{-t\Delta_\tau} - \mathcal{P}))\, dt\right]
\end{aligned}$$

Since $\frac{1}{\Gamma(s)} \sim s + \mathcal{O}(s^2)$ as $s \to 0$, this is equal to

$$ \operatorname*{Res}_{s=0} \int_0^\infty t^{s} {}^R\operatorname{Tr}(\omega'(\tau)(e^{-t\Delta_\tau} - \mathcal{P}))\, \frac{dt}{t} = \operatorname*{FP}_{t=0} {}^R\operatorname{Tr}(\omega'(\tau)(e^{-t\Delta_\tau} - \mathcal{P})) \tag{2.14} $$

In the interior, we know that $\omega'(\tau)e^{-t\Delta}$ has a short-time expansion of the form

$$ \omega'(\tau)(e^{-t\Delta_\tau} - \mathcal{P}) \sim \frac{\alpha_{-1}}{t} + \alpha_0 + o(1) $$

where the coefficients α_k are precisely those functions that occur on a closed surface, multiplied by $\omega'(\tau)$, e.g. $\alpha_0 = \frac{R}{24\pi}\omega'(\tau)$. Because $\omega'(\tau) - \omega_i'(\tau) =$

$\mathcal{O}(x_i^2)$, we then see from lemma 2.6 as well as the construction of the heat kernel in [35] for cusps (see [5, Appendix]) and [1] for funnels that

$$\operatorname*{FP}_{t=0} \operatorname{Tr}(\omega'(\tau) e^{-t\Delta_\tau}) = \frac{1}{24\pi} \int_M \omega'(\tau) R_\tau \, \mathrm{dA}_\tau \, .$$

Thus, we conclude that

$$\begin{aligned} &\operatorname*{FP}_{t=0} {}^R\operatorname{Tr}(\omega'(\tau)(e^{-t\Delta_\tau} - \mathcal{P})) \\ &= \frac{1}{24\pi} \int_M \omega'(\tau) R_\tau \, \mathrm{dA}_\tau - \frac{1}{\operatorname{Area}_\tau(M)} \int_M \omega'(\tau) \, \mathrm{dA}_\tau \end{aligned}$$

where the final term is replaced by zero if the volume of M is infinite, and, if not, can be rewritten

$$\frac{1}{\operatorname{Area}_\tau(M)} \int_M \omega'(\tau) \, \mathrm{dA}_\tau = \frac{1}{\operatorname{Area}_\tau(M)} \int_M \partial_\tau (e^{\omega(\tau)}) \, \mathrm{dA}_0 = \partial_\tau \log \operatorname{Area}_\tau(M).$$

Finally, since

$$\frac{\partial}{\partial \tau} \frac{\partial}{\partial s}\bigg|_{s=0} \zeta_{e^\omega g}(s) = -\frac{\partial}{\partial \tau} \log \det \Delta_\tau$$

this finishes the proof. □

To study the determinant of the Laplacian, it is natural to impose the following normalization condition

$$ {}^R\operatorname{Area}(g) = -2\pi\chi(M) \tag{2.15}$$

so that one can not increase the determinant 'artificially' by scaling the metric. This leads naturally to the following definition.

Definition 2.10. *A* F, hc *metric g satisfying* (2.15) *is said to be critical for the determinant of the Laplacian if for any $\psi \in \mathcal{C}_c^\infty(M)$ with $\int_M \psi dg = 0$, we have that*

$$\int_M \psi R_g dg = 0.$$

Thus, a critical F, hc *metric is precisely one with constant scalar curvature* -2.

Now suppose that $\omega(\tau) = \tau\omega$ for a fixed smooth function ω satisfying

$$\omega = \omega_0 + \widetilde{\omega}, \text{ with } \widetilde{\omega} = \mathcal{O}(x^2) \quad \text{and} \quad \omega_0 \text{ a constant,} \tag{2.16}$$

and let $g_1 = e^\omega g_0$. Integrating equation (2.13) and using (3.5), we have

$$\begin{aligned} \log \det \Delta_{g_1} - \log \det \Delta_{g_0} &= \int_0^1 \partial_\tau \log \det \Delta_{e^{\tau\omega} g_0} \, d\tau \\ &= \int_0^1 \left[-\frac{1}{24\pi} \int_M \omega R_\tau \, \mathrm{dA}_\tau + \partial_\tau \log \operatorname{Area}_\tau(M) \right] d\tau \\ &= \log \operatorname{Area}_1(M) - \log \operatorname{Area}_0(M) - \frac{1}{24\pi} \int_M \left(\omega R_0 + |\nabla_0 \omega|^2 \right) \mathrm{dA}_0 \end{aligned} \tag{2.17}$$

where (2.16) guarantees both that there is no boundary term from applying Green's theorem and that $|\nabla\omega|^2$ is integrable. In the same way, integrating (2.12) yields

$$
\begin{aligned}
F(\omega) = \log\det\Delta_{g_1} - \log\det\Delta_{g_0} &= -\frac{1}{24\pi}\,{}^{R}\!\!\int \left(\omega R_0 + |\nabla_0\omega|^2\right) \mathrm{dA}_0 \\
&= -\frac{\chi(M)}{6}\omega_0 - \frac{1}{24\pi}\int \left(\widetilde{\omega} R_0 + |\nabla_0\widetilde{\omega}|^2\right) \mathrm{dA}_0\,.
\end{aligned} \tag{2.18}
$$

3. Ricci Flow on surfaces with funnel, cusp metrics

Hamilton [15] (see also [9]) studied the Ricci flow on closed surfaces and showed that, if the Euler characteristic is negative, then a solution to the normalized Ricci flow exists for all time and converges exponentially to a hyperbolic metric in the conformal class of the original metric. Hamilton's result and approach were extended to non-compact surfaces with asymptotically hyperbolic cusp ends by Ji, Mazzeo, and Šešum [18]. In this section, we further extend their result to non-compact surfaces whose ends are asymptotic to funnels or hyperbolic cusps.

In this section we will assume less regularity on the metrics we work with than in the previous sections. Let M be a surface with boundary and choose a background F,hc metric $\mathfrak{g}$ that is exactly hyperbolic in a neighborhood of the cusp ends. For $\alpha \in (0,1)$ and a continuous function v define

$$
\|v\|_{0,\alpha} = \sup_{\zeta\in M} |v(\zeta)| + \sup\left\{ \frac{|v(\zeta) - v(\zeta')|^{\alpha}}{d(\zeta,\zeta')} : d(\zeta,\zeta') < 1 \right\},
$$

where the distance between two points is measured with respect to $\mathfrak{g}$. Let $\mathcal{C}^{0,\alpha}_{\mathrm{F},\mathrm{hc}}(M)$ denote the space of functions for which $\|v\|_{0,\alpha} < \infty$, where along the cusps the collapse of the injectivity radius is dealt with as in [18] by passing to a covering space. For $k \in \mathbb{N}$, we say that $v \in \mathcal{C}^{0,\alpha}_{\mathrm{F},\mathrm{hc}}(M)$ is an element of the Hölder space $\mathcal{C}^{k,\alpha}_{\mathrm{F},\mathrm{hc}}(M)$ if, whenever $V_1, \dots, V_k$ are vector fields of bounded pointwise length with respect to $\mathfrak{g}$, we have

$$
V_1 \cdots V_j v \in \mathcal{C}^{0,\alpha}_{\mathrm{F},\mathrm{hc}}(M) \quad \text{for } j \leq k.
$$

Notice that, along a cusp, vector fields of bounded pointwise length are linear combinations of the vector fields

$$
x\partial_x, \quad \frac{1}{x}\partial_\theta.
$$

In this section, we will work with metrics g_0 as in Definition 1. However, instead of assuming the function φ is smooth **up to the boundary**, we will assume that $\varphi \in \mathcal{C}^{k+2}_{\mathrm{F},\mathrm{hc}}(M) \cap \mathcal{C}^{\infty}(M)$ for some $k \geq 1$. We will also assume that for each $i \in \{1, \dots, n_{\mathrm{F}} + n_{\mathrm{hc}}\}$, there is a constant $\phi_i \in \mathbb{R}$ and $\delta > 0$ such that

$$
\varphi - \phi_i \in x_i^{\delta}\mathcal{C}^{k+2}_{\mathrm{F},\mathrm{hc}}(M). \tag{3.1}
$$

It will sometimes be convenient to take different decay conditions for cusp and funnel ends. In that case, we will use the notation $\delta = \delta_{\mathrm{F}}$ for $i \in \{1, \ldots, n_{\mathrm{F}}\}$ and $\delta = \delta_{\mathrm{hc}}$ for $i \in \{n_{\mathrm{F}} + 1, \ldots, n_{\mathrm{F}} + n_{\mathrm{hc}}\}$.

We point out that under these assumptions, the scalar curvature of g_0 satisfies

$$R_{g_0} - r_i \in x_i^{\delta} \mathcal{C}^k_{\mathrm{F,hc}}(M) \tag{3.2}$$

where $r_i = -2e^{-\phi_i}$ is the asymptotic value of the curvature at the end Y_i.

3.1. Ricci flow and renormalized area.

Let M be a surface, g a metric on M and R the scalar curvature of g. On a surface the curvature is determined by the scalar curvature, in particular the Ricci curvature of g is $\frac{1}{2}Rg$, and the normalized Ricci flow equation is

$$\begin{cases} \partial_t g(t) = (\mathscr{C} - R_t) g(t) \\ g(0) = g_0 \end{cases} \tag{3.3}$$

where $\mathscr{C}$ is a constant. This flow preserves the conformal class of g_0, so we can write

$$g(t) = e^{\omega(t)} g_0$$

for some smooth function ω which satisfies

$$\omega'(t) = \mathscr{C} - R_t. \tag{3.4}$$

It is useful to recall that under a conformal change of metric we have

$$\Delta_{g(t)} = e^{-\omega(t)} \Delta_0, \quad \mathrm{dA}_t = e^{\omega(t)} \, \mathrm{dA}_0, \quad R_{g(t)} = e^{-\omega(t)} (R_{g_0} + \Delta_{g_0} \omega(t)) \tag{3.5}$$

as then from (3.4) it is easy to derive equations for the evolution of these quantities[1],

$$\begin{gathered} \partial_t \Delta_{g(t)} = (R_{g(t)} - \mathscr{C}) \Delta_t, \quad \partial_t \, \mathrm{dA}_t = (\mathscr{C} - R_{g(t)}) \, \mathrm{dA}_t, \\ \partial_t R_t = -\Delta_{g(t)} R_{g(t)} + R_{g(t)} (R_{g(t)} - \mathscr{C}). \end{gathered} \tag{3.6}$$

In particular, the normalized Ricci flow can be written as a scalar equation

$$\frac{\partial \omega}{\partial t} = -e^{-\omega} (\Delta_{g_0} \omega + R_{g_0}) + \mathscr{C}, \quad \omega(0) \equiv 0. \tag{3.7}$$

As on a compact surface, one natural choice for the constant $\mathscr{C}$ is to take the (renormalized) average curvature. If the funnel ends are totally geodesic (i.e., $\delta_{\mathrm{F}} > 1$), so that the renormalized Gauss-Bonnet theorem holds, then the flow with this choice of $\mathscr{C}$ will preserve the renormalized area.

Lemma 3.1. *Suppose that M is a non-compact surface and $g(t)$ is a smooth family of metrics satisfying* (1) *and* (2) *for some $\varphi \in \mathcal{C}^{k+2}_{\mathrm{F,hc}}(M)$ satisfying*

[1] We use the positive definite Laplacian, Hamilton [15] uses the negative definite Laplacian.

(3.1) *with* $\delta_{\mathrm{F}} > 1$. *If we assume that* ${}^R\operatorname{Area}_0(M) \neq 0$ *and that* $\partial_t g(t) = (\mathscr{C} - R_t)g(t)$ *with*

$$\mathscr{C} = \overline{R} = \frac{4\pi\chi(M)}{{}^R\operatorname{Area}_0(M)},$$

then, for all t, *we have*

$${}^R\operatorname{Area}_t(M) = {}^R\operatorname{Area}_0(M).$$

If instead we assume that ${}^R\operatorname{Area}_0(M) = 0$ *and* $\chi(M) = 0$ *then, for any* $\mathscr{C} \in \mathbb{R}$, *a smooth solution* $g(t)$ *to* $\partial_t g(t) = (\mathscr{C} - R_t)g(t)$ *satisfies* ${}^R\operatorname{Area}_t(M) = 0$ *for all* t.

Proof. Working with an arbitrary value of $\mathscr{C}$, we find

$$\begin{aligned}
\frac{\partial}{\partial t}{}^R\operatorname{Area}_t(M) &= \operatorname*{FP}_{z=0}\frac{\partial}{\partial t}\int_M x^z\,\mathrm{dA}_t = \operatorname*{FP}_{z=0}\int_M x^z\omega'(t)\,\mathrm{dA}_t \\
&= \operatorname*{FP}_{z=0}\int_M x^z(\mathscr{C} - R_{g(t)})\,\mathrm{dA}_t \\
&= \mathscr{C}\left({}^R\operatorname{Area}_t(M)\right) - {}^R\!\!\int_M R_{g(t)}\,\mathrm{dA}_t \\
&= \mathscr{C}\left({}^R\operatorname{Area}_t(M)\right) - 4\pi\chi(M)
\end{aligned}$$

from which the result follows when $\chi(M) = {}^R\operatorname{Area}_0(M) = 0$. If $\mathscr{C} \neq 0$, then this implies that for some constant A,

$${}^R\operatorname{Area}_t(M) = Ae^{\mathscr{C}t} + \frac{4\pi\chi(M)}{\mathscr{C}}.$$

We can find A by setting $t = 0$,

$$ {}^R\operatorname{Area}_t(M) = \left[{}^R\operatorname{Area}_0(M) - \frac{4\pi\chi(M)}{\mathscr{C}}\right]e^{\mathscr{C}t} + \frac{4\pi\chi(M)}{\mathscr{C}} \tag{3.8}$$

and the result follows. □

One other natural choice is to take $\mathscr{C} = r_i$ so that the asymptotic behavior of the curvature at Y_i is preserved along the flow, see Corollary 3.6 below. This choice is particularly useful to study the behavior of the metric and the curvature at infinity.

3.2. **Asymptotic behavior of a solution at infinity.**

Given a metric g_0 satisfying (1) and (2) and $g(t)$ a solution to (3.3), we would like to know when these asymptotic behaviors will be preserved along the normalized Ricci flow. It turns out to be convenient to study the asymptotic behavior of the metric $g(t)$ in terms of the solution $\omega(t)$ of (3.7), since in this setting one can easily invoke the maximum principle. The following elementary lemma and proposition are essentially taken from [34, §3.4]. We include their proof for completeness.

Lemma 3.2. *Let $t \mapsto h(t)$ be a smooth family of complete Riemannian metrics on M for $t \in [0,T]$ with curvature uniformly bounded. Let $u, v \in \mathcal{C}^2([0,T]\times M)\cap\mathcal{C}^1([0,T]\times\overline{M})$ be two functions such that $u \geq v$ on $[0,T]\times\partial\overline{M}$ and $u(0,m) \geq v(0,m)$ for all $m \in M$. Given $A \in \mathbb{R}$, exactly one of the following two possibilities happens:*

(i) *$u(t,m) \geq v(t,m)$ for all $(t,m) \in [0,T]\times\overline{M}$ or else*
(ii) *there exists $(t,m) \in (0,T]\times M$ such that*

$$\begin{aligned} &u(t,m) < v(t,m), && (\Delta_{h(t)}u)(t,m) \leq (\Delta_{h(t)}v)(t,m), \\ &\nabla u(t,m) = \nabla v(t,m), && \tfrac{du}{dt}(t,m) \leq \tfrac{dv}{dt}(t,m) - A(v(t,m)-u(t,m)). \end{aligned}$$

Proof. Replacing u, v with $u-v, 0$, we can assume $v = 0$. Replacing u by $e^{At}u$, we can also assume that $A = 0$.

In that case, if (i) holds, then clearly (ii) cannot hold. Conversely, if (i) does not hold, then the minimum of u is negative and there exists $(t,m) \in (0,T]\times\overline{M}$ where this minimum is achieved. Since we assume that $u \geq v$ on $[0,T]\times\partial\overline{M}$, this point is in $(0,T]\times M$ and, at this point, we have

$$\begin{aligned} &u(t,m) < 0, && \nabla u(t,m) = 0, \\ &(\Delta_{h(t)}u)(t,m) \leq 0, && \tfrac{du}{dt}(t,m) \leq 0, \end{aligned}$$

so that (ii) holds.

□

Proposition 3.3. *With the same assumptions as in Lemma 3.2, suppose that u and v also satisfy*

$$\begin{aligned} \frac{du}{dt} &\geq -\Delta_{h(t)}u + \nabla_{X(t)}u + F(t,m,u), \\ \frac{dv}{dt} &\leq -\Delta_{h(t)}v + \nabla_{X(t)}v + F(t,m,v), \end{aligned}$$

for all $(t,m) \in [0,T]\times M$ where $t \mapsto X(t)$ is a smooth family of smooth vector fields and F is a function which is uniformly Lipschitz in the last variable. Then $u(t,m) \geq v(t,m)$ for all (t,m) in $[0,T]\times M$.

Proof. Substracting the second equation from the first equation and using the Lipschitz property of F, we get

$$\frac{d}{dt}(u-v) \geq -\Delta_{h(t)}(u-v) + \nabla_{X(t)}(u-v) - C|u-v| \tag{3.9}$$

where C is the Lipschitz constant of F. Choosing $A > C$ in Lemma 3.2, we see that (ii) cannot occur and hence (i) holds. □

We will first put this into use to study the asymptotic behavior of the scalar curvature along the flow.

Proposition 3.4. *Let ω be a smooth solution to* (3.7) *and suppose that the function φ of definition 1 satisfies* (3.1) *for some $\delta > 0$ and $k \geq 2$. Suppose*

$\omega(t)$ is in $\mathcal{C}^{k+2}_{\mathrm{F,hc}}(M)$ for $t \in [0,T]$ and assume that the constant $\mathscr{C}$ is chosen to be $r_i = -2e^{-\phi_i}$. Then, for all $t \in [0,T]$,

$$R_{g(t)} - r_i \in x_i^\delta \mathcal{C}^{k-2}_{\mathrm{F,hc}}(M).$$

Proof. The evolution equation of $R_{g(t)} - r_i$ is given by

$$\frac{\partial}{\partial t}(R_{g(t)} - r_i) = -\Delta_{g(t)}(R_{g(t)} - r_i) + R_{g(t)}(R_{g(t)} - r_i).$$

For $\nu > 0$, consider $\psi = x_i^\nu(R_{g(t)} - r_i)$. Then its evolution equation is given by

$$\frac{\partial \psi}{\partial t} = -\Delta_{g(t)}\psi + \nabla_{X(t)}\psi + f\psi, \quad \psi|_{Y_i} = 0,$$

where $X^p(t) = 2x_i^\nu g^{pq}\nabla_q x_i^{-\nu}$ is a family of vector fields in $\mathcal{C}^{k+2}_{\mathrm{F,hc}}(M;TM)$ and $f = -x_i^\nu \Delta_{g(t)} x_i^{-\nu} + R_{g(t)}$ is in $\mathcal{C}^k_{\mathrm{F,hc}}(M)$.

Since $R_{g(t)}$ is uniformly bounded, we can choose positive constants C and C_1 such that $v = Ce^{C_1 t}x_i^{\delta+\nu}$ satisfies

$$\frac{\partial v}{\partial t} = C_1 v \geq -\Delta_{g(t)}v + \nabla_{X(t)}v + fv,$$
$$\frac{\partial(-v)}{\partial t} = -C_1 v \leq -\Delta_{g(t)}(-v) + \nabla_{X(t)}(-v) + f(-v),$$

and $v(t,m) \geq |\psi(t,m)|$ for all $t \in [0,T]$ and m outside a collar neighborhood of Y_i. This last property is to insure we control what happens at the other ends of the surface. Choosing $C > 0$ big enough, we can also assume that $v(0,m) \geq |\psi(0,m)|$ for all $m \in M$. We can then apply Proposition 3.3 to conclude that

$$-v(t,m) \leq \psi(t,m) \leq v(t,m)$$

for all $(t,m) \in [0,T] \times M$. Thus, this gives that

$$-Ce^{C_1 t}x_i^\delta \leq R_{g(t)} - r_i \leq Ce^{C_1 t}x_i^\delta.$$

We can derive similar estimates for the derivatives of $R_{g(t)}$ (up to order $k-2$) by looking at their evolution equations, from which the result follows. □

The asymptotic value of $\omega(t)$ does not necessarily stay constant at the boundaries along the flow. Proposition 3.5 below describes how it decays.

If $\omega(t)$ does not vanish at Y_i, then the asymptotic value of the scalar curvature varies with t and we will denote it by $r_i(t)$. It is easy to see that for $\mathscr{C} < 0$

$$r_i'(t) = -r_i(t)(\mathscr{C} - r_i(t)) \implies \frac{|\mathscr{C} - r_i(t)|}{|r_i(t)|} = \frac{|\mathscr{C} - r_i(0)|}{|r_i(0)|}e^{\mathscr{C}t}$$

which implies that

$$\frac{\mathscr{C} - r_i(t)}{r_i(t)} = \frac{\mathscr{C} - r_i(0)}{r_i(0)}e^{\mathscr{C}t} \implies r_i(t) = \frac{r_i(0)\mathscr{C}}{r_i(0) + (\mathscr{C} - r_i(0))e^{\mathscr{C}t}} \tag{3.10}$$

Proposition 3.5. *Let ω be a smooth solution to* (3.7) *with $\varphi \in \mathcal{C}^{k+2}_{\mathrm{F,hc}}(M)$ for some $k \geq 2$ satisfying* (3.1) *for some $\delta > 0$. Suppose $\omega(t)$ is in $\mathcal{C}^{k+2}_{\mathrm{F,hc}}(M)$ for $t \in [0,T]$. Then for each i, there exists a smooth function $c_i : [0,T] \to \mathbb{R}$ such that*

$$\omega(t) - c_i(t) \in x_i^\delta \mathcal{C}^{k-2}_{\mathrm{F,hc}}(M) \ \forall\ t \in [0,T]. \tag{3.11}$$

Proof. Fix $i \in \{1, \ldots, n_{\mathrm{hc}} + n_{\mathrm{F}}\}$. By rescaling if necessary, we can assume that $\mathscr{C} = r_i$. In that case, we need to show that $\omega(t) \in x_i^\delta \mathcal{C}^k_{\mathrm{F,hc}}(M)$ for all $t \in [0,T]$. Let $\nu \in (0,\delta)$ be given. Then the function $\psi_i = x_i^\nu \omega$ satisfies the equation

$$\begin{gathered}\frac{\partial \psi_i}{\partial t} = -e^{-\omega}(\Delta_{g_0}\psi_i + x_i^\nu R_{g_0}) + x_i^\nu \mathscr{C} + e^{-\omega} f \psi_i + \nabla_{X(t)}\psi_i, \\ \psi|_{\partial \overline{M}} = 0,\end{gathered} \tag{3.12}$$

where $X(t) \in \mathcal{C}^{k+2}_{\mathrm{F,hc}}(M;TM)$ is a family of vector fields on M and $f \in \mathcal{C}^k_{\mathrm{F,hc}}(M)$. Notice in particular that $\sup_M |X(t)|_{g(0)} < \infty$ for all $t \in [0,T]$. We can rewrite this equation as

$$\frac{\partial \psi_i}{\partial t} = -\Delta_{g(t)}\psi_i + \nabla_{X(t)}\psi_i + f_1(t,m)\psi_i + f_0(t,m) \tag{3.13}$$

with $f_0(t,m) = -e^{-\omega(t,m)} x_i^\nu R_{g_0} + x_i^\nu \mathscr{C}$ and $f_1(t,m) = e^{-\omega(t,m)} f$. We know by Proposition 3.4 and (3.2) that

$$f_0(t,\cdot) \in x^{\nu+\delta}\mathcal{C}^\infty_{\mathrm{F,hc}}(M) \quad \forall\ t \in [0,T]. \tag{3.14}$$

Thus, since ω is uniformly bounded, we can choose positive constants C, C_1 sufficiently large so that $v = Ce^{C_1 t}x^{\delta+\nu}$ satisfies

$$\begin{gathered}\frac{\partial v}{\partial t} = C_1 v \geq -\Delta_{g(t)} v + \nabla_{X(t)} v + f_1 v + f_0, \\ \frac{\partial(-v)}{\partial t} = -C_1 v \leq -\Delta_{g(t)}(-v) + \nabla_{X(t)}(-v) + f_1(-v) + f_0,\end{gathered} \tag{3.15}$$

and $v(t,m) \geq |\psi_i(t,m)|$ for $t \in [0,T]$ and m outside a collar neighborhood of Y_i in $\overline{M}$. We can also choose $C > 0$ so that $v(0,m) \geq |\psi_i(0,m)|$ for all $m \in M$. We can then apply Proposition 3.3 to conclude that

$$-v(t,m) \leq \psi_i(t,m) \leq v(t,m), \quad \forall (t,m) \in [0,T] \times M.$$

Thus, this gives that

$$-Ce^{C_1 t}x^\delta \leq \omega \leq Ce^{C_1 t}x^\delta.$$

We can derive similar estimates for the derivatives of ω (up to order k-2) by looking at their evolution equations (which can be derived by using the identity $\nabla\Delta = \Delta\nabla + \frac{1}{2}R\nabla$), from which the result follows. $\square$

Corollary 3.6. *Let $\omega(t)$ be as in Proposition 3.5. Then the scalar curvature $R_{g(t)}$ of the metric $g(t) = e^{\omega(t)}g_0$ is such that*

$$R_{g(t)} - r_i(t) \in x_i^\delta \mathcal{C}^k_{\mathrm{F,hc}}(M) \quad \forall\ t \in [0,T],$$

where $r_i : [0,T] \to \mathbb{R}$ *is given by* (3.10)*. When* $\mathscr{C} = r_i(0)$*, then* r_i *is constant along the flow.*

Proof. This is a direct consequence of Proposition 3.4 and Proposition 3.5. □

When we will apply Polyakov's formula to a family of metrics $g(t) = e^{\omega(t)}g_0$ evolving according to the Ricci flow, it will be convenient to know that the conformal factor $\omega(t)$ remains smooth up to the boundary along the flow. This is the content of the next proposition.

Proposition 3.7. *Let* ω *be a smooth solution to* (3.7) *with initial metric* g_0 *satisfying* (1) *and* (2) *with* $\varphi \in \mathcal{C}^\infty_{\mathrm{F,hc}}(M) \cap \mathcal{C}^\infty(\overline{M})$ *satisfying* (3.1) *for some* $\delta > 0$*. Suppose that* $\omega(t)$ *is uniformly in* $\mathcal{C}^\infty_{\mathrm{F,hc}}(M)$ *for* $t \in [0,T]$*. Then*

$$\omega(t) \in \mathcal{C}^\infty(\overline{M}) \cap \mathcal{C}^\infty_{\mathrm{F,hc}}(M) \ \forall t \in [0,T].$$

Proof. Notice that if there are no cusp ends, then $\mathcal{C}^\infty(\overline{M}) \subset \mathcal{C}^\infty_{\mathrm{F,hc}}(M)$. However, if $\partial_{\mathrm{hc}}\overline{M} \neq \emptyset$, then there is no such inclusion. In fact, near a cusp end, we have that

$$f \in \mathcal{C}^\infty_{\mathrm{F,hc}}(M) \cap \mathcal{C}^\infty(\overline{M}) \quad \Longrightarrow f|_{\partial_{\mathrm{hc}}\overline{M}} \text{ is locally constant.} \tag{3.16}$$

To show that $\omega(t) \in \mathcal{C}^\infty(\overline{M})$, we will inductively construct the Taylor series of ω at the boundary. Thus, we need to show that there exists $\omega_k \in \mathcal{C}^\infty([0,T] \times \partial\overline{M})$ for $k \in \mathbb{N} \cup \{0\}$ such that

$$\left(\omega(t) - \sum_{k=0}^{N} \chi(x)\omega_k(t)x^k\right) = \mathcal{O}(x^{N+1}) \quad \forall\, N \in \mathbb{N} \cup \{0\} \tag{3.17}$$

in a collar neighborhood of $\partial\overline{M}$, where $\chi : [0,+\infty) \to [0,+\infty)$ is a smooth cut-off function with $\chi(x) = 1$ for $x < \frac{\epsilon}{2}$ and $\chi(x) = 0$ for $x > \frac{3\epsilon}{4}$.

If ω_k exists, notice by (3.16) that it is locally constant on $\partial_{\mathrm{hc}}\overline{M}$. Without loss of generality, we can assume that $\partial\overline{M}$ has only one boundary component which is associated either to a cusp or a funnel. We first need to define ω_0. If $\omega(t)$ were in $\mathcal{C}^\infty(\overline{M})$ as claimed, then the evolution equation for ω_0 would be

$$\frac{\partial \omega_0}{\partial t} = -e^{-\omega_0}r_i(t) + \mathscr{C}, \quad \omega_0(0) \equiv 0. \tag{3.18}$$

Thus, we can define ω_0 to be the unique solution to (3.18). Rescaling the flow and the metric if necessary, we can assume that $\mathscr{C} = r_i = -2$ so that $R_{g(t)} + 2 \in x^\delta \mathcal{C}^\infty_{\mathrm{F,hc}}(M)$. Then $\widetilde{\omega}_1(t) = \omega(t) - \omega_0(t)\chi(x)$ satisfies the evolution equation

$$
\begin{aligned}
\frac{\partial \widetilde{\omega}_1}{\partial t} &= -e^{-\omega}(\Delta_{g_0}\omega + R_{g_0}) + \mathscr{C} + \chi(x)\left(e^{-\omega_0}(-2) - \mathscr{C}\right) \\
&= -e^{-\omega}\Delta_{g_0}\omega + R_{g_0}(-e^{-\omega} + e^{-\chi(x)\omega_0}) \\
&\quad + \left(-e^{-\chi(x)\omega_0}R_{g_0} - 2\chi(x)e^{-\omega_0}\right) + \mathscr{C}(1-\chi(x)) \\
&= -\Delta_{g(t)}\widetilde{\omega}_1 + \widetilde{f}_1\widetilde{\omega}_1 + \widetilde{h}_1
\end{aligned} \tag{3.19}
$$

where $\widetilde{f}_1 = R_{g_0}e^{-\chi(x)\omega_0}\left(\frac{1-e^{-\widetilde{\omega}_1}}{\widetilde{\omega}_1}\right)$ is in $\mathcal{C}^\infty_{\mathrm{F,hc}}(M)$ for all $t \in [0,T]$ and has the same regularity as ω at the boundary, while $\widetilde{h}_1 \in x(\mathcal{C}^\infty_{\mathrm{F,hc}}(M) \cap \mathcal{C}^\infty(\overline{M}))$. Thus using a barrier function of the form $v = Ce^{C_1 t}x$ for some $C, C_1 > 0$ large enough, we can proceed as in the proof of Proposition 3.5, to show that $\overline{\omega}_1 = \frac{\widetilde{\omega}_1}{x} \in \mathcal{C}^\infty_{\mathrm{F,hc}}(M)$ for all $t \in [0,T]$. From (3.19), its evolution equation is of the form

$$
\frac{\partial \overline{\omega}_1}{\partial t} = -\Delta_{g(t)}\overline{\omega}_1 + \nabla_{X_1(t)}\overline{\omega}_1 + \overline{f}_1\overline{\omega}_1 + \overline{h}_1(t,m), \quad \overline{\omega}_1(0) \equiv 0, \tag{3.20}
$$

where $X_1(t) \in \mathcal{C}^\infty_{\mathrm{F,hc}}(M;TM)$ and $\overline{f}_1(t,\cdot)$ is in $\mathcal{C}^\infty_{\mathrm{F,hc}}(M)$ for all $t \in [0,T]$ and has the same regularity as ω at the boundary, while $\overline{h}_1 \in \mathcal{C}^\infty_{\mathrm{F,hc}}(M) \cap \mathcal{C}^\infty(\overline{M})$ for all $t \in [0,T]$.

Suppose now for a proof by induction that $\omega_0, \dots, \omega_{N-1}$ have been defined to satisfy (3.17) and that

$$
\overline{\omega}_N = \frac{\omega - \sum_{k=1}^{N-1}\chi(x)\omega_k x^k}{x^N} \in \mathcal{C}^\infty_{\mathrm{F,hc}}(M) \tag{3.21}
$$

satisfies the evolution equation

$$
\frac{\partial \overline{\omega}_N}{\partial t} = -\Delta_{g(t)}\overline{\omega}_N + \nabla_{X_N(t)}\overline{\omega}_N + \overline{f}_N\overline{\omega}_N + \overline{h}_N \tag{3.22}
$$

where $X_N(t) \in \mathcal{C}^\infty_{\mathrm{F,hc}}(M;TM)$ and with $\overline{f}_N(t,\cdot)$, $\overline{h}_N(t,\cdot)$ in $\mathcal{C}^\infty_{\mathrm{F,hc}}(M)$ having the same regularity as $\overline{\omega}_{N-1}$ at the boundary (with the convention that $\overline{\omega}_0 = \omega$). We then define $\omega_N \in \mathcal{C}^\infty([0,T]\times\partial\overline{M})$ to be the unique solution of the evolution equation

$$
\frac{\partial \omega_N}{\partial t} = \left(\overline{f}_N\big|_{\partial\overline{M}}\right)\omega_N + \overline{h}_N\big|_{\partial\overline{M}}, \quad \omega_N(0) \equiv 0. \tag{3.23}
$$

Then $\widetilde{\omega}_{N+1} = \overline{\omega}_N - \chi(x)\omega_N$ satisfies the evolution equation

$$
\begin{aligned}
\frac{\partial \widetilde{\omega}_{N+1}}{\partial t} &= -\Delta_{g(t)}\overline{\omega}_N + \nabla_{X_N(t)}\overline{\omega}_N + \overline{f}_N\overline{\omega}_N + \overline{h}_N \\
&\quad - \chi(x)\left(\overline{f}_N\big|_{\partial\overline{M}}\right)\omega_N - \chi(x)\left(\overline{h}_N\big|_{\partial\overline{M}}\right) \\
&= -\Delta_{g(t)}\overline{\omega}_N + \nabla_{X_N(t)}\overline{\omega}_N + \overline{f}_N(\overline{\omega}_N - \chi(x)\omega_N) \\
&\quad + (\overline{f}_N - \overline{f}_N\big|_{\partial\overline{M}})\chi(x)\omega_N + (\overline{h}_N - \chi(x)\,\overline{h}_N\big|_{\partial\overline{M}}) \\
&= -\Delta_{g(t)}\widetilde{\omega}_{N+1} + \nabla_{X_N(t)}\widetilde{\omega}_{N+1} + \overline{f}_N\widetilde{\omega}_{N+1} + \widetilde{h}_{N+1}
\end{aligned} \tag{3.24}
$$

where $\widetilde{h}_{N+1}(t,\cdot) \in x\mathcal{C}^{\infty}_{\mathrm{F,hc}}(M)$ has the same regularity as $\overline{\omega}_{N-1}$ at the boundary. Thus, using a barrier function of the form $v = Ce^{C_1 t}x$ for some $C, C_1 > 0$ large enough, we can again proceed as in the proof of Proposition 3.5 to show that $\overline{\omega}_{N+1}(t) := \frac{\widetilde{\omega}_{N+1}}{x}$ is in $\mathcal{C}^{\infty}_{\mathrm{F,hc}}(M)$. Moreover, it satisfies the evolution equation

$$\frac{\partial \overline{\omega}_{N+1}}{\partial t} = \Delta_{g(t)}\overline{\omega}_{N+1} + \nabla_{X_{N+1}(t)}\overline{\omega}_{N+1} + \overline{f}_{N+1}\overline{\omega}_{N+1} + \overline{h}_{N+1} \tag{3.25}$$

with $\overline{f}_{N+1}(t,\cdot), \overline{h}_{N+1}(t,\cdot) \in \mathcal{C}^{\infty}_{\mathrm{F,hc}}(M)$ having the same regularity up to the boundary as $\overline{\omega}_N$ and with $X_{N+1}(t) \in \mathcal{C}^{\infty}_{\mathrm{F,hc}}(M;TM)$.

In this way, we define inductively $\omega_k \in \mathcal{C}^{\infty}([0,T] \times \partial\overline{M})$ such that (3.17) is satisfied for all $N \in \mathbb{N}$.

□

3.3. **Short-time existence and uniqueness.** The short-time existence of a solution of (3.7) follows from the more general result of Shi [32] who established the short-time existence of the Ricci flow on a complete Riemannian manifold with bounded curvature. In our context, since the equation can be written in the scalar form (3.7), it is possible to prove uniqueness relatively easily using the maximum principle.

Proposition 3.8 (Short-time existence and uniqueness)**.** *Consider the equation* (3.7) *with initial metric* g_0 *satisfying* (1) *and* (2) *with* $\varphi \in \mathcal{C}^{k+2}_{\mathrm{F,hc}}(M)$ *satisfying* (3.1) *for some* $\delta > 0$ *and* $k \geq 2$*. In particular, there exists* $K > 0$ *such that* $|R_{g_0}| < K$ *everywhere on* M*. Then there exists* $T = T(K, \mathscr{C}) > 0$ *depending on* K *and* $\mathscr{C}$ *such that* (3.7) *has a unique smooth solution* $\omega(t)$ *on* $[0,T] \times M$ *satisfying*

$$\omega(t) - c_i(t) \in x_i^{\delta}\mathcal{C}^{k-2}_{\mathrm{F,hc}}(M)$$

for some smooth functions $c_i : [0,T] \to \mathbb{R}$.

Proof. Short-time existence follows from the short-time existence result of Shi [32] for the Ricci flow on complete manifolds with bounded curvature. The decay property of the solution follows from Proposition 3.5.

To prove uniqueness, suppose ω_1 and ω_2 are two solutions. By the proof of Proposition 3.5 (see also (3.18)), we know that $\omega_1|_{\partial\overline{M}} = \omega_2|_{\partial\overline{M}}$. In fact, by Proposition 3.5, we have that $v = \omega_1 - \omega_2$ is in $x^{\delta}\mathcal{C}^{k-2}_{\mathrm{F,hc}}(M)$ and satisfies

$$\begin{cases} \frac{\partial v}{\partial t} = -\Delta_{g_1(t)}v + F(t,m,\omega_1) - F(t,m,\omega_2), \quad g_1(t) = e^{\omega_1(t)}g_0, \\ v(0) \equiv 0, \end{cases} \tag{3.26}$$

where

$$F(t,m,\psi) = (-\Delta_{g_0}\omega_2(m,t) - R_{g_0}(m))e^{-\psi}, \quad t \in [0,T],\ m \in M,\ \psi \in \mathbb{R}. \tag{3.27}$$

Since ω_1 and ω_2 together with their derivatives are uniformly bounded on $[0,T]$, there exists a constant $A > 0$ such that

$$|F(t,m,\omega_1) - F(t,m,\omega_2)| \leq A|v(t,m)| \quad \forall\ (t,m) \in [0,T] \times M. \tag{3.28}$$

Consequently,

$$\frac{\partial v}{\partial t} \geq -\Delta_{g_1(t)} v - A|v|, \quad v(0) \equiv 0, \quad v|_{\partial M} = 0. \tag{3.29}$$

By Proposition 3.3 applied to v and 0, this means that $v(t,m) \geq 0$ for all $(t,m) \in [0,T] \times M$. Interchanging the rôles of ω_1 and ω_2, we can also show that $v \leq 0$, which implies that $v \equiv 0$, that is, $\omega_1 \equiv \omega_2$ on $[0,T] \times M$, establishing uniqueness. □

3.4. **Long-time existence.** To prove long-time existence, it suffices to get an a priori bound on the scalar curvature, for then we can apply Proposition 3.8 recursively to get long-time existence. As noticed by Hamilton [15], on a compact surface, such an a priori estimate follows from the existence of a potential function. This approach was recently generalized to surfaces with cusps ends by Ji, Mazzeo and Šešum [18].

To get long-time existence in our case, we need to generalize the construction of the potential function given in [18] to also include funnel ends.

Proposition 3.9 (Potential function)**.** *Let g be a Riemannian metric on M satisfying* (1) *and* (2) *with $\varphi \in \mathcal{C}^3_{\mathrm{F,hc}}(M)$ satisfying* (3.1) *for some $\delta_{\mathrm{F}} > 2$ and $\delta_{\mathrm{hc}} > 0$. Let $\mathscr{C} < 0$ be arbitrary if $\partial_{\mathrm{F}} M \neq \emptyset$ and $\mathscr{C} = \overline{R}$ if $\partial_{\mathrm{F}} \overline{M} = \emptyset$. Then there exists a unique function f and constants c_i such that*

$$-\Delta_g f = R - \mathscr{C}, \quad (f - \sum_{i=1}^{n_{\mathrm{F}}+n_{\mathrm{hc}}} c_i \log x_i) \in C^{2,\alpha}_{\mathrm{F,hc}}(M), \quad \sup_M |\nabla f|_g < \infty, \tag{3.30}$$

with $\int_M f dg = 0$ if $\partial_{\mathrm{F}} \overline{M} = \emptyset$ and

$$(f - c_1 \log x_1)|_{Y_1} = 0, \quad \frac{\partial}{\partial x_j}(f - \sum_{i=2}^{n_{\mathrm{F}}} c_i \log x_i)\Big|_{Y_j} = 0, \quad j \in \{2, \ldots, n_{\mathrm{F}}\}$$

otherwise.

Proof. If M is compact, this is very easy to prove. If (M,g) has only cusp ends, this was proved in [18], in fact with weaker decay assumptions on φ. To prove the proposition when there are funnel ends, the idea is to reduce to the case where there are only cusp ends (or when M is compact) via a doubling construction.

Thus, assume that $\partial_{\mathrm{F}} \overline{M} \neq \emptyset$. By (3.2), we know that for $i \in \{1, \ldots, n_{\mathrm{F}}\}$,

$$R_g - r_i \in x_i^{\delta_{\mathrm{F}}} C^1_{\mathrm{F,hc}}(M). \tag{3.31}$$

By assumption, there also exists a constant ϕ_i such that

$$(\varphi - \phi_i) \in x_i^{\delta_{\mathrm{F}}} C^3_{\mathrm{F,hc}}(M). \tag{3.32}$$

Then the function

$$\psi_i = e^{\phi_i}(r_i - \mathscr{C}) \log x_i \tag{3.33}$$

has its support in a collar neighborhood of Y_i and is such that

$$(-\Delta\psi_i - (R - \mathscr{C})) \in x_i^{\delta_{\mathrm{F}}} C^1_{\mathrm{F,hc}}(M). \tag{3.34}$$

Thus, if we set $\widetilde{f} = f - \sum_{i=1}^{n_{\mathrm{F}}} \psi_i$, we can rewrite $-\Delta f = R - \mathscr{C}$ as

$$-\Delta\widetilde{f} = h, \quad \text{with } h = (R - \mathscr{C} + \sum_{i=1}^{n_{\mathrm{F}}} \Delta\psi_i) \in \mathcal{C}^1_{\mathrm{F,hc}}(M). \tag{3.35}$$

In a collar neighborhood of $\partial_{\mathrm{F}}\overline{M}$, equation (3.35) takes the form

$$-e^{-\varphi} x_{\mathrm{F}}^2 \Delta_{g_E} \widetilde{f} = h \tag{3.36}$$

where Δ_{g_E} is the Laplacian associated to the incomplete cylindrical metric $g_E = dx^2 + \pi_F^* h_F$ where h_F is a metric on $\partial_{\mathrm{F}}\overline{M}$ and π_F is the projection from the collar neighborhood of $\partial_{\mathrm{F}}\overline{M}$ onto $\partial_{\mathrm{F}}\overline{M}$.

Thus, with respect to the metric

$$\widetilde{g} = e^{-\varphi\chi} x_{\mathrm{F}}^2 g, \tag{3.37}$$

where $\chi \in \mathcal{C}_c^\infty(\partial_F M \times [0,\epsilon)_{x_{\mathrm{F}}})$ is a nonnegative cut-off function equal to 1 for $x_{\mathrm{F}} < \frac{\epsilon}{2}$ and equal to zero for $x_{\mathrm{F}} > \frac{3\epsilon}{4}$, equation (3.36) can be rewritten as

$$-\Delta_{\widetilde{g}} \widetilde{f} = \widetilde{h}, \quad \widetilde{h} = \frac{h}{e^{-\varphi\chi} x_F^2}. \tag{3.38}$$

The metric $\widetilde{g}$ is incomplete and if we glue two copies of $\overline{M}$ along Y_1 to get

$$\mathbb{M}_1 = \overline{M} \cup_{Y_1} \overline{M}, \tag{3.39}$$

then the metrics $\widetilde{g}$ on each copy of $\overline{M}$ glue together to give a smooth metric $\widehat{g}_1$ on $\mathbb{M}_1$. The Riemannian manifold $(\mathbb{M}_1, \widehat{g}_1)$ has $2(n_{\mathrm{F}} - 1)$ ends where the metric is asymptotic to an incomplete cylinder and $2n_{\mathrm{hc}}$ ends were it is asymptotic to a cusp. Let

$$\partial_F \mathbb{M}_1 \cong \left(\bigcup_{i=2}^{n_{\mathrm{F}}} Y_i \right) \bigsqcup \left(\bigcup_{i=2}^{n_{\mathrm{F}}} Y_i \right) \tag{3.40}$$

be the part of the boundary associated to cylindrical ends. We can consider the double of $\mathbb{M}_1$ along $\partial_{\mathrm{F}}\mathbb{M}_1$

$$\mathbb{M}_2 = \mathbb{M}_1 \cup_{\partial_{\mathrm{F}}\mathbb{M}_1} \mathbb{M}_1. \tag{3.41}$$

The metric $\widehat{g}_1$ on each copy of $\mathbb{M}_1$ glue together to give a smooth metric $\widehat{g}_2$ on $\mathbb{M}_2$. Clearly, $(\mathbb{M}_2, \widehat{g}_2)$ is a complete surface with $4n_{\mathrm{hc}}$ cusp ends (or is a compact surface if $n_{hc} = 0$). Let $\widehat{x} \in \mathcal{C}^\infty(\overline{\mathbb{M}}_2)$ be a boundary defining function on $\overline{\mathbb{M}}_2$ whose restriction to each copy of $\overline{M}$ in $\overline{\mathbb{M}}_2$ is equal to x_{hc}. Let $\widehat{h}_1$ be the function on $\mathbb{M}_1$ whose restriction to one copy of M in $\mathbb{M}_1$ is $\widetilde{h}$ and whose restriction to the other copy is $-\widetilde{h}$. Let $\widehat{h}_2$ be the function whose restriction to each copy of $\mathbb{M}_1$ in $\mathbb{M}_2$ is $\widehat{h}_1$. By (3.31) and since $\delta_{\mathrm{F}} > 2$, we have that $\widehat{h}_2 \in \widehat{x}^{\delta_{\mathrm{hc}}}\mathcal{C}^{0,\alpha}_{\widehat{g}_2}(\mathbb{M}_2)$. Then on $\mathbb{M}_2$, one can consider the equation

$$-\Delta_{\widehat{g}_2}\widehat{f} = \widehat{h}_2 \quad \text{on } \mathbb{M}_2. \tag{3.42}$$

Clearly, by symmetry, $\int_{\mathbb{M}_2}\widehat{h}_2 d\widehat{g}_2 = 0$ so that we can apply the result of [18] to conclude that there exists a unique solution $\widehat{f}$ with constants $\widehat{c}_i$, $i \in \{1, \ldots, 2n_{\mathrm{hc}}\}$ such that

$$(\widehat{f} - \sum_{i=1}^{4n_{\mathrm{hc}}} \widehat{c}_i \log \widehat{x}_i) \in \mathcal{C}^{2,\alpha}_{\widehat{g}}(\mathbb{M})$$

and

$$-\Delta_{\widehat{g}_2}\widehat{f} = \widehat{h}_2, \quad \int_{\mathbb{M}_2} \widehat{f} d\widehat{g}_2 = 0, \quad \sup_{\mathbb{M}_2} |\nabla \widehat{f}|_{\widehat{g}_2} < \infty. \tag{3.43}$$

Since this solution is unique, we see by symmetry that the restriction of this solution to one of the copies of $\overline{M}$ in $\mathbb{M}_2$ will solve the equation $-\Delta_g \widetilde{f} = R - \mathscr{C}$ with bounded gradient, Dirichlet boundary condition on Y_1 and Neumann boundary condition on $\cup_{i=2}^{n_{\mathrm{F}}} Y_i$. Thus $f = \widetilde{f} + \sum_{i=1}^{n_{\mathrm{F}}} \psi_i$ will be the desired solution and is clearly unique.

□

It is also interesting to consider the following variant of the construction which only involves Neumann boundary conditions.

Proposition 3.10. *Let g be a Riemannian metric satisfying* (1) *and* (2) *with $\varphi \in \mathcal{C}^\infty(\overline{M}) \cap \mathcal{C}^3_{\mathrm{F,hc}}(M)$ satisfying* (3.1) *for $\delta_{\mathrm{F}} = 2$ and some $\delta_{\mathrm{hc}} > 0$. Suppose that $\partial_{\mathrm{F}} M \neq 0$ and that ${}^R\mathrm{Area}(g) \neq 0$ so that $\overline{R}$ is well-defined. Then there exists a unique f and constants c_i such that*

$$-\Delta_g f = R - \overline{R}, \quad (f - \sum_{i=1}^{n_{\mathrm{F}}+n_{\mathrm{hc}}} c_i \log x_i) \in C^{2,\alpha}_{\mathrm{F,hc}}(M), \quad \sup_M |\nabla f|_g < \infty,$$

with

$$\frac{\partial}{\partial x_{\mathrm{F}}}(f - \sum_{i=2}^{n_{\mathrm{F}}} c_i \log x_i)\Big|_{\partial_{\mathrm{F}}\overline{M}} = 0 \quad \text{and} \quad \int_M x_{\mathrm{F}}^2 f dg = 0.$$

Proof. We proceed as in the proof of proposition 3.9 to define the functions ψ_i, $\widetilde{h}$ and $\widetilde{f}$ and the metric $\widetilde{g}$. However, instead of $\mathbb{M}_1$ and $\mathbb{M}_2$, we consider directly the double of M along $\partial_{\mathrm{F}}\overline{M}$,

$$\mathbb{M} = \overline{M} \cup_{\partial_{\mathrm{F}}\overline{M}} \overline{M}.$$

The metrics $\widetilde{g}$ on each copy of M glue together to give a metric $\widehat{g}$ on $\mathbb{M}$ having only cusp ends. On $\mathbb{M}$, we consider the function $\widehat{h}$ whose restriction to each copy of $\overline{M}$ in $\mathbb{M}$ is $\widetilde{h}$ and the equation

$$-\Delta_{\widehat{g}}\widehat{f} = \widehat{h}. \tag{3.44}$$

Since $\varphi \in \mathcal{C}^\infty(\overline{M})$, we only need $\delta_{\mathrm{F}} = 2$ to deduce from (3.31) that $\widehat{h} \in \widehat{x}^{\delta_{\mathrm{hc}}}\mathcal{C}^1_{\widehat{g}}(\mathbb{M})$. A quick computation shows that ${}^R\int_M \Delta_g \psi_i dg = 0$ so that

$$\int_{\mathbb{M}} \widehat{h} d\widehat{g} = 2\int_M \widetilde{h} d\widetilde{g} = 2\int_M h dg = 2^R\int_M (R - \overline{R}) dg = 0. \tag{3.45}$$

This means that we can use the result of [18] to solve (3.44). Restricting to $M \subset \mathbb{M}$ and adding an appropriate constant gives the desired potential function.

□

With these potential functions, it is then easy to get long-time existence for the normalized Ricci-flow converging to a constant scalar curvature metric as $t \to +\infty$.

Theorem 3.11. (i) *(Ji-Mazzeo-Šešum) Suppose that $\partial_{\mathrm{F}}\overline{M} = \emptyset$ (finite volume case) and that $\chi(M) < 0$. Let g_0 be a metric on M as in proposition 3.9. Then the solution $g(t) = e^{\omega(t)}g_0$ to the normalized Ricci flow* (3.3) *with $\mathscr{C} = \overline{R}$ with initial metric g_0 exists for all $t > 0$ and converges exponentially fast to a complete metric of constant negative curvature in its conformal class.*

(ii) *Suppose that $\partial_{\mathrm{F}}M \neq \emptyset$ and that g_0 is a metric on M satisfying* (1) *and* (2) *with $\varphi \in \mathcal{C}^4_{\mathrm{F,hc}}(M)$ satisfying* (3.1) *for some $\delta_{\mathrm{F}} > 2$ and $\delta_{\mathrm{hc}} > 0$. Then the solution $g(t) = e^{\omega(t)}g_0$ to the normalized Ricci flow* (3.3) *with $\mathscr{C} < 0$ and with initial metric g_0 exists for all $t > 0$ and converges exponentially fast to a complete metric of constant negative curvature in its conformal class.*

(iii) *If we assume that ${}^R\mathrm{Area}(g_0)\chi(M) < 0$ and that g_0 is a smooth metric as in (ii) but with $\delta_{\mathrm{F}} = 2$ (instead of $\delta_{\mathrm{F}} > 2$) ,then the same result holds with $\mathscr{C} = \overline{R}$.*

Proof. Statement (i) of the theorem is the result of Ji-Mazzeo-Šešum [18]. With the potential function of proposition 3.9, the proof of statement (ii) is basically the same as the one originally given by Hamilton [15] in the compact case and by [18] in the cusp case. We will repeat it for the convenience of the reader.

Thus, let g_0 be as in statement (ii) of the theorem and let $g(t)$ be the solution of (3.3) with $\mathscr{C} < 0$. Let $f(t)$ denote the potential function of (3.30) associated to the metric $g(t)$. As in [15], one computes that

$$-\Delta \frac{\partial f}{\partial t} = -\Delta(-\Delta f + \mathscr{C} f) \tag{3.46}$$

Now, if $\psi_i = c_i \log x_i$ with $c_i : [0,T] \to \mathbb{R}$ for $i \in \{1, \dots, n_{\mathrm{F}}\}$ are the functions such that

$$\widetilde{f} = f - \sum_{i=1}^{n_{\mathrm{F}}} \psi_i \in \mathcal{C}^{2,\alpha}_{\mathrm{F,hc}}(M),$$

one sees from (3.33) (3.18) and (3.10) that the evolution equation for ψ_i is $\frac{\partial \psi_i}{\partial t} = \mathscr{C}\psi_i$ so that from (3.46), we have

$$- \Delta \frac{\partial \widetilde{f}}{\partial t} = -\Delta(-\Delta \widetilde{f} + \mathscr{C}\widetilde{f} - \sum_{i=1}^{n_{\mathrm{F}}} \Delta\psi_i). \tag{3.47}$$

Since each term satisfies the Neumann boundary conditions at $Y_2, \ldots, Y_{n_{\mathrm{F}}}$ and the Dirichlet boundary condition at Y_1 (modulo a constant for $\Delta\psi_1$), we see by considering the corresponding equation on $\mathbb{M}_2$ that there exists a function $K : [0, T] \to \mathbb{R}$ such that

$$\frac{\partial \widetilde{f}}{\partial t} = -\Delta \widetilde{f} + \mathscr{C}\widetilde{f} - \sum_{i=1}^{n_{\mathrm{F}}} \Delta\psi_i + K(t), \quad \frac{\partial f}{\partial t} = -\Delta f + \mathscr{C} f + K(t). \tag{3.48}$$

The trick is then to consider the function

$$h = -\Delta_{g(t)} f + |\nabla f|^2_{g(t)}. \tag{3.49}$$

As in [15], one computes that

$$\partial_t h = -\Delta_{g(t)} h - 2|Z|^2 + \mathscr{C} h \leq -\Delta_{g(t)} h + \mathscr{C} h, \tag{3.50}$$

where Z is the trace-free part of the second covariant derivative of f. From the construction of the potential function f, the function $h(t)$ as an asymptotic value $h_i(t)$ in the end Y_i which is determined by the function ψ_i. This value is uniformly bounded in t (as are $r_i(t)$ in (3.10) and $c_i(t)$ in (3.11)). This means we can therefore apply the maximum principle to (3.50) to get that there exists a constant K such that $h_{\max}(t) \leq K e^{\mathscr{C} t}$ for all t. This implies that

$$R_t = h - |\nabla f|^2 + \mathscr{C} \leq K e^{\mathscr{C} t} + \mathscr{C}. \tag{3.51}$$

To get a lower bound for $R - \mathscr{C}$, suppose that the minimum of $R - \mathscr{C}$ on M becomes negative at a certain time t_0 (if not we get $R - \mathscr{C} \geq 0$ as a lower bound). If $R_{\min}$ is the minimum of the curvature at this time, then we have that $R_{\min} \leq \mathscr{C}$. By (3.10), the only way the curvature can blow up is if the minimum of R becomes very negative and is attained in the interior of M. In that case, from the evolution equation of R, we get that

$$\frac{d}{dt} R_{\min} \geq R_{\min}(R_{\min} - \mathscr{C}) \geq \mathscr{C}(R_{\min} - \mathscr{C}), \tag{3.52}$$

from which we deduce that $R_{\min} - \mathscr{C} \geq C e^{\mathscr{C} t}$ for some constant C. Combining with our upper bound, we see that

$$C e^{\mathscr{C} t} \leq R - \mathscr{C} \leq K e^{\mathscr{C} t}. \tag{3.53}$$

In particular, this gives an a priori bound on the curvature from which we get long-time existence. In fact, from (3.53), we also see that R converges to the constant $\mathscr{C}$ exponentially fast as $t \to +\infty$. Looking at covariant derivatives of the evolution equation of R in (3.6) and boostrapping gives the corresponding a priori estimates for all higher derivatives of R. Integrating the flow, this gives that $g(t)$ converges exponentially fast to a metric g_∞ with constant scalar curvature $\mathscr{C}$.

For statement (iii), we use instead the potential function of proposition 3.10 and proceed in a similar way. We leave the details to the reader. $\square$

4. Ricci flow and the determinant of the Laplacian

Given a F,hc metric g_0 on a non-compact surface M, we can now optimize the determinant of its Laplacian within its conformal class. For surfaces of finite area, the analysis of the determinant on closed surfaces in [27] applies easily to the determinant defined with renormalized traces. For infinite area surfaces the situation is more delicate, but also includes situations where the Euler characteristic is nonnegative. The main difficulty is the fact that the renormalized integral of a positive density need not be positive (e.g., Lemma 1.3). To deal with this, we shall have to impose restrictions on the value of the asymptotic curvature in the funnel ends of the metrics we consider. A second difficulty is that our definition of the determinant of the Laplacian for a F,hc metric in §2.1 requires that the factors e^{φ} in (1) and (2) be in $\mathcal{C}^\infty(\overline{M})$, since we make use of the constructions of the heat kernel in [1] and [35]. Thanks to propostion 3.7, this will be preserved along the flow provided φ is also in $\mathcal{C}^\infty_{\mathrm{F},\mathrm{hc}}(M)$.

Theorem 4.1. *Suppose g_0 is a F,hc metric with totally geodesic ends on a non-compact surface M satisfying (1) and (2) with $\varphi \in \mathcal{C}^\infty(\overline{M}) \cap \mathcal{C}^\infty_{F,\mathrm{hc}}(M)$. Assume that*

$$ {}^R\mathrm{Area}(g_0) = -2\pi\chi(M) \quad \textit{and} \quad r_i = -2 \textit{ for } i \in \{1,\dots,n_{\mathrm{F}}\}. \tag{4.1}$$

When $\chi(M) = 0$, assume further that φ satisfies (3.1) with $\delta_{\mathrm{F}} > 2$. Then among all F,hc metrics g with totally geodesic ends, conformal to g_0, and satisfying (4.1) the determinant of the Laplacian is greatest at the hyperbolic metric in this class.

Proof. Assume first that $\chi(M) \neq 0$. Consider normalized Ricci flow starting at g_0 with normalization constant

$$\mathscr{C} = \overline{R} = \frac{4\pi\chi(M)}{{}^R\mathrm{Area}(g_0)} < 0.$$

We know from Lemma 3.1 and Theorem 3.11 that, because of (4.1), this flow exists for all time, preserves the renormalized area, and converges to a hyperbolic metric. We know from proposition 3.5 that $\omega'(t) = \omega_0(t) + \widetilde{\omega}(t)$ with $\widetilde{\omega}(t) = \mathcal{O}(x_{\mathrm{F}}^2)$ and $\omega_0 \in \mathcal{C}^\infty([0,+\infty))$. We also know from proposition 3.7 that $\omega(t) \in \mathcal{C}^\infty(\overline{M}) \cap \mathcal{C}^\infty_{\mathrm{F},\mathrm{hc}}(M)$. Keeping in mind Lemma 3.1, we can therefore apply the Polyakov formula of Theorem 2.9 to get

$$\begin{aligned}\partial_t \det(\Delta_{g(t)}) &= -\frac{1}{24\pi} {}^R\!\!\int \omega'(t) R_t \, \mathrm{dA}_t \\ &= \frac{1}{24\pi} {}^R\!\!\int (R_t - \mathscr{C})^2 \, \mathrm{dA}_t + \frac{\mathscr{C}}{12\pi} {}^R\!\!\int (R_t - \mathscr{C}) \, \mathrm{dA}_t \\ &= \frac{1}{24\pi} {}^R\!\!\int (R_t - \mathscr{C})^2 \, \mathrm{dA}_t \,.\end{aligned} \tag{4.2}$$

Because of (4.1), this last integral does not need renormalization and is non-negative, hence the maximum value of $\det(\Delta_{g(t)})$ is obtained at the limiting value of $\omega(t)$, and the theorem is proved in this case.

When $\chi(M) = 0$, consider the normalized Ricci flow starting at g_0 with normalization constant

$$\mathscr{C} = -2.$$

We know from Theorem 3.11 that this flow exists for all time, stays within the class of metrics we are considering, and converges to a hyperbolic metric. Clearly, (4.2) still holds in this case since

$${}^R\!\!\int (R_t + 2) dg_t = 4\pi\chi(M) + 2^R \operatorname{Area}(g_t) = 0.$$

Since $R_t + 2 = \mathcal{O}(x)$, we can thus conclude as before that

$$\partial_t \det(\Delta_{g(t)}) = \frac{1}{24\pi} {}^R\!\!\int (R_t - \mathscr{C})^2 \, \mathrm{dA}_t \geq 0.$$

Finally, since the functional $F(\omega)$ in (2.18) is concave when ω and $\nabla\omega$ are in L^2, notice that this implies the maximum of the determinant is unique and confirms that among all $F,$hc metrics g conformal to g_0 and satisfying the hypotheses of the theorem, there is a unique hyperbolic metric. $\square$

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