Astron. & Astrophys. 44, 417—420 (1975)

Relativistic Spin Precession in Two-body Systems

G. Börner, J. Ehlers and E. Rudolph

Max-Planck-Institut für Physik und Astrophysik, München

Received May 6, revised August 29, 1975

Summary. The spin precession of each component of a two-body system is calculated by means of the parametrized post-Newtonian approximation to metric theories of gravity without preferred frame effects. This calculation is carried out in view of an application to the recently discovered binary pulsar PSR 1913+16. For the case of general relativity we find, in agreement with Barker and O'Connell who used nonclassical methods, that in the case of arbitrary masses the spin precession rate of the first body is $\frac{M_2+\mu/3}{M_1+M_2}$ times what it would be if the first body were a test particle moving in the field of a fixed body with mass M_1+M_2, μ being the reduced mass.

Key words: binary pulsar — relativistic spin precession

1. Introduction and Discussion

The discovery of a pulsar in a binary system (PSR 1913+16) opens up new and exciting possibilities for observing relativistic effects and of testing theories of gravity. Recently, several authors have computed the value of the spin precession rate in a binary system using quantum field theoretical techniques such as graviton exchange. Since the effect concerns a macroscopic system, it should also be possible to derive it straightforwardly from classical general relativity. (The desirability of such a derivation – which turns out to be much simpler than the field-theoretic one – has also been emphasized by Barker and O'Connell.) Actually, it is just as easy to do the calculation not only for Einstein's theory, but for any metric theory of gravity which has no preferred frames (as defined in MTW[1]) and the references given there), using the parametrized post-Newtonian formalism. This method shows that an observation of the precession rate would provide the value of the same PPN-parameter γ which is measured by light deflection and radar time delay observations. This space curvature parameter γ equals 1 in Einstein's theory, and is $\frac{1+\omega}{2+\omega}$ in the Dicke-Brans-Jordan theory (see, e.g., MTW p. 1072, box 39.2).

As usual we distinguish between the orbital angular momentum of a body and its angular momentum due to the rotation around its own center of mass which we call spin. The spin precession rate $\boldsymbol{\Omega}$ of a body not subject to any torque, with respect to coordinate axes comoving with the body and oriented relatively to distant matter (for details, see MTW §§ 39.10 and 39.12 and Section 2 below), has been derived generally within the post-Newtonian approximation to metric theories of gravity assuming that

(A) the spin 4-vector S^μ is Fermi-Walker transported along the world line of the center-of-mass of the body.

In the case of a freely falling body (i.e., if the CM world line is a geodesic) and assuming the absence of preferred frame terms one has, according to MTW Eq. (40.33):

$$\frac{d\boldsymbol{S}}{dt}=\boldsymbol{\Omega}\times\boldsymbol{S}, \tag{1}$$

$$\boldsymbol{\Omega}=(\gamma+1/2)\boldsymbol{u}\times\nabla U+(\gamma+1)\nabla\times\boldsymbol{V}, \tag{2}$$

where

$$U(\boldsymbol{x},t)=G/c^2\int\frac{dm'}{|\boldsymbol{x}-\boldsymbol{x}'|} \tag{3}$$

$$\boldsymbol{V}(\boldsymbol{x},t)=G/c^2\int\frac{\boldsymbol{v}(\boldsymbol{x}',t)dm'}{|\boldsymbol{x}-\boldsymbol{x}'|}. \tag{4}$$

In these equations, c is the vacuum speed of light, G the Newtonian constant of gravity, $(\boldsymbol{x},t)=(x^i,t)$ are quasi-inertial, standard PPN coordinates (see MTW § 39.8), γ is the space curvature parameter, $\boldsymbol{u}$ is the CM velocity of the body with spin $\boldsymbol{S}\cdot\boldsymbol{V}$ is the velocity field within the bodies of the system, and dm' may for our purposes be identified with the Newtonian mass element at $(\boldsymbol{x}',t)$. The potentials U and $\boldsymbol{V}$ are to be computed by integration over all bodies belonging to the isolated system under consideration.

In (1) and (2) all quantities are to be taken along the CM world line of the body in question.

(MTW use the vector field $\boldsymbol{g}=-\frac{7}{2}\Delta_1\boldsymbol{V}-\frac{1}{2}\Delta_2\boldsymbol{W}$ with $\boldsymbol{W}$ defined in their Eq. (39.23d). It is easily seen, however,

[1]) C. W. Misner, K. S. Thorne, J. A. Wheeler, Gravitation, San Francisco 1973, here and hereafter quoted as MTW.

that

$$W = V - \nabla P \quad \text{with} \quad P = \int \frac{dm' \boldsymbol{v} \cdot (\boldsymbol{x} - \boldsymbol{x}')}{|\boldsymbol{x} - \boldsymbol{x}'|}.$$

Therefore $\nabla \times \boldsymbol{g} = -\frac{1}{2}(7\Delta_1 + \Delta_2)\nabla \times \boldsymbol{V}$. Moreover, in theories without a preferred universal frame one has $7\Delta_1 + \Delta_2 = 4\gamma + 4$ (MTW p. 1098). These facts lead to the simple formula (2) which covers, in particular, the Einstein and Dicke-Brans-Jordan theories. – It is worth emphasizing that (1) presupposes the vanishing of the *total* torque acting on the "gyroscope", not just the one due to non-gravitational forces.)

In the case of a binary system the relativistic spin precession rate of, say, the first body consists of three contributions associated with:

a) the motion of the first body around the center of mass of the system (deSitter-Fokker precession),

b) the spin-spin interaction between the two bodies, and

c) the dragging of the inertial frame at the first body due to the translational motion of the second body.

b) and c) are described by the vector potential $\boldsymbol{V}$. According to (2), $(2\gamma + 2)\,\boldsymbol{V}$ can be visualized as the velocity field of a fictitious ether which drags along the spin $\boldsymbol{S}$. (4) shows to which extent this "ether" is dragged along by matter. This picture further illustrates the role of $\gamma + 1$ as a *dragging coefficient* (see also MTW pp. 1119, 1120). The self-fields of the first body should be included (see Section 2) but as one would expect, they do not contribute, at this low order of approximation. Because of contribution c) a naive generalization of the result obtained in the case $M_1 \ll M_2$ leads to a wrong result.

Equation (1) holds only if the Newtonian torque due to the coupling of the quadrupole moment of the first body to the tidal field due to the second body is negligible compared to the right-hand-side of Eq. (1). For a pulsar in a system similar to PSR 1916+13 this is indeed the case: $\left(\frac{d\boldsymbol{S}}{dt}\right)_{\text{Newton}} = c^2 \int_{V_1} (\boldsymbol{x} \times \nabla U)\, dm$, gives for the quadrupole interaction $\approx G M_2 M_1 R_1 \delta R_1 / a^3$, where a is the distance between the two stars, and δR_1 is a measure of the slightly oblate shape of body 1. For a typical pulsar an upper limit to $\delta R_1 / R_1$ is (Ruderman, 1969):

$$\delta R_1 / R_1 \approx 10^{-5}.$$

Then using Eq. (12) for an order-of-magnitude estimate (inserting $M_1 = M_2 = 1 M_\odot$, $a = R_\odot$), we find

$$\frac{(dS/dt)_{\text{Quad}}}{(dS/dt)_{\text{rel}}} \approx \frac{\delta R_1}{R_1} \approx 10^{-5}.$$

2. Remarks about the Derivation and the Interpretation of the Spin Precession Formula

Usually (e.g. MTW § 40.7, S. Weinberg (1972), § 9), the spin precession formula is derived from the assumption *A* stated in the introduction. For a test body (i.e., a body with negligible gravitational self field) this assumption, as well as the geodesic motion of the center of mass, follow from Dixon's equations of motion for an extended body in a multipole approximation, providing that the influence of the spin on the orbit and tidal effects are neglected. A star is not a test body, however, and the application of (1)–(4) to stars requires a more detailed dynamical justification. Dixon (1974) has derived exact equations of motion for momentum and spin without a multipole approximation using the total field $g_{\mu\nu}$ (including the self field). Unfortunately, it is not at all obvious under which conditions his equations reduce to the geodesic law and to assumption A. Another possibility would be to derive these laws from the PPN equations of motion [see, e.g., MTW § 39.11 or, in the case of general relativity, Fock (1959) or Chandrasekhar (1965)]. In this paper we shall not attempt to give a derivation, but instead *assume* the validity of the geodesic law and assumption A *even for a star*.

The two main effects producing non-geodesic motion are the spin-orbit coupling, which appears already in Einstein's theory, and the Nordvedt effect of the scalar-tensor theory. Both of these effects require one to use Fermi transport instead of parallel transport of the spin vector. The error committed by disregarding these effects amounts to less than 1% even in extreme cases, as can be seen by using formulae (40.27), (40.33), (40.55) of MTW for an estimate of the influence of the Nordvedt effect and formula (40.49) of MTW for an estimate of the spin-orbit coupling (the parameters of the binary pulsar system (Barker and O'Connell, 1975) give an effect of $\leqq 1\%$). The spin precession formula which we obtain by neglecting these two effects is completely sufficient in the case of General Relativity. In addition we want to point out in this paper that the measurement of the spin precession could provide another method to determine the PPN-parameter γ. Such a determination of γ with an accuracy of 1% or better, however, requires to include the Nordvedt effect and the spin-orbit coupling. We leave an inclusion of these effects to future investigations.

A second point deserving a comment is the definition and physical meaning of the comoving coordinate-axes to which Eq. (1) refers. The standard PPN coordinates (x^i, x^0) are well defined throughout the system and are asymptotically inertial "at infinity". MTW attach an orthonormal basis field $\{e_{\tilde{\lambda}}\}$ to this coordinate system by taking $e_{\tilde{0}}$ tangent to the x^0-lines, and choosing $e_{\tilde{i}}$ tangent to the (x^i, x^0)-coordinate surface. Then they choose that comoving frame $\{e_{\hat{\lambda}}\}$ along the CM worldline of the spinning body which is related to $\{e_{\tilde{\lambda}}\}$ by a Lorentz boost, so that the transformation $e_{\tilde{\lambda}} \to e_{\hat{\lambda}}$ does not contain a spatial rotation. $\boldsymbol{\Omega}$ given by (2) is the angular velocity with which Fermi-transported spatial axes rotate relatively to the triad $\{e_{\hat{i}}\}$. Although these prescriptions give an unambiguous mathematical definition of $\boldsymbol{\Omega}$ they do not specify how $\boldsymbol{\Omega}$ could be measured by a comoving observer, since it is not clear how the $e_{\hat{\imath}}$ can be characterized physically. If the motion of our double star system were periodic (as it is in the Newtonian approximation) with (coordinate-time) period T, then the transformations $(x^i, x^0) \to (x^i, x^0 + kT)$, k integer, would be isometries of the space-time metric, and although the direction of the light from a star at infinity as judged by $\{e_{\hat{\imath}}\}$ would not be exactly constant during one period because of the varying light deflection caused by the system's stars, this direction would change periodically only. Therefore, the axes $e_{\hat{\imath}}$ would be fixed relative to the light from distant stars on the average, and consequently would measure the precession of $\boldsymbol{S}$ relative to the directions of distant stars over times large compared to one period. This interpretation can be generalized to the case of a non-periodic motion (perihelion shift). This consideration still does not show how $\boldsymbol{\Omega}$ could be measured by a distant observer. In order to analyse that one would have to relate $\boldsymbol{S}$ to the configuration of the body and, in the case of a pulsar to the light which the latter emits. We leave such an analysis to further investigation.

3. The Spin Precession Rate in a Binary System

In a two-body system the potentials (3) and (4) naturally consist of contributions due to the two bodies,

$$U = U_1 + U_2 , \qquad V = V_1 + V_2 \tag{5}$$

We assume that, at each instant, each body is axially symmetric and has an "equatorial" plane of symmetry orthogonal to the axis. Moreover we assume that the mass distribution and the velocity field within each body, $\boldsymbol{w}_i(\boldsymbol{x}, t)$, share the symmetry of the body and that $\boldsymbol{w}_i$ describes a rotation of each element of the body around the common axis of symmetry. Then $\boldsymbol{v}_i(\boldsymbol{x}, t) = \boldsymbol{u}_i(t) + \boldsymbol{w}_i(x, t)$ and

$$V_i = U_i \boldsymbol{u}_i + \frac{G}{c^2} \int \frac{\boldsymbol{w}_i' dm'}{|\boldsymbol{x} - \boldsymbol{x}'|} = U_i \boldsymbol{u}_i + \boldsymbol{W}_i . \tag{6}$$

Outside the second body we have therefore

$$V_2 = U_2 \boldsymbol{u}_2 + \frac{G}{2c^2} \frac{\boldsymbol{S}_2 \times (\boldsymbol{x} - \boldsymbol{x}_2)}{|\boldsymbol{x} - \boldsymbol{x}_2|^3} , \tag{7}$$

where $\boldsymbol{S}_2$ is the spin of that body, $\boldsymbol{x}_2$ is the position of its center of mass, and higher order multipole terms have been neglected.

According to (2) and (5) we obtain for the spin precession rate of the first body:

$$\boldsymbol{\Omega} = (\gamma + \tfrac{1}{2}) \boldsymbol{u}_1 \times (\nabla U_1 + \nabla U_2) + (\gamma + 1) \nabla \times (V_1 + V_2) , \tag{8}$$

where all expressions are to be taken at $\boldsymbol{x}_1$, the CM position of the first body.

The self-field terms of body 1 do not contribute because of our symmetry assumptions: Both $\nabla U_1(\boldsymbol{x}_1)$ and $\nabla \times V_1(\boldsymbol{x}_1)$ must be invariant under rotations about the symmetry axis of body 1, and $\nabla U_1(\boldsymbol{x}_1)$ must in addition be reflection-invariant. Hence, $\nabla U_1(\boldsymbol{x}_1) = 0$, and $\nabla \times V_1(\boldsymbol{x}_1)$, being proportional to $\boldsymbol{S}_1$, does not contribute to (1) and can thus be dropped from (8). It remains to evaluate

$$\boldsymbol{\Omega} = (\gamma + \tfrac{1}{2}) \boldsymbol{u}_1 \times \nabla U_2 + (\gamma + 1) \nabla \times V_2 . \tag{9}$$

For this purpose we take the center of mass of the system as the origin so that, with the notation $\boldsymbol{R} = \boldsymbol{X}_1 - \boldsymbol{X}_2$, $\boldsymbol{V} = \dot{\boldsymbol{R}}$, $R = |\boldsymbol{R}|$ one has $\boldsymbol{x}_1 = \dfrac{M_2}{M_1 + M_2} \boldsymbol{R}$ etc. Since far from the second body, $U_2 = \dfrac{GM_2}{c^2 |\boldsymbol{x} - \boldsymbol{x}_2|}$, we obtain, using (7) and (9):

$$\boldsymbol{\Omega} = \frac{G\mu}{c^2} (\gamma + 1 + [\gamma + 1/2] M_2/M_1) \frac{\boldsymbol{R} \times \boldsymbol{v}}{R^3} + \frac{(\gamma + 1)G}{2c^2 R^3} \left(\frac{3(\boldsymbol{S}_2 \cdot \boldsymbol{R})\boldsymbol{R}}{R^2} - \boldsymbol{S}_2 \right) , \tag{10}$$

where $\mu = \dfrac{M_1 M_2}{M_1 + M_2}$ is the reduced mass.

The last term, which gives the spin-spin interaction, decreases more rapidly with R^{-1} than the usually dominating first term.

For an elliptical Keplerian orbit $\boldsymbol{R} \times \boldsymbol{V}$ is conserved and normal to the orbital plane, parallel to the unit vector $\boldsymbol{n}$, say:

$$\boldsymbol{R} \times \boldsymbol{v} = (a(1 - e^2) G(M_1 + M_2))^{1/2} \boldsymbol{n} \tag{11}$$

a and e are the semimajor axis and the numerical excentricity, respectively, of the ellipse. Averaging over one period, neglecting the spin-spin interaction, and using $\langle R^{-3} \rangle = a^{-3}(1 - e^2)^{-3/2}$, we obtain

$$\langle \boldsymbol{\Omega} \rangle = \frac{G^{3/2} \mu}{c^2} \left(\gamma + 1 + [\gamma + 1/2] \frac{M_2}{M_1} \right) \frac{(M_1 + M_2)^{1/2}}{a^{5/2}(1 - e^2)} \boldsymbol{n} . \tag{12}$$

For the case of general relativity, $\gamma = 1$, this result agrees with the one obtained by Barker and O'Connell (1975) and by Dass and Cho (1975) who both used the theory of interacting spin 2 fields.

The possibility of observing such an effect in the binary pulsar PSR 1913 as a secular change in pulse shape remains to be discussed.

Acknowledgements. We would like to thank Dr. Hari Dass and Prof. F. Dyson for bringing this problem to our attention. Dr. Hari Dass also suggested that one might use the spin precession of the binary pulsar to test general relativity.

Note added in proofs. The result (12) can be rewritten in a form which is perhaps somewhat more elucidating. If ω denotes the rate of the periastron advance and $\nu = \dfrac{M_2}{M_1}$ is the mass ratio, then

$$\langle \boldsymbol{\Omega} \rangle = (2 + 2\gamma - \beta)^{-1} (\gamma + 1 + [\gamma + \tfrac{1}{2}]\nu) \frac{\nu}{(1 + \nu)^2} \omega .$$

Hence, measurement of both ω and $\langle \boldsymbol{\Omega} \rangle$ for fixed PPN-parameters γ, β gives the mass ratio.

References

Barker, B. M., O'Connell, R. F. 1975, The Gravitational Two Body Problem with Arbitrary Masses, Spins and Quadrupole Moments, Prepr., Lousiana State University

Barker, B. M., O'Connell, R. F. 1975, Relativistic Effects in the Binary Pulsar PSR 1913+16, preprint, Lousiana State University

Barker, B. M., O'Connell, R. F. 1975, *Astron. J.* **199**, L25

Chandrasekhar, S. 1965, *Astron. J.* **142**, 1488

Cho, C. F., Hari Dass, N. D. 1975, Gravitational two body problem — a source theory view point, preprint MPI Munich

Hari Dass, N. D., Radhakrishnan, V. 1975, The New Binary Pulsar and the Observation of Gravitational Spin Precession, preprint MPI, Munich

Dixon, W. G. 1974, Dynamics of extended bodies in general relativity, III, Equations of motion, *Phil. Trans. Roy. Soc. London* A**277**, 59–119

Fock, V. 1959, The Theory of Space, Time and Gravitation

Misner, C. W., Thorne, K. S., Wheeler, J. A. 1973, Gravitation

Ruderman, M. 1969, *Nature* **223**, 597

Weinberg, S. 1972, Gravitation and Cosmology

G. Börner
J. Ehlers
E. Rudolph
Max-Planck-Institut für Physik und Astrophysik
D-8000 München 40, Föhringer Ring 6
Federal Republic of Germany