Effects of curvature couplings for quantum fields on cosmic-string space-times

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(Received 15 January 1990)

The effect of ξR curvature coupling terms in the equation of motion of a massless scalar field are considered on a cosmic-string background. Realistic two- and four-dimensional models of a string, where the space-time curvature is spread over a region of nonzero size r_0, are constructed. In two and four dimensions, the nonzero ξ term modifies the behavior within a region centered about the string, whose radius is proportional to r_0; however, the constant of proportionality may be exponentially large. For the standard idealized string (a conical space-time) which has no intrinsic length scale, the effects of nonzero ξ do not appear.

I. MOTIVATION

In the space-time around a cosmic string,[1] the curvature is concentrated in a region around the string, and vanishes outside the string. Indeed, for the standard idealized model of a string, the space-time is conical, being flat everywhere apart from a δ-function singularity in the curvature on the tip of the cone. Although the curvature is concentrated on the string, its effects are felt far away, for example, in the focusing of geodesics. This paper is concerned with the effects of the curvature on the behavior of a quantum field in the cosmic-string space-time. The usual equation of motion of a massless scalar field

$$(\Box-\xi R)\varphi=0 \tag{1}$$

contains a coupling to the scalar curvature R, whose strength is determined by the dimensionless coupling constant ξ. It is not evident whether the effects of this coupling term on the quantum theory are limited to a region around the string itself, or whether they alter the behavior of the fields at large distances from the string. Many of the standard treatments of quantum fields around an idealized cosmic string[2–7] deal primarily with the case $\xi=0$, "since the support of R is a set of measure 0." In a number of these papers, it is assumed, for example, that the two-point function is independent of ξ. (A nice summary of existing work on quantum field theory around cosmic strings can be found in Ref. 8.)

A simple way to see the dependence on ξ is to consider the Born expansion of the Feynman two-point function, which obeys the equation of motion

$$(\Box-\xi R)G(x,y;\xi)=-\delta(x,y)\ . \tag{2}$$

Differentiating this equation with respect to ξ, multiplying by $G(x,z;\xi)$, and integrating over x one obtains

$$\frac{d}{d\xi}G(y,z;\xi)=-\int dx\, G(y,x;\xi)R(x)G(x,z;\xi)\ , \tag{3}$$

where the integral is over the volume of space-time. Repeatedly differentiating Eq. (3) with respect to ξ, and setting $\xi=0$, one obtains the Born expansion

$$\begin{aligned}G(y,z;\xi)=G(y,z;0)&-\xi\int dx\, G(y,x;0)R(x)G(x,z;0)\\&+\xi^2\int dx\int dx'G(y,x;0)R(x)G(x,x';0)\\&\times R(x')G(x',z;0)-\cdots\ .\end{aligned} \tag{4}$$

This equation can be interpreted as expressing the amplitude for scattering from y to z as a sum over all paths that scatter n times off the curvature, where the coefficient of ξ^n represents the amplitude for n scatterings taking place.[9]

The standard model for a cosmic-string space-time is the conical metric

$$ds^2=dt^2+dz^2+dr^2+r^2d\phi^2\ ,$$

where the angular range is $\phi\in[0,\alpha)$. Throughout this paper we shall work with a positive-definite metric; this presents no problems as the space-time is static. The standard two-point function for a massless scalar field on this space is[2]

$$G(x,x')=\frac{1}{8\pi^2}\frac{\kappa\sinh\kappa\eta}{rr'\sinh\eta(\cosh\kappa\eta-\cos\kappa\Delta\phi)}\ , \tag{5}$$

where $\kappa=2\pi/\alpha$, and

$$\cosh\eta=\frac{\Delta t^2+\Delta z^2+r^2+r'^2}{2rr'}\ , \tag{6}$$

with $\Delta t=t-t'$ and likewise for ϕ and z. We remain intentionally vague at this stage about the value(s) of ξ for which Eq. (5) holds.

Evidence that $G(x,x';\xi)$ is not independent of ξ may be obtained by inserting G into Eq. (3). The scalar curvature is given by

$$R=2(\kappa-1)\delta(r+0)/r\ ,$$

where the argument of the δ function is infinitesimally in-

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cremented to remove the end-point ambiguity. One obtains

$$\frac{d}{d\xi}G(x,x')=-\frac{1}{4\pi^2}\kappa(\kappa-1)\frac{\eta}{rr'\sinh\eta} .$$

The left-hand side is nonzero because the standard two-point function does not vanish at the tip of the cone, where $r=0$.

Further evidence that G depends upon ξ may be obtained as follows. Consider a small cylindrical four-volume of radius ε centered above the string. Place the point y outside this volume, and integrate the left-hand side of Eq. (2) over the volume. Integrating the $\Box G$ term by parts, one obtains a boundary term proportional to ε. The ξRG term integrates to give an ε-independent term. Sending ε to zero one obtains the relation

$$\xi G(r=0;r';\Delta t,\Delta z,\Delta\phi)=0 .$$

Thus one is forced to conclude that the two-point function given by (5) could only be correct for $\xi=0$, and that the two-point function must vanish at the origin if ξ does not vanish. Note that $(d/d\xi)G(r=0;r';\Delta t,\Delta z,\Delta\phi)$ is infinite. [Note furthermore that this argument implies that $G(x,x';\xi)$ is not an analytic function of ξ at $\xi=0$ and thus that the expansion in powers of ξ given in (4) may not be trustworthy.]

In the preceding argument, we assume that the gradient of G in the vicinity of the string singularity diverges more slowly than $1/\varepsilon$. That this is indeed the case may be seen either by differentiating the expression for G directly, or else by examining the mode-sum form of the two-point function (23).

If the two-point function depends upon ξ, then certain physical quantities obtained from G will also depend upon ξ. Using Eq. (5) as the two-point function for $\xi=0$ in Eq. (3) we obtain, for example,

$$\frac{d}{d\xi}\langle\hat{\varphi}^2\rangle\bigg|_{\xi=0}=-\frac{1}{4\pi^2}\kappa(\kappa-1)\frac{1}{r^2} ,$$

where the expectation value of the field operator squared has been calculated by the standard Hadamard method.[10]

II. REALISTIC MODELS FOR THE SPACE-TIME OF A COSMIC STRING

The idealized model of a cosmic-string space-time, in which the curvature is concentrated at the tip of a cone, is correct when one is far from a local string. However the string has a characteristic core radius given by $r_0\approx 1/M$, where M is the mass scale that characterizes the symmetry-breaking scale at which the string is formed.[11] For grand-unified-theory-(GUT-) scale strings one has $r_0\approx 10^{-30}$ cm. The curvature of an infinite straight cosmic string is actually spread out over a cylindrical region whose radius is of the order of r_0. On these length scales the tip of the conical singularity is not a point, but rather is a smooth cap.[12]

The metric of a conical space-time with a smooth cap can be written in the form

$$ds^2=dt^2+dz^2+P^2(r)dr^2+r^2d\phi^2 , \tag{7}$$

where the range of the angular coordinate is $\phi\in[0,\alpha)$. The function $P(r)$ has the property that

$$\lim_{r/r_0\to 0}P(r)=\alpha/2\pi=1/\kappa \quad\text{and}\quad \lim_{r/r_0\to\infty}P(r)=1 .$$

The first condition states that there is no conical singularity at $r=0$, and the second condition means that for large r the cone has a deficit angle $(2\pi-\alpha)$. The function P should be a smooth monotonic function, and the condition that the curvature be concentrated in a region of radius r_0 about the string implies that all of the derivatives of $P(r)$ should be small outside that region.

The curvature in this space-time is given by

$$R_{abcd}=2R(\phi_{[a}r_{b]}\phi_{[c}r_{d]}) , \tag{8a}$$

$$R_{ab}=\tfrac{1}{2}R(\phi_a\phi_b+r_ar_b) , \tag{8b}$$

$$R=\frac{2}{r}\frac{P'(r)}{P^3} , \tag{8c}$$

where a prime denotes d/dr, and we have introduced the obvious orthonormal tetrad. The Gauss-Bonnet theorem is easily verified, since the curvature integrated over a two-dimensional surface $t=z=0$ is

$$\int R\sqrt{{}^{(2)}g}\,d^2x=\int RPr\,d\phi\,dr=2\alpha(-1/P)_0^\infty$$
$$=4\pi-2\alpha ,$$

independent of the form of $P(r)$.

The wave operator in the space-time is given by

$$\Box\varphi=\left[\frac{\partial^2}{\partial t^2}+\frac{\partial^2}{\partial z^2}+\frac{1}{r^2}\frac{\partial^2}{\partial\phi^2}+\frac{1}{rP}\frac{\partial}{\partial r}\frac{r}{P}\frac{\partial}{\partial r}\right]\varphi . \tag{9}$$

In this paper we shall consider two models corresponding to particular choices of the function P. In the flower-pot (F) model, the curvature of space-time is concentrated on a ring of radius r_0. This corresponds to the choice of P

$$P(r)=\begin{cases}\dfrac{\alpha}{2\pi} & r<r_0-\epsilon ,\\ 1, & r>r_0+\epsilon ,\end{cases} \tag{10}$$

for ϵ/r_0 infinitesimal. The function $P(r)$ is assumed to vary smoothly in the region $|r-r_0|<\epsilon$. In the limit as $\epsilon\to 0$ the scalar curvature R approaches $2(k-1)\delta(r-r_0)/rP$.

In the "ballpoint pen" (B) model, the space-time is flat for $r>r_0$, and the curvature is constant in the interior region $r<r_0$. This model, proposed independently by Hiscock and Gott,[12] corresponds to replacing the conical singularity by a smooth spherical cap which is tangent to the cone at $r=r_0$. For this model one has

$$P(r)=\begin{cases}\left[\dfrac{r^2}{r_0^2}(1-\kappa^2)+\kappa^2\right]^{-1/2}, & r<r_0 , \quad (11)\\ 1, & r\geq r_0 . \quad (12)\end{cases}$$

The scalar curvature vanishes for $r>r_0$ and has a con-

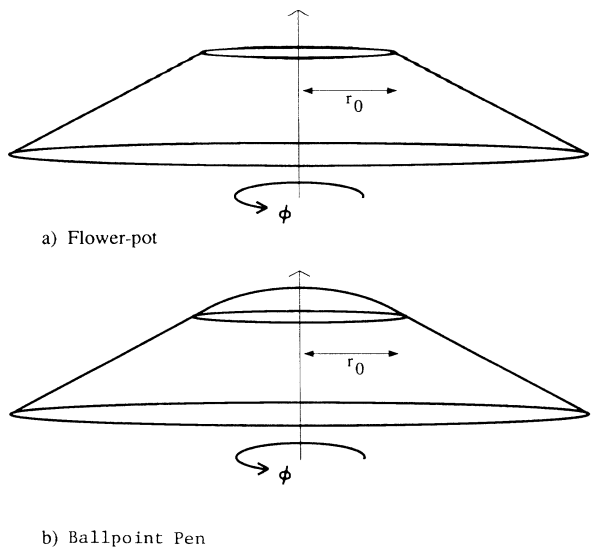

FIG. 1. These diagrams show two-dimensional projections of two models of a cosmic-string space-time. The axis of rotational symmetry is the vertical line about which the polar angle ϕ is measured. (a) In the "flower-pot" (F) model, the curvature is a δ function, concentrated on a ring of radius r_0. (b) In the "ballpoint pen" (B) model, the curvature is smoothly distributed on part of a sphere, whose radius is chosen so that it is tangent to the cone on a ring of radius r_0.

stant value $R=2(\kappa^2-1)/r_0^2$ for $r<r_0$. These two models of the space-time around a cosmic string are shown in Fig. 1.

III. GREEN FUNCTIONS ON FOUR-DIMENSIONAL MODELS

We begin by expressing the four-dimensional δ function as the Fourier sum

$$\delta(x,x')=\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}e^{i\omega\Delta t}\int_{-\infty}^{\infty}\frac{dk}{2\pi}e^{ik\Delta z}\times\sum_{n=-\infty}^{\infty}\frac{1}{\alpha}e^{in\kappa\Delta\phi}\frac{\delta(r-r')}{rP(r)}\ . \quad (13)$$

Using standard methods to expand the two-point function for the elliptic boundary-value problem $(\Box-\xi R)G=-\delta$ one obtains the expression

$$G(x,x')=\int\frac{d\omega}{2\pi}e^{i\omega\Delta t}\int\frac{dk}{2\pi}e^{ik\Delta z}\times\sum_{n=-\infty}^{\infty}\frac{1}{\alpha}e^{in\kappa\Delta\phi}\Psi_<^{(\omega kn)}(r_<)\Psi_>^{(\omega kn)}(r_>)\ , \quad (14)$$

where $r_<=\min(r,r')$ and $r_>=\max(r,r')$. The functions $\Psi_<(r)$ are regular as $r\to 0$ and the functions $\Psi_>(r)$ fall off as $r\to\infty$. Acting on the expression for G with the wave operator (9), one obtains the equation satisfied by Ψ (note that for notational simplicity we drop the ωkn superscript that labels Ψ):

$$\left[\frac{1}{rP}\frac{d}{dr}\frac{r}{P}\frac{d}{dr}-\omega^2-k^2-\frac{n^2\kappa^2}{r^2}-\frac{2\xi}{r}\frac{P'}{P^3}\right]\circ\Psi_<(r_<)\Psi_>(r_>)=\frac{-\delta(r-r')}{rP}\ . \quad (15)$$

If we multiply this equation by $rP(r)$ and integrate from $r=r'-\epsilon$ to $r=r'+\epsilon$ we obtain the Wronskian normalization condition

$$-1=\left[\frac{r}{P}\frac{d}{dr}\Psi_<(r_<)\Psi_>(r_>)\right]_{r=r'-\epsilon}^{r=r'+\epsilon}$$

$$\Longrightarrow\Psi_>'(r)\Psi_<(r)-\Psi_<'(r)\Psi_>(r)=\frac{-P(r)}{r}\ . \quad (16)$$

This condition determines the normalization of the product $\Psi_<(r)\Psi_>(r)$. Note that if r is greater than r_0, then $P(r)=1$.

In the case of the "flower pot," the function $\Psi_<$ is determined by choosing the solution of Eq. (15) which is well behaved at $r=0$ and integrating it out. The ξR term in the equation of motion is nonzero only in the infinitesimal region $|r-r_0|<\epsilon$. In the limit as ϵ vanishes, the effect of this ξR term can be seen by integrating the equation for $\Psi_<$ through the point $r=r_0$. One obtains the relation

$$\left[\frac{r}{P}\frac{d}{dr}\Psi_<+2\xi\frac{\Psi_<}{P}\right]_{r=r_0-\epsilon}^{r=r_0+\epsilon}=0 \quad (17)$$

which implies that the mode function $\Psi_<$ has a discontinuity in its slope at $r=r_0$.

The solutions of Eq. (15) are Bessel functions. Since the Bessel function $I_\nu(x)$ vanishes as $x\to 0$ and blows up as $x\to\infty$, and the Bessel function $K_\nu(x)$ falls off exponentially as $x\to\infty$ and blows up as $x\to 0$, one obtains as solutions for the "interior" mode functions

$$\Psi_<(r)=\begin{cases} I_{|n|}(sr/\kappa) & \text{for } r<r_0\ ,\\ AI_{\kappa|n|}(sr)+BK_{\kappa|n|}(sr) & \text{for } r>r_0\ ,\end{cases} \quad (18)$$

where $s^2=\omega^2+k^2$. The ratio of the constants $C=B/A$ is determined by the jump condition (17) to be $C(sr_0,n,\xi)$, where

$$C(x,n,\xi)=\frac{xI'_{\kappa|n|}(x)I_{|n|}(x/\kappa)-xI_{\kappa|n|}(x)I'_{|n|}(x/\kappa)-2\xi(\kappa-1)I_{\kappa|n|}(x)I_{|n|}(x/\kappa)}{xK_{\kappa|n|}(x)I'_{|n|}(x/\kappa)-xK'_{\kappa|n|}(x)I_{|n|}(x/\kappa)+2\xi(\kappa-1)K_{\kappa|n|}(x)I_{|n|}(x/\kappa)}\ . \quad (19)$$

The solutions for the "exterior" mode functions are determined by the condition that they fall off when $r\to\infty$. Together with the normalization condition (16) this yields

$$\Psi_>(r)=\frac{1}{A}K_{\kappa|n|}(sr)\quad\text{for } r>r_0\ . \quad (20)$$

The two-point function on the four-dimensional flower pot is now given by the expression

$$G_F(x,x')=\int \frac{d\omega}{2\pi}e^{i\omega\Delta t}\int \frac{dk}{2\pi}e^{ik\Delta z}\sum_{n=-\infty}^{\infty}\frac{1}{\alpha}e^{in\kappa\Delta\phi}K_{\kappa|n|}(sr_>)[I_{\kappa|n|}(sr_<)+C(sr_0,n,\xi)K_{\kappa|n|}(sr_<)] \ . \tag{21}$$

Note that in this expression it is assumed that both r and r' are greater than r_0. The only dependence upon ξ and r_0 here is through the function C. The two-point function of the conical space-time with $\xi=0$ can be obtained from this expression by setting $C=0$.

To understand the effects of the coupling to curvature, we consider C for small r_0. For this purpose it is convenient to use the relation that $\pi J_0(z)=\int_0^{\pi}\exp(iz\cos\theta)d\theta$, and to change variables to $v=r_< s$. One thus obtains

$$G_F(x,x')=\frac{1}{2\pi\alpha r_<^2}\int_0^{\infty}v\,dv\,J_0\left[\frac{\sqrt{\Delta t^2+\Delta z^2}\,v}{r_<}\right]\sum_{n=-\infty}^{\infty}e^{in\kappa\Delta\phi}K_{\kappa|n|}\left[v\frac{r_>}{r_<}\right][I_{\kappa|n|}(v)+C(\rho v,n,\xi)K_{\kappa|n|}(v)] \ , \tag{22}$$

where $\rho=r_0/r_<$. Because the factor of $K_{\kappa|n|}(v)$ falls off exponentially quickly for $v>\kappa|n|$, one can expand this expression for small ρ to explore the behavior of the two-point function far away from the cosmic string. Expanding $C(\rho v,n,\xi)$ for small values of ρ one finds that

$$C(\rho v,n,\xi)\sim\begin{cases}-(\rho v/2)^{2\kappa|n|}\dfrac{2\xi(\kappa-1)}{\kappa|n|+\xi(\kappa-1)}\dfrac{1}{\Gamma(\kappa|n|+1)\Gamma(\kappa|n|)} & \text{for } n\neq 0 \ ,\\[2ex] \dfrac{2\xi(\kappa-1)}{2\xi(\kappa-1)\ln(\rho v/2)-1} & \text{for } n=0 \ .\end{cases}$$

Thus for small ρ, corresponding to r and r' both much greater than r_0, one recovers the usual two-point function on the ideal cone with $\xi=0$:

$$G_C(x,x')=\frac{1}{2\pi\alpha r_<^2}\int_0^{\infty}v\,dv\sum_{n=-\infty}^{\infty}e^{in\kappa\Delta\phi}J_0\left[\frac{v\sqrt{\Delta t^2+\Delta z^2}}{r_<}\right]K_{\kappa|n|}\left[v\frac{r_>}{r_<}\right]I_{\kappa|n|}(v) \ . \tag{23}$$

This result proves that in the limit as r_0 becomes very small for fixed values of r and r', the two-point function on the flower pot approaches the two-point function for $\xi=0$ on the ideal cone. Note, however, that the value of the two-point function when one point is on the curvature singularity (i.e., $r=r_0$ and $r'>r$) does depend strongly upon the value of ξ, in agreement with what the Born expansion would suggest.

In the case $r_0\to 0$ it is now easy to obtain the simple form (5) for the two-point function which we gave in the Introduction. Using formula 6.578.11 from Ref. 13 one obtains

$$\begin{aligned}G_C(x,x')&=\frac{1}{2\pi\alpha}\int_0^{\infty}v\,dv\sum_{n=-\infty}^{\infty}e^{in\kappa\Delta\phi}J_0(v\sqrt{\Delta t^2+\Delta z^2})K_{\kappa|n|}(vr_>)I_{\kappa|n|}(vr_<)\\&=\frac{1}{4\pi\alpha}\frac{1}{r_>r_<\sinh\eta}\sum_{n=-\infty}^{\infty}(e^{in\kappa\Delta\phi-\kappa|n|\eta})\\&=\frac{1}{8\pi^2}\frac{\kappa\sinh\kappa\eta}{rr'\sinh\eta(\cosh\kappa\eta-\cos\kappa\Delta\phi)} \ .\end{aligned} \tag{24}$$

It is also straightforward to construct the two-point function on the ballpoint pen model, in which a spherical cap is pasted onto the end of the cone. For convenience, it is useful to define another radial coordinate θ on the cap, which is given by

$$\sin\theta=\frac{r}{r_0}\sqrt{1-1/\kappa^2} \ . \tag{25}$$

The two-point function has an expansion given by (14), where the inner-mode function obeys the equation (for $r<r_0$)

$$\left[-\frac{r_0^2s^2}{\kappa^2-1}-2\xi-\frac{n^2}{\sin^2\theta}+\frac{1}{\sin\theta}\frac{d}{d\theta}\sin\theta\frac{d}{d\theta}\right]\Psi_<(\theta)=0 \ , \tag{26}$$

where $s^2=\omega^2+k^2$. The solutions to this equation are the Legendre functions $P_\nu^{|n|}(\cos\theta)$ and $Q_\nu^{|n|}(\cos\theta)$, where

$$\nu(\nu+1)=-2\xi-\frac{r_0^2s^2}{\kappa^2-1} \ .$$

The first solution is nonsingular as $r\to 0$ and the second solution is singular. We choose the solution that is well behaved as $r\to 0$,

$$\Psi_< = \begin{cases} P_\nu^{|n|}(\cos\theta) & \text{for } r < r_0 \ , \\ AI_{\kappa|n|}(sr) + BK_{\kappa|n|}(sr) & \text{for } r > r_0 \ . \end{cases} \tag{27}$$

The constants A and B are now determined by the requirement that $\Psi_<$ and its derivative be continuous at $r = r_0$. This implies that the ratio $C = B/A$ is given by $C(sr_0, n, \xi)$, where

$$C(x,n,\xi) = \frac{\sqrt{\kappa^2-1}\, I_{\kappa|n|}(x) P_\nu'^{|n|}(1/\kappa) - x I'_{\kappa|n|}(x) P_\nu^{|n|}(1/\kappa)}{-\sqrt{\kappa^2-1}\, K_{\kappa|n|}(x) P_\nu'^{|n|}(1/\kappa) + x K'_{\kappa|n|}(x) P_\nu^{|n|}(1/\kappa)} \ . \tag{28}$$

The two-point function is now given by Eq. (21) with this new coefficient function C. As before, the only dependence upon ξ and r_0 is through C. Again, we can examine the limit $r_0 \to 0$ for nonzero values of ξ. One finds that as $r_0/r_<$ vanishes, the coefficient function C also vanishes, showing that the two-point function is affected by the curvature coupling term only for r of the order of r_0.

IV. GREEN FUNCTIONS ON TWO-DIMENSIONAL MODELS

In the two-dimensional case, the metric is given by the final two terms of Eq. (7), and the wave operator is given by the final two terms of Eq. (9). The two-dimensional δ function has an expansion identical to that given in Eq. (13) without the integrals over ω and k. The two-point function takes the form

$$G(x,x') = \sum_{n=-\infty}^{\infty} \frac{1}{\alpha} e^{in\kappa\Delta\phi} \Psi_<^{(\omega kn)}(r_<) \Psi_>^{(\omega kn)}(r_>) \ . \tag{29}$$

The mode functions obey Eq. (15) as in the four-dimensional case, with $\omega = k = 0$. However, there is an additional difficulty in two dimensions in that the two-point function of a massless field in two dimensions has an infrared divergence. The simplest way to avoid this problem in our context is to impose an additional boundary condition on a large cylinder of radius $r = a$, and to demand that the field and two-point function vanish on that cylinder.

We shall deal only with the simpler flower-pot model. Demanding that $\Psi_<$ be regular as $r \to 0$, and that $\Psi_>$ vanish on the outer boundary at $r = a$, one obtains

$$\Psi_< = \begin{cases} \left[\dfrac{r}{r_0}\right]^{|n|} & \text{for } r < r_0 \ , \\ A\left[\dfrac{r}{r_0}\right]^{\kappa|n|} + B\left[\dfrac{r}{r_0}\right]^{-\kappa|n|} & \text{for } r > r_0 \text{ and } n \neq 0 \ , \\ A + B\ln(r/r_o) & \text{for } r > r_0 \text{ and } n = 0 \ . \end{cases} \tag{30}$$

The continuity of Ψ at $r = r_0$ implies that $A + B = 1$ for $n \neq 0$ and that $A = 1$ for $n = 0$. The jump condition (17) fixes the remaining freedom in A and B. One finds that their ratio is

$$C = \frac{B}{A} = \begin{cases} \dfrac{\xi(1-\kappa)}{\kappa|n| + \xi(\kappa-1)} & \text{for } n \neq 0 \ , \\ 2\xi(\kappa-1) & \text{for } n = 0 \ . \end{cases} \tag{31}$$

The outer-mode function, which vanishes at $r = a$ is given by

$$\Psi_> = \begin{cases} \dfrac{1}{2A\kappa|n|}\left[1 + C\left[\dfrac{r_0^2}{a^2}\right]^{\kappa|n|}\right]^{-1}\left[\left[\dfrac{r}{r_0}\right]^{-\kappa|n|} - \left[\dfrac{rr_0}{a^2}\right]^{\kappa|n|}\right] & \text{for } r > r_0 \text{ and } n \neq 0 \ , \\ \dfrac{1}{A[C\ln(r_0/a) - 1]}\ln(r/a) & \text{for } r > r_0 \text{ and } n = 0 \ , \end{cases} \tag{32}$$

where the normalization condition (16) has been used.

The two-point function on the two-dimensional flower pot then becomes

$$G_F(x,x')=\frac{\kappa}{2\pi}\frac{2\xi(\kappa-1)\ln(r_<\!/r_0)+1}{2\xi(\kappa-1)\ln(a/r_0)+1}\ln(a/r_>)$$
$$+\sum_{n=-\infty}^{\infty}{}'\frac{e^{in\kappa\Delta\phi}}{4\pi|n|}\left[1+C\left[\frac{r_0}{a}\right]^{2\kappa|n|}\right]^{-1}\left[\left[\frac{r_<}{r_>}\right]^{\kappa|n|}+C\left[\frac{r_0^2}{r_<r_>}\right]^{\kappa|n|}\right]\left[1-\left[\frac{r_>^2}{a^2}\right]^{\kappa|n|}\right], \tag{33}$$

where r and r' are assumed to be greater than r_0, and the prime means that the $n=0$ term is to be omitted from the sum.

For r and r' fixed, the limit as $r_0\to 0$ reduces to the $\xi=0$ two-point function for the ideal two-dimensional cone. In this way the two-dimensional case is clearly analogous to the four-dimensional case discussed earlier. One obtains

$$G_{F,\xi=0}=\lim_{r_0\to 0}G_F=\frac{\kappa}{2\pi}\ln\left[\frac{a}{r_>}\right]+\sum_{n=-\infty}^{\infty}{}'\frac{e^{in\kappa\Delta\phi}}{4\pi|n|}\left[\left[\frac{r_<}{r_>}\right]^{\kappa|n|}-\left[\frac{r_>r_<}{a^2}\right]^{\kappa|n|}\right]. \tag{34}$$

This sum can be evaluated to yield

$$G_{F,\xi=0}=\lim_{r_0\to 0}G_F=\frac{1}{4\pi}\ln\left[\frac{1-2\left[\frac{r_>r_<}{a^2}\right]^{\kappa}\cos\kappa\Delta\phi+\left[\frac{r_>r_<}{a^2}\right]^{2\kappa}}{\left[\frac{r_>}{a}\right]^{2\kappa}-2\left[\frac{r_>r_<}{a^2}\right]^{\kappa}\cos\kappa\Delta\phi+\left[\frac{r_<}{a}\right]^{2\kappa}}\right]. \tag{35}$$

The limit as $a\to\infty$ now gives the standard form of the $\xi=0$ two-point function on a cone, complete with an infrared divergence due to the zero mode,

$$\lim_{a\to\infty}G_{F,\xi=0}=\lim_{\substack{r_0\to 0\\ a\to\infty}}G_F=-\frac{1}{4\pi}\ln[r_>^{2\kappa}-2(r_>r_<)^{\kappa}\cos\kappa\Delta\phi+r_<^{2\kappa}]+\frac{\kappa}{2\pi}\ln(a)\,. \tag{36}$$

The limit as $a\to\infty$ for ξ nonzero differs from this. In that limit one obtains

$$\lim_{a\to\infty}G_F=\frac{\kappa}{2\pi}\left[\ln\left[\frac{r_<}{r_0}\right]+\frac{1}{2\xi(\kappa-1)}\right]+\sum_{n=-\infty}^{\infty}{}'\frac{e^{in\kappa\Delta\phi}}{4\pi|n|}\left[\left[\frac{r_<}{r_>}\right]^{\kappa|n|}+C\left[\frac{r_0^2}{r_<r_>}\right]^{\kappa|n|}\right], \tag{37}$$

which can also be summed to yield

$$\lim_{a\to\infty}G_F=\frac{\kappa}{2\pi}\left[\ln\left[\frac{r_<}{r_0}\right]+\frac{1}{2\xi(\kappa-1)}\right]+\frac{1}{4\pi}\ln\left[\frac{1-2\left[\frac{r_0^2}{r_>r_<}\right]^{\kappa}\cos\kappa\Delta\phi+\left[\frac{r_0^2}{r_>r_<}\right]^{2\kappa}}{1-2\left[\frac{r_<}{r_>}\right]^{\kappa}\cos\kappa\Delta\phi+\left[\frac{r_<}{r_>}\right]^{2\kappa}}\right]$$
$$+\frac{1}{4\pi}\frac{\kappa}{\kappa-\xi(\kappa-1)}\left\{e^{i\kappa\Delta\phi}\left[\frac{r_0^2}{r_>r_<}\right]^{\kappa}{}_2F_1\left[1,1-\xi\frac{(\kappa-1)}{\kappa};2-\xi\frac{(\kappa-1)}{\kappa};e^{i\kappa\Delta\phi}\left[\frac{r_0^2}{r_>r_<}\right]^{\kappa}\right]\right.$$
$$\left.+e^{-i\kappa\Delta\phi}\left[\frac{r_0^2}{r_>r_<}\right]^{\kappa}{}_2F_1\left[1,1-\xi\frac{(\kappa-1)}{\kappa};2-\xi\frac{(\kappa-1)}{\kappa};e^{-i\kappa\Delta\phi}\left[\frac{r_0^2}{r_>r_<}\right]^{\kappa}\right]\right\}. \tag{38}$$

V. THE BORN EXPANSION

Our conclusion from the previous sections has been that a curvature coupling term only affects the behavior of the two-point function in the region near the string itself. This was in contrast to our expectation from the Born expansion calculated in Sec. I. Here we examine the discrepancy.

First let us show the relationship between our two-point function for the four-dimensional flower pot for general ξ and the Born expansion. Starting with the Born expansion it is easy to see that the coefficient ξ^N is

$$[-2(\kappa-1)]^N\int\frac{d\omega}{2\pi}e^{i\omega\Delta t}\int\frac{dk}{2\pi}e^{ik\Delta z}\sum_{n=-\infty}^{\infty}\frac{1}{\alpha}e^{in\kappa\Delta\phi}[\Psi_<^0(r_0)]^{N+1}[\Psi_>^0(r_0)]^{N-1}\Psi_>^0(r)\Psi_>^0(r')\,, \tag{39}$$

where $\Psi_<^0$ and $\Psi_>^0$ denote the solutions corresponding to $\xi=0$. The simple nature of this expansion arises from the δ-function nature of the curvature which enables all radial integrations to be performed immediately. The t, z, and ϕ integrations are all trivial.

On the other hand, we may obtain a power series in ξ for our flower-pot two-point function (21) by examining the Taylor series for $C(sr_0,n,\xi)$. We have

$$C(\xi)=C(sr_0,n,\xi)=\frac{B_0-2\xi(\kappa-1)I_{\kappa|n|}(sr_0)I_{|n|}(sr_0/\kappa)}{A_0+2\xi(\kappa-1)K_{\kappa|n|}(sr_0)I_{|n|}(sr_0/\kappa)}\ , \tag{40}$$

where A_0 and B_0 denote the coefficients A and B of Eq. (18) for $\xi=0$ and we have suppressed the dependence on s and n for notational simplicity. Thus,

$$\begin{aligned}C(\xi)&=[C(0)-2\xi A_0^{-1}I_{\kappa|n|}(sr_0)I_{|n|}(sr_0/\kappa)]\sum_{N=0}^{\infty}[-2\xi(\kappa-1)]^N[A_0^{-1}K_{\kappa|n|}(sr_0)]^N I_{|n|}(sr_0/\kappa)^N\\&=C(0)+A_0^{-2}\sum_{N=0}^{\infty}[-2\xi(\kappa-1)]^N[A_0^{-1}K_{\kappa|n|}(sr_0)]^{N-1}I_{|n|}(sr_0/\kappa)^{N+1}\ ,\end{aligned} \tag{41}$$

where in moving to the second form we have used the continuity of $\Psi^0_<$ at $r=r_0$. Equation (41) is equivalent to Eq. (39) on noting that $\Psi^0_<(r_0)=I_{|n|}(sr_0/\kappa)$ and recalling Eq. (20). Unfortunately, the corresponding calculation in two dimensions is complicated by the infrared divergence which gives rise to factors of C in the denominator; however, there is no reason to doubt that the expansion is perfectly valid here also.

Let us now turn to the case of the idealized cone. Dealing with the two-dimensional case first for simplicity, we note that, by Eq. (35),

$$G_C(r'=0;r;\Delta\phi)=-\frac{\kappa}{2\pi}\ln\left[\frac{r}{a}\right]\ . \tag{42}$$

Thus one finds

$$\int GRG=\frac{\kappa^2}{2\pi^2}(\kappa-1)\ln\left[\frac{r}{a}\right]\ln\left[\frac{r'}{a}\right]\ . \tag{43}$$

However, if one calculates the next term in the Born expansion, one obtains the meaningless expression

$$\int\int GRGRG=-\frac{\kappa^3}{2\pi^3}(\kappa-1)^2\ln(0)\ln\left[\frac{r}{a}\right]\ln\left[\frac{r'}{a}\right]\ .$$

Indeed, all the higher terms may be calculated and summed to yield the formal expression

$$G(x,x';\xi)=G(x,x';0)+\frac{1}{2\pi^2}\frac{\kappa^2(\kappa-1)\xi}{1+\frac{\kappa(\kappa-1)\xi}{\pi}\ln(0)}\ln\left[\frac{r}{a}\right]\ln\left[\frac{r'}{a}\right]\ . \tag{44}$$

An exactly analogous situation arises when one attempts to calculate higher terms in the four-dimensional Born expansion. In this case again it is only the axisymmetric terms that contribute because for other terms $\Psi^0_<(r)$ vanishes on the axis. From Eq. (39) the coefficient of ξ^N is

$$[-2(\kappa-1)]^N\int\frac{d\omega}{2\pi}e^{i\omega\Delta t}\int\frac{dk}{2\pi}e^{ik\Delta z}\times\frac{1}{\alpha}[K_0(0)]^{N-1}K_0(sr)K_0(sr')\ ,$$

which yields the finite expression given in Sec. I when $N=1$ but is infinite for $N\geq 2$. Summing the series we get

$$G(x,x';\xi)=G(x,x';0)-\frac{1}{4\pi^2}\frac{\kappa(\kappa-1)\xi}{1+2(\kappa-1)\xi K_0(0)}\frac{\eta}{rr'\sinh\eta}\ . \tag{45}$$

VI. EXPECTATION VALUE $\langle\hat{\varphi}^2\rangle$ FOR THE FLOWER-POT MODEL

For the ideal conical string space-time, the renormalized expectation value $\langle\hat{\varphi}^2\rangle_C$ may be easily found. Because the curvature vanishes (except at the isolated conical tip), the renormalization may be effected by subtracting the flat-space ($\kappa=1$) two-point function:

$$\begin{aligned}\langle\hat{\varphi}^2\rangle_C&=\lim_{x\to x'}[G_C(x,x';\kappa)-G_C(x,x';1)]\\&=\frac{1}{8\pi^2r^2}\lim_{\eta\to 0}\frac{1}{\sinh\eta}\left[\frac{\kappa\sinh\kappa\eta}{\cosh\kappa\eta-1}-\frac{\sinh\eta}{\cosh\eta-1}\right]\\&=\frac{\kappa^2-1}{48\pi^2r^2}\ .\end{aligned}$$

It is interesting to calculate the same expectation value for one of the more realistic string models presented in this paper; we shall investigate the flower-pot model. In that model, the renormalized expectation value of $\langle\hat{\varphi}^2\rangle$ is

$$\begin{aligned}\langle\hat{\varphi}^2\rangle_F&=[G_F(x,x';\kappa)-G_C(x,x';1)]\\&=[G_F(x,x';\kappa)-G_C(x,x';\kappa)]\\&\quad+[G_C(x,x';\kappa)-G_C(x,x';1)]\ ,\end{aligned}$$

where square brackets mean "take the coincidence" limit, i.e., $[Q(x,x')]=\lim_{x\to x'}Q(x,x')$.

The second term in square brackets above is just $\langle\hat{\varphi}^2\rangle_C$. From Eq. (22) one can see that the first term in square brackets above is given by

$$[G_F(x,x';\kappa)-G_C(x,x';\kappa)]=\frac{\kappa}{4\pi^2r^2}\int_0^\infty v\,dv\sum_{n=-\infty}^{\infty}K^2_{\kappa|n|}(v)C(vr_0/r,n,\xi)\ ,$$

where C is defined following Eq. (22). We now define a

dimensionless radial variable x by $x=r/r_0$, and consider the dimensionless radial function $\Psi(x)$ defined by

$$\Psi(x)=\frac{\langle\hat{\varphi}^2\rangle_F-\langle\hat{\varphi}^2\rangle_C}{\langle\hat{\varphi}^2\rangle_C}$$
$$=\frac{12\kappa}{\kappa^2-1}\int_0^\infty v\,dv\sum_{n=-\infty}^{\infty}K^2_{\kappa|n|}(v)C(v/x,n,\xi)\ . \quad (46)$$

The function $\Psi(x)$ provides a fractional measure of the amount by which the renormalized expectation value $\langle\hat{\varphi}^2\rangle$ differs between the ideal conical model and the more realistic flower-pot model of a cosmic string:

$$\langle\hat{\varphi}^2\rangle_F=\langle\hat{\varphi}^2\rangle_C[1+\Psi(x)]\ .$$

Our earlier results imply that Ψ must vanish at large radii, so that $\lim_{x\to\infty}\Psi(x)=0$. It is interesting to note, however, that $\Psi(x)$ vanishes *very* slowly for increasing x. Denoting by $\Psi_n(x)$ the nth term of Eq. (46), one has

$$\Psi(x)=\sum_{n=-\infty}^{\infty}\Psi_n(x)\ .$$

One can estimate the rate at which Ψ_n vanishes for large x by making use of formula 6.576.3 from Ref. 13, and the small-s expansion for $C(s,n,\xi)$ given immediately before Eq. (23). One finds that as $x\to\infty$ the functions $\Psi_n(x)$ vanish as

$$x\to\infty,\quad \Psi_n(x)\propto x^{-2\kappa|n|}\ \text{ and }\ \Psi_0(x)\propto 1/\ln(x)\ .$$

It is $\Psi_0(x)$ that is responsible for the logarithmically slow fall off of $\Psi(x)$ for large x.

We have numerically evaluated the function $\Psi(x)$ for a conformally coupled field ($\xi=\frac{1}{6}$) for several values of the angular deficit. The results shown in Figs. 2 and 3 were obtained from Eq. (46) by summing the terms $-20\le n\le 20$, and integrating over the region $0\le v\le 80$.

As can be seen from Fig. 2, the fall off of Ψ is indeed very slow, and as the angle of the cone approaches 2π one needs to get exponentially far away from the cone before $\langle\hat{\varphi}^2\rangle_F$ approaches $\langle\hat{\varphi}^2\rangle_C$. For a realistic cosmic string, corresponding to symmetry breaking at the GUT energy scale of 10^{16} GeV, the radius r_0 of the string would be approximately 10^{-30} cm, and the parameter $k\approx 1+10^{-6}$. It is clear from Fig. 2 that at a distance of $\exp(150)r_0\approx 10^{35}$ cm (greater than the present Hubble radius of the universe) the expectation value $\langle\hat{\varphi}^2\rangle$ still differs from $\langle\hat{\varphi}^2\rangle_C$ of an ideal conical string by more than 90%. Thus it is fair to say that the ideal cosmic string is not an accurate model of a realistic cosmic string, even quite far away from the core of the string. Of course at this enormous distance, $\langle\hat{\varphi}^2\rangle_F$ is unobservably small, so the difference is of strictly academic interest.

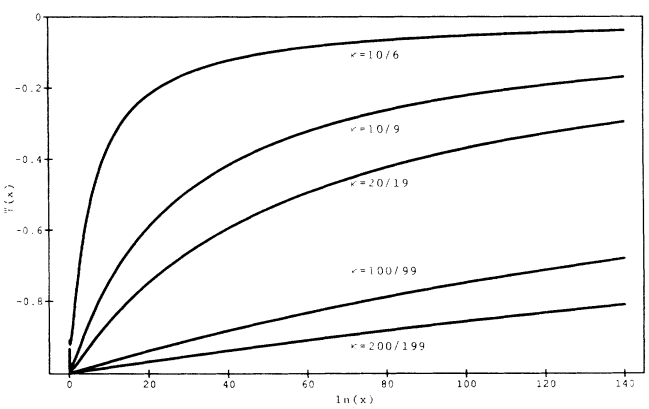

FIG. 2. Shown is a plot of the function $\Psi(x)$ defined in Eq. (46), which relates $\langle\hat{\varphi}^2\rangle_F$ on the "flower-pot" model to $\langle\hat{\varphi}^2\rangle_C$ on an ideal cone: $\langle\hat{\varphi}^2\rangle_F=\langle\hat{\varphi}^2\rangle_C[1+\Psi(x)]$. The scalar field is conformally coupled ($\xi=\frac{1}{6}$), and the parameter κ is related to the angular deficit of the cone. The horizontal axis shows the natural logarithm of the dimensionless radial variable $x=r/r_0$. It can be seen that $\Psi(x)$ vanishes logarithmically slowly for large x.

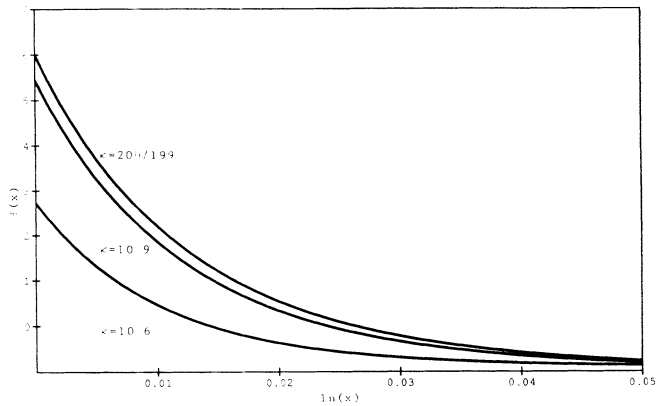

FIG. 3. This figure provides a detail of Fig. 2 for radii near the core of the cosmic string.

Figure 3 shows the function $\Psi(x)$ for small values of x, near the core of the string. In this case, it seems likely that the limit $\kappa\to 1$ exists; however, it is not clear if our numerical scheme has converged to that limit or not. For this reason, the results of Fig. 3 should be viewed with a skeptical eye.

VII. CONCLUSION

While Eqs. (44) and (45) are suggestive, for example, all space-time dependence is explicit, it is far from clear whether they can be given any real meaning. Indeed, it is not even clear whether the equation

$$(\Box-\xi R)\varphi=0$$

has any meaning for nonzero ξ when R has a distributional character. Rather than pursue this mathematical question we prefer to take the view that real cosmic strings possess an intrinsic length scale and in this case we have argued by example that the effects of the curvature coupling are restricted to a neighborhood of the string. Thus, for fixed r and r' one has

$$\lim_{r_0\to 0}G_F(r;r';\Delta t,\Delta z,\Delta\phi)=\lim_{r_0\to 0}G_B(r;r';\Delta t,\Delta z,\Delta\phi)$$
$$=G_C(r;r';\Delta t,\Delta z,\Delta\phi)\ .$$

Note, however, that this is not true if one point remains

on the region where the curvature of the string is found:

$$\lim_{r_0\to 0} G_F(r_0;r';\Delta t,\Delta z,\Delta\phi)=\lim_{r_0\to 0} G_B(r_0;r';\Delta t,\Delta z,\Delta\phi) \neq G_C(0;r';\Delta t,\Delta z,\Delta\phi)\ .$$

It is these latter expressions that are important for the Born expansion.

The effects of any curvature coupling appear to be confined to a region whose size is proportional to size r_0 (with an exponentially large proportionality factor). For this reason, the effects of curvature coupling for the idealized cone disappear, since there is no natural length scale for the effect. However, we repeat that this statement may simply not make sense, because the meaning of the wave equation in the presence of a curvature singularity is unclear. One extension of this work might be to study particle production during the formation of a realistic cosmic string, extending the work of Parker and Mendell and Hiscock.[14] The latter authors considered the effect of curvature coupling before and on the transition hypersurface, but as they used an idealzed string model, they did not consider the effects in the final cosmic string background.

ACKNOWLEDGMENTS

This work has been supported in part by NSF Grants Nos. 87-05107 and 89-03027, and by a grant from the Science and Engineering Research Council, United Kingdom.

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