Examples of the Vilkovisky-DeWitt effective action in one-loop quantum gravity

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We present two examples where the contribution of one-loop gravitons to the renormalized stress-energy tensor may be calculated explicitly from both the standard and the Vilkovisky-DeWitt effective action. The renormalization is carried out using the Hadamard renormalization procedure outlined by Allen, Folacci, and Ottewill [Phys. Rev. D **38**, 1069 (1988)]. The examples show that the standard and Vilkovisky-DeWitt formulations of quantum field theory lead to different physical predictions at one loop.

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I. INTRODUCTION

To obtain a consistent one-loop approximation to quantum gravity, it is necessary to include the effects of linearized perturbations of the gravitational field. In particular, gravitons will contribute to the one-loop effective stress-energy tensor a term of the same order as those from ordinary matter fields.

There is an ambiguity in the definition of this stress-energy tensor arising from an ambiguity in the one-loop effective action. The "standard" effective action arises from a straightforward loop expansion of the Einstein-Hilbert action. However, this theory possesses the undesirable features that the off-shell effective action depends upon the choice of the background-field gauge and the choice of gauge-breaking term in the action. Vilkovisky [1] and DeWitt [2] have proposed a modification which differs from the standard theory off shell and claims to be free of these defects. Their definition has great aesthetic appeal, advocating taking seriously the geometrical structure of field space. Unfortunately, it involves extreme computational complexity, and as the on-shell effective actions agree, it is not clear that this additional complexity is warranted.

The one-loop effective stress-energy tensor for gravitons arises from the *differentiated* effective action which probes the off-shell structure. In Ref. [3] the calculation of the renormalized one-loop effective stress-energy tensor was outlined according to both definitions for gravitons on a general Ricci-flat background. At that time we were unable to find a specific background with a degree of symmetry sufficient to allow for the explicit evaluation of the two stress-energy tensors, without being so great as to make them equal. It is the purpose of this paper to present such examples and thereby demonstrate that there is a genuine difference between the physical predictions of the two approaches.

The main example which we consider (Sec. II) concerns quantum field theory in Rindler space. To reach the graviton Green's function, we first calculate the scalar and electromagnetic Green's functions. We take the opportunity to provide a pedagogical approach to practical calculations using Hadamard renormalization [4].

The second example (Sec. III) is unphysical, corresponding to Minkowski spacetime containing a "gravitational conductor," but is striking in its simplicity and in the way in which it illustrates the prime importance of the Ward identities of the theory.

We should add that there has been much work [5] on the Vilkovisky-DeWitt (VDW) effective action, and many problems have arisen since Ref. [3] was written. These problems concern the definition beyond one loop, and so although of fundamental significance to the Vilkovisky-DeWitt program, they do not affect our work directly. We shall proceed with the faith that the elegance of the one-loop term reflects some deeper truth.

Throughout the paper we use units in which $\hbar = c = 1$ and follow the sign conventions of Misner, Thorne, and Wheeler [6]. Equation numbers prefixed by "GR" refer to formulas in the mathematical tables of Gradshteyn and Ryzhik [7].

II. RINDLER SPACE

In this section we first give the Green's function and calculate the Hadamard-renormalized stress-energy tensor for the scalar and electromagnetic fields on the Rindler-space background and check the answers against those previously obtained by more traditional methods

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(although the explicit formula for the electromagnetic Green's function in Feynman gauge has never, as far as we know, been determined before). We will then be in a position to perform the same calculation for the linearized gravitational field, with and without the VDW modification.

The line element for Rindler space is given by

$$ds^2=-\xi^2 d\tau^2+d\xi^2+dy^2+dz^2 , \tag{2.1}$$

where $\tau,y,z\in\mathbb{R}$ and $\xi\in(0,\infty)$. This line element may be related to that for the cosmic string, which may be written in coordinates (t,r,ϕ,z) as

$$ds^2=-dt^2+dr^2+r^2 d\phi^2+dz^2 , \tag{2.2}$$

where the angular coordinate ϕ runs from 0 to α. The relationship is given by making the identifications

$$r=\xi , \quad \phi=i\tau , \quad t=iy , \quad z=z ,$$

and taking the limit $\alpha\to\infty$. This will prove useful to us since we calculated the scalar, electromagnetic, and gravitational Green's functions on the cosmic string in Ref. [8] and so may write down the corresponding Green's functions on Rindler space via this identification. In Ref. [8] the Green's functions were expressed in terms of the parameter $\kappa=2\pi/\alpha$, and so the appropriate Rindler Green's functions are obtained by letting $\kappa\to 0$.

It will prove valuable for the ensuing discussion to have an explicit formula for the bivector of parallel transport $g^{\mu}{}_{\nu'}(x,x')$ in Rindler space. In Minkowski space this object is just $\mathrm{diag}(1,1,1,1)^{\mu}{}_{\nu'}$ in the usual Cartesian coordinates; upon transforming to Rindler coordinates and remembering that $g^{\mu}{}_{\nu'}(x,x')$ transforms as a bivector, we deduce that

$$g^{\mu}{}_{\nu'}(x,x')=\begin{array}{c} \begin{bmatrix} \frac{\xi'}{\xi}\cosh\Delta\tau & -\frac{1}{\xi}\sinh\Delta\tau & 0 & 0 \\ -\xi'\sinh\Delta\tau & \cosh\Delta\tau & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array}\begin{array}{l}\mu \\ \\ \\ \nu'\end{array} ,$$

where $\Delta\tau\equiv(\tau-\tau')$.

A. Scalar field

Making the identifications mentioned above, we find that the Green's function $G(x,x')=i\langle 0|T[\varphi(x)\varphi(x')]|0\rangle$ for a massless scalar field $\varphi(x)$ on Rindler space is

$$G(x,x')=\frac{i}{4\pi^2}\frac{1}{\xi\xi'\sinh\eta}\frac{\eta}{\eta^2-(\Delta\tau)^2} , \tag{2.3}$$

where η is the non-negative real function given by

$$\cosh\eta=\frac{\xi^2+\xi'^2+(y-y')^2+(z-z')^2}{2\xi\xi'} . \tag{2.4}$$

The next objective is to isolate the regular part $W(x,x')$ of $G(x,x')$, which is given by [4]

$$W(x,x')=\frac{8\pi^2}{i}G(x,x')-\frac{1}{\sigma} .$$

Here $\sigma(x,x')$ is one-half the square of the distance along a geodesic joining x to x'. In writing this equation we have used the fact that we are dealing with a massless theory in a flat space-time. Substitute Eq. (2.3) for $G(x,x')$ and expand the result to second order in powers of the coordinate differences $\Delta x^{\mu}\equiv(x-x')^{\mu}$; the expansion

$$\eta^2=\xi^{-2}[(\Delta\xi)^2+(\Delta y)^2+(\Delta z)^2]+O((\Delta x^{\mu})^3)$$

is useful in this regard. A lengthy calculation yields

$$W(x,x')=-\frac{1}{6\xi^2}\left\{1+\frac{\Delta\xi}{\xi}-\frac{11}{60\xi^2}[(\Delta y)^2+(\Delta z)^2]+\frac{49}{60\xi^2}(\Delta\xi)^2-\frac{1}{20}(\Delta\tau)^2\right\}+O((\Delta x^{\mu})^3) . \tag{2.5}$$

To calculate the stress-energy tensor, we must now determine the first three coefficients occurring in the covariant Taylor expansion

$$W(x,x')=w(x)+w_{\mu}(x)\sigma^{\mu}+\tfrac{1}{2}w_{\mu\nu}(x)\sigma^{\mu}\sigma^{\nu}+\cdots ,$$

where $\sigma^{\mu}(x,x')\equiv\sigma^{;\mu}(x,x')$. Substituting (2.5) into the formulas

$$w(x)=[W] ,$$
$$w_{\mu}(x)=[-g_{\mu}{}^{\rho'}W_{;\rho'}] ,$$
$$w_{\mu\nu}(x)=[g_{\mu}{}^{\rho'}g_{\nu}{}^{\tau'}W_{;\rho'\tau'}] ,$$

where the square brackets indicate the coincidence limit $x=x'$, we find that the only nonvanishing coefficients are

$$w=-\frac{1}{6\xi^2} , \quad w_{\xi}=-\frac{1}{6\xi^3} ,$$
$$w_{yy}=w_{zz}=\frac{11}{180\xi^4} , \quad w_{\xi\xi}=-\frac{49}{180\xi^4} , \quad w_{\tau\tau}=-\frac{9}{60\xi^2} .$$

A useful check on these expressions is provided by the identities [4]

$$w_{\mu}=-\tfrac{1}{2}w_{;\mu} , \quad w_{\mu\nu}{}^{;\nu}=\tfrac{1}{4}(\Box w)_{;\mu} , \quad w_{\mu}{}^{\mu}=0 ,$$

which follow from the anomalous wave equation for W on a flat space-time and the symmetry property $W(x,x')=W(x',x)$. Substituting our expressions for the Taylor coefficients into the formulas of Ref. [4], we find that the renormalized stress-energy tensor is

$$\langle 0|T_{\mu}{}^{\nu}|0\rangle_R=\frac{1}{1440\pi^2\xi^4}\{\mathrm{diag}(3,-1,-1,-1)_{\mu}{}^{\nu}+60(\zeta-\tfrac{1}{6})\mathrm{diag}(-3,1,-2,-2)_{\mu}{}^{\nu}\} , \tag{2.6}$$

where ζ denotes the coupling to the scalar curvature. Equation (2.6) is in agreement with the standard result.

It should be clear from this example that the Hadamard-renormalization procedure is both practical

and straightforward; it even contains its own built-in checks.

B. Electromagnetic field

Using the standard choice of gauge-breaking term [4], the Green's function

$$G^{\mu\nu'}(x,x')=i\langle 0|T[A^{\mu}(x)A^{\nu}(x')]|0\rangle ,$$

for the vector potential $A^{\mu}(x)$ satisfies the equation

$$\Box G^{\mu\nu'}(x,x')=-g^{\mu\nu'}\delta^4(x,x') , \tag{2.7}$$

and the Green's function $\tilde{G}(x,x')=i\langle 0|T[\bar{c}(x)c(x')]|0\rangle$ for the associated complex, scalar ghost field $c(x)$ satisfies the equation

$$\Box\tilde{G}(x,x')=-\delta^4(x,x') . \tag{2.8}$$

Becchi-Rouet-Stora (BRS) invariance yields the Ward identity

$$G^{\mu\nu'}{}_{;\mu}+\tilde{G}^{;\nu'}=0 , \tag{2.9}$$

for the theory.

From Ref. [8] we find that the tensor components of the Feynman-gauge Green's function for the electromagnetic field on Rindler space are all zero apart from

$$\begin{aligned} &G^{yy'}=G^{zz'}=G ,\\ &G^{\xi\xi'}=-\xi\xi' G^{\tau\tau'}=\frac{i}{4\pi^2}\frac{1}{\xi\xi'\sinh\eta}\frac{\eta\cosh\eta}{\eta^2-(\Delta\tau)^2} ,\\ &\xi' G^{\xi\tau'}=-\xi G^{\tau\xi'}=\frac{i}{4\pi^2}\frac{1}{\xi\xi'\sinh\eta}\frac{\Delta\tau\sinh\eta}{\eta^2-(\Delta\tau)^2} , \end{aligned} \tag{2.10}$$

where G is the scalar Green's function (2.3). The ghost Green's function $\tilde{G}$ coincides with the scalar Green's function G.

As before, the next stage in the calculation is to determine the regular part of the photon Green's function. It proves most economical to introduce the natural null complex tetrad as in Ref. [8]. In Rindler space this takes the form

$$\begin{aligned} e^{\mu}_{(1)}&=\frac{1}{\sqrt{2}}(-iy^{\mu}+z^{\mu})=\frac{1}{\sqrt{2}}(0,0,-i,1) ,\\ e^{\mu}_{(2)}&=\frac{1}{\sqrt{2}}(-iy^{\mu}-z^{\mu})=\frac{1}{\sqrt{2}}(0,0,-i,-1) ,\\ e^{\mu}_{(3)}&=\frac{1}{\sqrt{2}}(\tau^{\mu}+\xi^{\mu})=\frac{1}{\sqrt{2}}(1/\xi,1,0,0) ,\\ e^{\mu}_{(4)}&=\frac{1}{\sqrt{2}}(-\tau^{\mu}+\xi^{\mu})=\frac{1}{\sqrt{2}}(-1/\xi,1,0,0) , \end{aligned} \tag{2.11}$$

where $\{\tau^{\mu},\xi^{\mu},y^{\mu},z^{\mu}\}$ is the standard orthonormal tetrad for the space-time. The nonvanishing components of the photon Green's function with respect to this tetrad are found to be

$$\begin{aligned} &G^{(1)(2')}=G^{(2)(1')}=-G ,\\ &G^{(3)(4')}=\frac{i}{4\pi^2}\frac{1}{\xi\xi'\sinh\eta}\left[\frac{\eta\cosh\eta-\Delta\tau\sinh\eta}{\eta^2-(\Delta\tau)^2}\right] ,\\ &G^{(4)(3')}=\frac{i}{4\pi^2}\frac{1}{\xi\xi'\sinh\eta}\left[\frac{\eta\cosh\eta+\Delta\tau\sinh\eta}{\eta^2-(\Delta\tau)^2}\right] . \end{aligned}$$

Note that $G^{(2)(1')}$ and $G^{(4)(3')}$ can be obtained from $G^{(1)(2')}$ and $G^{(3)(4')}$ via the symmetry $G^{(a)(b')}(x,x')=G^{(b)(a')}(x',x)$.

We may now determine the tetrad components of the regular part of the photon Green's function by substituting for $G^{(a)(b')}$ in the formula [4]

$$\begin{aligned} W^{(a)(b)}(x,x')&\equiv g^{(b)}{}_{(b')}(x,x')W^{(a)(b')}(x,x')\\ &=\frac{8\pi^2}{i}g^{(b)}{}_{(b')}(x,x')G^{(a)(b')}(x,x')\\ &\quad-\frac{g^{(a)(b)}}{\sigma(x,x')} \end{aligned}$$

and expanding the result as a power series in the coordinate differences Δx^{μ}. Here

$$g^{(b)}{}_{(b')}(x,x')=\begin{array}{c} \begin{bmatrix} 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & \exp(-\Delta\tau) & 0\\ 0 & 0 & 0 & \exp(\Delta\tau) \end{bmatrix}^{(b)}\\ {}_{(b')} \end{array}$$

and

$$g^{(a)(b)}=\begin{bmatrix} 0 & -1 & 0 & 0\\ -1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 1 & 0 \end{bmatrix}^{(a)(b)}$$

are the tetrad components of the bivector of parallel transport and metric, respectively. A straightforward though tedious computation produces the result

$$\begin{aligned} &W^{(1)(2)}=W^{(2)(1)}=-W ,\\ &W^{(3)(4)}=\frac{1}{6\xi^2}\left\{5+5\frac{\Delta\xi}{\xi}+4\Delta\tau+\frac{19}{60\xi^2}[(\Delta y)^2+(\Delta z)^2]\right.\\ &\qquad\left.+\frac{281}{60}\frac{(\Delta\xi)^2}{\xi^2}+4\frac{\Delta\xi\,\Delta\tau}{\xi}+\frac{31}{20}\Delta\tau^2\right\}\\ &\qquad+O((\Delta x^{\mu})^3) , \end{aligned} \tag{2.12}$$

where W is given by the scalar expansion (2.5). The expansion for $W^{(4)(3)}$ is given by that for $W^{(3)(4)}$ with the substitution $\Delta\tau\to-\Delta\tau$.

The next step is to determine the first three coefficients occurring in the covariant Taylor expansion

$$\begin{aligned} W^{(a)(b)}(x,x')&=w^{(a)(b)}(x)+w^{(a)(b)}{}_{\mu}(x)\sigma^{\mu}\\ &\quad+\tfrac{1}{2}w^{(a)(b)}{}_{\mu\nu}(x)\sigma^{\mu}\sigma^{\nu}+\cdots . \end{aligned}$$

The coefficients of $W^{(1)(2)}=W^{(2)(1)}$ will obviously be just minus the scalar coefficients which we have already calculated. The nonvanishing coefficients of $W^{(3)(4)}$ are found by direct calculation to be

$$w^{(3)(4)}=\frac{5}{6\xi^2} , \quad w^{(3)(4)}{}_{\xi}=\frac{5}{6\xi^3} , \quad w^{(3)(4)}{}_{\tau}=\frac{2}{3\xi^2} ,$$

$$w^{(3)(4)}{}_{\xi\xi}=\frac{281}{180\xi^4}\ ,\quad w^{(3)(4)}{}_{\xi\tau}=w^{(3)(4)}{}_{\tau\xi}=\frac{4}{3\xi^3}\ ,$$
$$w^{(3)(4)}{}_{\tau\tau}=\frac{81}{60\xi^2}\ ,\quad w^{(3)(4)}{}_{yy}=w^{(3)(4)}{}_{zz}=-\frac{19}{180\xi^4}\ .$$

The coefficients of $W^{(4)(3)}$ are plus or minus those for $W^{(3)(4)}$ according as there are an even or odd number of occurrences of τ as a subscript.

At this point we can simultaneously revert to tensor indices and compute the symmetric and antisymmetric coefficients required for calculating the stress-energy tensor using

$$s^{\mu\nu}{}_{\rho\cdots\tau}=e^{(\mu}_{(a)}e^{\nu)}_{(b)}w^{(a)(b)}{}_{\rho\cdots\tau}$$

and

$$a^{\mu\nu}{}_{\rho\cdots\tau}=e^{[\mu}_{(a)}e^{\nu]}_{(b)}w^{(a)(b)}{}_{\rho\cdots\tau}\ .$$

We can now assure ourselves that the calculation so far is correct by checking that the coefficients we have computed satisfy those identities which arise from the symmetry constraint, wave equation, and Ward identity satisfied by the photon Green's function [4]. Finally, utilizing the formulas of Ref. [4], we find that the vacuum expectation value of the renormalized stress-energy tensor for the electromagnetic field on Rindler space is

$$\langle 0|T_\mu{}^\nu|0\rangle_R=\frac{11}{720\pi^2\xi^4}\mathrm{diag}(3,-1,-1,-1)_\mu{}^\nu\ , \tag{2.13}$$

in agreement with Ref. [9], where it was calculated by a quite different method using Hertz potentials.

C. Graviton field

The calculation of the standard graviton stress-energy tensor is more difficult than the photon calculation only in that there are twice as many indices to juggle. Using the standard choice of gauge-breaking term [3], the Green's function

$$G^{\mu\nu\rho'\tau'}(x,x')=(i/32\pi G)\langle 0|T[h^{\mu\nu}(x)h^{\rho'\tau'}(x')]|0\rangle\ ,$$

for the graviton field $h^{\mu\nu}(x)$ satisfies the equation

$$\Box G^{\mu\nu\rho'\tau'}(x,x')=-\gamma^{\mu\nu\rho'\tau'}\delta^4(x,x')\ , \tag{2.14}$$

where

$$\gamma^{\mu\nu\rho'\tau'}\equiv\tfrac{1}{2}(g^{\mu\rho'}g^{\nu\tau'}+g^{\mu\tau'}g^{\mu\rho'}-g^{\mu\nu}g^{\rho'\tau'})$$

and the Green's function

$$\tilde{G}^{\mu\rho'}(x,x')=(i/32\pi G)\langle 0|T[\bar{c}^\mu(x)c^{\rho'}(x')]|0\rangle\ ,$$

for the associated complex, vector, ghost field $c^\mu(x)$ satisfies the equation

$$\Box\tilde{G}^{\mu\rho'}(x,x')=-g^{\mu\rho'}\delta^4(x,x')\ . \tag{2.15}$$

BRS invariance yields the Ward identity

$$G^{\mu\nu\rho'\tau'}{}_{;\mu}-\tfrac{1}{2}G_\mu{}^{\mu\rho'\tau';\nu}+\tilde{G}^{\nu(\rho';\tau')}=0\ , \tag{2.16}$$

for the theory.

From Ref. [8] we find that the nonzero tensor components of the graviton Green's function on a Rindler-space background are

$$\begin{aligned}
&G^{yyy'y'}=-G^{yy\xi'\xi'}=\xi'^2G^{yy\tau'\tau'}=-G^{yyz'z'}=G^{yzy'z'}=-G^{\xi\xi z'z'}=\xi^2G^{\tau\tau z'z'}=G^{zzz'z'}=\tfrac{1}{2}G\ ,\\
&G^{y\xi y'\xi'}=-\xi\xi'G^{y\tau y'\tau'}=G^{\xi z\xi'z'}=-\xi\xi'G^{\tau z\tau'z'}=\tfrac{1}{2}G^{\xi\xi'}\ ,\\
&G^{y\xi y'\tau'}=G^{\xi z\tau'z'}=\tfrac{1}{2}G^{\xi\tau'}\ ,\\
&G^{\xi\xi\xi'\xi'}=\xi'^2G^{\xi\xi\tau'\tau'}=-\xi\xi'G^{\xi\tau\xi'\tau'}=\xi^2\xi'^2G^{\tau\tau\tau'\tau'}=\frac{i}{8\pi^2}\frac{1}{\xi\xi'\sinh\eta}\frac{\eta\cosh2\eta}{\eta^2-(\Delta\tau)^2}\ ,\\
&\xi'G^{\xi\xi\xi'\tau'}=-\xi\xi'^2G^{\xi\tau\tau'\tau'}=\frac{i}{8\pi^2}\frac{1}{\xi\xi'\sinh\eta}\frac{\Delta\tau\sinh2\eta}{\eta^2-(\Delta\tau)^2}
\end{aligned} \tag{2.17}$$

[where G and $G^{\mu\nu'}$ are the scalar and photon Green's functions (2.3) and (2.10), respectively], plus those obtainable from the symmetries

$$\begin{aligned}
G^{\mu\nu\rho'\tau'}(x,x')&=G^{\nu\mu\rho'\tau'}(x,x')\\
&=G^{\mu\nu\tau'\rho'}(x,x')\ ,\\
G^{\mu\nu\rho'\tau'}(x,x')&=G^{\rho'\tau'\mu\nu}(x',x)\ .
\end{aligned} \tag{2.18}$$

The ghost Green's function coincides with the photon Green's function. One can check explicitly that (2.17) satisfies both the wave equation (for $x\neq x'$) and the Ward identity.

The nonvanishing tetrad components of the graviton Green's function are

$$\begin{aligned}
&G^{(1)(1)(2')(2')}=2G^{(1)(2)(3')(4')}=G\ ,\\
&G^{(1)(3)(2')(4')}=G^{(2)(3)(1')(4')}=-\tfrac{1}{2}G^{(3)(4')}\ ,\\
&G^{(3)(3)(4')(4')}=\frac{i}{4\pi^2}\frac{1}{\xi\xi'\sinh\eta}\left[\frac{\eta\cosh2\eta-\Delta\tau\sinh2\eta}{\eta^2-(\Delta\tau)^2}\right]\ ,
\end{aligned}$$

plus those obtainable from the tetrad version of the symmetries (2.18).

Substituting for $G^{(a)(b)(c')(d')}$ in the formula [3]

$$W^{(a)(b)(c)(d)}(x,x')=\frac{8\pi^2}{i}g^{(c)}{}_{(c')}g^{(d)}{}_{(d')}G^{(a)(b)(c')(d')}(x,x')-\frac{\gamma^{(a)(b)(c)(d)}}{\sigma(x,x')}\ , \tag{2.19}$$

where

$$\gamma^{(a)(b)(c)(d)}=\tfrac{1}{2}(g^{(a)(c)}g^{(b)(d)}+g^{(a)(d)}g^{(b)(c)}-g^{(a)(b)}g^{(c)(d)}) ,$$

and expanding the result to second order in powers of the coordinate differences Δx^μ, we find that the only nonzero tetrad components of the regular part of the graviton Green's function are

$$W^{(1)(1)(2)(2)}=2W^{(1)(2)(3)(4)}=W ,$$

$$W^{(1)(3)(2)(4)}=W^{(2)(3)(1)(4)}=-\tfrac{1}{2}W^{(3)(4)} ,$$

$$W^{(3)(3)(4)(4)}=\frac{1}{6\xi^2}\left\{23+23\frac{\Delta\xi}{\xi}+32\Delta\tau+\frac{251}{60}\frac{(\Delta y)^2+(\Delta z)^2}{\xi^2}+\frac{1631}{60}\frac{(\Delta\xi)^2}{\xi^2}+32\frac{\Delta\xi\Delta\tau}{\xi}+\frac{481}{20}(\Delta\tau)^2\right\}+O((\Delta x^\mu)^3) , \tag{2.20}$$

plus all others obtainable from the symmetries

$$W^{(a)(b)(c)(d)}(x,x')=W^{(b)(a)(c)(d)}(x,x')=W^{(a)(b)(d)(c)}(x,x') ,$$

$$W^{(a)(b)(c)(d)}(x,x')=W^{(c)(d)(a)(b)}(x',x) .$$

[Here W and $W^{(3)(4)}$ are to be replaced by the expansions (2.5) and (2.12), respectively.]

Next, we determine the coefficients occurring in the covariant Taylor expansion

$$W^{(a)(b)(c)(d)}(x,x')=w^{(a)(b)(c)(d)}(x)+w^{(a)(b)(c)(d)}{}_{\mu}(x)\sigma^\mu+\tfrac{1}{2}w^{(a)(b)(c)(d)}{}_{\mu\nu}(x)\sigma^\mu\sigma^\nu+\cdots .$$

In fact, we only have to calculate the coefficients of $W^{(3)(3)(4)(4)}$, since those of the remaining components are electromagnetic or scalar coefficients which we have computed above. Direct calculation yields

$$\begin{aligned}
&w^{(3)(3)(4)(4)}=\frac{23}{6\xi^2} , \quad w^{(3)(3)(4)(4)}{}_{\xi}=\frac{23}{6\xi^3} ,\\
&w^{(3)(3)(4)(4)}{}_{\tau}=\frac{16}{3\xi^2} , \quad w^{(3)(3)(4)(4)}{}_{\xi\xi}=\frac{1631}{180\xi^4} ,\\
&w^{(3)(3)(4)(4)}{}_{\tau\tau}=\frac{711}{60\xi^2} ,\\
&w^{(3)(3)(4)(4)}{}_{\xi\tau}=w^{(3)(3)(4)(4)}{}_{\tau\xi}=\frac{32}{3\xi^3} ,\\
&w^{(3)(3)(4)(4)}{}_{yy}=w^{(3)(3)(4)(4)}{}_{zz}=\frac{251}{180\xi^4}
\end{aligned} \tag{2.21}$$

(with all others zero). Next, we obtain the symmetric and antisymmetric parts of these coefficients

$$s^{\mu\nu\rho\tau}{}_{\alpha\cdots\beta}=\tfrac{1}{2}(e^\mu_{(a)}e^\nu_{(b)}e^\rho_{(c)}e^\tau_{(d)}+e^\rho_{(a)}e^\tau_{(b)}e^\mu_{(c)}e^\nu_{(d)})w^{(a)(b)(c)(d)}{}_{\alpha\cdots\beta}$$

and

$$a^{\mu\nu\rho\tau}{}_{\alpha\cdots\beta}=\tfrac{1}{2}(e^\mu_{(a)}e^\nu_{(b)}e^\rho_{(c)}e^\tau_{(d)}-e^\rho_{(a)}e^\tau_{(b)}e^\mu_{(c)}e^\nu_{(d)})w^{(a)(b)(c)(d)}{}_{\alpha\cdots\beta}$$

and check that our answers satisfy the identities [3]

$$\begin{aligned}
&a^{\mu\nu\rho\tau\alpha}{}_{;\alpha}=0 , \quad s^{\mu\nu\rho\tau\alpha}{}_{\alpha}=0 ,\\
&s^{\mu\nu\rho\tau\alpha\beta}{}_{;\beta}=\tfrac{1}{4}\Box(s^{\mu\nu\rho\tau;\alpha}) ,\\
&s^{\mu}{}_{\mu\rho\tau;\nu}-2s_{\nu\mu\rho\tau}{}^{;\mu}+2a^{\mu}{}_{\mu\rho\tau\nu}-4a_{\nu\mu\rho\tau}{}^{\mu}\\
&\qquad=-4\tilde{a}_{\nu(\rho\tau)}+2\tilde{s}_{\nu(\rho;\tau)} ,\\
&s^{\sigma}{}_{\sigma\nu\rho;\tau\mu}-2s_{\sigma\mu\nu\rho;\tau}{}^{\sigma}+4a^{\sigma}{}_{\mu\nu\rho[\tau;\sigma]}-2a^{\sigma}{}_{\sigma\nu\rho[\tau;\mu]}\\
&\qquad-2s^{\sigma}{}_{\sigma\nu\rho\tau\mu}+4s_{\sigma\mu\nu\rho\tau}{}^{\sigma}=4\tilde{s}_{\mu(\nu\rho)\tau}-2\tilde{a}_{\mu(\nu\rho);\tau}-2\tilde{a}_{\mu(\nu|\tau|;\rho)} ,
\end{aligned} \tag{2.22}$$

the first three of which follow from the wave equation and the last two from the Ward identity. It is now a straightforward, if tedious, calculation to derive from the formulas of Ref. [3] the vacuum expectation value of the standard renormalized stress-energy tensor for the graviton field on Rindler space:

$$\langle 0|T_\mu{}^\nu|0\rangle_R=\frac{1}{720\pi^2\xi^4}\mathrm{diag}(753,-251,1459,1459)_\mu{}^\nu . \tag{2.23}$$

We now compute the VDW correction [3] to this stress-energy tensor, which requires knowledge of not only the graviton and photon Green's functions, but also of the convolution

$$\dot{G}^{(a)(b')}(x,x')=\int\sqrt{-g(x'')}d^4x''G^{(a)}{}_{(c'')}(x,x'')G^{(c'')(b')}(x'',x') ,$$

of photon Green's functions. To calculate $\dot{G}^{(a)(b')}(x,x')$ explicitly, we use the alternative representation [3]

$$\dot{G}^{(a)(b')}(x,x')=-\left[\frac{\partial}{\partial M^2}G^{(a)(b')}(x,x';M^2)\right]_{M^2=0} ,$$

where $G^{(a)(b')}(x,x';M^2)$ is the Green's function for a massive vector field which satisfies

$$(\Box - M^2)G^{\mu\nu'}(x,x';M^2) = -g^{\mu\nu'}\delta^4(x,x') \ . \tag{2.24}$$

We may solve (2.24) using the Euclidean mode decomposition method of Ref. [8]. [Note that, as we are taking the limit $\kappa \to 0$, we need to consider the special case $0<\kappa<1$ discussed in Sec. III of Ref. [8]. This is responsible for the apparently strange subscripts on the modified Bessel functions in Eq. (2.26).] We obtain the following nonvanishing, independent tetrad components of the Euclidean Green's function:

$$G_E^{(1)(2')}(M^2) = -\frac{1}{4\pi^2}\int_{-\infty}^{\infty} d\omega\, e^{i\omega\Delta\psi}\int_0^{\infty} dk\, kJ_0(k|\Delta\mathbf{y}|)I_{|\omega|}[(k^2+M^2)^{1/2}\xi_<]K_{|\omega|}[(k^2+M^2)^{1/2}\xi_>] \ , \tag{2.25}$$

$$\begin{aligned} G_E^{(3)(4')}(M^2) = {} & \frac{1}{4\pi^2}\int_0^{\infty} d\omega\, e^{i\omega\Delta\psi}\int_0^{\infty} dk\, kJ_0(k|\Delta\mathbf{y}|)I_{\omega-1}[(k^2+M^2)^{1/2}\xi_<]K_{\omega-1}[(k^2+M^2)^{1/2}\xi_>] \\ & + \frac{1}{4\pi^2}\int_{-\infty}^{0} d\omega\, e^{i\omega\Delta\psi}\int_0^{\infty} dk\, kJ_0(k|\Delta\mathbf{y}|)I_{1-\omega}[(k^2+M^2)^{1/2}\xi_<]K_{1-\omega}[(k^2+M^2)^{1/2}\xi_>] \ , \end{aligned} \tag{2.26}$$

where $\tau = -i\psi$ and $\mathbf{y}=(y,z)$.

Consider Eq. (2.25); using GR 6.541.1 we can write the product of the modified Bessel functions as an integral of ordinary Bessel functions:

$$\begin{aligned} G_E^{(1)(2')}(M^2) &= -\frac{1}{4\pi^2}\int_{-\infty}^{\infty} d\omega\, e^{i\omega\Delta\psi}\int_0^{\infty} dk\, kJ_0(k|\Delta\mathbf{y}|)\int_0^{\infty}\frac{dp\,p}{p^2+k^2+M^2}J_{|\omega|}(p\xi)J_{|\omega|}(p\xi') \\ &= -\frac{1}{2\pi^2}\int_0^{\infty} d\omega\cos\omega\,\Delta\psi\int_0^{\infty} dk\, kJ_0(k|\Delta\mathbf{y}|)\int_0^{\infty}\frac{dp\,p}{p^2+k^2+M^2}J_{\omega}(p\xi)J_{\omega}(p\xi') \ . \end{aligned} \tag{2.27}$$

We can now differentiate this equation with respect to M^2 and perform the k integration using GR 6.565.4, yielding

$$\frac{\partial G_E^{(1)(2')}(M^2)}{\partial M^2} = \frac{|\Delta\mathbf{y}|}{4\pi^2}\int_0^{\infty} d\omega\cos\omega\,\Delta\psi\int_0^{\infty}\frac{dp\,p}{(p^2+M^2)^{1/2}}K_{-1}[(p^2+M^2)^{1/2}|\Delta\mathbf{y}|]J_{\omega}(p\xi)J_{\omega}(p\xi') \ . \tag{2.28}$$

When $M^2=0$ the p integral can be computed using GR 6.578.6. We find that

$$\dot{G}_E^{(1)(2')} = \frac{1}{8\pi^2}\int_0^{\infty}\frac{d\omega}{\omega}e^{-\omega\eta}\cos\omega\,\Delta\psi \tag{2.29}$$

(we have also used GR 8.736.4, 8.754.4 to obtain the result in the form shown). This integral clearly contains the standard infrared divergence expected in the massless convolution [3]. Introducing an infrared cutoff, we have

$$\begin{aligned} \dot{G}^{(1)(2')} &= \frac{i}{8\pi^2}\lim_{\epsilon\to 0}\left[\int_{\epsilon}^{\infty}\frac{d\omega}{\omega}e^{-\omega\eta}\cosh\omega\,\Delta\tau\right] \\ &= \frac{-i}{16\pi^2}\lim_{\epsilon\to 0}[\mathrm{Ei}(-\epsilon(\eta+\Delta\tau)) \\ &\qquad + \mathrm{Ei}(-\epsilon(\eta-\Delta\tau))] \ , \end{aligned} \tag{2.30}$$

in the Lorentzian space-time, assuming $\eta^2>(\Delta\tau)^2$. Here $\mathrm{Ei}(x)$ is the exponential-integral function defined by

$$\mathrm{Ei}(x) = \int_{-\infty}^{x}\frac{e^t}{t}dt \ ,$$

where for $x>0$ the integral is understood to mean

$$\int_{-\infty}^{x} \equiv \lim_{\delta\to 0}\left[\int_{-\infty}^{-\delta} + \int_{\delta}^{x}\right] \ .$$

Making use of the expansion (GR 8.214)

$$\mathrm{Ei}(x) = C + \ln|x| + \sum_{k=1}^{\infty}\frac{x^k}{k\cdot k!} \tag{2.31}$$

(C is Euler's constant), (2.30) becomes

$$\dot{G}^{(1)(2')} = \frac{-i}{16\pi^2}\ln[\eta^2-(\Delta\tau)^2] + \mathrm{const} \tag{2.32}$$

(where the infrared divergence has been absorbed into the constant).

The next task is to repeat the above argument for $\dot{G}_E^{(3)(4')}$ given by (2.26). A similar analysis to the above reveals

$$\dot{G}_E^{(3)(4')} = -\frac{1}{16\pi^2}\int_0^{\infty} d\omega\left\{e^{\eta}\frac{e^{-\omega(\eta-i\Delta\psi)}}{\omega-1} + e^{-\eta}\frac{e^{-\omega(\eta+i\Delta\psi)}}{\omega+1}\right\} ,$$

which may be written in closed form on the Lorentzian space-time as

$$\dot{G}^{(3)(4')} = \frac{i}{16\pi^2}e^{-\Delta\tau}[\mathrm{Ei}(\eta+\Delta\tau) + \mathrm{Ei}(-\eta+\Delta\tau)] \ , \tag{2.33}$$

assuming $\eta^2>(\Delta\tau)^2$. Note that our answers are consistent with the equations

$$\Box\dot{G}^{(1)(2')} = G^{(1)(2')}(M^2=0)$$

and

$$\left[\Box - \frac{2}{\xi^2}\frac{\partial}{\partial\tau} - \frac{1}{\xi^2}\right]\dot{G}^{(3)(4')} = G^{(3)(4')}(M^2=0) \ ,$$

which follow from differentiating the tetrad version of (2.24) with respect to M^2 and then setting $M^2=0$.

We must now calculate $\dot{W}^{(a)(b)}$, which is given by the formula [3]

$$\dot{W}^{(a)(b)}=g^{(b)}{}_{(b')}\left[\frac{8\pi^2}{i}\dot{G}^{(a)(b')}-\dot{V}^{(a)(b')}\ln\sigma\right],$$

and since the Rindler space-time is flat, we know that $\dot{V}^{(a)(b')}=\frac{1}{2}g^{(a)(b')}$. Noting that, in Rindler coordinates,

$$\sigma=\xi\xi'(\cosh\eta-\cosh\Delta\tau),$$

we obtain

$$\dot{W}^{(1)(2)}=\frac{1}{2}\ln\left\{\frac{\xi\xi'(\cosh\eta-\cosh\Delta\tau)}{\eta^2-(\Delta\tau)^2}\right\}+\text{const}$$
$$=\text{const}+\ln\xi-\frac{1}{2}\frac{\Delta\xi}{\xi}+\frac{1}{24}(\Delta\tau)^2-\frac{5}{24}\frac{(\Delta\xi)^2}{\xi^2}+\frac{1}{24}\frac{(\Delta y)^2+(\Delta z)^2}{\xi^2}+O((\Delta x^\mu)^3)$$

and

$$\dot{W}^{(3)(4)}=\frac{1}{2}\{\text{Ei}(\Delta\tau+\eta)+\text{Ei}(-\eta+\Delta\tau)-\ln[\xi\xi'(\cosh\eta-\cosh\Delta\tau)]\}$$
$$=\text{const}-\ln\xi+\frac{1}{2}\frac{\Delta\xi}{\xi}+\Delta\tau+\frac{5}{24}(\Delta\tau)^2+\frac{11}{24}\frac{(\Delta\xi)^2}{\xi^2}+\frac{5}{24}\frac{(\Delta y)^2+(\Delta z)^2}{\xi^2}+O((\Delta x^\mu)^3).$$

Taking derivatives and coincidence limits in the usual fashion, we obtain the following Taylor-series coefficients:

$$\dot{w}^{(1)(2)}=\ln\xi+\text{const},\quad \dot{w}^{(3)(4)}=-\ln\xi+\text{const},$$
$$\dot{w}^{(1)(2)}{}_{\xi}=-\frac{1}{2\xi},\quad \dot{w}^{(3)(4)}{}_{\xi}=\frac{1}{2\xi},\quad \dot{w}^{(3)(4)}{}_{\tau}=1,$$
$$\dot{w}^{(1)(2)}{}_{yy}=\dot{w}^{(1)(2)}{}_{zz}=\frac{1}{12\xi^2},$$
$$\dot{w}^{(1)(2)}{}_{\tau\tau}=-\frac{5}{12},\quad \dot{w}^{(1)(2)}{}_{\xi\xi}=-\frac{5}{12\xi^2},$$
$$\dot{w}^{(3)(4)}{}_{yy}=\dot{w}^{(3)(4)}{}_{zz}=\frac{5}{12\xi^2},$$
$$\dot{w}^{(3)(4)}{}_{\xi\xi}=\frac{11}{12\xi^2},\quad \dot{w}^{(3)(4)}{}_{\tau\tau}=\frac{11}{12},$$

with the only other nonzero coefficients of $\dot{W}^{(a)(b)}$ being those of $\dot{W}^{(2)(1)}=\dot{W}^{(1)(2)}$ and those of $\dot{W}^{(4)(3)}$, which are plus or minus those of $\dot{W}^{(3)(4)}$ according as there are an even or odd number of occurrences of τ in the subscript. From these coefficients we can then construct

$$\dot{s}^{\mu\nu}{}_{\alpha\cdots\beta}=e^{(\mu}_{(a)}e^{\nu)}_{(b)}\dot{w}^{(a)(b)}{}_{\alpha\cdots\beta},$$
$$\dot{a}^{\mu\nu}{}_{\alpha\cdots\beta}=e^{[\mu}_{(a)}e^{\nu]}_{(b)}\dot{w}^{(a)(b)}{}_{\alpha\cdots\beta},$$

and check that [3]

$$\dot{s}^{\mu\nu\alpha}{}_{\alpha}=s^{\mu\nu},\quad \dot{a}^{\mu\nu\alpha}{}_{;\alpha}=0,$$
$$\dot{s}^{\mu\nu\alpha\beta}{}_{;\beta}=\tfrac{1}{4}(\Box\dot{s}^{\mu\nu})^{;\alpha} \tag{2.34}$$

(where $s^{\mu\nu}$ is a coefficient associated with the massless photon Green's function). We finally have all the coefficients required to compute the VDW correction term contained in Ref. [3] and obtain the result

$$\langle 0|T_\mu{}^\nu|0\rangle_R^C=\frac{1}{24\pi^2\xi^4}\text{diag}(45,-15,-8,-8)_\mu{}^\nu. \tag{2.35}$$

Thus the VDW approach to quantization genuinely does make a difference to the physical output of the theory.

By adding on the correction term (2.35) to the standard result (2.23), we obtain the renormalized stress-energy tensor in the VDW approach:

$$\langle 0|T_\mu{}^\nu|0\rangle_R^{\text{VDW}}=\frac{1}{720\pi^2\xi^4}\text{diag}(2103,-701,1219,1219)_\mu{}^\nu. \tag{2.36}$$

Note that the trace of this quantity is simply

$$\langle 0|T_\mu{}^\mu|0\rangle_R^{\text{VDW}}=\frac{16}{3\pi^2\xi^4}.$$

It is interesting to compare the expressions we have found for the renormalized stress-energy tensors for all three types of quantum field. We have the following results for the energy densities of the quantum fields:

$$-\langle T_0{}^0\rangle_R\approx\frac{1}{480\pi^2\xi^4}\begin{cases}-1 & \text{spin 0 } (\zeta=\frac{1}{6}),\\ -22 & \text{spin 1},\\ -502 & \text{spin 2 (standard)},\\ -1402 & \text{spin 2 (VDW)}.\end{cases} \tag{2.37}$$

As one might expect, the graviton contribution to the total vacuum polarization strongly dominates that from conformally invariant particles whether we use the standard or VDW approach; however, the effect is much more pronounced in the VDW approach.

III. CASIMIR EFFECT

The simplest example of nontrivial space-time structure affecting the energy density of the vacuum is obtained by introducing an infinite conducting plate into Minkowski space. Although this example has been extensively studied in the past for both scalar and electromagnetic fields (see, for example, Ref. [10] for the electromagnetic case), it has never before been subjected to

the Hadamard-renormalization procedure; the following calculations are remarkable both in their simplicity and in the way in which they highlight the fundamental role played by the Ward identities in the theory. We also consider here for the first time the Casimir effect for gravitons, according to both the standard and VDW approaches. Although this case is unphysical, the results are so striking from a theoretical point of view as to warrant their consideration.

We work in Cartesian coordinates (t,x,y,z). The plate is positioned in the $z=0$ plane, and we require that the field satisfy Dirichlet boundary conditions upon it, in a sense to be made precise below.

A. Scalar field

For a massless scalar field satisfying $\varphi=0$ on $z=0$, it follows immediately by the method of images that the Green's function is

$$G(x,x')=G_0(x,x')-G_0(x,\bar{x}') , \tag{3.1}$$

where G_0 denotes the Green's function for uncluttered Minkowski space and

$$x\equiv(t,x,y,z) , \quad \bar{x}\equiv(t,x,y,-z) .$$

Thus

$$G(x,x')=\frac{i}{8\pi^2}\left\{\frac{1}{\sigma(x,x')}-\frac{1}{\sigma(x,\bar{x}')}\right\} , \tag{3.2}$$

where $\sigma(x,x')$ is one-half the square of the distance along a geodesic joining x to x' in Minkowski space, i.e.,

$$\sigma(x,x')=\tfrac{1}{2}\{-(\Delta t)^2+(\Delta x)^2+(\Delta y)^2+(\Delta z)^2\} . \tag{3.3}$$

It is clear that (3.2) satisfies the wave equation $\Box G(x,x')=-\delta^4(x,x')$ and vanishes whenever either z or z' is zero. The regular part [4] of the Green's function is

$$W(x,x')=-\frac{1}{\sigma(x,\bar{x}')} , \tag{3.4}$$

which, using (3.3), may be expanded as follows in terms of the coordinate differences Δx^μ:

$$W(x,x')=-\frac{1}{2z^2}\left\{1+\frac{\Delta z}{z}+\frac{1}{4z^2}[(\Delta t)^2-(\Delta x)^2-(\Delta y)^2+3(\Delta z)^2]+O((\Delta x^\mu)^3)\right\} . \tag{3.5}$$

In order to compute the Hadamard-renormalized stress-energy tensor, we require the first three coefficients occurring in the covariant Taylor expansion of $W(x,x')$. Following the procedure outlined in Sec. II, we deduce from (3.5) that the only nonvanishing coefficients are

$$w=-\frac{1}{2z^2} , \quad w_z=-\frac{1}{2z^3} ,$$
$$w_{tt}=-w_{xx}=-w_{yy}=\frac{1}{3}w_{zz}=-\frac{1}{4z^4} . \tag{3.6}$$

One can easily check that these expressions satisfy the various identities which follow from the anomalous wave equation and symmetry constraint for $W(x,x')$.

Substituting (3.6) into the relevant formulas of Ref. [4], we find that the renormalized stress-energy tensor is

$$\langle 0|T_\mu{}^\nu|0\rangle_R=\frac{1-6\zeta}{16\pi^2z^4}(\delta_\mu{}^\nu-\hat{z}_\mu\hat{z}^\nu) , \tag{3.7}$$

where ζ is the coupling constant associated with the massless scalar field and $\hat{z}^\mu\equiv\delta_3{}^\mu$ is the unit normal to the conducting plate. The above expression is both conserved and consistent with the symmetry of the system; it also vanishes when the field is conformally coupled to the background geometry ($\zeta=\frac{1}{6}$), as expected.

B. Electromagnetic field

The photon Green's function in uncluttered Minkowski space is

$$G_0^{\mu\nu'}(x,x')=\frac{i}{8\pi^2}\frac{g^{\mu\nu'}}{\sigma(x,x')} ,$$

where $g_\mu{}^{\nu'}=\mathrm{diag}[1,1,1,1]_\mu{}^{\nu'}$ is the bivector of parallel transport for Minkowski space. Using the method of images, we deduce that when a plane conductor is introduced at $z=0$, the photon Green's function becomes

$$G_\mu{}^{\nu'}(x,x')=\frac{i}{8\pi^2}\left\{\frac{g_\mu{}^{\nu'}}{\sigma(x,x')}-\frac{\bar{g}_\mu{}^{\nu'}}{\sigma(x,\bar{x}')}\right\} , \tag{3.8}$$

where

$$\bar{g}_\mu{}^{\nu'}=\frac{\partial x^{\nu'}}{\partial\bar{x}^{\alpha'}}g_\mu{}^{\alpha'}=g_\mu{}^{\nu'}-2\hat{z}_\mu\hat{z}^{\nu'} \tag{3.9}$$

is the reflection of $g_\mu{}^{\nu'}$ in the $z=0$ plane. It is straightforward to verify that (3.8) satisfies the wave equation

$$\Box G_\mu{}^{\nu'}(x,x')=-g_\mu{}^{\nu'}\delta^4(x,x')$$

and Ward identity

$$G_\mu{}^{\nu';\mu}+G^{;\nu'}=0 ,$$

with G given by (3.2); it also ensures that the electromagnetic potential $A^\mu(x)$ will obey Dirichlet boundary conditions

$${}^*F^{\mu\nu}\hat{z}_\nu=0 ,$$

on the plane $z=0$, where ${}^*F^{\mu\nu}\equiv\frac{1}{2}\epsilon^{\mu\nu}{}_{\rho\lambda}F^{\rho\lambda}$ is the dual of the antisymmetric field-strength tensor $F^{\rho\lambda}=2A^{[\lambda;\rho]}$.

The regular part of the photon Green's function is thus

$$W_\mu{}^{\nu'}(x,x')=-\frac{\bar{g}_\mu{}^{\nu'}}{\sigma(x,\bar{x}')}=\bar{g}_\mu{}^{\nu'}W(x,x') , \tag{3.10}$$

where $W(x,x')$ is the regular part (3.4) of the scalar Green's function. By virtue of (3.9) the first three coefficients occurring in the covariant Taylor series of $g^\nu{}_{\nu'}W_\mu{}^{\nu'}$ are just

$$w_{\mu}{}^{\nu}{}_{\alpha\cdots\beta}(x)=(\delta_{\mu}{}^{\nu}-2\hat{z}_{\mu}\hat{z}^{\nu})w_{\alpha\cdots\beta}(x)\ , \tag{3.11}$$

where the scalar coefficients $w_{\alpha\cdots\beta}(x)$ are given explicitly by (3.6). The splitting of these coefficients into their symmetric and antisymmetric parts is trivial in this case since $w^{[\mu\nu]}{}_{\alpha\cdots\beta}$ clearly vanishes; we can thus proceed directly to the next step in the procedure, which is to check that our expressions for the coefficients obey the various identities [4] which arise from the symmetry constraint, wave equation, and Ward identity for the photon Green's function. With this done we can finally utilize the formulas of Ref. [4] to obtain the renormalized stress-energy tensor for the system; we find that it vanishes as expected for a conformally invariant field in the neighborhood of an infinite conducting plate.

C. Graviton field

In uncluttered Minkowski space the graviton Green's function is

$$G_0^{\mu\nu\rho'\tau'}(x,x')=\frac{i}{8\pi^2}\frac{\gamma^{\mu\nu\rho'\tau'}}{\sigma(x,x')}\ , \tag{3.12}$$

where

$$\gamma_{\mu\nu}{}^{\rho'\tau'}=g_{(\mu}{}^{\rho'}g_{\nu)}{}^{\tau'}-\tfrac{1}{2}g_{\mu\nu}g^{\rho'\tau'}\ .$$

If we now introduce the boundary at $z=0$, then by the method of images the graviton Green's function becomes

$$G_{\mu\nu}{}^{\rho'\tau'}(x,x')=\frac{i}{8\pi^2}\left\{\frac{\gamma_{\mu\nu}{}^{\rho'\tau'}}{\sigma(x,x')}-\frac{\bar{\gamma}_{\mu\nu}{}^{\rho'\tau'}}{\sigma(x,\bar{x}')}\right\}\ , \tag{3.13}$$

where

$$\begin{aligned}\bar{\gamma}_{\mu\nu}{}^{\rho'\tau'}&\equiv\frac{\partial x^{\rho'}}{\partial\bar{x}^{\alpha'}}\frac{\partial x^{\tau'}}{\partial\bar{x}^{\beta'}}\gamma_{\mu\nu}{}^{\alpha'\beta'}\\&=g_{(\mu}{}^{\rho'}g_{\nu)}{}^{\tau'}-\tfrac{1}{2}g_{\mu\nu}g^{\rho'\tau'}+4\hat{z}_{\mu}\hat{z}_{\nu}\hat{z}^{\rho'}\hat{z}^{\tau'}\\&\quad-4g_{(\mu}{}^{(\rho'}\hat{z}_{\nu)}\hat{z}^{\tau')}\ .\end{aligned} \tag{3.14}$$

One can verify that (3.13) satisfies the wave equation

$$\Box G_{\mu\nu}{}^{\rho'\tau'}(x,x')=-\gamma_{\mu\nu}{}^{\rho'\tau'}\delta^4(x,x')$$

and Ward identity

$$G_{\mu\nu}{}^{\rho'\tau';\mu}-\tfrac{1}{2}G_{\mu}{}^{\mu\rho'\tau'}{}_{;\nu}+G_{\nu}{}^{(\rho';\tau')}=0\ ,$$

with $G_{\nu}{}^{\rho'}$ given by (3.8); it also ensures that the graviton field $h^{\mu\nu}(x)$ will satisfy Dirichlet boundary conditions

$${}^*\dot{R}^{\mu\nu\rho\lambda}\hat{z}_{\nu}\hat{z}_{\lambda}=0\ ,$$

on the plane $z=0$, where

$${}^*\dot{R}^{\mu\nu\rho\lambda}=\tfrac{1}{4}\epsilon^{\mu\nu}{}_{\alpha\beta}\epsilon^{\rho\lambda}{}_{\gamma\delta}\dot{R}^{\alpha\beta\gamma\delta}\ ,$$

and $\dot{R}^{\alpha\beta\gamma\delta}$ denotes the first-order variation of $R^{\alpha\beta\gamma\delta}$.

The regular part of the graviton Green's function is

$$\begin{aligned}W_{\mu\nu}{}^{\rho'\tau'}(x,x')&=-\frac{\bar{\gamma}_{\mu\nu}{}^{\rho'\tau'}}{\sigma(x,\bar{x}')}\\&=\bar{\gamma}_{\mu\nu}{}^{\rho'\tau'}W(x,x')\ ,\end{aligned} \tag{3.15}$$

and so the first three Taylor coefficients of $g^{\rho}{}_{\rho'}g^{\tau}{}_{\tau'}W_{\mu\nu}{}^{\rho'\tau'}$ may be expressed in terms of the scalar coefficients (3.6) according to the formula

$$\begin{aligned}w_{\mu\nu}{}^{\rho\tau}{}_{\alpha\cdots\beta}(x)=(&\delta_{(\mu}{}^{\rho}\delta_{\nu)}{}^{\tau}-\tfrac{1}{2}g_{\mu\nu}g^{\rho\tau}+4\hat{z}_{\mu}\hat{z}_{\nu}\hat{z}^{\rho}\hat{z}^{\tau}\\&-4\delta_{(\mu}{}^{(\rho}\hat{z}_{\nu)}\hat{z}^{\tau)})w_{\alpha\cdots\beta}(x)\ ,\end{aligned} \tag{3.16}$$

where we have used (3.14). We see immediately that

$$\begin{aligned}s^{\mu\nu\rho\tau}{}_{\alpha\cdots\beta}&=\tfrac{1}{2}(w^{\mu\nu\rho\tau}{}_{\alpha\cdots\beta}+w^{\rho\tau\mu\nu}{}_{\alpha\cdots\beta})\\&=w^{\mu\nu\rho\tau}{}_{\alpha\cdots\beta}\ ,\\a^{\mu\nu\rho\tau}{}_{\alpha\cdots\beta}&=\tfrac{1}{2}(w^{\mu\nu\rho\tau}{}_{\alpha\cdots\beta}-w^{\rho\tau\mu\nu}{}_{\alpha\cdots\beta})=0\ ,\end{aligned}$$

and can easily check that the identities (2.22) are satisfied. Substituting for the above coefficients in the formula of Allen, Folacci, and Ottewill [3] for the standard renormalized stress-energy tensor, we find that

$$\langle 0|T_{\mu}{}^{\nu}(x)|0\rangle_R=\frac{-3}{4\pi^2z^4}(\delta_{\mu}{}^{\nu}-\hat{z}_{\mu}^{3}\hat{z}_{3}^{\nu})\ . \tag{3.17}$$

In order to determine the VDW correction to this expectation value, we must once again consider the massive photon Green's function for the theory. In uncluttered Minkowski space this Green's function is [11]

$$\begin{aligned}&G_0^{\mu\nu'}(x,x';M^2)\\&\quad=\frac{g^{\mu\nu'}}{16\pi i}\left[\frac{2M^2}{\sigma(x,x')}\right]^{1/2}H_1^{(2)}\{[-2M^2\sigma(x,x')]^{1/2}\}\ ,\end{aligned}$$

from which we deduce that

$$\begin{aligned}\dot{G}_0^{\mu\nu'}(x,x')&=-\left[\frac{\partial}{\partial M^2}G_0^{\mu\nu'}(x,x';M^2)\right]_{M^2=0}\\&=\frac{ig^{\mu\nu'}}{16\pi^2}\{\ln\sigma(x,x')+\text{const}\}\ ,\end{aligned}$$

where the constant includes the standard infrared divergence. So, when a plane conductor is introduced to the space-time at $z=0$, we obtain

$$\begin{aligned}\dot{G}_{\mu}{}^{\nu'}(x,x')=\frac{i}{16\pi^2}\{&g_{\mu}{}^{\nu'}\ln\sigma(x,x')-\bar{g}_{\mu}{}^{\nu'}\ln\sigma(x,\bar{x}')\\&+\text{const}\times(g_{\mu}{}^{\nu'}-\bar{g}_{\mu}{}^{\nu'})\}\ ,\end{aligned} \tag{3.18}$$

using the method of images once again. It is straightforward to check that this expression satisfies the identity

$$\Box\dot{G}_{\mu}{}^{\nu'}=G_{\mu}{}^{\nu'}\ ,$$

where $G_{\mu}{}^{\nu'}$ is the massless photon Green's function (3.8).

The regular part of $\dot{G}_{\mu}{}^{\nu'}$ is

$$\begin{aligned}\dot{W}_{\mu}{}^{\nu}&=g^{\nu}{}_{\nu'}\left[\frac{8\pi^2}{i}\dot{G}_{\mu}{}^{\nu'}-\dot{V}_{\mu}{}^{\nu'}\ln\sigma\right]\\&=-\tfrac{1}{2}(\delta_{\mu}{}^{\nu}-2\hat{z}_{\mu}\hat{z}^{\nu})\ln(\sigma+2zz')+\text{const}\times\hat{z}_{\mu}\hat{z}^{\nu}\ ,\end{aligned}$$

using (3.18) and the fact that $\dot{V}_{\mu}{}^{\nu'}=\tfrac{1}{2}g_{\mu}{}^{\nu'}$ in a flat space-time. Expanding in powers of the coordinate differences Δx^{μ}, we have

$$\dot{W}_\mu{}^\nu = \text{const} \times \hat{z}_\mu \hat{z}^\nu - \tfrac{1}{2}(\delta_\mu{}^\nu - 2\hat{z}_\mu \hat{z}^\nu)\left\{ \ln 2 + 2\ln z - \frac{\Delta z}{z} + \frac{(\Delta x)^2 + (\Delta y)^2 - (\Delta t)^2 - (\Delta z)^2}{4z^2} + O((\Delta x^\mu)^3) \right\}$$

and obtain the coefficients

$$\dot{w}_\mu{}^\nu = \text{const} \times \hat{z}_\mu \hat{z}^\nu - \tfrac{1}{2}(\delta_\mu{}^\nu - 2\hat{z}_\mu \hat{z}^\nu)(\ln 2 + 2\ln z) ,$$

$$\dot{w}_\mu{}^\nu{}_\alpha = (\delta_\mu{}^\nu - 2\hat{z}_\mu \hat{z}^\nu)\frac{\hat{z}_\alpha}{2z} ,$$

$$\dot{w}_\mu{}^\nu{}_{\alpha\beta} = -(\delta_\mu{}^\nu - 2\hat{z}_\mu \hat{z}^\nu)(g_{\alpha\beta} - 2\hat{z}_\alpha \hat{z}_\beta)\frac{1}{4z^2} .$$

Again, these coefficients are purely symmetric; one can also check that they satisfy the identities (2.34). We are now in a position to compute the VDW correction to the standard stress-energy tensor (3.17); using the appropriate formula in Ref. [3], we find

$$\langle 0|T_\mu{}^\nu(x)|0\rangle_R^C = \frac{3}{4\pi^2 z^4}(\delta_\mu{}^\nu - \hat{z}_\mu \hat{z}^\nu) ,$$

so that remarkably $\langle 0|T_\mu{}^\nu|0\rangle_R^{\text{VDW}} = 0$. We conclude once again that the VDW modification does make a difference to the physical output of a quantum field theory.

IV. CONCLUSION

We have shown by means of explicit examples that the one-loop theories of quantum general relativity about a given Ricci flat (indeed flat) background obtained from the standard and Vilkovisky-DeWitt effective actions differ. While one may challenge the use of the word "physical" to describe the predictions of the one-loop approximation to a nonrenormalizable theory, one may still hope that in the absence of a full theory the one-loop theory may provide some insight into quantum gravitational effects. In this light the differences we have found are significant. In any case, in a theory as complicated as that of the Vilkovisky-DeWitt effective action, exact solutions such as those we have given are an invaluable guide.

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