Analytic results for the gravitational radiation from a class of cosmic string loops

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Cosmic string loops are defined by a pair of periodic functions **a** and **b**, which trace out unit-length closed curves in three-dimensional space. We consider a particular class of loops, for which **a** lies along a line and **b** lies in the plane orthogonal to that line. For this class of cosmic string loops one may give a simple analytic expression for the power γ radiated in gravitational waves. We evaluate γ exactly in closed form for several special cases: (1) **b** a circle traversed M times; (2) **b** a regular polygon with N sides and interior vertex angle $\pi - 2\pi M/N$; (3) **b** an isosceles triangle with semiangle θ. We prove that case (1) with $M = 1$ is the absolute minimum of γ within our special class of loops, and identify all the stationary points of γ in this class.

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I. INTRODUCTION

Cosmic strings are topological defects that may have formed at phase transitions as the Universe expanded and cooled [1–4]. Loops of cosmic string oscillate and emit gravitational radiation. This process is of particular importance because most of the observational limits on cosmic string networks in the early Universe are obtained by considering the effects of this gravitational radiation ([4,5] and references therein).

The power emitted in gravitational radiation by a cosmic string loop depends upon its shape and velocity. In the center-of-mass frame, a cosmic string loop is specified by the position $\mathbf{x}(t,\sigma)$ of the string as a function of two variables: time t and a spacelike parameter σ that runs from 0 to L. (The total energy of the loop is μL where μ is the mass per unit length of the string.) When gravitational back reaction is neglected, the string loop satisfies equations of motion whose most general solution in the center-of-mass frame is

$$\mathbf{x}(t,\sigma) = \tfrac{1}{2}\left[\mathbf{a}(t+\sigma) + \mathbf{b}(t-\sigma)\right]. \tag{1.1}$$

Here $\mathbf{a}(u) \equiv \mathbf{a}(u+L)$ and $\mathbf{b}(v) \equiv \mathbf{b}(v+L)$ are a pair of periodic functions, satisfying the "gauge condition" $|\mathbf{a}'(u)| = |\mathbf{b}'(v)| = 1$, where a prime denotes differentiation with respect to the function's argument. The average power radiated by an oscillating string loop is given by

$$P = \gamma G \mu^2 c, \tag{1.2}$$

where G is Newton's constant and c is the speed of light. Quite suprisingly, the dimensionless quantity γ depends only upon the shape of the cosmic string loop determined by the functions **a** and **b**; it is independent of the length L of the loop [3,6]. From here on, we therefore set $L = 1$.

In a recent paper we presented a new formula for γ. The formula is an exact analytic closed form for any piecewise linear cosmic string loop [7]. Although numerical values of γ are easily obtained by that method, the large number of terms makes it difficult to write out the analytic formula explicitly for most loops. The conventions and notation used in the present work are adopted from this earlier work. In particular, for the rest of this paper we use units with $c = 1$, and use $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, and $\hat{\mathbf{k}}$ to denote a right-handed triad of orthogonal unit vectors in flat $\mathbb{R}^3$.

In this paper we evaluate γ for a particular class of cosmic string loops. For loops in this class, the **a** loop traces out the following path. When u vanishes, $\mathbf{a}(u)$ also vanishes. As u increases, $\mathbf{a}(u)$ moves smoothly up the z axis with unit velocity $\hat{\mathbf{k}}$, until $u = 1/2$. From $u = 1/2$ until $u = 1$, $\mathbf{a}(u)$ moves back down to the origin with unit velocity $-\hat{\mathbf{k}}$. For loops in the class that we consider, the **b** loop lies entirely in the x-y plane, i.e., in the plane perpendicular to the path of the **a** loop. See Fig. 1 for an example. Note that the **b** loops in this class are *not* required to be piecewise linear.

The position (1.1) of the cosmic string loop in $\mathbb{R}^3$ may be easily visualized. Consider a string loop at time $t = 0$. The x and y components of $\mathbf{x}(\sigma,0)$ are given by (half) the x and y components of $\mathbf{b}(-\sigma)$. The z component of $\mathbf{x}(\sigma,0)$ is given by (half) the z component of $\mathbf{a}(\sigma)$. Thus, when viewed from large positive z, the $x-y$ projection of the string loop looks like the **b** loop (scaled by a factor of 1/2). Of course, the string loop does not lie in a plane. As σ increases from 0 to 1/2, the z component of $\mathbf{x}(\sigma,0)$ increases smoothly from 0 to 1/4. As σ continues to increase from 1/2 to 1, the z component of $\mathbf{x}(\sigma,0)$ de-

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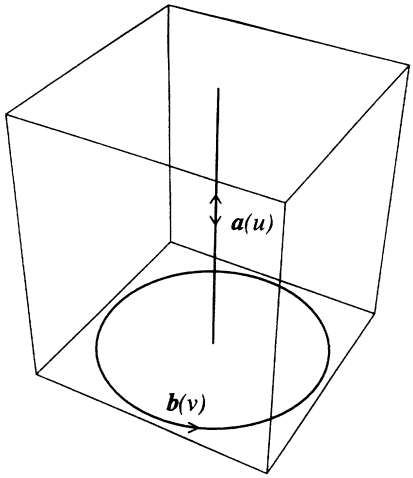

FIG. 1. A 2-segment **a** loop perpendicular to a circular **b** loop with $M = 1$. The cosmic string loop defined by these **a** and **b** loops has $\gamma \approx 39.002\,454$. This is the minimum value of γ attained by any loop in the limited class which we consider.

creases smoothly from 1/4 back to 0. As the value of t changes, the z component of a point on the string loop oscillates between 0 and 1/4 while the x-y projection of the string loop remains unchanged. From this, it can be seen immediately that string loops in this class will self-intersect if and only if the **b** loop intersects itself. In this paper self-intersecting string loops are assumed *not* to intercommute; i.e., the strings are assumed to "pass through" one another without interaction.

A short outline of the paper is as follows. In Sec. II we derive a simple analytic formula for γ for loops in the particular class described above. This formula expresses γ as a convolution of the **b** function with itself. In Sec. III this formula is exploited to determine γ for a several particular loops. In the first case, the **b** loop is a circle traversed M times in the x-y plane. In the second case, the **b** loop traces out a regular, N-sided polygon with interior vertex angle $\pi - 2\pi M/N$ in the x-y plane. In the third case, the **b** loop is an isosceles triangle in the x-y plane, with semiangle θ. In Sec. IV we find all of the extrema of γ for loops in the limited class described above. We prove that the minimum value of γ is attained only by the string loop of the first case with $M = 1$, for which

$$\gamma = 16 \int_0^{2\pi} \frac{1 - \cos x}{x} dx \approx 39.002\,454. \tag{1.3}$$

This is followed by a short conclusion, in which we also discuss the rate at which the values of γ for the piecewise linear polygon case with $M = 1$ approach the limiting value (1.3) of γ for the perfectly circular case.

II. FORMULA FOR THE TWO-PLANE CASE

In this section we find a simple general formula for γ, valid for our particular class of cosmic string loops. The string loops in this class are defined by **a** and **b** loops with the following properties. The **a** loop is composed of two straight segments joined at "kinks" where the tangent vector $\mathbf{a}'(u)$ is discontinuous. This loop is taken to lie along the z axis with one kink positioned at the origin. (Without loss of generality, the parameter u is chosen to be zero at this kink.) The other kink (at $u = 1/2$) is positioned above the first kink and has coordinates (0,0,1/2). The **b** loop is constrained to lie in the plane perpendicular to the **a** loop (i.e., the x-y plane). The formula for γ found in this section will be valid for all cosmic string loops in this limited class.

The starting point for the calculation in this section is the following equation, which is derived in Sec. III of [7]:

$$\gamma = 4 \int_0^1 du \int_0^1 dv \int_0^1 d\tilde{u} \int_{-\infty}^{\infty} d\tilde{v}\, \psi(u,v,\tilde{u},\tilde{v})\, D\left[\epsilon(\Delta t)\delta((\Delta t)^2 - |\Delta \mathbf{x}|^2)\right]. \tag{2.1}$$

In this equation, δ is the Dirac delta function and $\epsilon(x) = 2\theta(x) - 1$ where $\theta(x)$ is the unit step function. The $\tilde{u}$ and $\tilde{v}$ integrations are over the entire world tube swept out in space-time by the string loop as it oscillates. The u and v integrations are over a region on the world tube swept out by the string loop during a single oscillation. The functions $\Delta t \equiv (u + v - \tilde{u} - \tilde{v})/2$ and $\Delta \mathbf{x} \equiv (\mathbf{a}(u) + \mathbf{b}(v) - \mathbf{a}(\tilde{u}) - \mathbf{b}(\tilde{v}))/2$ describe the temporal and spatial separation of the two points on the string world tube with coordinates (u,v) and $(\tilde{u},\tilde{v})$, respectively. The function ψ is defined by

$$\begin{aligned}\psi(u,v,\tilde{u},\tilde{v}) = \tfrac{1}{8}[&(\mathbf{a}'(u)\cdot\mathbf{a}'(\tilde{u}) - 1)(\mathbf{b}'(v)\cdot\mathbf{b}'(\tilde{v}) - 1) \\ &+(\mathbf{a}'(u)\cdot\mathbf{b}'(\tilde{v}) - 1)(\mathbf{b}'(v)\cdot\mathbf{a}'(\tilde{u}) - 1) - (\mathbf{a}'(u)\cdot\mathbf{b}'(v) - 1)(\mathbf{b}'(\tilde{v})\cdot\mathbf{a}'(\tilde{u}) - 1)],\end{aligned} \tag{2.2}$$

and D is the linear differential operator

$$D = U(u,v,\tilde{u},\tilde{v})\partial_u + V(u,v,\tilde{u},\tilde{v})\partial_v - \tilde{U}(u,v,\tilde{u},\tilde{v})\partial_{\tilde{u}} - \tilde{V}(u,v,\tilde{u},\tilde{v})\partial_{\tilde{v}}. \tag{2.3}$$

The functions $U, V, \tilde{U}$, and $\tilde{V}$ are determined by requiring that D satisfy the four equations

$$D\Delta t(u,v,\tilde{u},\tilde{v}) = 1, \qquad D\Delta \mathbf{x}(u,v,\tilde{u},\tilde{v}) = \mathbf{0}, \tag{2.4}$$

which may be written as the matrix equation

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ \mathbf{a}'(u) & \mathbf{b}'(v) & \mathbf{a}'(\tilde{u}) & \mathbf{b}'(\tilde{v}) \end{pmatrix} \begin{pmatrix} U \\ V \\ \tilde{U} \\ \tilde{V} \end{pmatrix} = \begin{pmatrix} 2 \\ \mathbf{0} \end{pmatrix}. \tag{2.5}$$

The generic solution to this set of equations is given in Eq. (3.12) of [7].

We restrict our attention to the limited class of loops described above for the remainder of the paper. With this restriction, (2.1) will lead directly to a simple formula for γ. The function $\mathbf{a}(u)$ for this limited class of string loops may be written as

$$\mathbf{a}(u) = \begin{cases} u\hat{\mathbf{k}} & \text{for} \quad u \in [0, \frac{1}{2}), \\ (1-u)\hat{\mathbf{k}} & \text{for} \quad u \in [\frac{1}{2}, 1), \end{cases} \tag{2.6}$$

where $\hat{\mathbf{k}}$ is the unit vector along the z axis. From this equation and the fact that the $\mathbf{b}$ loop lies entirely in the plane perpendicular to $\hat{\mathbf{k}}$, it follows immediately that ψ takes the form

$$\psi(u, v, \tilde{u}, \tilde{v}) = \begin{cases} 0 & \text{for} \quad u \in [0, \frac{1}{2}),\ \tilde{u} \in [0, \frac{1}{2}) \ \text{ or } \ u \in [\frac{1}{2}, 1),\ \tilde{u} \in [\frac{1}{2}, 1), \\ \frac{1}{4}(1 - \mathbf{b}'(v)\cdot\mathbf{b}'(\tilde{v})) & \text{for} \quad u \in [0, \frac{1}{2}),\ \tilde{u} \in [\frac{1}{2}, 1) \text{ or } u \in [\frac{1}{2}, 1),\ \tilde{u} \in [0, \frac{1}{2}). \end{cases} \tag{2.7}$$

In addition, for $u \in [0, \frac{1}{2})$ and $\tilde{u} \in [\frac{1}{2}, 1)$, Eq. (2.5) for the coefficients in the differential operator D becomes

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ \hat{\mathbf{k}} & \mathbf{b}'(v) & -\hat{\mathbf{k}} & \mathbf{b}'(\tilde{v}) \end{pmatrix} \begin{pmatrix} U \\ V \\ \tilde{U} \\ \tilde{V} \end{pmatrix} = \begin{pmatrix} 2 \\ \mathbf{0} \end{pmatrix} \tag{2.8}$$

with the obvious solution $U = \tilde{U} = 1$, $V = \tilde{V} = 0$, i.e.,

$$D = \partial_u - \partial_{\tilde{u}}. \tag{2.9}$$

It is clear that the same operator also satisfies Eq. (2.4) for $u \in [\frac{1}{2}, 1)$ and $\tilde{u} \in [0, \frac{1}{2})$. [It is interesting that this solution works whether or not the determinant of the matrix in (2.5) vanishes.] Substituting (2.7) and (2.9) for ψ and D into (2.1) we find that

$$\gamma = \int_0^1 dv \int_{-\infty}^{\infty} d\tilde{v}\, (1 - \mathbf{b}'(v)\cdot\mathbf{b}'(\tilde{v})) \left\{ \int_0^{\frac{1}{2}} du \int_{\frac{1}{2}}^1 d\tilde{u} + \int_{\frac{1}{2}}^1 du \int_0^{\frac{1}{2}} d\tilde{u} \right\} (\partial_u - \partial_{\tilde{u}}) \left[\epsilon(\Delta t)\delta((\Delta t)^2 - |\Delta\mathbf{x}|^2) \right]. \tag{2.10}$$

For the specific form of $\mathbf{a}(u)$, the second integral in curly braces immediately reduces to the first under the change of variables $\tilde{u} = 1 - u'$, $u = 1 - \tilde{u}'$. In addition, we may perform the same change of variables in the second term of the derivative operator D. Doing so leads directly to

$$\gamma = 4 \int_0^1 dv \int_{-\infty}^{\infty} d\tilde{v}\, (1 - \mathbf{b}'(v)\cdot\mathbf{b}'(\tilde{v})) Q(v, \tilde{v}), \tag{2.11}$$

where

$$Q(v, \tilde{v}) = \int_0^{\frac{1}{2}} du \int_{\frac{1}{2}}^1 d\tilde{u}\, \partial_u \left[\epsilon(\Delta t)\delta((\Delta t)^2 - |\Delta\mathbf{x}|^2) \right]. \tag{2.12}$$

The next step will be to carry out the integrations over u and $\tilde{u}$ in $Q(v, \tilde{v})$.

The partial derivative in (2.12) allows the integration over u to be performed immediately. In doing so, the function in square brackets in (2.12) must be evaluated at both the upper and lower limits of the u integration, yielding

$$\begin{aligned} Q(v, \tilde{v}) &= \int_{\frac{1}{2}}^1 d\tilde{u} \left[\epsilon(\Delta t)\delta((\Delta t)^2 - |\Delta\mathbf{x}|^2) \right]_{u=0}^{u=\frac{1}{2}} \\ &\equiv T(v, \tilde{v}) - B(v, \tilde{v}), \end{aligned} \tag{2.13}$$

where T is the integral of the quantity in square brackets evaluated at the top limit, and B is the integral of the quantity in square brackets evaluated at the bottom limit. Consider first the top limit $T(v, \tilde{v})$. Because $\tilde{u} \in [\frac{1}{2}, 1)$ we have

$$\Delta t = \tfrac{1}{2} - \tilde{u} + v - \tilde{v}, \tag{2.14}$$

$$\Delta\mathbf{x} = (\tilde{u} - \tfrac{1}{2})\hat{\mathbf{k}} + \mathbf{b}(v) - \mathbf{b}(\tilde{v}). \tag{2.15}$$

Introducing $u' = \tilde{u} - \frac{1}{2}$, the integral over $\tilde{u}$ in $T(v, \tilde{v})$ can be written as

$$T(v,\tilde{v}) = \int_0^{\frac{1}{2}} du'\, \epsilon(v-\tilde{v}-u')\delta\left(\tfrac{1}{4}\left[-2u'(v-\tilde{v}) + (v-\tilde{v})^2 - (\mathbf{b}(v)-\mathbf{b}(\tilde{v}))^2\right]\right). \tag{2.16}$$

Turning to the lower limit $B(v,\tilde{v})$ we have

$$\Delta t = -\tilde{u} + v - \tilde{v}, \tag{2.17}$$

$$\Delta \mathbf{x} = (\tilde{u}-1)\hat{\mathbf{k}} + \mathbf{b}(v) - \mathbf{b}(\tilde{v}). \tag{2.18}$$

We now use the freedom to rewrite (2.11) as an integral with v running from $-\infty$ to ∞ and $\tilde{v}$ running from 0 to 1. (This may readily be seen as follows. Rewrite the integral over $\tilde{v}$ as a sum over n from $-\infty$ to ∞ of an integral from 0 to 1 of the integrand with $\tilde{v}$ replaced by $\tilde{v}+n$. Then use the explicit form of the integrand and the periodicity of $\mathbf{b}$ to combine the n with the v to transform the v integration into an integration from $-\infty$ to ∞ [7].) If we now make the simultaneous changes of variable $\tilde{v} = v'$, $v = \tilde{v}'+1$, and $\tilde{u} = 1-u'$ we recover exactly (2.11) (with v and $\tilde{v}$ replaced everywhere by v' and $\tilde{v}'$) except that the sign of the argument of the ϵ function has changed. However, since the ϵ function is odd and we are dealing with the bottom limit in (2.13), we recover exactly (2.16) for $-B(v,\tilde{v})$. Hence, $Q(v,\tilde{v}) = 2T(v,\tilde{v})$.

Using (2.13), (2.16), and $T = -B$, $Q(v,\tilde{v})$ may be rewritten in the form

$$Q(v,\tilde{v}) = \frac{4}{|v-\tilde{v}|}\int_0^{\frac{1}{2}} du'\, \epsilon(v-\tilde{v}-u')\delta\left(u' - \frac{(v-\tilde{v})^2 - \left[\mathbf{b}(v)-\mathbf{b}(\tilde{v})\right]^2}{2(v-\tilde{v})}\right). \tag{2.19}$$

This integration may be carried out immediately, yielding

$$Q(v,\tilde{v}) = \frac{4}{|v-\tilde{v}|}\epsilon\left(\frac{(v-\tilde{v})^2 + (\mathbf{b}(v)-\mathbf{b}(\tilde{v}))^2}{2(v-\tilde{v})}\right) \times \theta\left(\frac{(v-\tilde{v})^2 - (\mathbf{b}(v)-\mathbf{b}(\tilde{v}))^2}{2(v-\tilde{v})}\right)\theta\left(\frac{1}{2} - \frac{(v-\tilde{v})^2 - (\mathbf{b}(v)-\mathbf{b}(\tilde{v}))^2}{2(v-\tilde{v})}\right). \tag{2.20}$$

This answer may be considerably simplified.

First, because the numerator of the argument of the ϵ function in $Q(v,\tilde{v})$ is manifestly positive, the function is just equal to $\epsilon(v-\tilde{v})$. This term simply has the effect of removing the absolute value signs from the denominator of the prefactor. Next, consider the two θ functions appearing in (2.20).

(1) $|v-v'|$ is the arclength along the $\mathbf{b}$ loop from $\mathbf{b}(v)$ to $\mathbf{b}(\tilde{v})$ (traversed in a particular sense possibly more than once) which must be greater than the straight line distance $|\mathbf{b}(v)-\mathbf{b}(\tilde{v})|$ between the two points. Thus again the numerator is a positive function and the first θ function is just equal to $\theta(v-\tilde{v})$.

(2) The argument of the second θ function can be written as

$$\frac{(1-(v-\tilde{v}))(v-\tilde{v}) + (\mathbf{b}(v)-\mathbf{b}(\tilde{v}))^2}{2(v-\tilde{v})}. \tag{2.21}$$

We may neglect the denominator which is always positive when the first θ function is nonvanishing. In addition, we need only consider the behavior of the numerator for $v-\tilde{v} > 0$. First we note that for $(v-\tilde{v}) \in (0,1)$ it is manifestly positive. On the other hand for $(v-\tilde{v}) > 1$ we note that $(v-\tilde{v})$ is the arclength along the $\mathbf{b}$ loop from $\mathbf{b}(v)$ to $\mathbf{b}(\tilde{v})$ traversed in a particular sense (possibly more than once) while $(v-\tilde{v})-1$ is the arclength along the $\mathbf{b}$ loop from $\mathbf{b}(v)$ to $\mathbf{b}(\tilde{v})$ traversed in the opposite sense (possibly more than once). Thus $(v-\tilde{v}) \geq |\mathbf{b}(v)-\mathbf{b}(\tilde{v})|$ and $(v-\tilde{v})-1 \geq |\mathbf{b}(v)-\mathbf{b}(\tilde{v})|$ and so

$$(1-(v-\tilde{v}))(v-\tilde{v}) + (\mathbf{b}(v)-\mathbf{b}(\tilde{v}))^2 \leq 0 \quad \text{for} \quad (v-\tilde{v}) > 1. \tag{2.22}$$

Combining these results we see that (2.20) simplifies to yield

$$Q(v,\tilde{v}) = \frac{4}{(v-\tilde{v})}\theta(v-\tilde{v})\theta(1-(v-\tilde{v})). \tag{2.23}$$

Replacing $Q(v,\tilde{v})$ in (2.11) with (2.23) we obtain our principal result:

$$\gamma = 16\int_0^1 dv \int_{v-1}^{v} d\tilde{v}\, \frac{1-\mathbf{b}'(v)\cdot\mathbf{b}'(\tilde{v})}{(v-\tilde{v})} = 16\int_0^1 dv \int_0^1 dx\, \frac{1-\mathbf{b}'(v)\cdot\mathbf{b}'(v-x)}{x}. \tag{2.24}$$

In the next section we shall evaluate this remarkably simple expression for a number of particular loops.

III. RESULTS

A. Circular b loop

Our first application of formula (2.24) for γ is the case where the **b** loop winds M times around a perfect circle of radius $1/2\pi M$ (see Fig. 1). Here M is any positive integer. In this case the function $\mathbf{b}(v)$ may be written as

$$\mathbf{b}(v) = \frac{1}{2\pi M}[\cos(2\pi Mv)\hat{\mathbf{i}} + \sin(2\pi Mv)\hat{\mathbf{j}}]. \tag{3.1}$$

The dot product appearing in (2.24) gives

$$\begin{aligned}\mathbf{b}'(v)\cdot\mathbf{b}'(v-x) &= [\cos(2\pi Mv)\cos(2\pi M(v-x)) \\ &\quad + \sin(2\pi Mv)\sin(2\pi M(v-x))] \\ &= \cos(2\pi Mx).\end{aligned} \tag{3.2}$$

Thus, γ is given by

$$\begin{aligned}\gamma &= 16\int_0^1 dv \int_0^1 dx\,\frac{1-\cos(2\pi Mx)}{x} \\ &= 16\int_0^1 dx\,\frac{1-\cos(2\pi Mx)}{x} \\ &= 16\int_0^{2\pi M} dx\frac{1-\cos x}{x} \\ &= 16[C + \ln(2\pi M) - \mathrm{Ci}(2\pi M)],\end{aligned} \tag{3.3}$$

where $C = 0.577\,216\ldots$ is Euler's constant and $\mathrm{Ci}(x)$ is the cosine integral function defined by Eq. (5.2.2) of Ref. [8]. The numerical value of γ for this string loop is $\gamma \approx 39.002\,454$ when $M = 1$. In Sec. IV we show that this is the minimum value of γ for any loop in the limited class of loops for which the formula (2.24) is valid.

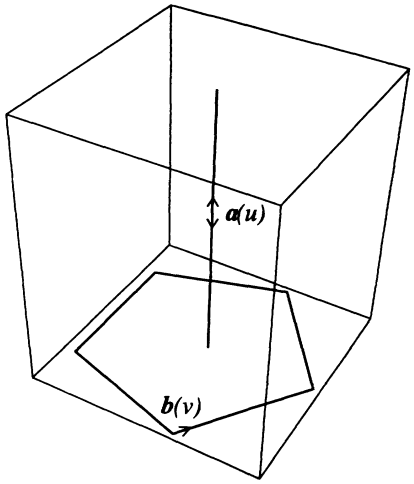

FIG. 2. A 2-segment **a** loop perpendicular to a **b** loop in the shape of a regular polygon with $M = 1$ and $N = 5$. The cosmic string loop defined by these **a** and **b** loops has $\gamma \approx 43.100\,206$.

B. Polygon b loop

As the next application of formula (2.24) we consider the case where the **b** loop takes the shape of a regular, N-sided polygon (see Fig. 2). Because the length of a given side of the polygon is $1/N$, it is convenient to break the integrations in (2.24) into N equally spaced intervals:

$$\gamma = 16\sum_{i,j=0}^{N-1}\int_{i/N}^{(i+1)/N} dv \int_{j/N}^{(j+1)/N} dx\frac{1-\mathbf{b}'(v)\cdot\mathbf{b}'(v-x)}{x}. \tag{3.4}$$

For each i and j, the v and x integrations are over specific segments on the polygon. The integrations may be simplified by introducing a new pair of coordinates, $\bar{v}$ and $\bar{x}$, which will each be in the range $[0, 1/N]$

$$\bar{v} = v - \frac{i}{N}, \qquad \bar{x} = x - \frac{j}{N}. \tag{3.5}$$

With this change of variables, γ becomes

$$\gamma = 16\sum_{i,j=0}^{N-1}\int_0^{1/N} d\bar{v}\int_0^{1/N} d\bar{x}\frac{1-\mathbf{b}'(i/N+\bar{v})\cdot\mathbf{b}'(i/N+\bar{v}-j/N-\bar{x})}{j/N+\bar{x}}. \tag{3.6}$$

The vector $\mathbf{b}'(i/N+\bar{v})$ is a constant vector (tangent to the ith segment on the polygon) for each value of i. When $\bar{x} < \bar{v}$, the vector $\mathbf{b}'(i/N+\bar{v}-j/N-\bar{x})$ will be the constant vector tangent to the $(i-j)$th segment on the polygon. When $\bar{x} > \bar{v}$, it will be tangent to the $(i-j-1)$th segment. Thus, the $\bar{x}$ integration may be broken into two ranges where the dot product $\mathbf{b}'(i/N+\bar{v})\cdot\mathbf{b}'(i/N+\bar{v}-j/N-\bar{x})$ is constant in each range:

$$\gamma = 16 \sum_{i,j=0}^{N-1} \int_0^{1/N} d\bar{v} \Bigg\{ \int_0^{\bar{v}} d\bar{x} \frac{1 - \mathbf{b}'(i/N + \bar{v}) \cdot \mathbf{b}'(i/N + \bar{v} - j/N - \bar{x})}{j/N + \bar{x}} + \int_{\bar{v}}^{1/N} d\bar{x} \frac{1 - \mathbf{b}'(i/N + \bar{v}) \cdot \mathbf{b}'(i/N + \bar{v} - j/N - \bar{x})}{j/N + \bar{x}} \Bigg\}. \tag{3.7}$$

For each value of i and j, the constant dot products in (3.7) may be evaluated. Because $\mathbf{b}'(i/N + \bar{v})$ is tangent to the ith segment on the polygon and $\mathbf{b}'(i/N + \bar{v} - j/N - \bar{x})$ is tangent to either the $(i-j)$th segment (for $\bar{v} > \bar{x}$) or the $(i-j-1)$th segment (for $\bar{x} > \bar{v}$), it follows that their dot product is given by

$$\mathbf{b}'(i/N + \bar{v}) \cdot \mathbf{b}'(i/N + \bar{v} - j/N - \bar{x}) = \begin{cases} \cos\left(\frac{2\pi}{N} j\right) & \text{for } \bar{v} > \bar{x}, \\ \cos\left(\frac{2\pi}{N}(j-1)\right) & \text{for } \bar{v} < \bar{x}, \end{cases} \tag{3.8}$$

where $2\pi/N$ is the angle through which one must rotate a segment of the polygon to bring it parallel to the next segment. The right-hand side (RHS) of (3.8) is independent of i. Thus, the sum over i simply gives a factor of N. The integrations over $\bar{x}$ may be done giving

$$\gamma = 16N \sum_{j=0}^{N-1} \int_0^{1/N} d\bar{v} \Bigg\{ \left(1 - \cos\left(\frac{2\pi}{N} j\right)\right) (\ln(j/N + \bar{v}) - \ln(j/N)) + \left(1 - \cos\left(\frac{2\pi}{N}(j+1)\right)\right) (\ln(j/N + 1/N) - \ln(j/N + \bar{v})) \Bigg\}. \tag{3.9}$$

The $\bar{v}$ integration may now be carried out. After combining terms, this yields

$$\gamma = 16\left(1 - \cos\left(\frac{2\pi}{N}\right)\right) + 16 \sum_{j=1}^{N-1} \left[\left(1 + j\cos\left(\frac{2\pi}{N}(j+1)\right) - (j+1)\cos\left(\frac{2\pi}{N} j\right)\right) \ln\left(\frac{j+1}{j}\right) + \cos\left(\frac{2\pi}{N} j\right) - \cos\left(\frac{2\pi}{N}(j+1)\right) \right]. \tag{3.10}$$

Finally, we note that all the terms not multiplied by $\ln\left(\frac{j+1}{j}\right)$ in (3.10) cancel exactly in the sum over j. Thus, we arrive at the final form

$$\gamma = 16 \sum_{j=1}^{N-1} \left(1 + j\cos\left(\frac{2\pi}{N}(j+1)\right) - (j+1)\cos\left(\frac{2\pi}{N} j\right)\right) \ln\left(\frac{j+1}{j}\right), \tag{3.11}$$

or equivalently

$$\gamma = 32\left(1 - \cos\left(\frac{2\pi}{N}\right)\right)\left(\frac{1}{2} N \ln N + \sum_{j=2}^{N-1} j \ln(j)\, \cos\left(\frac{2\pi}{N} j\right)\right). \tag{3.12}$$

For the first few values of N, Eqs. (3.11) and (3.12) give

$$\gamma = \begin{cases} 64\ln 2 \approx 44.36 & \text{for } N = 2, \\ 72\ln 3 - 48\ln 2 \approx 45.83 & \text{for } N = 3, \\ 64\ln 2 \approx 44.36 & \text{for } N = 4. \end{cases} \tag{3.13}$$

Note that the $N = 2$ case is identical to the Garfinkle and Vachaspati case (Eq. (3.9) of Ref. [9]) with $\alpha = \pi/2$, and has the same value of γ as they obtained. Numerical values for γ are shown as a function of N in Fig. 3.

It is interesting to note that (3.11) may be trivially modified to obtain a formula for γ for a set of (self-intersecting) loops related to the polygon case. Consider again a **b** loop which is composed of N straight, equal length segments. However, in this case, instead of placing each segment at an angle of $2\pi/N$ relative to the previous segment (and thereby getting an N-sided polygon), each successive segment is placed at an angle of $2M\pi/N$ relative to the previous segment (where M is a positive integer not equal to N). This causes the **b** loop to wind M times around the origin. The only change required in (3.11) is to replace the factor $2\pi/N$ appearing in the cosine terms by $2M\pi/N$:

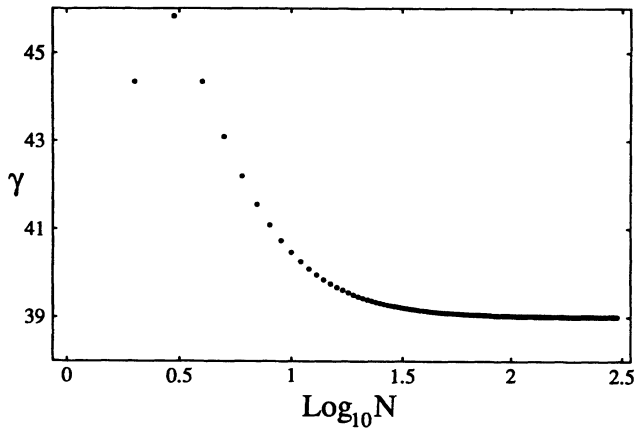

FIG. 3. Values of γ for cosmic string loops with 2-segment **a** loops perpendicular to **b** loops in the shape of regular N-sided polygons with $M = 1$. These γ values converge like $1/N^2$ to the value of γ for the circular **b** loop case.

$$\gamma = 16 \sum_{j=1}^{N-1} \left(1 + j \cos\left(\frac{2\pi M}{N}(j+1) \right) - (j+1) \cos\left(\frac{2\pi M}{N} j \right) \right) \ln\left(\frac{j+1}{j} \right), \tag{3.14}$$

or, equivalently,

$$\gamma = 32 \left(1 - \cos\left(\frac{2\pi M}{N} \right) \right) \times \left(\frac{1}{2} N \ln N + \sum_{j=2}^{N-1} j \ln(j) \cos\left(\frac{2\pi M}{N} j \right) \right). \tag{3.15}$$

If $M = 1$ we have the polygon case considered above. If for instance, $M = 2$ and $N = 5$, the **b** loop would have the shape of a pentagram (see Fig. 4).

C. Isosceles b loop

Our final application of (2.24) is the case where the **b** loop takes the shape of an isosceles triangle. This is a one-parameter family of loops where the parameter is the half-angle θ between the two equal length sides of the triangle (see Fig. 5). The length of the two equal sides of the triangle is denoted by l_1. The length of the base of the triangle is denoted l_2. Using the constraint

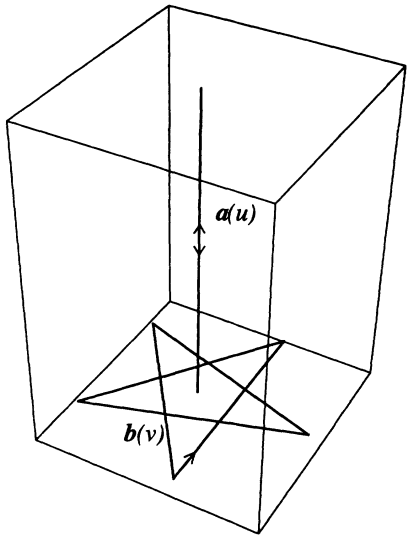

FIG. 4. A 2-segment **a** loop perpendicular to a **b** loop in the shape of a regular polygon with $M = 2$ and $N = 5$. The cosmic string loop defined by these **a** and **b** loops has $\gamma \approx 56.980\,278$.

$2l_1 + l_2 = 1$, one can easily write l_1 and l_2 in terms of the semiangle θ of the triangle:

$$l_1 = \frac{1}{2(\sin\theta + 1)}, \qquad l_2 = 2l_1 \sin\theta. \tag{3.16}$$

The parameter along the **b** loop is taken to run from 0 to l_1 along the first side of the triangle, from l_1 to $2l_1$ along the second side, and from $2l_1$ to 1 along the third. If we label the unit tangent vectors to the first, second, and third sides of the triangle by $\mathbf{b}'_1, \mathbf{b}'_2$, and $\mathbf{b}'_3$, respectively, the dot products between the various sides are given in terms of θ by

$$\mathbf{b}'_1 \cdot \mathbf{b}'_2 = -\cos(2\theta), \qquad \mathbf{b}'_1 \cdot \mathbf{b}'_3 = \mathbf{b}'_2 \cdot \mathbf{b}'_3 = -\sin\theta. \tag{3.17}$$

The dot product of any unit tangent vector with itself is simply 1.

We are now in a position to use Eq. (2.24) to find γ for this family of string loops. The integrations over v and $\tilde{v}$ may be broken into ranges corresponding to the three sides of the triangle, yielding

$$\gamma = 16 \left\{ \int_0^{l_1} dv \left[\int_{v-1}^{-l_2-l_1} d\tilde{v} \frac{1 - \mathbf{b}'_1 \cdot \mathbf{b}'_1}{v - \tilde{v}} + \int_{-l_2-l_1}^{-l_2} d\tilde{v} \frac{1 - \mathbf{b}'_1 \cdot \mathbf{b}'_2}{v - \tilde{v}} + \int_{-l_2}^{0} d\tilde{v} \frac{1 - \mathbf{b}'_1 \cdot \mathbf{b}'_3}{v - \tilde{v}} + \int_0^{v} d\tilde{v} \frac{1 - \mathbf{b}'_1 \cdot \mathbf{b}'_1}{v - \tilde{v}} \right] \right.$$
$$+ \int_{l_1}^{2l_1} dv \left[\int_{v-1}^{-l_2} d\tilde{v} \frac{1 - \mathbf{b}'_2 \cdot \mathbf{b}'_2}{v - \tilde{v}} + \int_{-l_2}^{0} d\tilde{v} \frac{1 - \mathbf{b}'_2 \cdot \mathbf{b}'_3}{v - \tilde{v}} + \int_0^{l_1} d\tilde{v} \frac{1 - \mathbf{b}'_2 \cdot \mathbf{b}'_1}{v - \tilde{v}} + \int_{l_1}^{v} d\tilde{v} \frac{1 - \mathbf{b}'_2 \cdot \mathbf{b}'_2}{v - \tilde{v}} \right]$$
$$\left. + \int_{2l_1}^{1} dv \left[\int_{v-1}^{0} d\tilde{v} \frac{1 - \mathbf{b}'_3 \cdot \mathbf{b}'_3}{v - \tilde{v}} + \int_0^{l_1} d\tilde{v} \frac{1 - \mathbf{b}'_3 \cdot \mathbf{b}'_1}{v - \tilde{v}} + \int_{l_1}^{2l_1} d\tilde{v} \frac{1 - \mathbf{b}'_3 \cdot \mathbf{b}'_2}{v - \tilde{v}} + \int_{2l_1}^{v} d\tilde{v} \frac{1 - \mathbf{b}'_3 \cdot \mathbf{b}'_3}{v - \tilde{v}} \right] \right\}. \tag{3.18}$$

The numerator of each integral in (3.18) has the form $1 - \mathbf{b}'_i \cdot \mathbf{b}'_j$. Each of these numerators is now constant. If $i = j$, the numerator (and hence the integral) vanishes. When $i \neq j$, Eq. (3.17) may be used to write the dot products in terms of θ. The integrals in (3.18) are now easy to evaluate and yield

$$\gamma = -16\{(1+\cos(2\theta))(2(l_1+l_2)\ln(l_1+l_2) - l_2\ln(l_2) + 2l_1\ln(l_1) - 2l_1\ln(2l_1)) + (1+\sin\theta)(2l_1\ln(2l_1) + l_2\ln(l_2) + l_2\ln(l_2) + 2l_1\ln(2l_1))\}. \tag{3.19}$$

Using (3.16) to write l_1 and l_2 in terms of θ and simplifying gives the final form

$$\gamma = 32\Big[2\ln 2\cos^2\theta - \sin^2(\theta)\ln(\sin\theta) + (2-\sin\theta)(1+\sin\theta)\ln(1+\sin\theta) - (1-\sin\theta)(1+2\sin\theta)\ln(1+2\sin\theta)\Big]. \tag{3.20}$$

This formula for γ is plotted in Fig. 6 for a range of angles θ. It should be noted that (3.20) correctly reduces to the $M = 1$, $N = 2$ result (3.14) of the previous section when $\theta = 0°$ or $90°$, and to the $M = 1$, $N = 3$ result when $\theta = 30°$.

IV. MINIMUM γ LOOP AND STATIONARY POINTS OF γ

In this section we obtain the minimum value and stationary points of γ for *all* loops in our limited class. Since $\mathbf{b}(v)$ is a periodic function on the interval $v \in [0,1)$, its derivative with respect to v is also a periodic function, and may be expressed as the Fourier series

$$\mathbf{b}'(v) = \sum_{n=-\infty}^{\infty} e^{2\pi i n u}\left(a_n\hat{\mathbf{i}} + b_n\hat{\mathbf{j}}\right). \tag{4.1}$$

Since the function $\mathbf{b}'(v)$ is real, one has $a_n = a^*_{-n}$ and $b_n = b^*_{-n}$ with an asterisk denoting complex conjugation. The periodicity of $\mathbf{b}$ also implies that a_0 and b_0 vanish, since

$$\int_0^1 \mathbf{b}'(v)dv = \mathbf{b}(1) - \mathbf{b}(0) = a_0\hat{\mathbf{i}} + b_0\hat{\mathbf{j}} = \mathbf{0}. \tag{4.2}$$

For the remainder of this section we use the notation $\sum'_n$ to indicate the sum from minus infinity to infinity with the $n = 0$ term excluded.

The Fourier coefficients a_n and b_n are not arbitrary; they satisfy an infinite number of constraints which follow from the gauge condition that the length of the vector $|\mathbf{b}'(v)| = 1$. One of these constraints may be expressed in a useful "integrated" form:

$$\begin{aligned} 1 &= \int_0^1 |\mathbf{b}'(v)|^2 dv \\ &= \int_0^1 \sum_n{}' \sum_m{}' e^{2\pi i(n+m)v}(a_n a_m + b_n b_m)dv \\ &= \sum_n{}'(a_n a_n^* + b_n b_n^*), \end{aligned} \tag{4.3}$$

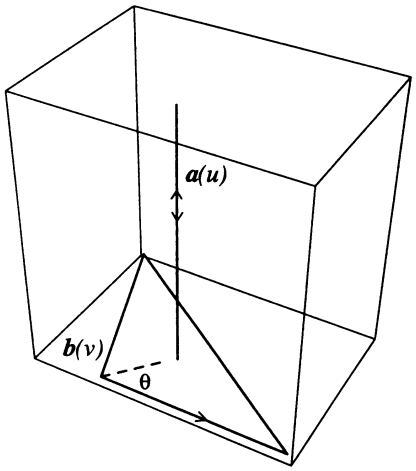

FIG. 5. A 2-segment **a** loop perpendicular to a **b** loop in the shape of an isosceles triangle with semiangle θ. The semiangle θ lies in the range $\theta \in [0, \pi/2]$.

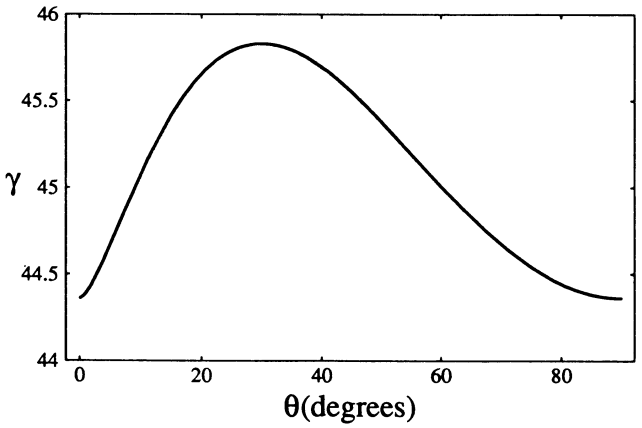

FIG. 6. Values of γ plotted as a function of the semiangle θ for the cosmic string loops described in Fig. 5. The cases $\theta = 0$ and $\theta = 90°$ yield the same result, $\gamma = 64\ln 2 \approx 44.361\,419$, as given by Garfinkle and Vachaspati (Eq. (3.9) of [9] with $\alpha = \pi/2$).

where in the final line we have used the orthogonality of the exponential functions on the unit interval and the reality conditions $a_n = a^*_{-n}$ and $b_n = b^*_{-n}$. One may define a non-negative real quantity $c_n^2 = 2(a_n a_n^* + b_n b_n^*)$. Note that for any cosmic string loop one has $c_n^2 = c_{-n}^2$, and that the normalization condition

$$\sum_{n=1}^{\infty} c_n^2 = 1 \tag{4.4}$$

is implied by the integrated constraint (4.3).

Now consider the value of γ given by Eq. (2.24):

$$\begin{aligned}\gamma &= 16 \int_0^1 dv \int_0^1 \frac{dx}{x} \left[1 - \mathbf{b}'(v) \cdot \mathbf{b}'(v-x)\right] \\ &= 16 \int_0^1 \frac{dx}{x} \int_0^1 dv \left[1 - {\sum_n}' {\sum_m}' e^{2\pi i((n+m)v - mx)} (a_n a_m + b_n b_m)\right] \\ &= 16 \int_0^1 \frac{dx}{x} \left[1 - {\sum_n}' e^{2\pi i n x} (a_n a_n^* + b_n b_n^*)\right]. \end{aligned} \tag{4.5}$$

For any cosmic string loop, the function $\mathbf{b}$ satisfies the integrated constraint (4.3). Substituting this for the *first* term in the previous equation (4.5), one obtains

$$\begin{aligned}\gamma &= 16 {\sum_n}' \int_0^1 dx \frac{(1 - e^{2\pi i n x})}{x} (a_n a_n^* + b_n b_n^*) \\ &= \sum_{n=1}^{\infty} \lambda_n c_n^2. \end{aligned} \tag{4.6}$$

This is a quadratic form for γ; its "eigenvalues" are

$$\begin{aligned}\lambda_n &\equiv 16 \mathrm{Re}\left[\int_0^1 \frac{(1 - e^{2\pi i n x})}{x} dx\right] \\ &= 16 \int_0^{2\pi n} \frac{(1 - \cos x)}{x} dx \\ &= 16\left[C + \ln(2\pi n) - \mathrm{Ci}(2\pi n)\right] \\ &= 16\left[C + \ln(2\pi n) - \int_0^{\infty} \frac{t e^{-2\pi n t}}{1 + t^2} dt\right] \\ &\approx 16\left[C + \ln(2\pi n) - \frac{1}{4\pi^2 n^2} + O\left(\frac{1}{n^4}\right)\right], \end{aligned} \tag{4.7}$$

where $C = 0.577\,216...$ is Euler's constant and $\mathrm{Ci}(x)$ is the cosine integral function defined by Eq. (5.2.2) of Ref. [8]. From the integral forms it is clear that the eigenvalues increase without bound: $\lambda_n < \lambda_{n+1}$. The first few eigenvalues are approximately

$$(\lambda_1, \lambda_2, \lambda_3, \ldots) \approx (39.0025, 49.8297, 56.2636, 60.8473, 64.4086, 67.3208, \ldots). \tag{4.8}$$

From the quadratic form of γ (4.6) one may easily find the absolute minimum and stationary points of γ.

Since any cosmic string loop in our limited class has $\sum_{n=1}^{\infty} c_n^2 = 1$ and $\gamma = \sum_{n=1}^{\infty} c_n^2 \lambda_n$, it is immediately clear that the minimum value of γ is attained when $c_1^2 = 1$ and all other $c_{n\neq 1}^2$ vanish. (The problem is identical to that of minimizing the energy of a quantum mechanical system with orthonormal eigenstates $|n>$, where the general normalized state is denoted $\sum_n c_n |n>$ and the energy of the nth state is λ_n.) The loop with $c_1^2 = 1$ is precisely the $M = 1$ case considered in Sec. III A, where the $\mathbf{b}$ loop is a circle traversed once as v increases from zero to one. The remaining stationary points of γ are immediately seen to be loops with $c_k^2 = 1$ for some integer k and all other coefficients vanishing. This corresponds to the $M = k$ case considered in Sec. III A where the $\mathbf{b}$ loop is a circle traversed k times as v increases from zero to one.

V. CONCLUSION

In this paper we have found a simple formula for γ, the power radiated in gravitational waves, for a limited class

of cosmic string loops. We have applied this formula to a variety of loop configurations within the limited class and have shown that the minimum value for γ in this class is attained only when the **b** loop has the shape of a perfect circle (traversed once).

The results for γ in the piecewise linear cases (polygon and isosceles **b** loops) have been checked against the general analytic formula given in Ref. [7], using a symbolic manipulator (MATHEMATICA) to evaluate the formula of Ref. [7]. We have also used these exact results to test a floating-point implementation (written in C) of the exact formula of Ref. [7]. This floating-point implementation was described in some detail in [7] and is publicly available. The tests have shown that the floating-point implementation typically yields results accurate to between 6 and 8 decimal places, the difference being due to accumulated round-off error.

In [7] we presented an analytic formula yielding γ for piecewise linear loops, and argued that any smooth loop could be well approximated by an N-segment piecewise linear loop with N sufficiently large. The exact formula in this paper allows us to determine the rate of convergence for large N, at least for a limited class of loops. The results for the circular **b** loop case considered in Sec. III A (with $M = 1$) may be used to examine how well smooth loops are approximated by piecewise linear loops. The regular polygon **b** loops ($M = 1$) of Sec. III B are increasingly circular in shape as $N \to \infty$. The difference between the γ value for a cosmic string loop with a **b** loop in the shape of an N-sided polygon (γ_N) and the result from the case where the **b** loop is a perfect circle (γ_∞) is shown as a function of N in Fig. 7. The piecewise linear result quickly converges to the smooth result (see also Fig. 3). The convergence goes like $1/N^2$ for $N > 10$.

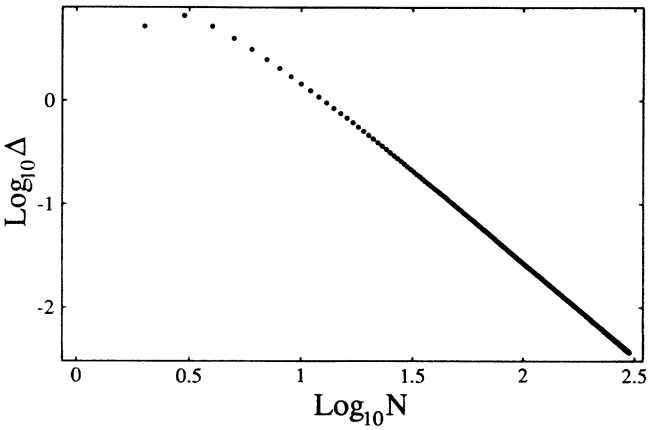

FIG. 7. The convergence of $\Delta \equiv \gamma_N - \gamma_\infty$, where γ_N is the power radiated by a cosmic string loop with a 2-segment **a** loop perpendicular to an N-sided regular polygon ($M = 1$) **b** loop, and γ_∞ is the power radiated when the **b** loop is a circle ($M = 1$). The value of Δ converges to zero like $1/N^2$ for large N.

ACKNOWLEDGMENTS

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