

The initial singularity in solutions of the Einstein–Vlasov system of Bianchi type I

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The dynamics of solutions of the Einstein–Vlasov system with Bianchi I symmetry is discussed in the case of massive or massless particles. It is shown that in the case of massive particles the solutions are asymptotic to isotropic dust solutions at late times. The initial singularity is more difficult to analyze. It is shown that the asymptotic behavior there must be one of a small set of possibilities, but it is not clear whether all of these possibilities are realized. One solution is exhibited in the case of massless particles, which behaves quite differently near the singularity from any Bianchi I solution with perfect fluid as a matter model. In particular, the matter is not dynamically negligible near the singularity for this solution. © 1996 American Institute of Physics. [S0022-2488(96)03901-4]

I. INTRODUCTION

The simplest of all anisotropic cosmological models are those of Bianchi type I. They are the space–times that admit a three-dimensional Abelian symmetry group whose orbits are space-like. (For general information on Bianchi models see Ref. 1.) Just how simple their dynamics is depends significantly on the nature of the matter content of the space–time. For a perfect fluid with a linear equation of state, it has been known for a long time how to analyze the dynamics.^{2,3} For a noninteracting mixture of two fluids with linear equations of state, the time evolution is also well understood and is asymptotic near the singularity and at large times to that of a single fluid.⁴ The case of a fluid with a nonlinear equation of state is discussed in an Appendix to the present paper. The dynamics does not differ much from the picture in the linear case. When a magnetic field is added to the fluid, things are already more complicated. In fact, as was shown by Collins,⁵ a Bianchi type I model with fluid and magnetic field resembles a model of the more complicated Bianchi type II with fluid alone. It is also interesting to note that models of type VI₀ with perfect fluid and a magnetic field have a dynamical behavior resembling the notoriously complicated “Mixmaster” behavior of Bianchi type IX models.⁶ Thus, changing the matter model can have effects on the complexity of the dynamics comparable with those encountered when passing to more general symmetry types.

A matter model for which the details of the global dynamics of Bianchi type I space–times has not previously been studied mathematically is the collisionless gas, described by the Vlasov equation. The only general facts that are known are that, with an appropriate choice of time orientation, (i) the space–time is future geodesically complete (when maximally extended toward the future); and (ii) there is a crushing singularity in the past where, except in the vacuum case, the curvature invariant $R_{\alpha\beta}R^{\alpha\beta}$ tends to infinity.

These fundamental facts were proved in Ref. 7, where it was shown that they hold for any Bianchi type other than IX and for a general class of matter models. The aim of this paper is to refine (i) and (ii) in the case of Bianchi type I symmetry and matter described by the Vlasov equation so as to get more detailed information about the asymptotics of the expanding phase and the nature of the initial singularity. An aspect of the situation that makes this more difficult than in the case of many other matter models is that for general initial data it is not possible to derive an explicit closed system of ordinary differential equations that describes the dynamics. This is because certain integrals that occur cannot be evaluated. In one special case, where massless

particles are considered and the initial phase space density has the form of the characteristic function of a ball, these integrals have been computed by Lukash and Starobinski.⁸ However, the explicit expressions they obtain are sufficiently complicated that they do not seem to make a rigorous analysis of the global dynamics any easier. On the other hand, they would probably be useful for numerical calculations, since they would allow costly numerical evaluation of integrals to be avoided.

The dynamics at late times of the models with massive particles can be described precisely. All solutions become isotropic and can be approximated by dust solutions in this limit (Theorem 5.4). On the other hand, the results of this paper do not give a complete picture of the dynamics near the initial singularity of the space–times being studied. They merely reduce the possible types of asymptotic behavior to a small number of alternatives. Improving on this is likely to require new techniques. These results leave open the possibility that Bianchi I space–times with a matter content described by kinetic theory may show complicated oscillatory behavior, and thus may be very different from those with other types of matter content studied up to now. The mechanism that allows for this complexity is simply the presence of anisotropy in the pressure that may respond to changes in the geometry. It may be that the only reason that the dynamics is so simple in the case of a perfect fluid is that this mechanism is excluded by a special symmetry assumption (the isotropy of the pressure). The one conclusion that emerges and that applies to all solutions considered here is that the ratio of the mean pressure to the energy density tends to one-third as the singularity is approached. This means that in a certain weak sense the dynamics for particles of unit mass m is approximated near the singularity by that for massless particles. For this reason both cases are often considered together in the following, although the main emphasis is on the case $m = 1$.

The results will now be summarized. There are, broadly speaking, two possible types of asymptotic behavior of solutions of the Einstein–Vlasov system with Bianchi I symmetry near the singularity. They will be referred to as convergent and oscillatory. Let λ_i denote the eigenvalues of the second fundamental form of the homogeneous hypersurfaces. Then the mean curvature of the homogeneous hypersurfaces is given by $\text{tr } k = \lambda_1 + \lambda_2 + \lambda_3$. Define the generalized Kasner exponents by $p_i = \lambda_i / \text{tr } k$. In the convergent type, the p_i tend to limits as the singularity is approached. There are three different cases, depending on these limiting values. The first case is that where the limiting values are $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. The well-known homogeneous and isotropic solutions of the Einstein–Vlasov system⁹ are of this type. The second is that the limiting values are $(0, \frac{1}{2}, \frac{1}{2})$ or some permutation thereof. The existence of solutions of this kind in the case of massless particles is shown in Sec. VI. These limiting values of the generalized Kasner exponents are not realized by any Bianchi type I space–time when the matter model is a perfect fluid (see the Appendix). The third is that the limiting values satisfy the Kasner relation $p_1^2 + p_2^2 + p_3^2 = 1$. Any solution for which one of the eigenvalues becomes negative at some time has this asymptotic behavior, and so there are plenty of examples. This is proved in Theorem 5.1. Note that the special case of this result when two of the p_i are equal is closely related to the homogeneous special case of a result of Rein¹⁰ for plane symmetric space–times. In the oscillatory type the p_i undergo infinitely many oscillations, in a sense that will now be specified. There are two cases to be considered, according to whether two of the eigenvalues are always equal or not. Consider first the case where two eigenvalues are equal and suppose, without loss of generality, that $\lambda_2 = \lambda_3$. Associate to any solution a string of symbols (which may be finite or infinite, depending on the solution) as follows. Moving backward from some fixed time, add an x to the string each time that $\lambda_1 - \lambda_2$ changes from being ≤ 0 to being > 0 and add a y each time it changes from being ≥ 0 to being < 0 . That this makes sense follows from the fact, proved in Sec. II, that the set of times where $\lambda_1 = \lambda_2$ can have no limit point unless this equality holds at all times. Thus, a finite time interval can only contribute a finite number of symbols. The solution is said to undergo infinitely many oscillations if the resulting string of symbols is infinite. Similarly, if there is some time where all eigenvalues are distinct, then a string

of symbols is associated to the solution by adding x , y , or z each time λ_1 , λ_2 , or λ_3 , respectively, becomes strictly larger than the other two eigenvalues.

Unfortunately it could not be shown whether any oscillatory solutions exist. If they did, then the behavior of Bianchi I models with kinetic theory as matter model would be much more complicated than in the case of a perfect fluid. If it could be shown that they existed, the question would remain whether the sequences of symbols they produce have some regularity or whether they are chaotic. In the absence of analytical techniques capable of deciding this question, it would be desirable to carry out a numerical investigation. This could provide evidence as to the existence (or otherwise) and nature of oscillatory behavior. It might also suggest new approaches to proving theorems about the global dynamics.

To each type of solution discussed above corresponds a characteristic behavior of the pressures. The solutions considered in the following all have diagonal energy-momentum tensors, and so three pressures P_i are defined by three diagonal components. The remaining diagonal component is the energy density ρ . The quantities $R_i = P_i/\rho$ must have a sum that converges to unity at the singularity. When the limiting values of the p_i are $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ or $(0, \frac{1}{2}, \frac{1}{2})$, then the limiting values of the R_i are $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ or $(0, \frac{1}{2}, \frac{1}{2})$, respectively. When the sum of the squares of the limiting values of the p_i is equal to unity, then the R_i tend to $(0, \frac{1}{2}, \frac{1}{2})$ or a permutation thereof, unless one p_i has the limiting value zero. In the latter case the R_i tend to $(1, 0, 0)$ or a permutation thereof.

The paper is organized as follows. In Sec. II some basic facts about the solutions are collected. Section III is concerned with a simplified system, which in some cases models the asymptotic behavior of the solutions of the original system. Some estimates for the pressures are derived in Sec. IV. Section V contains the main results. Section VI contains proofs of the existence or nonexistence of solutions with certain kinds of asymptotic behavior.

II. BASIC FACTS

The Einstein–Vlasov system is the system of equations that describes the kinetic theory of self-gravitating particles in general relativity. A thorough introduction to general relativistic kinetic theory and to the collisionless case, in particular, can be found in Ref. 11. For particles all of the same mass $m \geq 0$, the system can be written in the following form in the case of Bianchi type I symmetry:

$$-k_{ij}k^{ij} + (\text{tr } k)^2 = 16\pi\rho, \quad (2.1)$$

$$T_{0i} = 0, \quad (2.2)$$

$$\partial_t g_{ij} = -2k_{ij}, \quad (2.3)$$

$$\partial_t k_{ij} = \text{tr } k k_{ij} - 2k_{ik}k_j^k - 8\pi T_{ij} - 4\pi\rho g_{ij} + 4\pi \text{tr } T g_{ij}, \quad (2.4)$$

$$\frac{\partial f}{\partial t} + 2k_j^i v^j \frac{\partial f}{\partial v^i} = 0, \quad (2.5)$$

$$\rho = \int f(t, v^k) (m^2 + g_{rs} v^r v^s)^{1/2} (\det g)^{1/2} dv^1 dv^2 dv^3, \quad (2.6)$$

$$T_{ij} = \int f(t, v^k) v_i v_j (m^2 + g_{rs} v^r v^s)^{-1/2} (\det g)^{1/2} dv^1 dv^2 dv^3. \quad (2.7)$$

Equations (2.1)–(2.4) are the Einstein equations written in 3+1 form, (2.5) is the Vlasov equation, and (2.6) and (2.7) are the definition of the energy-momentum tensor in terms of the matter fields needed to complete the system. In these equations g_{ij} is the induced metric of the homogeneous

hypersurfaces, k_{ij} is the second fundamental form, f is the phase space density of particles, ρ is the energy density, T_{0i} and T_{ij} are components of the energy-momentum tensor, and $\text{tr } k$ is the mean curvature $g^{ij}k_{ij}$. With the exception of f all these quantities depend only on the time coordinate t . This time coordinate is Gaussian, i.e., it is constant on each homogeneous hypersurface and defines a parametrization by proper time when restricted to any geodesic normal to these hypersurfaces. The space–time metric is of the form

$$ds^2 = -dt^2 + g_{ij}(t)dx^i dx^j. \tag{2.8}$$

In the following it is always assumed, when talking about solutions of (2.1)–(2.7), that the function $f(t, v)$ is non-negative and has compact support for each fixed t . It is assumed that f is C^1 except in Sec. III, where f may be a distribution. The case of primary interest here is the case $m = 1$. Since, however, solutions of (2.1)–(2.7) with $m = 1$ resemble solutions with $m = 0$ close to the singularity, it is useful to also allow the case $m = 0$ from the beginning. Note that the real distinction here is between massless particles on the one hand and, on the other hand, massive particles, which all have the same mass. In the latter case it is convenient to choose the mass of a particle as a unit of mass so that the numerical value of the mass of a particle in the system of units used is unity.

For a given Bianchi I geometry, the Vlasov equation can be solved explicitly. (This is not possible for the other Bianchi types. The reason is explained in Ref. 12.) The result is that if f is expressed in terms of the covariant components v_i then it is independent of time. This means that if t_0 is some fixed time and $f_0(v_i) = f(t_0, v_i)$, then (2.6) can be rewritten as

$$\rho = \int f_0(v_i)(m^2 + g^{rs}v_r v_s)^{1/2}(\det g)^{-1/2} dv_1 dv_2 dv_3, \tag{2.9}$$

and (2.7) can be rewritten in a similar way. The explicit solution allows certain special subclasses of solutions of (2.1)–(2.7) to be identified. The first of these will be referred to as reflection-symmetric, and is defined by the conditions that

$$f_0(v_1, v_2, v_3) = f_0(-v_1, -v_2, v_3) = f_0(v_1, -v_2, -v_3), \tag{2.10}$$

and that the initial values of g_{ij} and k_{ij} are diagonal. Equation (2.10) implies that T_{ij} is diagonal. It then follows from (2.3) and (2.4) that g_{ij} and k_{ij} are always diagonal. The second case, which will be referred to as LRS (locally rotationally symmetric) is obtained by supplementing (2.10) by the conditions that

$$f_0(v_1, v_2, v_3) = F((v_1)^2 + (v_2)^2, v_3), \tag{2.11}$$

for some function F and that $g_{11} = g_{22}$, $k_{11} = k_{22}$ initially. Equation (2.11) implies that $T_{11} = T_{22}$ and it follows from (2.3) and (2.4) that $g_{11} = g_{22}$ and $k_{11} = k_{22}$ everywhere. A solution will also be called LRS if it satisfies the definition obtained from that just given by a permutation of the indices 1, 2, 3. A solution will be called isotropic if

$$f_0(v_1, v_2, v_3) = F((v_1)^2 + (v_2)^2 + (v_3)^2), \tag{2.12}$$

and if g_{ij} and k_{ij} are proportional to δ_{ij} on the initial hypersurface, and hence everywhere. If a solution satisfies the conditions on g_{ij} and k_{ij} in the definition of reflection-symmetric, LRS or isotropic for all t in some interval, but no assumption is made on f then the solution will be said to have reflection-symmetric, LRS or isotropic geometry, respectively, on that interval. It follows from (2.4) that T_{ij} has a corresponding symmetry property.

In this paper only reflection-symmetric solutions of (2.1)–(2.7) are considered. The alternative notation $a^2 = g_{11}/g_{11}(t_0)$, $b^2 = g_{22}/g_{22}(t_0)$, $c^2 = g_{33}/g_{33}(t_0)$ is used when it is convenient. Yet another form of (2.6) can then be obtained by doing a change of variables in (2.9),

$$\rho = \int f_0(aw_1, bw_2, cw_3)(m^2 + \delta^{rs}w_r w_s)^{1/2} dw_1 dw_2 dw_3. \quad (2.13)$$

The geometric interpretation of the w_i is that they are the components of the momentum in an orthonormal frame. Similarly,

$$T_i^i = \int f_0(aw_1, bw_2, cw_3)w_i^2(m^2 + \delta^{rs}w_r w_s)^{-1/2} dw_1 dw_2 dw_3. \quad (2.14)$$

In order to study the dynamics of the solutions of the system (2.1)–(2.7) in detail, it is useful to introduce certain dimensionless variables that remain finite at the singularity. It follows from (2.1) that $\text{tr } k$ never vanishes, except in the case of flat space–time, which is excluded from consideration in the following. By replacing t by $-t$ if necessary, it can be arranged that $\text{tr } k < 0$ everywhere. It will be assumed that this has been done. Define

$$\hat{k}_{ij} = k_{ij}/\text{tr } k, \quad (2.15)$$

$$\hat{\rho} = \rho/(\text{tr } k)^2, \quad (2.16)$$

$$\hat{T}_{ij} = T_{ij}/(\text{tr } k)^2, \quad (2.17)$$

$$\tau(t) = - \int_{t_0}^t \text{tr } k(t') dt'. \quad (2.18)$$

In terms of these variables equations (2.1) and (2.4) become

$$-\hat{k}_i^i \hat{k}_i^i + 1 = 16\pi\hat{\rho}, \quad (2.19)$$

$$\partial_\tau \hat{k}_j^i = -12\pi\hat{\rho}(\hat{k}_j^i - \frac{1}{3}\delta_j^i) + 8\pi\hat{T}_j^i + 4\pi(\hat{k}_j^i - \delta_j^i)\text{tr } \hat{T}. \quad (2.20)$$

The following lemma provides some information about the range of τ .

Lemma 2.1: Let a solution of (2.1)–(2.7) with $m=0$ or $m=1$ be given, which is the maximal globally hyperbolic development of initial data on the hypersurface $t=t_0$. Then $\text{tr } k(t)$ is a monotonic function defined on an interval (t_1, ∞) . By translating t , it can be assumed that $t_1=0$. Then $\lim_{t \rightarrow 0} \text{tr } k = -\infty$ and $\lim_{t \rightarrow \infty} \text{tr } k = 0$. Moreover, $-3/t \leq \text{tr } k(t) \leq -1/t$.

Proof: That $\text{tr } k(t)$ is monotonic and defined on an interval of the form (t_1, ∞) with $\lim_{t \rightarrow t_1} \text{tr } k = -\infty$ was shown for the case $m=1$ in Ref. 13. Essentially the same argument applies for $m=0$. In the latter case the coefficients of the characteristic system are only Lipschitz instead of C^1 , but this causes no problems. Now it follows from (2.4) that $\frac{1}{3}(\text{tr } k)^2 \leq \partial_t(\text{tr } k) \leq (\text{tr } k)^2$. Comparing the solution with the ordinary differential equations corresponding to these inequalities then gives the desired estimates (cf. Ref. 7).

This result implies that the integral defining τ diverges as $t \rightarrow 0$ and as $t \rightarrow \infty$. Hence, the solution of (2.19)–(2.20) exists globally in τ .

The solution of (2.19)–(2.20), of course, contains only a small part of the information of that contained in the solution of (2.1)–(2.7). The former is only a projection of the latter. Nevertheless, it will be seen that a lot of information about the solution of the full equations can be obtained by studying this projection. Consider now the set K of triples of real numbers $\hat{k}_1^1, \hat{k}_2^2, \hat{k}_3^3$ that satisfy $\sum_i (\hat{k}_i^i)^2 \leq 1$ and $\sum_i \hat{k}_i^i = 1$. This is a compact subset of \mathbf{R}^3 . In fact, it is a disk in a plane. A solution

of (2.19)–(2.20) defines a point of K at each time τ . It is on the boundary of K in the plane if f is identically zero and in the interior otherwise. This point depending on τ , considered as a mapping from \mathbf{R} to K , will be referred to as the projection of the given solution. The projection of a vacuum solution, which lies on the boundary of K , is constant. The vacuum solutions are, of course, the well-known Kasner solutions and the boundary of K may be referred to as the “Kasner circle.” Let C denote the point $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. If a solution has isotropic geometry on a time interval, then its projection lies at the point C during this time. Conversely, if its projection lies at C on a given time interval, then it can be made to have isotropic geometry by a time-independent rescaling of the spatial coordinates. Let L_1, L_2 , and L_3 be the subsets of K defined by $\hat{k}_2^2 = \hat{k}_3^3, \hat{k}_1^1 = \hat{k}_3^3$, and $\hat{k}_1^1 = \hat{k}_2^2$, respectively. A solution has LRS geometry on a time interval (up to a constant rescaling of the spatial coordinates as above), if and only if its projection lies on one of the lines L_1, L_2 , or L_3 during this time. Let L_i^+ denote the open half of L_i which ends at the point with coordinates $(-\frac{1}{3}, \frac{2}{3}, \frac{2}{3})$ or a permutation thereof, and let L_i^- denote the opposite half-line, which ends at the point with coordinates $(1, 0, 0)$ or a permutation thereof. Let V_i^+ and V_i^- denote these end points. Let A_1 be the open region bounded by L_2^+, L_3^+ and the boundary of K , and let A_2 and A_3 be defined by cyclically permuting $(1, 2, 3)$ in this definition. Let B_i be the subset of K where $\hat{k}_i^i \leq 0$.

The components of the metric satisfy the evolution equations

$$\frac{dg_{ii}}{d\tau} = 2\hat{k}_i^i g_{ii}, \quad \frac{d}{d\tau} \left(\frac{g_{ii}}{g_{jj}} \right) = 2(\hat{k}_i^i - \hat{k}_j^j) \left(\frac{g_{ii}}{g_{jj}} \right), \tag{2.21}$$

which imply that g_{ii} or their ratios increase or decrease exponentially if certain sign conditions are satisfied by the \hat{k}_i^i . There are, of course, corresponding statements for the scale factors a, b , and c . Given an initial datum f_0 for f the quantities ρ and T_i^i are determined uniquely by the g_{ii} by means of Eqs. (2.13) and (2.14). The quantities $\hat{\rho}$ and \hat{T}_i^i are given in terms of ρ and T_i^i and $\text{tr } k$ by the definitions (2.16) and (2.17), while the off-diagonal components \hat{T}_i^j are zero by assumption. Thus, (2.19), (2.20), and (2.21), together with the equation

$$\partial_\tau(\text{tr } k) = -(\text{tr } k)(1 - 12\pi\hat{\rho} + 4\pi \text{tr } \hat{T}), \tag{2.22}$$

derive from (2.4), form a closed system of ordinary differential equations, which, for fixed f_0 , formally determine the quantities \hat{k}_i^i, g_{ii} , and $\text{tr } k$ as functions of τ in terms of initial data. If the coefficients of this system were locally Lipschitz, it would follow from the standard uniqueness theorem for ordinary differential equations that they determine them uniquely. It will now be shown that, in fact, for $m=1$ this dependence is analytic. To do this it is convenient to use the expressions for ρ and T_i^i of the type (2.9). Analyticity is a consequence of the following lemma.

Lemma 2.2: Let W be a mapping of $U \times \mathbf{R}^3$ to \mathbf{R} , where U is an open subset of \mathbf{R}^3 . Suppose that W extends to a C^1 mapping \tilde{W} of $\tilde{U} \times \mathbf{R}^3$ to \mathbf{C} , where \tilde{U} is an open neighborhood of U in \mathbf{C}^3 , and that $\tilde{W}(\cdot, y)$ satisfies the Cauchy–Riemann equations for each fixed $y \in \mathbf{R}^3$. Finally, suppose that each $z \in \tilde{W}$ has an open neighborhood V such that the supports of the functions $\tilde{W}(z, \cdot)$ are contained in a common compact subset K of \mathbf{R}^3 . Then the function $F(x) = \int_{\mathbf{R}^3} W(x, y) dy$ is analytic.

Proof: It suffices to show that the function $\tilde{F}(z) = \int_{\mathbf{R}^3} \tilde{W}(z, y) dy$ is complex analytic, and this is true if \tilde{F} is C^1 and satisfies the Cauchy–Riemann equations.¹⁴ The assumptions on the smoothness and support of \tilde{W} justify differentiation under the integral, and so the Cauchy–Riemann equations for \tilde{F} follow from the Cauchy–Riemann equations for \tilde{W} .

A consequence of the analyticity of the coefficients in the system of ordinary differential equations is that the solutions are analytic. It follows that if a solution with $m=1$ has LRS or isotropic geometry on some nonempty open time interval, then it must have LRS or isotropic geometry, respectively, for all values of τ . The same conclusion holds if there is a sequence of times having a limit point where the geometry is LRS or isotropic, respectively.

III. THE ASYMPTOTIC SYSTEM

In this section a certain system of ordinary differential equations is introduced and the qualitative behavior of its solutions analyzed. This system is used later to study the asymptotic behavior of solutions of the system (2.1)–(2.7). This system can be obtained formally from (2.1)–(2.7) by replacing the C^1 function $f(t, v_1, v_2, v_3)$ by a measure of the form $f(t, v_1) \delta(v_2 - \bar{v}_2(t)) \delta(v_3 - \bar{v}_3(t))$, where δ is a Dirac measure, and taking $m=0$. Solutions of this system can be interpreted as certain distributional solutions of the Einstein–Vlasov system for massless particles. These are intermediate between smooth solutions and the even more singular solutions, which are in one to one correspondence with dust solutions. (For the correspondence between dust and distributional solutions of the Vlasov equation see Ref. 15.) The mathematical results that will now be derived are independent of this interpretation.

The system of ODEs to be considered is the special case of the equations (2.19) and (2.20) obtained by setting $\hat{T}_1^1 = \hat{T}_2^2 = 0$ and $\hat{T}_3^3 = \hat{\rho}$. Note that, in contrast to the general case of (2.19) and (2.20), these specialized equations suffice to determine all unknowns occurring in them from initial data. The explicit form of (2.20) in this case is

$$\partial_\tau \hat{k}_1^1 = -8\pi \hat{\rho} \hat{k}_1^1, \quad (3.1)$$

$$\partial_\tau \hat{k}_2^2 = -8\pi \hat{\rho} \hat{k}_2^2, \quad (3.2)$$

$$\partial_\tau \hat{k}_3^3 = -8\pi \hat{\rho} (\hat{k}_3^3 - 1). \quad (3.3)$$

If initial data are chosen at some time that satisfy the condition $\sum_i (\hat{k}_i^i) = 1$, then the solution also satisfies it. Only solutions with this property are considered here. It follows from (3.1) and (3.2) that $\partial_\tau (\hat{k}_2^2 / \hat{k}_1^1) = 0$ whenever $\hat{k}_1^1 \neq 0$. Let r be the constant value of $\hat{k}_2^2 / \hat{k}_1^1$. Then

$$\hat{k}_2^2 = r \hat{k}_1^1, \quad (3.4)$$

$$\hat{k}_3^3 = 1 - (1+r) \hat{k}_1^1. \quad (3.5)$$

Substituting (2.19), (3.4), and (3.5) into (3.1) gives

$$\partial_\tau \hat{k}_1^1 = (\hat{k}_1^1)^2 [(1+r+r^2) \hat{k}_1^1 - (1+r)]. \quad (3.6)$$

Proposition 3.1: Let $(\hat{k}_1^1, \hat{k}_2^2, \hat{k}_3^3)$ be a solution of (3.1)–(3.3) satisfying $\sum_i (\hat{k}_i^i) = 1$ and $\sum_i (\hat{k}_i^i)^2 < 1$. Define $r = \hat{k}_2^2 / \hat{k}_1^1$ whenever $\hat{k}_1^1 \neq 0$. Then

- (i) if \hat{k}_1^1 is zero initially it is always zero;
- (ii) if \hat{k}_1^1 is initially (and hence always) nonzero, then r is constant;
- (iii) when $\hat{k}_1^1 \neq 0$ it is a monotonic function with $\lim_{\tau \rightarrow -\infty} \hat{k}_1^1 = (1+r)/(1+r+r^2)$ and $\lim_{\tau \rightarrow \infty} \hat{k}_1^1 = 0$; and
- (iv) in that case $\lim_{\tau \rightarrow -\infty} (\hat{k}_3^3 - \hat{k}_2^2) = -r(r+2)/(1+r+r^2)$.

Proof: Statement (i) follows from (3.1). Statement (ii) has been demonstrated above. Statement (iii) follows from (3.6). The last conclusion is then an immediate consequence of the definitions.

Of course analogous statements hold if \hat{k}_1^1 and \hat{k}_2^2 are interchanged, since these two quantities occur symmetrically in the hypotheses.

IV. PRESSURE ESTIMATES

The results of this section are all variations on the theme that if the space–time is expanding in a certain direction, then the pressure in that direction tends to decrease. It is assumed throughout

that the geometry is reflection-symmetric. A solution of (2.1)–(2.7) with $m=0$ satisfies $\text{tr } T/\rho=1$. This condition never holds when $m=1$, but the next lemma shows that it does hold asymptotically.

Lemma 4.1: Suppose that some solution of (2.1)–(2.7) with $m=1$ is defined on an interval $(-\infty, \tau_1)$, with f not identically zero. Then $\lim_{\tau \rightarrow -\infty} \text{tr } T/\rho=1$.

Proof: It was shown in Ref. 7 that $\lim_{\tau \rightarrow -\infty} \rho = \infty$. Thus, the result will follow if it can be shown that $\rho \rightarrow \infty$ implies $\text{tr } T/\rho \rightarrow 1$. To do this, choose some radius $L > 0$ and write $\rho = \rho_1 + \rho_2$ and $\text{tr } T = (\text{tr } T)_1 + (\text{tr } T)_2$, where the first summand is the integral over the region $|w| < L$ of the integrand in (2.13) or (2.14), respectively, and the second is the integral over the complementary region. Using the fact that $(1+x^2)^{1/2} - x^2/(1+x^2)^{1/2} = (1+x^2)^{-1/2}$, it can be seen that $\rho_2 - (\text{tr } T)_2 \leq (1+L^2)^{-1} \rho_2$. Hence

$$(\text{tr } T)_2 \geq \frac{L^2}{1+L^2} \rho_2 \tag{4.1}$$

and

$$\text{tr } T \geq \text{tr } T_2 \geq \frac{L^2}{1+L^2} \left(\rho - \left(\frac{4\pi}{3} \right) L^3 (1+L^2)^{1/2} \|f_0\|_\infty \right). \tag{4.2}$$

By choosing L sufficiently large, the quantity $L^2/(1+L^2)$ can be made as close to unity as desired. For fixed L , the quantity in brackets on the right-hand side of (4.2) approaches ρ as ρ becomes large. This suffices to give the conclusion of the lemma.

For a given initial datum f_0 , the equation (2.14) defines the pressures T_i^j as functions of a , b , and c . The following results concern the qualitative behavior of these functions.

Lemma 4.2: If f_0 is not identically zero and $a \leq C' \min\{1, b, c\}$ for some constant $C' > 0$, there exists a constant $C > 0$ such that $T_1^1 \geq C a^{-2} b^{-1} c^{-1}$ and $T_2^2/T_1^1 \leq C(a/b)^{4/3}$. In the case $m=0$ the conclusion holds under the weaker hypothesis that $a \leq C' \min\{b, c\}$.

Proof: Let p be a point of \mathbf{R}^3 where $f_0 \neq 0$, whose first coordinate w_1 is nonzero. Let δ be a positive number such that f is bounded below by some positive constant η on the closed cube W of side 2δ centered at p and such that w_1 does not vanish anywhere on this cube. Consider now the image W' of the cube W under the mapping $(w_1, w_2, w_3) \mapsto (a^{-1}w_1, b^{-1}w_2, c^{-1}w_3)$. On W' the functions w_2/w_1 and w_3/w_1 are bounded. Under the assumptions of the lemma they are bounded by the same constant on W' . It follows that $|w_1|/(1+|w|^2)^{1/2}$ is bounded below on any such cube by a positive constant that is independent of a , b , and c , which satisfy the hypotheses of the lemma. The integral defining T_1^1 can be bounded from below by the integral of the same quantity over W . It follows that

$$T_1^1 \geq C \eta a^{-2} b^{-1} c^{-1}, \tag{4.3}$$

and this proves the first part of the lemma. To get a lower bound for T_1^1/T_2^2 , the domain of integration in the definition of these two quantities will be divided into the regions $|w_2| > R|w_1|$ and $|w_2| < R|w_1|$, where R is a positive number that will be specified later. Corresponding to this decomposition of the domain of integration, there are decompositions $T_1^1 = T_{11}^1 + T_{12}^1$ and $T_2^2 = T_{21}^2 + T_{22}^2$. The volume of the region where $|w_2| > R|w_1|$ and $f(a w_1, b w_2, c w_3) \neq 0$ can be bounded by an expression of the form $CR^{-1}c^{-1}b^{-2}$, and so $T_{21}^2 \leq CR^{-1}c^{-1}b^{-3}$. On the other hand, $T_{22}^2 \leq R^2 T_{12}^1$. Thus,

$$T_2^2 \leq CR^{-1}c^{-1}b^{-3} + R^2 T_{12}^1 \leq (CR^{-1}(a/b)^2 + R^2) T_{11}^1, \tag{4.4}$$

where in the last step (4.3) has been used. Choosing $R = (a/b)^{2/3}$ gives $T_2^2 \leq C(a/b)^{4/3} T_{11}^1$, and this proves the result for T_2^2/T_1^1 .

Lemma 4.3: Suppose that some solution of (2.1)–(2.7) is defined on the interval $(-\infty, \tau_1)$, with f not identically zero. If $\hat{k}_1^1 - \hat{k}_2^2 \geq A$ and $\hat{k}_1^1 - \hat{k}_3^3 \geq A$ on this interval for some $A > 0$, then $T_i^i/T_1^1 \leq Ce^{4A\tau/3}$ for $i = 2, 3$.

Proof: It suffices to note that under the assumptions of this lemma there will be a time interval $(-\infty, \tau_2)$ where the hypotheses of Lemma 4.2 hold, so that (4.4) can be applied.

The time derivatives of the quantities \hat{T}_i^i cannot, in general, be expressed in terms of the dimensionless quantities (2.9)–(2.11), so as to get a closed system of ordinary differential equations. However, they can be estimated in terms of these quantities. Note first that

$$\frac{d\hat{T}_i^i}{d\tau} = -(\text{tr } k)^{-3} \frac{dT_i^i}{dt} + 2\hat{T}_i^i[1 - 12\pi\hat{\rho} + 4\pi \text{tr } \hat{T}]. \tag{4.5}$$

Next, a change of variables in (2.7) gives, in the diagonal case,

$$T_i^i = g^{ii} \int f_0(v_1, v_2, v_3)(v_i)^2(m^2 + g^{rs}v_r v_s)^{-1/2}(\det g)^{-1/2} dv_1 dv_2 dv_3. \tag{4.6}$$

Hence,

$$\begin{aligned} \frac{dT_1^1}{dt} &= (3k_1^1 + k_2^2 + k_3^3)T_1^1 + g^{11} \int f_0(v_1, v_2, v_3)(v_1)^2 F(v_1, v_2, v_3) \\ &\quad \times (m^2 + g^{rs}v_r v_s)^{-3/2}(\det g)^{-1/2} dv_1 dv_2 dv_3, \end{aligned} \tag{4.7}$$

where

$$F(v_1, v_2, v_3) = (-g^{11}k_1^1(v_1)^2 - g^{22}k_2^2(v_2)^2 - g^{33}k_3^3(v_3)^2). \tag{4.8}$$

Note now that

$$|F(v_1, v_2, v_3)| \leq (|k_1^1| + |k_2^2| + |k_3^3|)(m^2 + g^{rs}v_r v_s), \tag{4.9}$$

and so the integral in (4.7) can be bounded in modulus by $3 \text{tr } k T_1^1$. Putting this information into (4.5) gives the desired bound.

Lemma 4.4: Consider a maximally extended solution of (2.1)–(2.7) with $m = 1$ and f not identically zero. If $g_{ii} \rightarrow \infty$ as $\tau \rightarrow \infty$, then $\lim_{\tau \rightarrow \infty} T_i^i/\rho = 0$. If, on the other hand, g_{11} is bounded above and all g_{ii} are bounded below by a positive constant on an interval of the form $[\tau_1, \infty)$, then T_1^1/ρ is bounded below by a positive constant on that interval.

Proof: It follows from (2.13) and (2.14) that $T_1^1 \leq Ca^{-2}\rho$, and this proves the first statement. To get the other conclusion, choose a cube C_1 as in the proof of Lemma 4.2. Then $T_1^1 \geq Ca^{-1}b^{-1}c^{-1}$ while $\rho \leq Ca^{-1}b^{-1}c^{-1}$. Hence, $T_1^1/\rho \geq C > 0$.

V. THE MAIN RESULTS

Lemma 5.1 (compactness lemma): Let a sequence of reflection-symmetric global solutions of Eqs. (2.1)–(2.7) be given. Then there exists a subsequence such that \hat{k}_j^j and $\hat{\rho}$ converge uniformly on compact sets of \mathbf{R} . Here \hat{T}_i^i also converges uniformly on compact subsets (after possibly passing to a subsequence once more), and the limiting quantities satisfy (2.19) and (2.20).

Proof: The quantities \hat{k}_j^j are contained in the compact set K and so are, in particular, uniformly bounded. By (2.19), $\hat{\rho}$ is uniformly bounded. It follows that \hat{T}_j^j is uniformly bounded. Equation (2.20) now shows that $\partial_\tau \hat{k}_j^j$ is uniformly bounded. By Ascoli’s theorem there exists a subsequence such that \hat{k}_j^j converges uniformly on the interval $[-1, 1]$. Applying the theorem again shows that this subsequence has a subsequence such that \hat{k}_j^j converges uniformly on $[-2, 2]$. Continuing in

this way, we obtain a collection of subsequences indexed by a positive integer n with the properties that for the n th subsequence \hat{k}_j^i converges uniformly on $[-n, n]$, and each sequence is a subsequence of the previous one. The diagonal sequence has the property that \hat{k}_j^i converges on each compact subset of the real line. By the Hamiltonian constraint $\hat{\rho}$ also converges uniformly on compact sets along this subsequence. In the diagonal case the derivatives $\partial_\tau \hat{T}_i^i$ are bounded, as was shown in Sec. IV and applying Ascoli’s theorem as before gives the remaining conclusions.

Theorem 5.1: Let a global solution of Eqs. (2.1)–(2.7) be given for which f is not identically zero. If at some time τ_1 , the projection of the solution lies in the set B_i for some i , then (i) the projection lies in B_i for all $\tau \leq \tau_1$; (ii) if there exists some $\tau_2 > \tau_1$ such that the projection of the solution lies in the complement of B_i , then it lies in the complement of all B_j for $\tau > \tau_2$; and (iii) as $\tau \rightarrow -\infty$ the projection converges to a point of the boundary of the region K that is not one of the points V_i^- .

Proof: Suppose without loss of generality that $i = 1$. It follows from (2.20) that

$$\partial_\tau \hat{k}_1^1 = -4\pi(\hat{\rho} - \text{tr } \hat{T})(3\hat{k}_1^1 - 1) - 8\pi \text{tr } \hat{T} \hat{k}_1^1 + 8\pi \hat{T}_1^1. \tag{5.1}$$

If at some time $\hat{k}_1^1 \leq 0$, then the first and second terms on the right-hand side of (5.1) are non-negative while the third term is positive. Hence $\partial_\tau \hat{k}_1^1 > 0$. This implies the first conclusion of the theorem. Moreover, it means that if the projection once leaves B_1 it can never reenter it. A similar statement, of course, applies to any other B_j , and this gives (ii). To prove (iii) note first that $\partial_\tau \hat{k}_1^1$ is bounded below by a positive constant as long as $\hat{\rho}$ is. This shows that $\liminf_{\tau \rightarrow -\infty} \hat{\rho} = 0$. Equation (5.1) also implies that the integral of $\hat{\rho}$ on the interval $(-\infty, \tau_1]$ must be finite so that for each i the integral of the right-hand side of (2.20) is absolutely convergent. Hence, each \hat{k}_i^i tends to a limit as $\tau \rightarrow -\infty$. By what has already been said it can only be a point of the boundary of K . The monotonicity of \hat{k}_1^1 shows that this limit cannot be one of the points V_i^- .

Theorem 5.2: Let a global solution of Eqs. (2.1)–(2.7) be given for which f is not identically zero. If at some time τ_1 the projection of the solution lies in the set A_i for some i , then as τ decreases either (i) the projection converges to a point of the boundary of K as $\tau \rightarrow -\infty$; or (ii) it reaches L_j^+ for some j or C at a finite time before τ_1 ; or (iii) it stays in A_i for all $\tau < \tau_1$ and it has a point of one of the lines L_j^+ or the point C as an accumulation point.

Proof: Suppose without loss of generality that $i = 1$. When the projection lies in A_1 the inequalities $k_1^1 > k_2^2$, $k_1^1 > k_3^3$, and $\hat{k}_1^1 > \frac{1}{3}$ hold. Suppose that on the time interval $(-\infty, \tau_1)$ the inequalities $\hat{k}_1^1 - \hat{k}_2^2 \geq A$ and $\hat{k}_1^1 - \hat{k}_3^3 \geq A$ are satisfied for some $A > 0$. Then, by Lemma 4.3, it follows that on this time interval T_1^1/T_2^2 and T_1^1/T_3^3 can be bounded below by a decreasing function, which tends to ∞ as $\tau \rightarrow -\infty$. Moreover, by Lemma 4.1, $\text{tr } T/\rho \rightarrow 1$ as $\tau \rightarrow -\infty$. Now, define a sequence of solutions of (2.1)–(2.7) by $u_n(\tau) = u(\tau - n)$, $\tau \in (-\infty, \tau_1)$, where u denotes any of the functions that make up the solution and n is a positive integer. By Lemma 5.1 there exists a subsequence such that \hat{k}_j^i and $\hat{\rho}$ converge uniformly on compact subsets. By the statements made above, $\text{tr } \hat{T}$ must tend to the same limit as $\hat{\rho}$ along this sequence. Also, \hat{T}_1^1 tends to this same limit and \hat{T}_2^2 and \hat{T}_3^3 tend to zero. Applying Lemma 5.1 again shows that the limits of these sequences satisfy (2.19) and (2.20). Because of the values of the limits they, in fact, satisfy the asymptotic system (3.1)–(3.3). The solution of the asymptotic system obtained inherits the properties that $\hat{k}_1^1 \geq \hat{k}_2^2$ and $\hat{k}_1^1 \geq \hat{k}_3^3$. The only solutions of the asymptotic system that satisfy these inequalities on an interval of the form $(-\infty, \tau_1)$ are the vacuum solutions. If for some choice of subsequence this vacuum solution is not that corresponding to the point V_1^- , then the projection of the original solution must converge to that point, by Theorem 5.1. Otherwise every subsequence of the sequence of translated solutions has a subsequence that converges to the same solution of the asymptotic system. Hence, the whole sequence converges to this solution and the projection of the original solution converges to V_1^- . In both cases the solution of the original system converges to a point of the boundary of K .

It remains to consider the case where the above estimate is not satisfied for any $A > 0$. If the solution does not reach L_j^+ for some j or C in finite time, then it stays in A_i for all $\tau < \tau_1$. Then it must have as an accumulation point either a point on L_j^+ for some j , C , or V_j^+ for some j . In the first two cases this gives case (iii) of the conclusion of the theorem. In the third case the solution enters B_j , and so by Theorem 5.1, case (i) of the conclusion holds.

In case (iii) of this theorem we can also consider a limit of translates of the solutions whose existence is guaranteed by Lemma 5.1. If the ratios a/b and a/c tended to zero as $\tau \rightarrow -\infty$ for the original solution, then by Lemma 4.2 the ratios T_1^1/T_2^2 and T_1^1/T_3^3 would tend to infinity and the solution would belong to case (i). Thus, in case (iii) it can be assumed without loss of generality (after possibly interchanging the indices 2 and 3) that b/a is bounded as $\tau \rightarrow -\infty$. Hence $\int_{-\infty}^{\tau_1} (\hat{k}_1^1 - \hat{k}_2^2)$ is finite. Since $\partial_\tau(\hat{k}_1^1 - \hat{k}_2^2)$ is bounded, it follows that $\hat{k}_1^1 - \hat{k}_2^2 \rightarrow 0$ as $\tau \rightarrow -\infty$. Hence, the solution obtained as a limit of translates has LRS geometry. It also satisfies $\rho = \text{tr } T$. Information about the asymptotics of LRS solutions can thus be used to obtain information about the asymptotics of the solutions, which fit into case (iii) of Theorem 5.2 but do not fit into case (i).

Theorem 5.3: Let a solution of Eqs. (2.19)–(2.20) be given that satisfies the LRS condition $k_2^2 = k_3^3$. If at some time τ_1 the projection satisfies $\hat{k}_1^1 < \frac{1}{3}$, then either (i) the projection of the solution tends to the point $(-\frac{1}{3}, \frac{2}{3}, \frac{2}{3})$ as $\tau \rightarrow -\infty$; (ii) it tends to the point $(0, \frac{1}{2}, \frac{1}{2})$ as $\tau \rightarrow -\infty$; (iii) it reaches the point $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ at a finite time before τ_1 ; or (iv) it tends to the point $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ as $\tau \rightarrow -\infty$.

Proof: Suppose first that $\hat{k}_1^1 \leq \frac{1}{3} - A$ for some $A > 0$. Then by Lemma 4.2 the ratio of $T_2^2 = T_3^3$ to T_1^1 increases without limit. Passing to a limit of translates in the familiar way gives a solution of the equation $\partial_\tau \hat{k}_1^1 = -8\pi \hat{k}_1^1 \hat{\rho}$ for which \hat{k}_1^1 satisfies the same inequality as before. There are only two such solutions, namely that for which $\hat{\rho} = 0$ and that for which $\hat{k}_1^1 = 0$. In the first of these cases the original solution must enter the region B_1 , and hence by Theorem 5.1 belong to case (i) of the conclusion of the present theorem. The only way of avoiding this is if, no matter which subsequence is chosen, the limiting value of \hat{k}_1^1 is zero. Hence, the projection of the original solution must converge to the point $(0, \frac{1}{2}, \frac{1}{2})$. Thus, the solution belongs to case (ii) of the conclusion. Now consider the case where there is no $A > 0$ with the given property. If, despite this, the ratio of T_2^2 to T_1^1 tends to infinity, we can argue as before to show that the solution belongs to case (i) or (ii). If, on the other hand, this ratio remains bounded, then the ratio a/b must remain bounded, and hence if \hat{k}_1^1 remains smaller than $\frac{1}{3}$ forever then $\int_{-\infty}^{\tau_1} (\frac{1}{3} - \hat{k}_1^1)$ is finite. It then follows as in the discussion following the proof of Theorem 5.2 that $\hat{k}_1^1 \rightarrow \frac{1}{3}$. Thus the solution either belongs to case (iii) or case (iv).

Theorem 5.4: Let a solution of the equations (2.1)–(2.7) with $m = 1$ and f not identically zero be given. Then $\hat{k}_i^i \rightarrow \frac{1}{3}$ and $T_i^i/\rho \rightarrow 0$ for each i as $\tau \rightarrow \infty$.

Proof: In Ref. 13, it was shown that the scale factors a , b , and c are bounded below by a positive constant on any interval of the form $[\tau_1, \infty)$. Using (5.1), this statement can be strengthened. Suppose that \hat{k}_1^1 were negative on an interval of the form $[\tau_1, \infty)$. Then it would follow, as in the proof of Theorem 5.1, that the integral of $\hat{\rho}$ on this interval was finite. But $\hat{\rho}$ is increasing on this interval, a contradiction. It follows that each \hat{k}_i^i must become zero after a finite time, and once this happens it must immediately become positive and stay positive. Thus, for τ_1 sufficiently large, a , b , and c are increasing. Consider now the behavior of the quantity $\min\{a, b, c\}$. Suppose first that it tends to infinity as $\tau \rightarrow \infty$. Then, by Lemma 4.4 the ratios T_i^i/ρ tend to zero as $\tau \rightarrow \infty$. Construct a limit of translates as in the proof of Theorem 5.2, except that this time the translations should be done in the opposite direction. Then the limiting solution satisfies $\partial_\tau \hat{k}_i^i = -12\pi \hat{\rho}(\hat{k}_i^i - \frac{1}{3})$. This is the equation that is satisfied by a Bianchi I solution of the Einstein equations coupled to dust. It is well known and also easy to see directly that in the case of dust each \hat{k}_i^i converges to $\frac{1}{3}$ as $\tau \rightarrow \infty$. Because a convergent subsequence can be extracted from any subsequence of the sequence of translates by integers, it follows that $\hat{k}_i^i \rightarrow \frac{1}{3}$ for the original solution of the Einstein–Vlasov system as well. Next, consider the case where a is bounded on an interval of the form $[\tau_1, \infty)$ while $\min\{b, c\} \rightarrow \infty$ as $\tau \rightarrow \infty$. Then, by Lemma 4.4 the ratios T_2^2/ρ and T_3^3/ρ converge to zero as $\tau \rightarrow \infty$ while T_1^1/ρ remains bounded away from zero. Equation (5.1) implies that $\partial_\tau \hat{k}_1^1$ is

bounded below by a positive constant if $\hat{k}_1^1 < \frac{1}{3}$ and T_2^2/T_1^1 and T_3^3/T_1^1 are less than $1-A$ for some constant $A > 0$. However, this contradicts the boundedness of a . Since the volume tends to infinity as $\tau \rightarrow \infty$, at least one of a , b , or c must tend to infinity. It follows that to complete the proof we may assume without loss of generality that a and b are bounded while c tends to infinity. By Lemma 4.4, $T_3^3/\rho \rightarrow 0$, while T_1^1/ρ and T_2^2/ρ are bounded below by a positive constant. Now the integral $\int_{\tau_1}^{\infty} \hat{k}_i^i(\tau) d\tau$ is finite for $i = 1, 2$ and $\partial_{\tau} \hat{k}_i^i$ is bounded. Hence, \hat{k}_1^1 and \hat{k}_2^2 tend to zero as $\tau \rightarrow \infty$ and $\hat{k}_3^3 \rightarrow 1$. But the given behavior of the pressures shows that for $\hat{k}_3^3 \geq \frac{1}{3}$ and sufficiently large times $\partial_{\tau} k_3^3$ is negative, a contradiction. This completes the proof.

VI. A COMPACTIFICATION

This section is devoted to a finer examination of LRS solutions of the Einstein–Vlasov system with massless particles. For LRS solutions with $k_2^2 = k_3^3$, let $k = \hat{k}_1^1$, $q = b/a$, $Q = T_1^1/\rho$. Then $\hat{k}_2^2 = \frac{1}{2}(1-k)$ and $\hat{\rho} = (1/16\pi)(\frac{1}{2} + k - \frac{3}{2}k^2)$. The essential equations describing the dynamics are

$$\partial_{\tau} k = \frac{1}{4}(1+3k)(1-k)(Q-k), \tag{6.1}$$

$$\partial_{\tau} q = \frac{1}{2}(1-3k)q. \tag{6.2}$$

The quantity Q can be expressed entirely in terms of q and the initial data as follows:

$$Q = q^2 \left[\frac{\int f_0(v_i) v_1^2 (q^2 v_1^2 + v_2^2 + v_3^2)^{-1/2} dv_1 dv_2 dv_3}{\int f_0(v_i) (q^2 v_1^2 + v_2^2 + v_3^2)^{1/2} dv_1 dv_2 dv_3} \right]. \tag{6.3}$$

Substituting (6.3) into (6.1) makes the equations (6.1) and (6.2) into an autonomous system of ordinary differential equations for k and q . Lemma 4.2 shows that $Q(q) = O(q^{4/3})$ as $q \rightarrow 0$. This means that the system (6.1)–(6.2) can be extended in a C^1 manner to the boundary $q = 0$. Moreover, Q does not contribute to the linearization of the extended system at the critical point $q = 0$, $k = 0$. The eigenvectors of the linearization are directed along the k and q axes, with eigenvalues $-\frac{1}{4}$ and $\frac{1}{2}$, respectively. It follows (see, e.g., Ref. 16) that the dynamical system has an unstable manifold that is a curve tangent to the k axis at the point $(0,0)$. This shows that for any initial value f_0 it is possible to find LRS solutions of the Einstein–Vlasov system, where the distribution function has the initial value f_0 and where the quantities \hat{k}_i^i converge to $(0, \frac{1}{2}, \frac{1}{2})$ as $\tau \rightarrow -\infty$. Note that the stable manifold is just the k axis, and so does not give rise to any smooth solutions of the Einstein–Vlasov system. The information about the linearization also determines the nature of the phase portrait near the singular point, and shows that there are solutions for which \hat{k}_1^1 approaches zero, but turns back before reaching it. A typical feature of Bianchi models is that the matter becomes dynamically negligible near the singularity. No attempt will be made here to make this notion precise, but one aspect of it is that the projection of the solution should tend to a point of the boundary of K as $\tau \rightarrow -\infty$. The solution whose existence has just been shown is an exception to the rule. For anisotropic Bianchi I models with a perfect fluid, no exceptional solutions of this kind exist. However, they do occur for other Bianchi types.¹⁷

The critical points of the system (6.1)–(6.2) in the region where $0 < q$ and $-\frac{1}{3} < k < 1$ are the points of the form $(\frac{1}{3}, q_0)$, where q_0 has the property that $Q(q_0) = \frac{1}{3}$. Differentiating (6.3) and estimating the result in an elementary way leads to the inequality $qQ' \geq Q(1-Q)$. This shows that the function Q is strictly increasing for $q > 0$. Taking account of the limiting values of Q , it follows that there is precisely one value q_0 for which $Q(q_0) = \frac{1}{3}$. Moreover, at this point $qQ' \geq \frac{2}{9}$. The eigenvalues of the linearization at the corresponding critical point are $-\frac{1}{6} \pm \frac{1}{2} \sqrt{\frac{1}{9} - q_0 Q'(q_0)}$. They both have negative real parts and the critical point is a sink. In particular, no solution emerges from this critical point. Thus, it is seen that if the projection of any LRS solution for massless particles approaches the point C as $\tau \rightarrow -\infty$, then the projection must stay at C for all time, i.e., the

solution must have isotropic geometry. This should be compared with the results of Newman¹⁸ on isotropic singularities in solutions of the Einstein equations coupled to a radiation fluid.

APPENDIX: FLUIDS WITH NONLINEAR EQUATION OF STATE

Consider a perfect fluid with equation of state $p=f(\rho)$ that satisfies the following general assumptions: (i) f is a continuous function from $[0,\infty)$ to itself with $f(0)=0$, which is C^1 for $\rho>0$; (ii) $0\leq f'(\rho)\leq 1$ for all $\rho>0$; and (iii) there exists a constant $C<1$ such that $p\leq C\rho$ for $\rho<1$.

Assumptions (i) and (ii) are standard. The third assumption is, when (i) and (ii) are satisfied, equivalent to the assumption made in Ref. 7 that the equation of state is not asymptotically stiff at low densities. In the case of a linear equation of state $f(\rho)=k\rho$, the assumptions (i)–(iii) are satisfied if and only if $0\leq k<1$. In a Bianchi I space–time, it follows from the momentum constraint (2.2) that the four-velocity of the fluid is orthogonal to the hypersurfaces of homogeneity. Hence, the energy density ρ measured by an observer whose world line is orthogonal to the hypersurfaces of homogeneity, is the same as that measured by a comoving observer. Equations (2.19) and (2.20) are valid, as in the case of the Einstein–Vlasov system. For a fluid, it can be assumed without loss of generality that the solution is reflection symmetric, because given any initial data, it suffices to do a linear transformation of the coordinates that simultaneously diagonalizes the metric and second fundamental form in order to transform the given data to data for a reflection-symmetric space–time.

In the case of a fluid $T_j^i=p\delta_j^i$ and hence $\hat{T}_j^i=\hat{p}\delta_j^i$, where $\hat{p}=p/(\text{tr } k)^2$. If the equation of state is linear, then \hat{p} can be expressed as a function of $\hat{\rho}$ alone, and (2.19) and (2.20) then reduce to a system of ordinary differential equations that suffice to determine \hat{k}_j^i from initial data. For a nonlinear equation of state, this is no longer the case. The equations (2.19) and (2.20) no longer form a closed system and must instead be considered as the projection of a bigger system, as in the case of the Vlasov equation. This is one reason why the linear case has been studied preferentially in the literature. Nevertheless, it turns out that the projection can be analyzed very effectively in the general case.

The first question that needs to be addressed is that of global existence in τ , i.e., the equivalent of Lemma 2.1 for a fluid. This follows from the results of Ref. 7. The assumption (iii) has been used at this stage. A direct calculation shows that the quantity $(\hat{k}_1^1-\hat{k}_2^2)/(\hat{k}_1^1-\hat{k}_3^3)$ is independent of τ whenever $\hat{k}_1^1-\hat{k}_3^3$ is nonzero. Moreover, if $\hat{k}_1^1-\hat{k}_3^3$ is zero at some time, it remains zero. Hence, the projection of each solution is constrained to move on a straight line in K passing through the center C . This is already a much stronger statement than could be proved in the case of the Vlasov equation. To find out how the projection moves on this straight line, the time derivative of the dimensionless version of the density will be calculated. For a fluid it is given by

$$\partial_\tau\hat{\rho}=(\hat{\rho}-\hat{p})(1-24\pi\hat{\rho}). \quad (\text{A1})$$

Noting that the Hamiltonian constraint implies that $24\pi\hat{\rho}\leq 1$ with equality only at the point C , it can now be seen that the projection moves monotonically from the boundary of K at $\tau=-\infty$ to the center C of K at $\tau=\infty$. This qualitative behavior is independent of the equation of state, satisfying (i)–(iii). The only difference is in the speed with which the projection moves along the radial line at different times. Equation (A1) also makes clear that this picture changes completely if the equations of state considered here are replaced by the limiting case of a stiff fluid, $p=\rho$.

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