COMMENT

Comment on the 'chaotic' singularity in some magnetic Bianchi VI₀ cosmologies

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Abstract. A description of the magnetic Bianchi VI₀ cosmologies of LeBlanc, Kerr and Wainwright in the formalisms both of Belinskii, Khalatnikov and Lifshitz, and of Misner is given. This allows a qualitative understanding of the Mixmaster-like singularity in those models.

LeBlanc, Kerr and Wainwright (LKW) [1] have recently used the dynamical systems approach popularized by Wainwright and Hsu [2] to study the class of Bianchi VI₀ cosmologies containing a magnetic field and a perfect fluid. As part of their complete analysis of this system, they reported that the generic singularity (for most fluid equations of state) was of the Mixmaster type [3, 4]. The purpose of this comment is to elucidate the nature of this feature of the models by re-expressing the results in the Belinskii, Khalatnikov and Lifshitz (BKL) and Misner's minisuperspace (MSS) terminologies. While nothing new is thereby revealed, those more conversant with these formalisms will find LKW's somewhat surprising result less mysterious. In most of the following, we shall restrict attention to the electrovac case, only remarking at the end on the effect of inclusion of a perfect fluid.

First consider the 'standard Mixmaster model' (SMM) of (diagonal) vacuum, Bianchi IX. The (BKL) metric components a^2, b^2, and c^2 are related to the MSS variables Ω and $\beta_\pm$ describing volume and anisotropy by

$$\alpha = \ln a = \Omega - 2\beta_+,$$
$$\zeta = \ln b = \Omega + \beta_+ + \sqrt{3}\beta_-, \tag{1}$$
$$\gamma = \ln c = \Omega + \beta_+ - \sqrt{3}\beta_-.$$

Einstein's equations may be obtained by variation of the super-Hamiltonian $\tilde{\mathcal{N}}\mathcal{H}‡$ with

$$2\mathcal{H} = -p_\Omega^2 + p_+^2 + p_-^2 + U(\Omega, \beta_\pm) \tag{2}$$

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‡ The rescaled lapse is $\tilde{\mathcal{N}} = \mathcal{N}e^{-3\Omega}$ where $dt = \mathcal{N}\,d\tau$ with τ our choice of time coordinate and t comoving proper time. In LKW, $\cdot \equiv d/dt$ while $\tilde{\mathcal{N}} = 1$ yields the BKL time coordinate.

where p_Ω and $p_\pm$ are, respectively, canonically conjugate to Ω and $\beta_\pm$ and

$$U = e^{4\Omega} V(\beta_+, \beta_-)$$

$$= e^{4\alpha} + e^{4\zeta} + e^{4\gamma} - 2e^{2(\zeta+\gamma)} - 2e^{2(\alpha+\gamma)} - 2e^{2(\alpha+\zeta)}. \tag{3}$$

Arbitrary constants in front of U may be absorbed by a suitable choice of spatial coordinates. In (2), the MSS potential, U, is related to the spatial scalar curvature through $U = e^{6\Omega} \, {}^3R$ where $e^{6\Omega}$ is the determinant of the spatial metric with scalar curvature ${}^3R$. Direct comparison of (2) and (3) with the LKW form of the Hamiltonian constraint suggests the identification

$$\Sigma_\pm = \frac{p_\pm}{-p_\Omega}; \qquad N_1 = c_1 \frac{e^{2\alpha}}{-p_\Omega}, \tag{4}$$

etc. LKW's equations of motion are then obtained with $\tilde{N} = -p_\Omega^{-1}$, so that the time coordinate is Ω. The c_i contain factors from the structure constants. Thus change of Bianchi type can lead to vanishing of some of the N_i and sign changes of others [2].

The approach to the singularity ($\Omega \to -\infty$) in the SMM has long been known to be characterized by an infinite sequence of Kasner models (vacuum Bianchi I) described by the single BKL parameter u [3, 4]. The Kasner solution is just that obtained for $U = 0$ in (2). The super-Hamiltonian becomes that for a free particle so that the momenta p_Ω and $p_\pm$ are constant. The Kasner epoch changes when a scattering (bounce) off the MSS potential U occurs†. The values of u in successive Kasner eras are related by the BKL map [3], equivalent to the Kasner map described by LKW [1]. All discussions of chaos in the SMM derive from the well known sensitivity to initial conditions of this map (see, for example, [5]). Here we need consider only the features of the SMM which are required to derive the BKL (or Kasner) map since the map will result in any model in which these same properties are present. In fact, only the first three terms in U are needed‡. Since the Kasner solution is characterized (in the approach toward the singularity) by two contracting metric components and one expanding metric component, one of α, ζ, or γ will dominate. For convenience, choose the dominant one to be α so that $U \approx e^{4\Omega - 8\beta_+}$. The canonical transformation generated by $x = \Omega - 2\beta_+$ and $y = -2\Omega + \beta_+$ puts the Hamiltonian constraint (2) into the form

$$3p_x^2 - 3p_y^2 + p_-^2 + e^{4x} = 0. \tag{5}$$

Equation (5) describes scattering off a potential with relationships among the momenta before and after the scattering of constant p_y and p_- with $p_x \to -p_x$. Since the BKL parameter u may be expressed in terms of these momenta [3], the scattering rules lead immediately to the BKL map. It is important to note that, even though the exact solution corresponding to (5) is known, only the momentum rules are used.

With the identifications (4), the Hamiltonian constraint yields the identity

$$p_\Omega^{-2} U = 1 - \Sigma \tag{6}$$

(where $\Sigma = \Sigma_+^2 + \Sigma_-^2$ [1]) so that

$$\frac{\mathrm{d}}{\mathrm{d}\Omega} \ln p_\Omega = 2 - q \tag{7}$$

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† The overall $e^{4\Omega}$ dependence of U means that, as $\Omega \to -\infty$, $U \to 0$. However, the Kasner solution (expressible as $\beta_\pm = \beta_\pm^0 + \Sigma_\pm(\Omega - \Omega_0)$) allows terms in U to be of order unity (e.g. if $\Omega = 2\beta_+$).

‡ One can argue that the remaining terms are always dominated by one of the first three except near the $120°$ corners of the MSS potential [4]. In this discussion of generic Mixmaster behaviour, the 'anomalous' corner behaviour will be ignored. See, however, [3, 6].

where $q = 2\Sigma$. The LKW equations for the N_i then yield the relationship of these variables to the MSS or BKL ones. For example, $N_1 = n_1/p_\Omega$ in [1]

$$\frac{dN_1}{d\Omega} = (q - 4\Sigma_+)N_1 \tag{8}$$

with (7) has the solution (using $d\beta_\pm/d\Omega = \Sigma_\pm$)

$$n_1 = n_1^0 e^{2\Omega - 4\beta_+}. \tag{9}$$

Similar constructions can be performed for N_2 and N_3.

The most attractive feature of the LKW formalism is its ability to handle all the Bianchi types with the same equations. Thus extension to the magnetic (but otherwise empty) Bianchi VI_0 equations is straightforward. To make the connection to the BKL and MSS formulations, however, we must include some properties of the structure constants (i.e. in the c_i of (4)). Thus, for Bianchi VI_0, $c_1 = 0$ and $c_2 c_3 < 0$ while $H_1 \equiv h_1/p_\Omega$ is the only magnetic field component consistent with this model [1]. The magnetic field contribution to the Hamiltonian constraint leads to a modification of (6) such that $p_\Omega^{-2} U = 1 - \Sigma - \frac{3}{2} H_1^2$ and $q = 2\Sigma + \frac{3}{2} H_1^2$. These identifications allow the LKW equation for H_1 [1]

$$\frac{dH_1}{d\Omega} = (q - 1 - 2\Sigma_+)H_1 \tag{10}$$

to yield the solution

$$h_1 = h_1^0 e^{\Omega - 2\beta_+}. \tag{11}$$

It is now possible to transcribe the LKW constraint for this model into the MSS language as

$$2\mathcal{H} = -p_\Omega^2 + p_+^2 + p_-^2 + \left[e^{4\zeta} + e^{4\gamma} + 2e^{2(\zeta+\gamma)}\right] + \xi e^{2\alpha}. \tag{12}$$

We note that, compared to the SMM constraint (2), the Kasner (momentum) part is unchanged. The first two terms in the brackets, identical to those in SMM, are two of those required for the BKL map derivation. The third term is the irrelevant cross term (with irrelevant changed sign). The final term in (12), from the magnetic field, is essentially the square root of the corresponding term in the SMM potential. However, the arbitrary constants, the absent cross terms and the difference in the power of e^α have no effect on the BKL map derivation. The same canonical transformation to x and y can be made, yielding a model for the scattering from one Kasner epoch to another of

$$3p_x^2 - 3p_y^2 + p_-^2 + \xi e^{2x} = 0 \tag{13}$$

rather than (5). Even though (as pointed out by LKW) the exact solution has changed, the rules describing the change in momenta at the bounce have not†. Thus the BKL map may be derived as a property of the magnetic Bianchi VI_0 vacuum model.

Finally, we note that a perfect fluid will enter the Hamiltonian constraint (2) as a term $U \to U + \rho_0 e^{3(2-\Gamma)\Omega}$ for equation of state $p = (\Gamma - 1)\rho$. As $\Omega \to -\infty$, this term goes to zero (if $\Gamma \neq 2$ in the range discussed by LKW) and can never revive due to cancellations such as those in other terms of U that produce Mixmaster bounces. Only if $\Gamma = 2$ will the fluid play a role in the singularity approach (as has been found by LKW).

Thus, the connection between the LKW formalism and the BKL and MSS descriptions of Bianchi cosmologies clarifies, for those familiar with the latter, the similarity between the singularity dynamics of the SMM and magnetic Bianchi VI_0 models by showing that the differences between the two fail to invalidate the derivation of the BKL map. Of course, the

† Of course, (5) is exactly reproduced to describe bounces for either ζ or γ approximately zero.

BKL approximate description of the SMM contains more than just the parameter u [3, 7]. While the exact solution to the bounce model equation is still not needed for this expanded description, differences between the two models might prove interesting.

Acknowledgments

References

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