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Asymptotically hyperbolic non-constant mean curvature solutions of the Einstein constraint equations

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Abstract. We describe how the iterative technique used by Isenberg and Moncrief to verify the existence of large sets of non-constant mean curvature solutions of the Einstein constraints on closed manifolds can be adapted to verify the existence of large sets of asymptotically hyperbolic non-constant mean curvature solutions of the Einstein constraints.

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Professor Andrzej Trautman was one of the first to think carefully about the asymptotic behaviour of solutions of Einstein's equations. One of us (JI) very fondly remembers learning about spinors and their role in general relativity from Professor Trautman during the mid 1970s at the University of Maryland. We are pleased to honour Professor Trautman on the occasion of his 64th birthday.

1. Introduction

For many years, the Einstein constraint equations have been studied primarily on either closed manifolds or on open manifolds with asymptotically Euclidean boundary conditions. Such a concentration makes sense if one focuses on the Cauchy problem for cosmological spacetimes or on the Cauchy problem for asymptotically flat spacetimes in a neighbourhood of a Cauchy surface which goes to spacelike infinity.

Recent work by Friedrich [11] has shown that one can very usefully study asymptotically flat spacetimes using a Cauchy problem based on spacelike hypersurfaces which approach null infinity rather than spacelike infinity. The prototype for such hypersurfaces is the 'constant mass' hyperboloid hypersurface in Minkowski spacetime which has constant negative intrinsic curvature. More generally, these spacelike hypersurfaces do not have constant negative curvature, but they necessarily approach (at least locally) a constant negative curvature hypersurface asymptotically. Hence, in studying the constraint equations on hypersurfaces of this sort, one imposes asymptotically hyperbolic boundary conditions (rather than the more familiar asymptotically Euclidean boundary conditions). We shall discuss these in detail below.

Regardless of the topology or the boundary conditions on the hypersurface, the constraint equations are much simpler to study for initial data with constant mean curvature (CMC) than for non-CMC initial data. This is because, in the CMC case, three out of the four constraint equations are essentially trivial, and so one need only work with one nonlinear partial differential equation; while in the non-CMC case, there are four coupled PDEs which

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must be handled. As a consequence of this difference, while CMC solutions of the constraint equations are essentially fully understood (on closed manifolds [9, 12], for asymptotically Euclidean data [6, 7], and for asymptotically hyperbolic data [2, 3]), it is only during the past few years that much has been learned about non-constant mean curvature solutions [8, 14].

In this paper, we discuss results which show that iterative techniques developed by Isenberg and Moncrief to study and produce non-CMC solutions of the constraint equations on closed manifolds [14] can be adapted to do the same for non-CMC solutions which are asymptotically hyperbolic. A careful proof of these new asymptotically hyperbolic results is presented elsewhere [18]. Here, we discuss a bit more informally how the adaptation of our techniques from the closed manifold case to the asymptotically hyperbolic case has been carried out. We start in section 2 with a very brief review of the LCBY conformal formulation of the Einstein constraint equations, which is the starting point for all of our analysis. In section 3, we review what we know so far about solutions of the constraints on closed manifolds, emphasizing how our iterative technique works to verify the existence of classes of non-CMC solutions on such manifolds. In section 4, we specify what asymptotically hyperbolic manifolds and geometries are, and we discuss the weighted Sobolev and weighted Hölder spaces which we use for our studies of the constraints on these manifolds. We also discuss in section 4 some key PDE analytical results concerning certain elliptic operators on these weighted function spaces. Then in section 5, we state our results concerning asymptotically hyperbolic non-CMC solutions of the constraints and sketch how these results are proven using the iterative techniques once they are adapted to asymptotically hyperbolic geometries. We conclude in section 6 with remarks on future directions for research.

2. A brief review of the conformal formulation of the Einstein constraint equations

The vacuum Einstein constraint equations for initial data (γ, K) consisting of a Riemannian metric γ and a symmetric $\binom{0}{2}$-tensor K take the form

$$\nabla_a K^a{}_b - \nabla_b (K^c{}_c) = 0 \tag{1a}$$

$$R - K^{ab} K_{ab} + (K^c{}_c)^2 = 0 \tag{1b}$$

where R is the scalar curvature for γ. To produce solutions of these equations on a given three-dimensional manifold Σ^3, as well as to study and parametrize these solutions, it is very useful to reformulate equations (1) using the LCBY conformal method, developed by Lichnerowicz, Choquet-Bruhat and York [9]. The idea is to split the data (γ, K) into two parts: the first part—the conformal data (λ, σ, τ)—consists of a Riemannian metric λ, a symmetric tensor σ which is trace-free ($\lambda_{ab}\sigma^{ab} = 0$) and divergence-free ($\nabla_a \sigma^{ab} = 0$) with respect to λ, and a scalar function τ on Σ^3. The second part—the determined data (W, ϕ)—consists of a vector field W and a positive-definite scalar function ϕ on Σ^3. Then to obtain a solution of the constraint equations (1), one first chooses (λ, σ, τ) and one then attempts to solve the equations

$$\nabla_a (LW)^a{}_b = \tfrac{2}{3}\phi^6 \nabla_b \tau \tag{2a}$$

$$\nabla^2 \phi = \tfrac{1}{8} R\phi - \tfrac{1}{8}(\sigma^{ab} + LW^{ab})(\sigma_{ab} + LW_{ab})\phi^{-7} + \tfrac{1}{12}\tau^2 \phi^5 \tag{2b}$$

for W and ϕ. Here ∇ is the covariant derivative compatible with λ, R is its scalar curvature, and L is the conformal Killing operator

$$LW_{ab} := \nabla_a W_b + \nabla_b W_a - \tfrac{2}{3}\lambda_{ab} \nabla_c W^c. \tag{3}$$

If, for some chosen set of the conformal data (λ, σ, τ), one can find a solution (W, ϕ) for equations (2), then the reconstituted initial data

$$\gamma_{ab} = \phi^4 \lambda_{ab} \tag{4a}$$

$$K^{cd} = \phi^{-10}(\sigma^{cd} + LW^{cd}) + \tfrac{1}{3}\phi^{-4}\lambda^{cd}\tau \tag{4b}$$

are a solution of the vacuum constraint equations (1).

One readily verifies that the second-order operator $\nabla \cdot L : W \mapsto \nabla_a (LW)^a{}_b$ is elliptic. Hence, in applying the conformal reformulation to the constraint equations (1), one transforms them into a manifestly (nonlinear) elliptic system of four PDEs for four unknown functions.

It is *not* true that for every choice of the conformal data (λ, σ, τ), there is a solution (W, ϕ) for equations (2). For example, if one works on the closed manifold $\Sigma^3 = \mathbf{S}^3$, and if one chooses λ to be a round sphere metric with $R = 8$, one chooses σ identically zero, and one chooses $\tau = \sqrt{8}$, then the system (2) reduces to

$$\nabla_a (LW)^a{}_b = 0 \tag{5a}$$

$$\nabla^2 \phi = \phi - \tfrac{1}{8}(LW)^2 \phi^{-7} + \phi^5. \tag{5b}$$

The former equation (5*a*) implies that $LW = 0$, from which it follows that (5*b*) takes the form

$$\nabla^2 \phi = \phi + \phi^5. \tag{6}$$

This equation admits no positive-definite solution on $\mathbf{S}^3$. So there is no solution to (2) for this choice of conformal data.

For which choices of (λ, σ, τ) does a solution to (2) exist? We review some of what is known in the next section.

3. A brief review of existence results for solutions of the Einstein constraint equations

As noted in the introduction, studies of the constraint equations have traditionally focused on two cases: solutions on closed manifolds, and asymptotically Euclidean solutions. In each of these cases, the existence and the parametrization of solutions is well understood if the mean curvature $K^c{}_c$ is taken to be constant; for non-constant mean curvature, there is less—but growing—understanding.

With the constant mean curvature condition imposed, the task of determining which conformal data (λ, σ, τ) permit equations (2) to be solved, and therefore which conformal data maps to a solution of the constraint equations (1), is simplified considerably. This is because if one chooses τ to be a constant—a necessary and sufficient condition for the mean curvature $K^c{}_c$ to be constant—then equation (2*a*) becomes $\nabla_a (LW)^a{}_b = 0$, which admits $(LW)_{ab} = 0$ as a solution. Consequently, the constraint equations reduce to one (semi-linear, elliptic) PDE (the 'Lichnerowicz equation')

$$\nabla^2 \phi = \tfrac{1}{8} R\phi - \tfrac{1}{8}\sigma^{ab}\sigma_{ab}\, \phi^{-7} + \tfrac{1}{12}\tau^2 \phi^5 \tag{7}$$

to be solved for the positive-definite function ϕ.

In both the closed manifold and the asymptotically Euclidean cases, necessary and sufficient conditions on the conformal data (λ, σ, τ) are known for the Lichnerowicz equation to admit solutions. In both cases, the behaviour of the scalar curvature R under conformal transformations of λ is crucial. For asymptotically Euclidean data (defined via weighted Sobolev spaces), Cantor [7] has shown that the Lichnerowicz equation (with $\tau = 0$, which must hold for asymptotically Euclidean data if one assumes constant mean

curvature) admits a solution if and only if there exists a conformal transformation of λ which produces $R = 0$; further, Brill and Cantor [6] give an integral condition for such a conformal transformation to exist. For closed manifolds, the combined work of Choquet-Bruhat, York and O'Murchadha [9] and Isenberg [12] shows that the criteria for existence depends upon three things: (i) the Yamabe class†—$\mathcal{Y}^+$, $\mathcal{Y}^0$ or $\mathcal{Y}^-$—of λ; (ii) whether τ is zero or not and (iii) whether σ^2 is identically zero or not. For example, if $\lambda \in \mathcal{Y}^+$, $\tau \neq 0$ and $\sigma^2 \equiv 0$, there is no solution; while if $\lambda \in \mathcal{Y}^-$, $\tau \neq 0$ and $\sigma^2 \not\equiv 0$, then there is a solution. In total, there are 12 possibilities. One finds that for those possibilities which imply (with $\phi > 0$, and with λ conformally transformed so that R is constant) a definite sign for the right-hand side of the Lichnerowicz equation, the equation admits no solution; otherwise a solution does exist. A careful statement and proof of these results for constant mean curvature data on a closed manifold is given in [13].

While the case for non-constant mean curvature is much less complete, considerable progress has been made in recent years, especially for data on closed manifolds. The analysis is more difficult, because in the non-CMC case one must work with the full, coupled system (2) rather than just the Lichnerowicz equation (7). However, using a sequence scheme which we will describe below, Moncrief and Isenberg have been able to show that equation (2) admits solutions (on a closed manifold) for each of the following classes of conformal data: (I) $\lambda \in \mathcal{Y}^-$, $\tau^2 > 0$ and $|\nabla\tau| < C(\lambda, \sigma)$, where $C(\lambda, \sigma)$ is a constant depending on λ and σ [14]; (II) $\lambda \in \mathcal{Y}^+$ and $|\nabla\tau| < C(\lambda, \sigma)$ [15] and (III) $\lambda \in \mathcal{Y}^0$, $\tau^2 > 0$ and $|\nabla\tau| < C(\lambda, \sigma)$ [15].

The sequence method is based on the semi-decoupled sequence of PDEs

$$\nabla_a (LW_n)^a{}_b = \tfrac{2}{3}(\phi_{n-1})^6 \, \nabla_b \tau \tag{8a}$$

and

$$\nabla^2 \phi_n = \tfrac{1}{8} R\phi_n - \tfrac{1}{8}\big(\sigma^{ab} + (LW_n)^{ab}\big)\big(\sigma_{ab} + (LW_n)_{ab}\big)(\phi_n)^{-7} + \tfrac{1}{12}\tau^2(\phi_n)^5 \,. \tag{8b}$$

These equations are semi-decoupled in the sense that if one knows ϕ_{n-1}, then equation (8a) is a (linear, elliptic) PDE for W_n alone and then once one knows W_n, equation (8b) is a (quasilinear, Lichnerowicz-type) PDE for ϕ_n alone. The idea is to (i) show that there is a sequence of solutions $\{(\phi_n, W_n)\}$ to the sequence of equations (8); (ii) show that the sequence $\{(\phi_n, W_n)\}$ converges to some $\{(\phi_\infty, W_\infty)\}$ and, finally, (iii) show that $\{(\phi_\infty, W_\infty)\}$ is a solution to equations (2) for the chosen set of conformal data.

To set the stage for discussing how the sequence method has been adapted to the study of equations (2) with asymptotically hyperbolic data, we will describe in a bit more detail how these three steps are carried out for conformal data on a closed manifold. First we consider how one shows that the sequence $\{(\phi_n, W_n)\}$ exists. The choice of ϕ_0 is free (within certain bounds; see [14]). Once ϕ_0 is chosen, the vector field W_1 is to be obtained from equation (8a) with $n = 1$. To show that indeed the linear elliptic equation (8a) does determine W_1 for an arbitrary (sufficiently smooth) choice of ϕ_0 and τ, one needs to verify that the operator $\nabla \cdot L$ is invertible on the space of vector fields being considered. Standard elliptic theory (see, e.g., Besse [5]) shows that if one works with either the standard Sobolev spaces $H_k^p(\Sigma^3)$ or the standard Hölder spaces $C^{k,\alpha}(\Sigma^3)$ of vector fields on a closed manifold, then $\nabla \cdot L$ is invertible; so for ϕ_0 and τ in appropriate Sobolev or Hölder spaces of functions on Σ^3, W_1 exists. Elliptic theory on closed Σ^3 also shows that there exist

† The Yamabe theorem [20] shows that every metric may be conformally transformed so that the corresponding scalar curvature is constant. For a given metric λ, one can transform to any constant, with a fixed sign charateristic of that metric. Hence λ is contained in a unique Yamabe class—$\mathcal{Y}^+$, $\mathcal{Y}^0$ or $\mathcal{Y}^-$—depending upon this sign.

constants C_1, C_2, C_3, and C_4 such that

$$\|W_1\|_{H^p_{k+2}} \leqslant C_1\|\nabla \cdot LW_1\|_{H^p_k} + C_2\|W_1\|_{L^1} \tag{9a}$$

and

$$\|W_1\|_{C^{k+2,\alpha}} \leqslant C_3\|\nabla \cdot LW_1\|_{C^{k,\alpha}} + C_4\|W_1\|_{C^0} \tag{9b}$$

hold; moreover, if the metric λ has no conformal Killing vector fields, one can replace inequalities (9) by

$$\|W_1\|_{H^p_{k+2}} \leqslant C_5\|\nabla \cdot LW_1\|_{H^p_k} \tag{10a}$$

and

$$\|W_1\|_{C^{k+2,\alpha}} \leqslant C_6\|\nabla \cdot LW_1\|_{C^{k,\alpha}} \tag{10b}$$

for some constants C_5 and C_6. Combining these inequalities (10) with appropriate embedding inequalities (see, e.g., [5]), together with equation (8*a*), we obtain the pointwise inequality

$$|LW_1| \leqslant C\Big(\max_{\Sigma^3}\phi_0\Big)^6\ \Big(\max_{\Sigma^3}|\nabla\tau|\Big), \tag{11}$$

which plays an important role in step (ii) of the sequence method proof.

It should be clear that the same existence and regularity results hold for values of n other than $n = 1$. Thus, given a sufficiently nice ϕ_{n-1}, one obtains W_n from equation (8*a*), and W_n satisfies (pointwise)

$$|LW_n| \leqslant C\Big(\max_{\Sigma^3}\phi_{n-1}\Big)^6\ \Big(\max_{\Sigma^3}|\nabla\tau|\Big), \tag{12}$$

for some constant C independent of n.

Now for a given vector field W_n, equation (8*b*) is the Lichnerowicz equation, to be solved for ϕ_n. Hence, keeping in mind that τ is no longer constant, one may attempt to use the techniques which work to prove existence for constant mean curvature conformal data [13]. For conformal data satisfying the three conditions noted earlier in this section, the sub- and super-solution technique readily applies, so long as the data are contained in appropriate Sobolev and Hölder spaces [14, 15]. Interestingly, we have recently been able to show that, in fact, for *any* non-CMC conformal data in appropriate function spaces, a solution ϕ_n for equation (8*b*) exists [15]. These results rely upon the sub- and super-solution theorem for equations of the form $\nabla^2\phi = f(\phi, x)$ on closed manifolds [14].

Once it is established that the sequence $\{(\phi_n, W_n)\}$ exists, one needs to show that the sequence converges. The way that this is done, to prove the theorems on closed manifolds, is via a contraction mapping argument, which proceeds as follows: first, using equation (8*b*) for consecutive values of n, we obtain equations of the form

$$\nabla^2(\phi_{n+1} - \phi_n) = \mathcal{F}[\phi_{n-1}, \phi_n, \phi_{n+1}, x] \tag{13}$$

where $\mathcal{F}$ is a non-local functional of ϕ_{n-1}, ϕ_n and ϕ_{n+1} (see equations (45), (46) in [14]). Next, one establishes n-independent upper and lower bounds on all ϕ_n (in practice, this is done by finding n-independent upper and lower bounds on the sequence of sub- and super-solutions $\{((\phi_-)_n, (\phi_+)_n)\}$ which one uses to prove the existence of the sequence ϕ_n of solutions to equation (8*b*)). Then, using these upper and lower bounds together with the pointwise inequalities (12), one shows that (13) can be written as

$$\nabla^2(\phi_{n+1} - \phi_n) - \mathcal{G}[\phi_{n+1} - \phi_n] = \mathcal{H}[\phi_n - \phi_{n-1}] \tag{14}$$

where

$$\mathcal{G}[\phi_{n+1} - \phi_n] \geqslant \Lambda(\phi_{n+1} - \phi_n) \tag{15a}$$

for a constant Λ which depends upon τ alone, and

$$\mathcal{H}[\phi_n - \phi_{n-1}] \leqslant \Theta \max_{\Sigma^3}(\phi_n - \phi_{n-1}) \tag{15b}$$

where Θ is a constant depending on the conformal data (λ, σ, τ). Both Λ and Θ are independent of n. It then follows from the maximum principle applied to (14) that

$$|\phi_{n+1} - \phi_n| \leqslant \frac{\Theta}{\Lambda} \max_{\Sigma^3} |\phi_n - \phi_{n-1}|. \tag{16}$$

Hence, if the conformal data are such that $\Theta/\Lambda < 1$, then (16) defines a contraction mapping, which implies convergence of the sequence $\{\phi_n\}$ to some positive function ϕ_∞. Since $\{\phi_n\}$ converges, it follows immediately from the linear equation (8*a*) that the sequence $\{W_n\}$ converges to some vector field $\{W_\infty\}$ as well.

So long as the conformal data is chosen with a sufficiently high degree of differentiability, e.g. $\lambda \in C^3(\Sigma^3)$, $\sigma \in H_2^p$ and $\tau \in H_2^p$ with $p > 3$—it is fairly straightforward to show that the limits (ϕ_∞, W_∞) of the converging sequence $\{(\phi_n, W_n)\}$ satisfy the constraint equations (2). One first shows that (ϕ_∞, W_∞) constitutes a weak solution; then one uses standard bootstrap arguments to argue that (ϕ_∞, W_∞) is sufficiently differentiable, i.e. C^2—that it constitutes a strong solution of (2). This completes the proof of the existence of solutions corresponding to certain families of conformal data on closed manifolds.

We would like to show that the sequence method just sketched can be adapted for use in producing and studying solutions of the constraints which are asymptotically hyperbolic. Before doing this, we need to carefully define asymptotically hyperbolic geometries and discuss the relevant function spaces and differential operators on these geometries.

4. Analysis on asymptotically hyperbolic geometries

While the intuitive idea of an asymptotically hyperbolic geometry is that of a Riemannian metric γ on a non-compact manifold Σ^3 with γ asymptotically approaching a constant negative curvature metric h as one approaches 'infinity' on Σ^3, it is more useful to work with a definition based on conformal compactification:

Definition 1. A Riemannian geometry (Σ^3, γ) is *asymptotically hyperbolic* if and only if there exists a triple (Λ^3, ρ, ψ) where

(i) Λ^3 is a smooth manifold with boundary.
(ii) $\rho : \Lambda^3 \to \mathbf{R}$ is a smooth non-negative function, with $\rho(x) = 0$ if and only if $x \in \partial\Lambda^3$ and with $\mathrm{d}\rho(x) \neq 0$ for $x \in \partial\Lambda^3$.
(iii) $\psi : \mathrm{int}(\Lambda^3) \to \Sigma^3$ is a smooth diffeomorphism, with $\rho^2\psi^*(\gamma)$ a smooth Riemannian metric on $\mathrm{int}(\Lambda^3)$ which extends smoothly to Λ^3.

One readily verifies that if (Σ^3, γ) is asymptotically hyperbolic in the sense of this definition, then indeed the intuitive sense of asymptotically hyperbolic is realized. Note that the function $\rho \circ \psi^{-1} : \Sigma^3 \to \mathbf{R}$ can be used effectively as an 'inverse radial coordinate' which approaches zero as one moves toward the asymptotic region on Σ^3. It is sometimes called the 'defining function' for the asymptotic region.

To study differential operators like the Laplacian and $\nabla \cdot L$ on an asymptotically hyperbolic geometry (Σ^3, γ), one needs to effectively specify boundary conditions on

the tensor fields upon which the operators act. There are a number of ways in which this can be done; the most useful way for our work here is through the use of weighted Hölder and weighted Sobolev spaces. These spaces are defined in the usual way, with the norms containing an indexed weight factor $\rho^{-\delta}$, where ρ is the defining function discussed above. That is, if we use u to denote a covariant tensor field of fixed rank, D to denote the γ-compatible covariant differential, and D^j to denote the jth iteration of D, then the weighted Hölder spaces† C_k^δ (for any non-negative integer k and any real δ) are defined via the weighted norms

$$\|u\|_{C_k^\delta} := \sum_{j=0}^{k} \sup_{\Sigma^3} |\rho^{-\delta} D^j u|_\gamma \,. \tag{17}$$

Similarly the weighted Sobolev spaces $H_k^{p,\delta}$ are defined via the weighted Sobolev norms

$$\|u\|_{H_k^{p,\delta}} := \sum_{j=0}^{k} \|\rho^{-\delta} D^j u\|_{L^p} \tag{18}$$

where $\|\cdot\|_{L^p}$ indicates the L^p norm on (Σ^3, γ).

The index 'δ' in both C_k^δ and $H_k^{p,\delta}$ indicates the required asymptotic fall-off rate for $D^j u$. Specifically, since ρ goes to zero asymptotically, the function $\rho^{-\delta}$ blows up asymptotically for positive δ; hence $\sup_{\Sigma^3} |\rho^{-\delta} D^j u|_\gamma$ is finite for $\delta > 0$ only if $|D^j u|_\gamma$ goes to zero quickly enough. Negative δ allows $|D^j u|_\gamma$ to go to zero more slowly, if at all. In general, larger δ means that a faster fall-off rate is required.

For carrying out the iteration method proof (especially the bootstrap steps at the end) it is important to know the embedding theorems for these function spaces, which describe how they all relate to each other. In summary, for tensor fields on a three-dimensional manifold, one finds [16]

(i) $H_k^{p,\delta} \subset H_l^{q,\epsilon}$ if $1 \leqslant q \leqslant p \leqslant \infty, \quad l \leqslant k, \quad$ and $\quad \delta - \epsilon > 3\left(\dfrac{1}{q} - \dfrac{1}{p}\right)$ (19*a*)

(ii) $C_k^\delta \subset C_l^\epsilon$ if $l \leqslant k$ and $\epsilon \leqslant \delta$ (19*b*)

and

(iii) $H_{k+s}^{p,\delta} \subset C_k^\delta$ if $sp > 3$. (19*c*)

One also has the very useful multiplication law for tensor fields u and v contained in weighted subspaces:

$$\|uv\|_{H_k^{p,\delta+\epsilon}} \leqslant C\|u\|_{H_k^{p,\delta}} \|v\|_{H_k^{p,\epsilon}} \qquad \text{if} \quad kp > 3 \tag{20}$$

for some constant C, from which it follows that if $u \in H_k^{p,\delta}$ and $v \in H_k^{p,\epsilon}$ and if $kp > 3$, then $uv \in H_k^{p,\delta+\epsilon}$.

There are four PDE analytical results which play an important role in the sequence method proof on closed manifolds, and which we therefore must consider on asymptotically hyperbolic geometries if we were to extend this method to these geometries: the invertibility of the operators $\nabla \cdot L$ and ∇^2, the regularity estimates (10) for $\nabla \cdot L$ (leading to the pointwise estimate (11) for $|LW|$), the maximum principle for the Laplacian, and the sub- and super-solution theorem for PDEs of the form $\nabla^2 \phi = F(\phi, x)$, e.g. the Lichnerowicz equation. We will now consider each of these issues in turn.

† Note that for convenience, here we only define Hölder spaces with the Hölder index α as zero. More general spaces can be defined, but they are not needed here.

The first two—the invertibility and regularity estimates for $\nabla \cdot L$ and ∇^2 on asymptotic geometries—are closely tied because the proof of the first property depends upon the validity of a version of the second and because once one verifies the first property the second one follows. Interestingly, it is only during this past year—through the work of Lee [17] combined with the earlier work of Andersson and Chruściel [2]—that these basic results have been established to the degree of generality which we need.

The explicit statement of the invertibility result is as follows:

Proposition 1. Let $1 < p < \infty$ and let $k \geqslant 0$.
If $|\delta - 1 + 2/p| < \sqrt{3}$, then

$$\nabla \cdot L : H^{p,\delta}_{k+2} \to H^{p,\delta}_{k} \tag{21a}$$

is invertible.

If $0 < \delta/2 + 1/p < 1$, then

$$\nabla^2 : H^{p,\delta}_{k+2} \to H^{p,\delta}_{k} \tag{21b}$$

is invertible.

What Lee shows [17] is that proposition 1 holds for all values of $p \in (1, \infty)$ so long as it holds for $p = 2$. (Note that the conditions which δ is required to satisfy in proposition 1 follow largely from the embedding condition (19*a*).) A key step in establishing the $p = 2$ result is the verification, for the appropriate values of δ in proposition 1, of the 'asymptotic elliptic estimate' [17]

$$(\lambda - \mathrm{o}(\epsilon))\|u\|_{H^{2,\delta}_0(\Sigma_\epsilon)} \leqslant \|\mathcal{D}u\|_{H^{2,\delta}_0(\Sigma_\epsilon)} \tag{22}$$

for $\mathcal{D} = \nabla^2$ and $\mathcal{D} = \nabla \cdot L$; here $\Sigma_\epsilon := \left\{x \in \Sigma^3 | \rho(x) < \epsilon\right\}$, where $\rho(x)$ is the defining function for the asymptotically hyperbolic geometry, and $\mathrm{o}(\epsilon)$ represents any continuous function which vanishes as $\epsilon \to 0$. From equation (22), one obtains [1]

$$\|u\|_{H^{2,\delta}_{k+2}} \leqslant C\left(\|\mathcal{D}u\|_{H^{2,\delta}_k} + \|u\|_{H^{2,\delta}_k(W)}\right) \tag{23}$$

where W is some compact set in Σ^3. Proposition 1 then follows from (23). For more details, see [18] and the references cited therein.

As noted, once invertibility is established for an elliptic operator, the regularity estimate is a consequence. So, as a corollary to proposition 1, we have, for $p > 1$ and $|\delta - 1 + 2/p| < \sqrt{3}$,

$$\|W\|_{H^{p,\delta}_{k+2}} \leqslant C\|\nabla \cdot LW\|_{H^{p,\delta}_k} \tag{24a}$$

and, for $p > 1$ and $|\delta - 1 + 2/p| < 1$,

$$\|\phi\|_{H^{p,\delta}_{k+2}} \leqslant C\|\nabla^2\phi\|_{H^{p,\delta}_k}. \tag{24b}$$

Let us now consider the maximum principle for the Laplacian ∇^2 on an asymptotically hyperbolic geometry. The maximum principle can take a number of different forms [12]. The version we need says the following.

Proposition 2. Let $\xi : \Sigma^3 \to \mathbf{R}$ be a positive-definite continuous function with $\xi(x) \geqslant m > 0$. Let $\lambda : \Sigma^3 \to \mathbf{R}$ be a continuous function with $|\lambda(x)| \leqslant M$. If $\psi : \Sigma^3 \to \mathbf{R}$ is a bounded C^2 function in the interior of Σ^3 and if it satisfies the equation

$$\nabla^2\psi - \xi\psi = \lambda \tag{25a}$$

then we have

$$|\psi| \leqslant \frac{M}{m}. \tag{25b}$$

Note that this result follows fairly directly from a recently proven asymptotic behaviour lemma of Graham and Lee [10], together with the maximum principle on compact manifolds.

The remaining result we need is a sub- and super-solution theorem for the Laplacian on an asymptotically hyperbolic geometry. The result is as follows:

Proposition 3. Let $0 < \epsilon < 2(1 - 1/p)$. Let f be a functional such that for every function $u : \Sigma^3 \to \mathbf{R}$ with $u - 1 \in H_0^{p,\epsilon}(\Sigma^3)$, we have $f(u; \cdot) \in H_0^{p,\epsilon}(\Sigma^3)$. Assume that there exists a pair of functions $\psi_- : \Sigma^3 \to \mathbf{R}^+$ and $\psi_+ : \Sigma^3 \to \mathbf{R}^+$ such that

(i) ψ_- and ψ_+ are both piecewise C^2 (i.e. they are C^2 outside of a union of submanifolds of lower dimension),
(ii) $(\psi_- - 1) \in H_1^{p,\epsilon}$ and $(\psi_+ - 1) \in H_1^{p,\epsilon}$,
(iii) $\psi_-(x) \leqslant \psi_+(x)$ for all $x \in \Sigma^3$ and
(iv) $\nabla^2\psi_- \geqslant f(\psi_-, x)$, $\nabla^2\psi_+ \leqslant f(\psi_+, x)$.

Then, there exists a unique function $\psi : \Sigma^3 \to \mathbf{R}^+$ such that

(i) $\psi - 1 \in H_3^{p,\epsilon}$,
(ii) $\psi_-(x) \leqslant \psi(x) \leqslant \psi_+(x)$ for all $x \in \Sigma^3$ and
(iii) $\nabla^2\psi = f(\psi, x)$.

The argument used to provide proposition 3 is much like that used in the proof of proposition 4.1 in [18]. There is one key extra step one needs for the result here: at a certain point in the argument—where one wants to show that ψ_1, the first element of the sequence which will converge to the solution ψ, satisfies $\psi_1 \leqslant \psi_+$—one invokes the maximum principle. In a sense, one seems to need a version of the maximum principle which would hold for weak solutions of (25). However, one may instead apply the C^2 maximum principle (proposition 2) in those regions where ψ_+ is C^2, and then use continuity to show that $\psi_1 \leqslant \psi_+$ everywhere on Σ^3.

5. Main result

Our main result prescribes conditions on a set of conformal data (λ, σ, τ) which are sufficient to guarantee that equations (2) can be solved for ϕ and W, and also guarantee that the fields (γ, K), which one obtains by combining (λ, σ, τ) and (ϕ, W) as per equations (4), constitute (constraint-satisfying) asymptotically hyperbolic initial data for a solution of Einstein's equations. While we have defined above (definition 1) what an asymptotically hyperbolic geometry (Σ^3, γ) is, we have not yet defined what asymptotically hyperbolic initial data (Σ^3, γ, K) is. The idea is that such initial data should correspond to the intrinsic and extrinsic geometry of a spacelike hypersurface which asymptotically goes to null infinity in an asymptotically flat spacetime. One finds [2] that the following definition is consistent with this idea:

Definition 2. A set of initial data (Σ^3, γ, K) is *asymptotically hyperbolic* if

(i) (Σ^3, γ) is an asymptotically hyperbolic geometry (in the sense of definition 1).
(ii) $\mathrm{tr}_\gamma K$ is bounded away from zero asymptotically (i.e. outside some γ-ball, $\mathrm{tr}_\gamma K$ is non-zero).
(iii) The trace-free part of K^{ab} is order ρ^3 asymptotically (i.e. if $K^{ab} - \frac{1}{3}(\mathrm{tr}_\gamma K)\gamma^{ab}$ is presumed to be differentiable to order l, then $K^{ab} - \frac{1}{3}(\mathrm{tr}_\gamma K)\gamma^{ab} \in C^{l,\delta}$ for $\delta \geqslant 3$).

Now it is possible that we could choose conformal data with fairly general asymptotic properties and then seek solutions (ϕ, W) which shift the asymptotic properties of the

resulting initial data (γ, K) so that they match definition 2. However, it is more straightforward to build the conditions of definition 2 directly into the conformal data, and then seek solutions (ϕ, W) which more or less leave these asymptotic conditions unchanged. So we will use

Definition 3. A set of conformal data $(\Sigma^3, \lambda, \sigma, \tau)$ satisfies the *asymptotically hyperbolic assumption* if the initial data $(\Sigma^3, \lambda, \sigma + \frac{1}{3}\lambda\tau)$ is asymptotically hyperbolic in the sense of definition 2.

We now state our main result, which describes some additional conditions on (λ, σ, τ) which guarantee that we can solve (2) for (ϕ, W) and thereby produce an asymptotically hyperbolic solution of the Einstein constraint equations:

Theorem 1. Let $(\Sigma^3, \lambda, \sigma, \tau)$ be a set of conformal data which satisfies the asymptotically hyperbolic assumption, plus the following additional conditions:

(i) λ has scalar curvature $R_\lambda < -r$ for some positive constant r.
(ii) $\sigma \in H_1^{p,\epsilon}$ for $p > 1$ and for $0 < \epsilon < 2 - 2/p$.
(iii) τ has no zeros, $\tau - \sqrt{\frac{3}{2}r} \in H_p^{1,\hat{\epsilon}}$ for $\epsilon < \hat{\epsilon} < 2 - 2/p$ and $\left\|\tau - \sqrt{\frac{3}{2}r}\right\|_{C^1} < \beta$ for a certain constant β which one can calculate from $(\Sigma^3, \lambda, \sigma)$.

Then, there exists a unique solution (ϕ, W) of equations (2), with $\phi - 1 \in H_3^{p,\delta}$ for $\delta < \epsilon$ and $W \in H_3^{p,\epsilon}$. The resulting initial data are asymptotically hyperbolic (in the sense of definition 2).

This theorem has been proven using the sequence method. We now discuss in rough terms (using the analytic results from section 4) how this works. See [18] for a more complete discussion of the details of the proof.

The first step, we recall, is to establish the existence of the sequence $\{(\phi_n, W_n)\}$ which satisfies the sequence of equations (8). One may choose ϕ_0 freely, within bounds we will note below. It then follows from proposition 1 that for the given conformal data and for the values of ϵ hypothesized in theorem 1, the operator $\nabla \cdot L$ is invertible, and so we obtain W_1.

To obtain ϕ_1, we need to find a solution to equation (8*b*) with W_1 inserted in the right-hand side. It follows from the sub- and super-solution theorem (proposition 3) that so long as we can find a sub-solution $(\phi_1)_-$ and a super-solution $(\phi_1)_+$ satisfying the hypotheses of proposition 3, then we have ϕ_1. Using the same calculations as appear in [14] for the closed manifold case (the hypotheses that R_λ is bounded negative and τ is bounded away from zero are needed here), we readily find *constants* $(m_1)_-$ and $(m_1)_+$ which satisfy hypotheses (i) and (iii) in proposition 3 to be sub- and super-solutions. However, these constants do not have the necessary asymptotic behaviour (as required by hypotheses (ii) in proposition 3). To fix this, we use

$$(\phi_1)_+ := \mathrm{Min}\{(m_1)_+, 1 + \rho^s\} \tag{26a}$$

and

$$(\phi_1)_- := \mathrm{Max}\{(m_1)_-, 1 - \rho^s\} \tag{26b}$$

and show (see lemma 3.7 of [18])

Claim 1. There exists $s > 0$ such that $(\phi_1)_-$ and $(\phi_1)_+$ belong to $H_0^{p,\epsilon}$ and hence satisfy the hypotheses of proposition 3 to be sub- and super-solutions with the desired asymptotic properties.

Note that proposition 3 states that if one finds appropriate sub- and super-solutions, then one has a unique solution to the equation of interest. Hence we obtain ϕ_1.

The same arguments work sequentially for all n, so indeed we obtain the sequence $\{(\phi_n, W_n)\}$.

We next need to verify that this sequence converges. To do this, we rely upon a contraction mapping argument very similar to the one used for the closed Σ^3 case. A key prerequisite for the contraction mapping argument to work is the existence of upper and lower bounds on the elements of the sequence $\{\phi_n\}$ which are independent of n. This is guaranteed by the existence of an n-independent upper bound on $\{(\phi_n)_+\}$ and an n-independent lower bound on $\{(\phi_n)_-\}$. Since the functions $1+\rho^s$ and $1-\rho^s$ are bounded above and below for positive s and small ρ (ρ is small in the asymptotic region where $1 \pm \rho^s$ are used), one only needs to establish an n-independent bound on the sequences of constants $\{(m_n)_+\}$ and $\{(m_n)_+\}$. But these sequences of constants are essentially the same as those which serve as sub- and super-solutions for $\{\phi_n\}$ in the closed manifold case. Hence the argument used in section 5, step 3 of [14] can be used here to establish these bounds, which we call ϕ_+ and ϕ_-. Note that the bounds within which ϕ_0 must be chosen (referred to earlier) are these constants ϕ_+ and ϕ_-.

Unfortunately, for a number of reasons (including the fact that an asymptotically hyperbolic geometry does not have a finite volume) the rather straightforward calculation leading to (12) for the closed case (see section 5, step 1 of [14]) does not work. One can, however, still prove the following:

Claim 2. For $p > 1$, there exists a constant $C = C(p)$ such that for all $W \in H_2^{p,\delta}$ with $|\delta - 1 + 2/p| < \sqrt{3}$, one has

$$|LW| \leqslant C \sup |\nabla \cdot LW|. \tag{27}$$

The proof of this claim is fairly intricate; it proceeds as a proof by contradiction, with the focus being on showing that if one could find a sequence of vector fields $V_k \in H_2^{p,\delta}$ (with $|\delta - 1 + 2/p| < \sqrt{3}$) for which

$$\sup_{\Sigma^3}\big(|V_k| + |LV_k|\big) = 1 \tag{28a}$$

yet

$$\lim_{k\to\infty}\big[\sup |\nabla \cdot LV_k|\big] = 0 \tag{28b}$$

then one would have a contradiction. If such a sequence does not exist, then claim 2 follows. It should not be a surprise that one could very readily produce a contradiction if we were to assume that there exists a sequence of points $\{x_k\}$ such that $|V_k(x_k)| + |LV_k(x_k)| > \frac{1}{2}$, yet $\{x_k\}$ is contained in a compact subset of Σ^3. The much harder work comes in examining what happens if the $\{x_k\}$ move out to infinity asymptotically. The proof works because as one moves towards infinity, the spatial geometry becomes (at least locally) close to a copy of a piece of hyperbolic half-space. Details of the proof are found in theorem 3.1 of [18].

With the n-independent pointwise estimate for $|LW_n|$ established, one may proceed with the contraction mapping argument as in the closed case. One derives equation (14)

$$\nabla^2(\phi_{n+1} - \phi_n) - \mathcal{G}[\phi_{n+1} - \phi_n] = \mathcal{H}[\phi_n - \phi_{n-1}] \tag{14}$$

and establishes the estimates (15) for $\mathcal{G}$ and $\mathcal{H}$. Then one uses the maximum principle (proposition 2) to deduce that

$$|\phi_{n+1} - \phi_n| \leqslant \frac{\Theta}{\Lambda} \max_{\Sigma^3} |\phi_n - \phi_{n-1}|. \tag{16}$$

Hypothesis (iii) of theorem 1, for a certain constant β—see chapter 3 of [18]—guarantees that $\Theta/\Lambda < 1$. It then follows that the sequence $\{\phi_n\}$ converges. The convergence of the sequence $\{W_n\}$ immediately follows from the invertibility of $\nabla \cdot L$.

The last step of the proof of theorem 1 involves showing that the limit (ϕ_∞, W_∞) of the sequence $\{(\phi_n, W_n)\}$ is a solution of the constraint equations (2), and also showing that the data $\gamma_{ab} = \phi^4\lambda_{ab}$ and $K^{cd} = \phi^{-10}(\sigma^{cd} + LW^{cd}) + \frac{1}{3}\phi^{-4}\lambda^{cd}\tau$ are asymptotically hyperbolic in the sense of definition 2. To show that we have a solution, we again rely on standard bootstrap arguments. Note that while the differentiability assumptions in theorem 1 are weaker than we have used in the closed manifold case (see theorem 1 in [14]) the regularity results which we cite in section 4 guarantee that $(\phi_n - 1) \in C^{2,\delta}$ and $W_n \in H_2^{p,\epsilon}$ for $0 < \delta < \epsilon < 2 - 2/p$. The contraction mapping argument for $\{(\phi_n, W_n)\}$ only guarantees *a priori* that we have C^0 convergence of the sequence. However, given this differentiability for $\{\phi_n\}$ and $\{W_n\}$, we may bootstrap ϕ_∞ into $C^{2,\delta}$ and W_∞ into $H_4^{p,\epsilon}$. Thus, after using the derivation of (ϕ_∞, W_∞) to show that (ϕ_∞, W_∞) is a weak solution of (2), we can argue that it constitutes a strong solution as well.

Since $\phi_\infty - 1 \in C^{2,\delta}$ with $0 < \delta < 2 - 2/p$, and $p > 1$, we see that $\phi_\infty \to 1$ asymptotically. Hence, since (Σ^3, γ) is asymptotically hyperbolic, it follows that $(\Sigma^3, \phi^4\lambda)$ is as well. Our assumption that τ is bounded away from zero by a positive constant guarantees that $\mathrm{tr}_\gamma K = \tau$ satisfies the second condition in definition 2. Finally, the third condition in definition 2 follows from our assumption on σ in definition 3 together with the demonstration that $W_\infty \in H_4^{p,\epsilon}$.

This completes our rough sketch of the proof of our main result, theorem 1.

6. Conclusion

The result we discuss here—theorem 1—demonstrates the existence of a substantial open set of asymptotically hyperbolic initial data which satisfy the Einstein constraint equations and have non-constant mean curvature. Theorem 1 does, however, invoke strong restrictions on the sets of conformal data $(\Sigma^3, \lambda, \sigma, \tau)$ which it shows map to solutions: (i) the scalar curvature of λ must be negative; (ii) the mean curvature τ must be non-zero; (iii) the gradient of τ is strongly controlled.

The first of these restrictions is not very severe, since it has been shown [3, 4] that every asymptotically hyperbolic geometry is conformally related to one with scalar curvature $R = -1$. One would like to remove the other two, however. Can one do so, and does some form of our sequence method serve to prove the existence of solutions?

Some preliminary work indicates that we can, at least, show that the sequence $\{(\phi_n, W_n)\}$ exists with these restrictions on τ removed†. Whether we can then show that this sequence converges is far from clear. Work continues in this direction.

We are also interested in seeing if our method can be used to produce non-CMC asymptotically hyperbolic solutions of the constraints with the polyhomogeneous behaviour found by Andersson *et al* in the CMC case [3]. There is no reason to suspect that it cannot.

Besides these theoretical questions, we are interested in studying whether our method might be useful as a practical tool for producing solutions numerically. There is interest among numerical relativists in considering non-constant mean curvature initial data. It may be that the sequence $\{(\phi_n, W_n)\}$ could be useful for this.

† We use non-constant $(m_n)_+$ and $(m_n)_-$ in doing this.

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