

Further results on geometric operators in quantum gravity

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Abstract. We investigate some properties of geometric operators in canonical quantum gravity in the connection approach à la Ashtekar, which are associated with the volume, area and length of spatial regions. We give the motivations for the construction of analogous discretized lattice quantities, compute various quantum commutators of the type [area, volume], [area, length] and [volume, length], and find that they are generally non-vanishing.

Although our calculations are performed mostly within a lattice-regularized approach, some are—for special, fixed spin-network configurations—identical to corresponding continuum computations. Comparison with the structure of the discretized theory leads us to conclude that anomalous commutators may be a general feature of operators constructed along similar lines within a continuum loop representation of quantum general relativity; the validity of the lattice approach remains unaffected.

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1. Introduction

The introduction of geometric quantum operators, like those measuring areas and volumes, has proven fruitful in the study of non-perturbative canonical gravity in $3 + 1$ dimensions. We are referring here to attempts at defining a theory of canonical gravity in a so-called loop representation, where the basic variables are one-dimensional holonomies along curves in spatial 3-slices Σ [1]. The present paper deals with the case where the holonomy variables are obtained by integrating a real $su(2)$ -valued connection form A on Σ . A is part of a canonical Yang–Mills-type variable pair (A, E) , which is a real version [2] of the well known Ashtekar variables [3].

In the study of pure gravity without matter coupling, the geometric operators *per se* are primarily of interest at the kinematic level, that is, before the quantum diffeomorphism constraints have been imposed on wavefunctions. They are not observables in that they do not commute with those constraints. Because of their obvious geometric interpretation, and relatively simple form in the quantum theory, they have been applied in a variety of contexts. A volume operator was first studied in [4] and part of its spectrum analysed in [5, 6, 8]. It has been used in the construction of the quantum Hamiltonian in the real connection approach, both on the lattice [9] and in a continuum regularization [10]. The area operator was also investigated in [4] and more complete versions of its spectrum were given later in [11, 12]. It has been applied in estimating the radiation of black holes [13] and making contact with semiclassical geometries, obtained from coarse-graining quantum states [14].

In the continuum quantum theory, wavefunctions at the kinematical level are labelled by so-called spin-network states [15], which are particular totally anti-symmetrized linear combinations of Wilson loops (gauge-invariant quantities with respect to local $SU(2)$ rotations, obtained by taking traces of holonomies of closed curves). This is simply a variation on the old theme of loop representation on the space of connections modulo gauge. The advantage of spanning the Hilbert space \mathcal{H} of square-integrable functions on this space by the spin networks is that in this basis the geometric operators can be diagonalized easily, namely, on finite-dimensional subspaces of \mathcal{H} .

Things look somewhat similar when one tries to construct a lattice regularization of connection gravity. The most obvious ansatz is to proceed as in Hamiltonian lattice gauge theory and use a cubic lattice with discretized Hamiltonian link variables [16, 5, 6]. In particular, one may go to the gauge-invariant sector of Hilbert space and employ a basis of spin-network states. The only significant difference with the continuum is that the configuration space underlying the Hilbert space is finite dimensional (for a finite lattice) and the lattice links (edges) themselves are a discrete approximation of space, instead of being embedded in a given manifold Σ .

In the construction of geometric operators in the continuum, roughly speaking one proceeds as follows (see, for example, [11]): first, one smears out the bare operators which are polynomials in \hat{E} and therefore contain multiple derivatives at a point. There is no unique way of doing this and we cannot comment here on the virtue of the different procedures people have adopted. One then defines the regularized operator \hat{O}_{reg} (describing the n -volume of some finite spatial region), evaluates it on a quantum state ψ_γ and looks at the entire expression $\hat{O}_{\text{reg}}\psi_\gamma$ in the limit as the relevant regularization parameters are taken to zero. The resulting expression is usually finite (i.e. no further renormalization is necessary). This is closely related to the kind of quantum representation one is considering, where typical quantum excitations are taken to be finite linear combinations of spin-network states associated with finite embedded graphs γ in Σ . The smeared-out operators act non-trivially only at points $x \in \Sigma$ which happen to be crossed by an edge of the graph γ underlying ψ_γ .

For the volume operators, the action reduces to a sum over intersection points of the graph that happen to lie in the given spatial region whose volume is to be measured. For the area operators, it reduces to a sum over intersection points of edges of the graph with the relevant 2-surface whose area is to be determined. For finite and well behaved graphs these sums are *finite*. Moreover, the ‘remainder’ of the operator action at each of the finite number of contributing points is rather simple: it corresponds to a finite rearrangement of how the incoming flux lines of ψ_γ can be contracted gauge-invariantly at x . (A similar construction can be performed for the length operator, but leads to the counter-intuitive result that the length of most curves is zero. This happens because a one-dimensional piece of curve, whose length is to be measured, generically does not have any intersections with the set of vertices of a graph $\gamma \in \Sigma$. One way of ‘fixing’ this problem by introducing a spatial smearing for the length operator is described in [17].) The spectra of all volume, area and length operators investigated up to now in the continuum are discrete.

In the lattice theory, one may define discretized analogues of the geometric operators (see, for example, [6]). Their structure is very similar to that of the ‘finite remainder’ of the continuum operators described in the previous paragraph. Many calculations one performs on the lattice can be considered as coming from a continuum calculation on a quantum state where the underlying graph happens to be a cubic lattice. This will be explained in more detail in the main part of the paper. Differences in interpretation do however arise, since the lattice theory is only a finite-dimensional approximation to the real theory, which

will only be attained in some infinite-volume limit where the details of the discretization become unimportant.

The remainder of this paper is organized as follows. In the next section we set up the lattice description and define various ways of discretizing volume, area and length functions on the lattice. In section 3, we compute various commutators of the corresponding geometric lattice operators and give some explicit examples of spin-network configurations where the commutators do not vanish. We also discuss a selection rule for states with non-negative volume. In section 4 we investigate some implications of the former result and explain why the presence of anomalous commutators in the lattice theory is natural, and show that the lattice commutators obtain their expected form in the limit as the lattice spacing a is taken to zero. Our calculations imply that anomalous commutators are also present in the continuum theory. This is worrying, since there no continuum limit is usually deemed necessary. We argue that the origin of non-commutativity is not a quantum effect, but lies in the choice of non-local basic variables in the continuum quantum theory.

2. Defining geometric lattice operators

We start with a brief summary of the basic ingredients of Hamiltonian lattice gauge theory à la Kogut and Susskind [18]. For computational simplicity, we take the lattice Λ to be a cubic N^3 -lattice with periodic boundary conditions. The basic quantum operators associated with each lattice link l are a group-valued $SU(2)$ -link holonomy \hat{V} (represented by multiplication by V), together with its inverse \hat{V}^{-1} , and a pair of canonical momentum operators \hat{p}_i^+ and \hat{p}_i^- , where i is an adjoint index. The operator $\hat{p}_i^+(n, \hat{a})$ is based at the vertex n and is associated with the link l oriented in the positive \hat{a} -direction. In contrast, $\hat{p}_i^-(n + \hat{1}_{\hat{a}}, \hat{a})$ is based at the vertex displaced by one lattice unit in the \hat{a} -direction and associated with the inverse link $l^{-1}(\hat{a}) = l(-\hat{a})$. In mathematical terms, the momenta \hat{p}^+ and \hat{p}^- correspond to the left- and right-invariant vector fields on the group manifold associated with a given link. The wavefunctions are elements of $\times_l L^2(SU(2), dg)$, with the product taken over all links and the canonical Haar measure dg on each copy of the group $SU(2)$. The basic commutators are

$$\begin{aligned} [\hat{V}_A^B(n, \hat{a}), \hat{V}_C^D(m, \hat{b})] &= 0, \\ [\hat{p}_i^+(n, \hat{a}), \hat{V}_A^C(m, \hat{b})] &= -\frac{1}{2}i \delta_{nm} \delta_{\hat{a}\hat{b}} \tau_{iA}^B \hat{V}_B^C(n, \hat{a}), \\ [\hat{p}_i^-(n, \hat{a}), \hat{V}_A^C(m, \hat{b})] &= -\frac{1}{2}i \delta_{n, m+1} \delta_{\hat{a}\hat{b}} \hat{V}_A^B(n, \hat{a}) \tau_{iB}^C, \\ [\hat{p}_i^\pm(n, \hat{a}), \hat{p}_j^\pm(m, \hat{b})] &= \pm i \delta_{nm} \delta_{\hat{a}\hat{b}} \epsilon_{ijk} \hat{p}_k^\pm(n, \hat{a}), \\ [\hat{p}_i^+(n, \hat{a}), \hat{p}_j^-(m, \hat{b})] &= 0, \end{aligned} \quad (2.1)$$

where ϵ_{ijk} are the structure constants of $SU(2)$. The normalization for the $SU(2)$ generators τ_i is such that $[\tau_i, \tau_j] = 2\epsilon_{ijk} \tau_k$ and $A_a = A_a^i \tau_i / 2$.

In order to relate discrete lattice expressions with their continuum counterparts, one uses power-series expansions in the so-called lattice spacing a , which is an unphysical parameter with the dimension of length. For the basic classical lattice variables, these are

$$\begin{aligned} V_A^B(\hat{b}) &= 1_A^B + a G A_{bA}^B + O(a^2), \\ p_i^\pm(\hat{b}) &= a^2 G^{-1} E_i^b + O(a^3). \end{aligned} \quad (2.2)$$

Note that Newton's constant G appears since the dimensions of the basic gravitational variables A and E differ from those of the corresponding Yang–Mills phase-space variables. We choose the components of the metric to be dimensionless, $[g_{ab}] = L^0$, which leads to

$[A_{\text{grav}}] = L^{-3}$ and $[E_{\text{grav}}] = L^0$, as opposed to the usual $[A_{\text{YM}}] = L^{-1}$ and $[E_{\text{YM}}] = L^{-2}$ in gauge theory.

Using the expansions (2.2), one obtains unambiguous continuum limits of composite classical lattice expressions by extracting the coefficient of the lowest-order term in the a -expansion. The converse is not true: there is no unique lattice discretization of a continuum expression, since one may always add to the lattice version terms of higher order in a which do not contribute in the continuum limit. In practice, one's choice of a lattice operator is usually motivated by simplicity and considerations of symmetry and will typically affect the convergence behaviour of the (quantum) theory.

Following this prescription, let us now write down the lattice equivalents of the volume, area and length functions. In the continuum theory, these are simply spatial integrals of the square root of the determinant of g_{ab} (or the determinant of the induced metric on the relevant two- or one-dimensional submanifold),

$$\begin{aligned} \mathcal{V}^{\text{cont}}(\mathcal{R}) &= \int_{\mathcal{R}} d^3x \sqrt{g} = \int_{\mathcal{R}} d^3x \sqrt{\frac{1}{3!} \epsilon_{abc} \epsilon^{ijk} E_i^a E_j^b E_k^c}, \\ \mathcal{A}^{\text{cont}}(\mathcal{S}) &= \int_{\mathcal{S}} d^2x \sqrt{^{(2)}g} = \int_{\mathcal{S}} d^2x \sqrt{E^{3i} E_i^3}, \\ \mathcal{L}^{\text{cont}}(\mathcal{C}) &= \int_{\mathcal{C}} dx \sqrt{^{(1)}g} = \int_{\mathcal{C}} dx \sqrt{\frac{1}{\det E} (E_j^2 E^{2j} E_k^3 E^{3k} - (E^{2j} E_j^3)^2)}, \end{aligned} \quad (2.3)$$

where for simplicity we have chosen the surface \mathcal{S} to be normal to the 3-direction everywhere and the curve \mathcal{C} to lie along the 1-direction. We are not concerned here with how the subspaces \mathcal{R} , \mathcal{S} , \mathcal{C} of Σ are defined (for example, they could be determined through some matter distribution), since our discussion is in any case restricted to the kinematical theory, where the diffeomorphism symmetry has not yet been taken into account. As in [6], let us define the lattice function $D(n)$ as

$$\begin{aligned} D(n) &:= \epsilon_{abc} \epsilon^{ijk} p_i(n, \hat{a}) p_j(n, \hat{b}) p_k(n, \hat{c}) \\ &:= \frac{1}{8} \epsilon_{abc} \epsilon^{ijk} (p_i^+(n, \hat{a}) + p_i^-(n, \hat{a})) (p_j^+(n, \hat{b}) + p_j^-(n, \hat{b})) (p_k^+(n, \hat{c}) + p_k^-(n, \hat{c})), \end{aligned} \quad (2.4)$$

where the sum is taken over all positive lattice directions, i.e. $\hat{a} = 1, 2, 3$. In the continuum limit, this goes over to

$$D(n) \xrightarrow{a \rightarrow 0} a^6 \epsilon_{abc} \epsilon^{ijk} E_i^a E_j^b E_k^c + \mathcal{O}(a^7). \quad (2.5)$$

To arrive at (2.4), we have substituted the continuum momenta E_i^a by the averaged lattice momenta $p_i(n, \hat{a}) := \frac{1}{2}(p_i^+(n, \hat{a}) + p_i^-(n, \hat{a}))$, which ensures that the final expression for $D(n)$ is invariant under relabelling of axes and no direction is preferred. This seems to be the only simple choice of lattice function with these properties and was the one adopted in [6] to analyse some spectral properties of the volume operator.

We may interpret $\mathcal{V}(n) := \sqrt{\frac{1}{6} D(n)}$ as the volume associated with the dual unit cube of Λ^* centred at n or as the lattice version of $\det E$ at the point n (there really is no distinction between 'local' and 'smeared over a unit cube' on the lattice). The latter interpretation is more closely related to the continuum formulation, where the finite operators (after the regulator has been removed) act non-trivially at points $x \in \gamma$, or rather, on segments of the graph γ meeting at x . More generally, we define

$$\mathcal{V}(\mathcal{R}) = \sum_{n \in \mathcal{R}} \sqrt{\frac{1}{3!} D(n)} \quad (2.6)$$

for the volume of a lattice region \mathcal{R} (a choice that will be justified in section 4). There are several possibilities for discretizing local area. Following the viewpoint of the dual lattice, the area of a unit 2-surface ‘perpendicular’ to the \hat{a} -direction may be defined as

$$\mathcal{A}_1(n, \hat{a}) = \sqrt{\frac{1}{2}(p_i^+(n, \hat{a})p^{+i}(n, \hat{a}) + p_i^-(n + 1_{\hat{a}}, \hat{a})p^{-i}(n + 1_{\hat{a}}, \hat{a}))} \quad (2.7)$$

(no summation over \hat{a}), which should be interpreted as the area of a unit surface in Λ^* , dual to the link $l = (n, \hat{a})$. Alternatively, if one prefers to think of the operator action as taking place at the vertices n , one may define

$$\mathcal{A}_2(n, \hat{a}) = \sqrt{p_i(n, \hat{a})p^i(n, \hat{a})} \equiv \sqrt{\frac{1}{4}(p_i^+(n, \hat{a}) + p_i^-(n, \hat{a}))(p^{+i}(n, \hat{a}) + p^{-i}(n, \hat{a}))} \quad (2.8)$$

or

$$\mathcal{A}_3(n, \hat{a}) = \sqrt{\frac{1}{2}(p_i^+(n, \hat{a})p^{+i}(n, \hat{a}) + p_i^-(n, \hat{a})p^{-i}(n, \hat{a}))}. \quad (2.9)$$

The functional form of \mathcal{A}_2 is the one that appears as a special case (i.e. when evaluated on states γ that lie on an embedded lattice Λ^{emb} in Σ of the area operator in the continuum theory [11]). In the limit as $a \rightarrow 0$, the expansions of the discretized area functions are all of the form

$$\mathcal{A}_i(n, \hat{a}) = a^2 \sqrt{E^{3i} E_i^3} + \mathcal{O}(a^3). \quad (2.10)$$

For finite lattice areas \mathcal{S} , perpendicular to \hat{a} , we define $\mathcal{A}_l(\mathcal{S}) = \sum_{n \in \mathcal{S}} \mathcal{A}_i(n, \hat{a})$.

Due to its complicated functional form, the definition of the length of a unit link is even more ambiguous. Thinking in terms of link length on the dual lattice, one would associate it with the dual unit plaquette in Λ . To obtain a symmetric expression, one possibility is to sum over the bi-vectors based at each of the four corners of the plaquette. Thus the continuum expression $E_i^a E^{ai} E_j^b E^{bj} - (E_i^a E^{bi})^2$ would (up to powers of a) be represented by $\frac{1}{4}$ times the sum of the four terms

$$\begin{aligned} C(n, \hat{a}, \hat{b}) &:= p_i^+(n, \hat{a})p^{+i}(n, \hat{a})p_j^+(n, \hat{b})p^{+j}(n, \hat{b}) - (p_i^+(n, \hat{a})p^{+i}(n, \hat{b}))^2, \\ C(n + 1_{\hat{a}}, -\hat{a}, \hat{b}) &:= p_i^-(n + 1_{\hat{a}}, \hat{a})p^{-i}(n + 1_{\hat{a}}, \hat{a})p_j^+(n + 1_{\hat{a}}, \hat{b})p^{+j}(n + 1_{\hat{a}}, \hat{b}) \\ &\quad - (p_i^-(n + 1_{\hat{a}}, \hat{a})p^{+i}(n + 1_{\hat{a}}, \hat{b}))^2, \\ C(n + 1_{\hat{a}} + 1_{\hat{b}}, -\hat{a}, -\hat{b}) &:= p_i^-(n + 1_{\hat{a}} + 1_{\hat{b}}, \hat{a})p^{-i}(n + 1_{\hat{a}} + 1_{\hat{b}}, \hat{a}) \\ &\quad \times p_j^-(n + 1_{\hat{a}} + 1_{\hat{b}}, \hat{b})p^{-j}(n + 1_{\hat{a}} + 1_{\hat{b}}, \hat{b}) \\ &\quad - (p_i^-(n + 1_{\hat{a}} + 1_{\hat{b}}, \hat{a})p^{-i}(n + 1_{\hat{a}} + 1_{\hat{b}}, \hat{b}))^2, \\ C(n + 1_{\hat{b}}, \hat{a}, -\hat{b}) &:= p_i^+(n + 1_{\hat{b}}, \hat{a})p^{+i}(n + 1_{\hat{b}}, \hat{a})p_j^-(n + 1_{\hat{b}}, \hat{b})p^{-j}(n + 1_{\hat{b}}, \hat{b}) \\ &\quad - (p_i^+(n + 1_{\hat{b}}, \hat{a})p^{-i}(n + 1_{\hat{b}}, \hat{b}))^2, \end{aligned} \quad (2.11)$$

(no sums over \hat{a}, \hat{b}). To obtain the complete expression for the link length, one still has to divide by the density factor. Two different ways of symmetrizing lead to

$$\begin{aligned} \mathcal{L}_1(n, \hat{a}, \hat{b}) &= \left[\frac{3}{2}(C(n, \hat{a}, \hat{b})D(n)^{-1} + C(n + 1_{\hat{a}}, -\hat{a}, \hat{b})D(n + 1_{\hat{a}})^{-1} \right. \\ &\quad \left. + C(n + 1_{\hat{a}} + 1_{\hat{b}}, -\hat{a}, -\hat{b})D(n + 1_{\hat{a}} + 1_{\hat{b}})^{-1} \right. \\ &\quad \left. + C(n + 1_{\hat{b}}, \hat{a}, -\hat{b})D(n + 1_{\hat{b}})^{-1} \right]^{1/2}, \end{aligned} \quad (2.12)$$

or

$$\begin{aligned} \mathcal{L}_2(n, \hat{a}, \hat{b}) = & \left[6(C(n, \hat{a}, \hat{b}) + C(n + 1_{\hat{a}}, -\hat{a}, \hat{b}) \right. \\ & + C(n + 1_{\hat{a}} + 1_{\hat{b}}, -\hat{a}, -\hat{b}) + C(n + 1_{\hat{b}}, \hat{a}, -\hat{b})) \\ & \left. \times (D(n) + D(n + 1_{\hat{a}}) + D(n + 1_{\hat{a}} + 1_{\hat{b}}) + D(n + 1_{\hat{b}}))^{-1} \right]^{1/2}. \end{aligned} \quad (2.13)$$

Alternatively, a simpler version containing only link variables based at n is given by

$$\mathcal{L}_3(n, \hat{a}, \hat{b}) = \sqrt{6 D(n)^{-1} (p_i(n, \hat{a}) p^i(n, \hat{a}) p_j(n, \hat{b}) p^j(n, \hat{b}) - (p_i(n, \hat{a}) p^i(n, \hat{b}))^2)}, \quad (2.14)$$

where the averaged link momenta $p_i(n, \hat{a})$ have been used. In the limit as $a \rightarrow 0$, all three expressions have an a -expansion of the form

$$\mathcal{L}_i(n, \hat{a}, \hat{b}) = a \sqrt{\frac{1}{\det E} (E^a_j E^{aj} E^b_k E^{bk} - (E^{aj} E^b_j)^2)} + O(a^2). \quad (2.15)$$

For finite curves \mathcal{C} perpendicular to the \hat{a} - and \hat{b} -directions, say, we use as an obvious definition $\mathcal{L}_i(\mathcal{C}) = \sum_{n \in \mathcal{C}} \mathcal{L}_i(n, \hat{a}, \hat{b})$.

One may think of yet another way of discretizing length, namely, by expressing the term under the square root in terms of the dreibeins, $g_{11} = e^i_1 e_{1i}$, say, and using a lattice equivalent of the continuum identity

$$e^i_a \equiv \frac{1}{2\sqrt{\det E}} \epsilon_{abc} \epsilon^{ijk} E^b_j E^c_k = 2 \left\{ A^i_a, \int d^3x \sqrt{\det E} \right\}. \quad (2.16)$$

One motivation for this choice is the substitution of terms containing negative powers of the square root of the metric by Poisson bracket terms $\{A, \int \sqrt{\det E}\}$, which are not obviously singular for vanishing $\det E$ [10]. As a discretized analogue one may use, for example,

$$-2 \left\{ V(n, \hat{a})_A^B, \sum_n \sqrt{\frac{1}{6} D(n)} \right\} V(n, \hat{a})^{-1}{}_B{}^C \tau_{iC}{}^A \xrightarrow{a \rightarrow 0} \frac{a}{\sqrt{G}} e^i_a(x) + O(a^2). \quad (2.17)$$

The g_{11} -term can be discretized by the following expression:

$$2 \left\{ V(n, \hat{1})_A^B, \sum_n \sqrt{\frac{1}{6} D(n)} \right\} \left\{ V(n, \hat{1})^{-1}{}_B{}^A, \sum_m \sqrt{\frac{1}{6} D(m)} \right\} \xrightarrow{a \rightarrow 0} \frac{a^2}{G} e^i_1 e_{1i} + O(a^3). \quad (2.18)$$

However, it turns out that this construction is not particularly convenient on the lattice because of the appearance of the link holonomies $V(n, \hat{a})$. Their quantum analogues, unlike the \hat{p} -operators, change the flux-line (or spin) assignments of the spin-network states they act upon. Equivalently, the quantized expression (2.18) does not commute with the Laplacians (4.3) and therefore cannot be diagonalized on the same finite-dimensional sub-Hilbert spaces as the local geometric operators we have considered so far. Unlike in the continuum, we cannot shrink away the link holonomies appearing in the quantum operators independently from those appearing in the wavefunctions. Thus the analysis of this type of length operator is structurally more complicated and we will not consider it at present.

3. Some explicit calculations

In order to avoid unnecessary degeneracy of geometric operators, we will consider lattice spin-network states whose flux line assignments are non-vanishing. For our purposes it will be sufficient to study the local behaviour of such states around a vertex n . Following [5, 6], we will label the flux line assignments of the six links meeting at n by a six-component

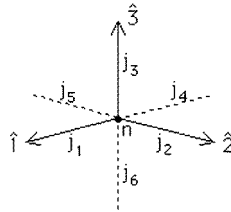


Figure 1. Labelling of flux lines at a lattice intersection n .

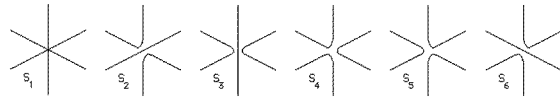


Figure 2. Spin-network states for $\vec{j} = (1, 1, 1, 1, 1, 1)$.

vector \vec{j} , where j_1, j_2, j_3 correspond to the links in the positive 1-, 2- and 3-direction, and j_4, j_5, j_6 to those in the three negative directions, with $j_i = 1, 2, \dots$ (figure 1). The value of j_i is twice the spin characterizing the irreducible representation of $SU(2)$ associated with the link. Recall that to specify a spin-network state locally, one needs in addition to \vec{j} a choice of gauge-invariant contractor for the flux lines meeting at n . Given \vec{j} , the space of all possible contractors at n is finite dimensional.

Let us consider the simplest type of a 6-valent intersection, namely, $\vec{j} = (1, 1, 1, 1, 1, 1)$. There is one linear relation between the six spin-network states S_i (figure 2) one can construct from these initial data, namely, $\tilde{S} := S_1 - S_2 - S_3 + S_4 + S_5 - S_6 = 0$.

Using the area operators \hat{A}_2 for measuring the local areas in the three main directions, one finds a set of simultaneous eigenstates with the following eigenvalues.

Table 1.

	$S_1 - 2S_2$	$S_1 - 2S_3$	$S_1 - 2S_6$	S_1	$S_4 - S_5$	\tilde{S}
$\hat{A}_2(n, \hat{1})$	$\sqrt{\frac{3}{4}}$	$\sqrt{\frac{1}{4}}$	$\sqrt{\frac{1}{4}}$	$\sqrt{\frac{3}{4}}$	$\sqrt{\frac{1}{4}}$	$\sqrt{\frac{1}{4}}$
$\hat{A}_2(n, \hat{2})$	$\sqrt{\frac{1}{4}}$	$\sqrt{\frac{1}{4}}$	$\sqrt{\frac{3}{4}}$	$\sqrt{\frac{3}{4}}$	$\sqrt{\frac{1}{4}}$	$\sqrt{\frac{1}{4}}$
$\hat{A}_2(n, \hat{3})$	$\sqrt{\frac{1}{4}}$	$\sqrt{\frac{3}{4}}$	$\sqrt{\frac{1}{4}}$	$\sqrt{\frac{3}{4}}$	$\sqrt{\frac{1}{4}}$	$\sqrt{\frac{1}{4}}$

Using the area operators \hat{A}_3 , every state is an eigenstate, with eigenvalue $\sqrt{\frac{3}{4}}$. Now computing the action of the operator $\hat{D}(n)$ on these states, one obtains the following eigenvectors:

eigenvalue 0 : $S_1 - 2S_2, S_1 - 2S_3, S_1 - 2S_6, \tilde{S}$

eigenvalue $\frac{3}{2}\sqrt{\frac{3}{2}}$: $\frac{1}{2}S_1 + S_2 + S_3 + (-1 + \frac{1}{2}i\sqrt{6})S_4 + (-1 - \frac{1}{2}i\sqrt{6})S_5 + S_6$

eigenvalue $-\frac{3}{2}\sqrt{\frac{3}{2}}$: $\frac{1}{2}S_1 + S_2 + S_3 + (-1 - \frac{1}{2}i\sqrt{6})S_4 + (-1 + \frac{1}{2}i\sqrt{6})S_5 + S_6$.

Even taking into account that $\tilde{S} \sim 0$, the two eigenvectors with eigenvalues $\pm\frac{3}{2}\sqrt{\frac{3}{2}}$ are not eigenvectors of the areas \hat{A}_2 , i.e. $\hat{D}(n)$ and \hat{A}_2 must necessarily be non-commuting

operators. For simplicity, instead of $[\hat{\mathcal{V}}(n), \hat{\mathcal{A}}_2(n, \hat{3})]$, we just calculate the commutator of the polynomial expressions appearing under the square roots (already omitting terms proportional to the Laplacian), i.e.

$$\begin{aligned} [\hat{D}(n), \hat{p}_i^+(n, \hat{3})\hat{p}^{-i}(n, \hat{3})] &= 12i \hat{p}_i(n, \hat{1})\hat{p}_j(n, \hat{2})\hat{p}^{+lj}(n, \hat{3})\hat{p}^{-il}(n, \hat{3}) \\ &= 6\epsilon^{ijk} \hat{p}_i(\hat{1})\hat{p}_j(\hat{2})\hat{p}_k^+(\hat{3}) + 12i \hat{p}_i(\hat{1})\hat{p}_j(\hat{2})\hat{p}^{+lj}(\hat{3}) \\ &\quad \times (\hat{p}^{+il}(\hat{1}) + \hat{p}^{+il}(\hat{2}) - \hat{p}^{-il}(\hat{1}) - \hat{p}^{-il}(\hat{2})), \end{aligned} \quad (3.1)$$

which is not the zero operator. In the second and third lines of the equation we have not written out the vertex n explicitly and have used the quantized version of the discrete form of Gauss' law,

$$p_i^+(n, \hat{1}) + p_i^+(n, \hat{2}) + p_i^+(n, \hat{3}) - p_i^-(n, \hat{1}) - p_i^-(n, \hat{2}) - p_i^-(n, \hat{3}) = 0, \quad (3.2)$$

to eliminate one of the six momentum operators based at n . One easily checks that in the continuum limit, taking into account the expansions

$$p_i^\pm(n, \hat{b}) = a^2 G^{-1} E_i^b \pm a^3 G^{-1} \nabla_b E_i^b + O(a^4) \quad (3.3)$$

(no sum over b), the flux conservation relation (3.2) to lowest order in a is proportional to the usual expression for the Gauss law.

Now, if one were to define the volume operator by $\hat{\mathcal{V}} = \sum \sqrt{(1/3!)|\hat{D}(n)|}$ (which, however, as we will explain in due course, is not the natural thing to do), one would obtain a degenerate eigenspace for the eigenvalue $+\frac{3}{2}\sqrt{\frac{3}{2}}$. It is easy to show that in the above example (modulo the zero-norm state \tilde{S}) a basis for this eigenspace is given by $\{S_1, S_4 - S_5\}$. Therefore, on the particular set of states with $\vec{j} = (1, 1, 1, 1, 1, 1)$, the operator $|\hat{D}(n)|$ does commute with all the areas. Vanishing commutation relation with $|\hat{D}(n)|$ is a weaker condition than with $\hat{D}(n)$ alone. The spectrum of $|\hat{D}(n)|$ coincides with the square root of the spectrum of \hat{D}^2 , and we have

$$[\hat{D}^2, \hat{P}] = \hat{D}[\hat{D}, \hat{P}] + [\hat{D}, \hat{P}]\hat{D}. \quad (3.4)$$

It follows that for an arbitrary operator \hat{P} , $[\hat{D}, \hat{P}] = 0 \Rightarrow [|\hat{D}|, \hat{P}] = 0$, but not the other way round. Substituting in the squared area operator, one obtains

$$[\hat{D}^2, \hat{p}_i^+(\hat{3})\hat{p}^{-i}(\hat{3})] = 2[\hat{D}, \hat{p}_i^+(\hat{3})\hat{p}^{-i}(\hat{3})]\hat{D} + \text{higher-order terms in } \hbar. \quad (3.5)$$

We already know that $[\hat{D}, \hat{p}_i^+(\hat{3})\hat{p}^{-i}(\hat{3})]$ is a non-vanishing operator, but it could, in principle, vanish on all states that are annihilated by \hat{D} (in a volume eigenbasis), leading to $[\hat{D}, \hat{p}_i^+(\hat{3})\hat{p}^{-i}(\hat{3})]\hat{D} = 0$. However, the following example demonstrates that this does not happen.

The simplest Hilbert subspace of loop configurations we have found where the commutator (3.4) is non-vanishing is one with 10 incoming flux lines at the vertex n , namely, $\vec{j} = (2, 2, 1, 2, 2, 1)$. By taking appropriate linear combinations of Wilson loop states (i.e. by antisymmetrizing over flux lines of multiply occupied links), one finds 11 spin-network states, subject to two Mandelstam identities, i.e. the subspace they span is nine dimensional. Let us call these states $\{S_i, i = 1, \dots, 11\}$, as illustrated in figure 3. Simultaneous eigenstates of the three local area operators $\hat{\mathcal{A}}_2$ are the linear combinations

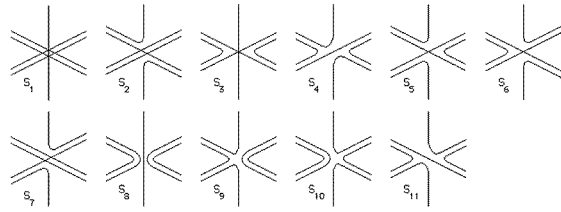


Figure 3. Spin-network states for $\vec{j} = (2, 2, 1, 2, 2, 1)$.

T_i , defined by

$$\begin{aligned}
 T_1 &= S_1 - 2S_7, & T_2 &= S_1 - 2S_2, & T_3 &= S_5 - S_6, \\
 T_4 &= S_1 - 2S_3, & T_5 &= S_1, & T_6 &= S_1 - 6S_3 + 6S_8, \\
 T_7 &= S_1 + S_2 - 3S_3 - 3S_5 - 3S_6 - 3S_7 + 12S_{11}, \\
 T_8 &= S_1 - 3S_2 - 3S_3 + 12S_4 - 3S_5 - 3S_6 + S_7, & (3.6) \\
 T_9 &= S_5 - S_6 - 2S_9 + 2S_{10}, \\
 T_{10} &= S_1 - S_2 - S_3 + S_5 + S_6 - S_7, \\
 T_{11} &= S_3 - S_4 - S_8 + S_9 + S_{10} - S_{11}.
 \end{aligned}$$

Both T_{10} and T_{11} are zero-norm states, which will be set identically to zero in what follows. The table of eigenvalues for the remaining states is shown below.

Table 2.

	T_1	T_2	T_3	T_4	T_5	T_6	T_7	T_8	T_9
$\hat{A}_2(n, \hat{1})$	$\sqrt{\frac{3}{2}}$	$\sqrt{2}$	$\sqrt{\frac{3}{2}}$	$\sqrt{\frac{3}{2}}$	$\sqrt{2}$	$\sqrt{\frac{1}{2}}$	$\sqrt{\frac{1}{2}}$	$\sqrt{\frac{3}{2}}$	$\sqrt{\frac{1}{2}}$
$\hat{A}_2(n, \hat{2})$	$\sqrt{2}$	$\sqrt{\frac{3}{2}}$	$\sqrt{\frac{3}{2}}$	$\sqrt{\frac{3}{2}}$	$\sqrt{2}$	$\sqrt{\frac{1}{2}}$	$\sqrt{\frac{3}{2}}$	$\sqrt{\frac{1}{2}}$	$\sqrt{\frac{1}{2}}$
$\hat{A}_2(n, \hat{3})$	$\sqrt{\frac{1}{4}}$	$\sqrt{\frac{1}{4}}$	$\sqrt{\frac{1}{4}}$	$\sqrt{\frac{3}{4}}$	$\sqrt{\frac{3}{4}}$	$\sqrt{\frac{3}{4}}$	$\sqrt{\frac{1}{4}}$	$\sqrt{\frac{1}{4}}$	$\sqrt{\frac{1}{4}}$

Now calculating the eigenvalues of the operator $\hat{D}(n)$ on the T_i , one finds that the eigenvalue-zero space is fivefold degenerate. The remaining eigenstates are

$$\begin{aligned}
 \text{eigenvalue } \frac{3}{2}\sqrt{\frac{5}{2}} : & & V_1 &= 5 T_4 - i\sqrt{\frac{5}{2}} T_9 \\
 \text{eigenvalue } -\frac{3}{2}\sqrt{\frac{5}{2}} : & & V_2 &= 5 T_4 + i\sqrt{\frac{5}{2}} T_9 \\
 \text{eigenvalue } \frac{3}{2}\sqrt{\frac{23}{2}} : & & V_3 &= 8 T_5 - 3 T_6 + 3i\sqrt{\frac{23}{2}} T_3 \\
 \text{eigenvalue } -\frac{3}{2}\sqrt{\frac{23}{2}} : & & V_4 &= 8 T_5 - 3 T_6 - 3i\sqrt{\frac{23}{2}} T_3.
 \end{aligned}$$

From the point of view of the operator $|\hat{D}(n)|$, the sets $\{V_1, V_2\}$ and $\{V_3, V_4\}$ span the degenerate eigenspaces with eigenvalues $\frac{3}{2}\sqrt{\frac{5}{2}}$ and $\frac{3}{2}\sqrt{\frac{23}{2}}$, respectively. Still, the second of these eigenspaces depends on more than two of the T_i , by which it is shown that the \hat{A}_2 and $|\hat{D}(n)|$ cannot be diagonalized simultaneously.

Let us now explain why *no* modulus should appear under the square root in the definition of the volume operator. In order to understand this, one has to go back to the definition of

the continuum canonical momentum $E_i^a(x)^\dagger$. This is a densitized inverse dreibein, given by $E_i^a = (\det e_b^j) e_i^a$ where e_i^a is the inverse dreibein satisfying

$$e_i^a e_{aj} = \delta_{ij}, \quad e_{aj} e_b^j = g_{ab}, \quad (3.7)$$

with the determinant $\det e_b^j$ taking values $\pm\sqrt{\det g}$. One therefore derives for classical, non-degenerate metrics the inequality $\det E = (\det e)^2 > 0$. With $\det E$ positive, the volume is simply $\int d^3x \sqrt{\det E}$.

However, in Yang–Mills-type quantum representations with canonical commutation relations

$$[\hat{A}_a^i(x), \hat{E}_j^b(y)] = \delta_a^b \delta_j^i \delta(x, y) \quad (3.8)$$

or commutation relations derived from (3.8), $\widehat{\det E} > 0$ is *not* automatic (indeed, in Yang–Mills phase space, there is no such restriction). This is borne out by the fact that all non-zero eigenvalues of $\hat{D}(n)$ come in pairs $\pm d$ of opposite sign [6, 7]. In gravity, we therefore have to *impose* a quantum analogue of $\det E > 0$ in the quantum theory. On the lattice this is straightforward—go to a basis where all operators $\hat{D}(n)$ are diagonal and eliminate all eigenstates with non-positive eigenvalues. We do not know whether a similar restriction is compatible with the continuum regularization of the volume operator.

Let us briefly discuss the commutation relations of the simplest of the length operators, $\hat{\mathcal{L}}_3$. We choose to abbreviate its polynomial part by

$$L(n, \hat{a}, \hat{b}) := p_i(n, \hat{a}) p^i(n, \hat{a}) p_j(n, \hat{b}) p^j(n, \hat{b}) - (p_i(n, \hat{a}) p^i(n, \hat{b}))^2 \quad (3.9)$$

(no sums over \hat{a} and \hat{b}), so that the entire discretized length function is given by

$\mathcal{L}_3(n, \hat{a}, \hat{b}) = \sqrt{6 D(n)^{-1} L(n, \hat{a}, \hat{b})}$. The commutator between $\hat{D}(n)$ and $\hat{L}(n, \hat{a}, \hat{b})$ is non-vanishing, but since its form is not particularly illuminating, we will not write it explicitly. Note, however, that this implies a factor-ordering ambiguity in the definition of the quantum operator $\hat{\mathcal{L}}_3(n, \hat{a}, \hat{b})$, which was not present for the area and volume operators. The simplest explicit case with $[\hat{L}(n, \hat{a}, \hat{b}), \hat{D}(n)] \neq 0$ we have found is the set of spin-network states with $\vec{j} = (2, 1, 1; 2, 1, 1)$, and for the set of states with $\vec{j} = (2, 2, 1; 2, 2, 1)$ one finds that in addition also $[\hat{L}(n, \hat{a}, \hat{b}), \hat{D}(n)^2] \neq 0$.

In order not to have to address the factor-ordering problem, for the case of length and area functions, we confine ourselves to a computation of the classical Poisson brackets, which is identical with the quantum result to lowest order in \hbar . Moreover, let us omit the square roots and first calculate

$$\left\{ \frac{L(n, \hat{1}, \hat{2})}{D(n)}, p_k(\hat{3}) p^k(\hat{3}) \right\} = -\frac{L(n, \hat{1}, \hat{2})}{D(n)^2} \{D(n), p_k(\hat{3}) p^k(\hat{3})\}, \quad (3.10)$$

which by virtue of (3.5) is non-vanishing. Slightly more complicated is the computation of the Poisson brackets of the length in the 3-direction and the area perpendicular to the 1-direction, say. It is proportional to

$$\begin{aligned} \left\{ \frac{L(n, \hat{1}, \hat{2})}{D(n)}, p_k(\hat{1}) p^k(\hat{1}) \right\} &= -\frac{L(n, \hat{1}, \hat{2})}{D(n)^2} \{D(n), p_k(\hat{1}) p^k(\hat{1})\} \\ &\quad + \frac{1}{D(n)} \{L(n, \hat{1}, \hat{2}), p_k(\hat{1}) p^k(\hat{1})\} \\ &= -\frac{1}{D(n)^2} (6 L(n, \hat{1}, \hat{2}) p_i(\hat{3}) p_j(\hat{1}) p^{lj+}(\hat{1}) p^{il-}(\hat{3}) \\ &\quad + D(n) \epsilon_{jkl} p_i(\hat{2}) p_j(\hat{2}) p^{i+}(\hat{1}) p^{l+}(\hat{1}) p^{k-}(\hat{1})), \end{aligned} \quad (3.11)$$

† I thank T Thiemann for a discussion on this point.

with the round brackets denoting symmetrization. Assuming suitable regularity conditions (like, for instance, $D(n) > 0$), we conclude that lengths and areas, do not, in general, commute.

4. Why does non-commutativity arise?

Having seen some instances of non-commutativity of geometric operators in the previous section, one may wonder whether this was to be expected. Naively, it is rather surprising since the geometric operators contain only information about components of the spatial metric, which classically Poisson-commute. One obvious difference in the connection approach is the fact that the classical dreibein variables E_i^a (or composite quantities containing them) in the quantum theory are represented by differential operators. This is the opposite of what happens in quantum representations based on the metric formulation, where the operators \hat{g}_{ab} usually act by multiplication.

However, on the lattice, the non-commutativity we have found is not a quantum effect, but simply a consequence of the regularization. Not even the basic classical variables of lattice gauge theory obey the canonical Poisson brackets of their continuum counterparts. Only in the continuum limit as $a \rightarrow 0$ does one rederive the expected result. Consider, for example, the classical Poisson relation

$$\{p_i^+(n, \hat{a}), V_A^C(m, \hat{b})\} = -\frac{1}{2} \delta_{nm} \delta_{\hat{a}\hat{b}} \tau_{iA}^B V_B^C(n, \hat{a}). \tag{4.1}$$

Replacing the link variables by their expansions (2.2), one obtains

$$\{a^2 G^{-1} E_i^a + O(a^3), \frac{1}{2} a G A_b^j \tau_{jA}^C + O(a^2)\} = -\frac{1}{2} \delta_{nm} \delta_{\hat{a}\hat{b}} \tau_{iA}^B (\mathbb{1}_B^C + O(a)). \tag{4.2}$$

Dividing both sides by a^3 and using $(1/a^3) \delta_{mn} \xrightarrow{a \rightarrow 0} \delta^3(x, y)$, one finds in the limit the canonical Poisson brackets $\{E_i^a(x), A_b^j(y)\} = -\delta_{ij} \delta_b^a \delta^3(x, y)$. Likewise, the non-vanishing lattice brackets $\{p^\pm, p^\pm\}$ are to lowest order in a equivalent to the continuum brackets $\{E, E\} = 0$. In this case, the non-commutativity is clearly a result of the regularization implicit in the definition of the basic lattice variables and present both classically and quantum mechanically.

It is therefore not surprising when composite quantities depending on link momenta do not commute in the lattice discretization. Sometimes one can find discretized versions of continuum expressions that have this property *before* the limit $a \rightarrow 0$ is taken. It may then be convenient to use them, since in this case a property of the continuum theory (commutativity of two functions) is implemented exactly on the lattice, which probably improves its convergence behaviour. Typical examples are the sums of squares

$$\sum_i p_i^\pm(n, \hat{a}) p_i^{\pm i}(n, \hat{a}) \tag{4.3}$$

which commute with all other functions of the $p_i^\pm(m, \hat{b})$'s. The corresponding quantum operators are, of course, proportional to the Laplacian on the group manifold associated with the links (n, \hat{a}) and $(n + 1_{\hat{a}}, \hat{a})$, respectively.

Applying the same reasoning to the Poisson bracket of the local volume and area function, schematically one has

$$\{\sqrt{p^3}, \sqrt{p^2}\} = \frac{1}{\sqrt{p^5}} \sum (\dots) p^4 \xrightarrow{a \rightarrow 0} \{\sqrt{E^3}, \sqrt{E^2}\} = 0. \tag{4.4}$$

To arrive at the second relation, we have divided through by a^5 on both sides. To summarize, the non-commutativity of discretized analogues of geometric functions or their quantum operators is not *a priori* particularly surprising or worrying.

What conclusions may we draw from the above discussion for the continuum theory? It turns out that the calculations and results obtained in the lattice discretization are special cases of calculations that may be done in the continuum theory. That is, after choosing a particular geometric quantum operator \hat{O} and a specific state ψ in the continuum, the finite expression for \hat{O} acting on ψ , after the regulator has been removed, may be identical to that of a particular lattice calculation for a corresponding operator \hat{O}^{latt} .

On the lattice, due to the discretization, one is more restricted in the kind of quantities one can compute at a ‘point’. For example, local area can only be measured in the three directions given by the lattice axes meeting at an intersection n . This happens because on the lattice there are only a finite number of degrees of freedom associated with a unit cube, namely, nine (independent holonomy components) before and six after going to the quotient with respect to local gauge transformations. This corresponds to the six degrees of freedom contained in the metric g_{ab} before the imposition of diffeomorphism symmetry. Hence there is a maximum of six independent metric quantities that classically can be associated with a unit cell of the lattice. For example, one may choose them to be the volume, three areas and two length functions.

Furthermore, due to the special geometry we have chosen for the lattice, we can only compare graph or loop configurations that are at most six-valent (and such that pairs of incoming links are collinear and the entire set is not coplanar at the intersection point). Obviously, in order to prove a statement like ‘two operators do not commute’, it is sufficient to exhibit a single instance of when they do not. For example, consider the area operator $\hat{\mathcal{A}}_2^{\text{cont}}$ corresponding to some finite spatial 2-surface \mathcal{S} [11]. Now, take any continuum graph γ with a six-valent intersection (of the type just described) at some point $x \in \mathcal{S}$ such that two of the collinear pairs of incoming edges are tangent to \mathcal{S} in x , and such that the third pair in x is collinear with the unit normal defining the surface. For simplicity, the graph γ is supposed to have no further intersections with itself or with \mathcal{S} (in the present discussion, we ignore cases where global constraints prevent such a construction).

Evaluating the area operator on the finite set of spin-network states associated with the graph γ and some definite flux assignment to its edges, the only non-vanishing contributions come from the intersection at x , and the calculation of eigenstates and eigenvalues is isomorphic to the corresponding one on the lattice. On the same set of states one may evaluate the volume operator of an arbitrary spatial region \mathcal{R} containing x , as defined in [19, 20], and again obtain the same result as one would have in the lattice calculation, using $\hat{\mathcal{V}}$. It therefore follows immediately from our calculations in section 3 that $[\hat{\mathcal{A}}^{\text{cont}}(\mathcal{S}), \hat{\mathcal{V}}^{\text{cont}}(\mathcal{R})] \neq 0$ in the continuum.

Next considering the local area operators associated with the three main directions on the lattice, one finds that they all mutually commute, independent of which of the three definitions we choose. We will not be bothered here with defining area functions for surfaces that lie obliquely in the lattice. There is no obstruction in principle to doing this, a necessary requirement being that (in the limit as $a \rightarrow 0$) there are no preferred directions on the lattice. This is analogous to the restoration of rotational symmetry of observables in the usual lattice gauge theory on a fixed Euclidean lattice in the continuum limit.

In this restricted lattice framework we therefore cannot reproduce the result derived by Ashtekar, Corichi and Zapata, who found non-commutativity when evaluating pairs of area operators on certain configurations of spin-network states in the continuum [21]. The non-vanishing commutators between volumes and areas were found independently by the

author during her lattice investigations. The origins of these different instances of non-commutativity are of course related.

As for the commutators involving the length functions, results of the lattice computations cannot be compared immediately with the only continuum result available [17], since the finite operators are not the same, as explained earlier. However, we would find it rather surprising if the continuum length operator defined there did commute with the volume, say. It is probably relatively straightforward to find instances of explicit spin-network configurations (with non-vanishing volume) where they do not. After all, for an n -valent intersection, there can only exist a small number (of order n) of independent, mutually commuting operators one may construct from the right- and left-invariant vector fields on the corresponding n -fold copy of $SU(2)$.

One other comment concerns different definitions of the volume operator in the continuum, which again have to do with the modulus sign under the square root. In the version of the volume operator used in [8], in evaluating the action on a spin-network state, the modulus is taken for each individual term contributing to the sum at a given vertex, then the sum is taken, then the square root. In contrast, the volume operator envisaged in [19, 20, 9] has the modulus signs outside the entire sum, not each individual term. (This and other differences between the two types of volume operators are also discussed in [20].) The presence and different position of the modulus signs of course give rise to different finite operators, and in turn affect the quantum commutators with other geometric operators. Since *no* modulus signs are present classically, it is *a priori* not even clear that—*if* one were also to introduce something like a small- a expansion in the continuum—the correct (vanishing) commutators could be reproduced. In this regard the lattice formulation has a definite advantage, since one is not forced to introduce the modulus anywhere, as we explained in the previous section.

The presence of anomalous commutators of geometric operators is a potentially worrying result from the point of view of the continuum theory, where there is no analogue of the lattice expansion parameter a and it is usually claimed that no further continuum limits have to be taken [11, 9] (for a criticism of this approach, see the discussion in [22]). One could try to argue that the non-commutativity of classically commuting quantities is a quantum effect and therefore not unexpected. Considering the lattice analogy, this is not really convincing. Rather, in both cases non-commutativity arises at an intermediate step of the regularization, at which the differential operators are not truly local, but smeared out, and therefore depend implicitly on the holonomies (involved in the parallel transport of the momenta) [21]. In the lattice theory, this property does not survive in the continuum limit (in which commutativity is restored), whereas in the continuum the operators remain non-commutative as the operator regulator is removed. Their algebra is therefore still that of non-local quantities. This difference between the lattice and the continuum formulation arises since in the latter one regulates the operators separately, i.e. on a given space of quantum states, whereas in the lattice theory the regularization of operators and states is controlled by a single small parameter, the lattice spacing a .

Note that a difference between the lattice and continuum approach is that at a kinematical level (i.e. before spatial diffeomorphisms are taken into account—this is the usual setting for discussing geometric operators), in the lattice regularization there is no problem in defining operators measuring local information about the metric. In the continuum, in contrast, because of the way the quantum theory has been set up, only operators corresponding to *finite* areas and volumes can be regularized and therefore defined properly.

This does not invalidate our discussion about the commutativity or otherwise of geometric operators, since above we have only exploited the fact that certain calculations in both approaches are identical.

One may also define length, area and volume operators for entire lattice regions, in analogy with the continuum construction. For the case of the volume function this is straightforward [6]: simply sum over all vertices n contained in the lattice region \mathcal{R} ,

$$\mathcal{V}(\mathcal{R}_{\text{latt}}) = \sum_{n \in \mathcal{R}_{\text{latt}}} \sqrt{\frac{1}{3!} D(n)}. \tag{4.5}$$

When expanding in powers of a , this becomes

$$\sum_{\mathcal{R}_{\text{latt}}} a^3 \left(\sqrt{\frac{1}{3!} \epsilon_{abc} \epsilon^{ijk} E_i^a E_j^b E_k^c} + \mathcal{O}(a) \right) \xrightarrow{a \rightarrow 0} \int_{\mathcal{R}} d^3x \sqrt{\frac{1}{3!} \epsilon_{abc} \epsilon^{ijk} E_i^a E_j^b E_k^c}. \tag{4.6}$$

For the area function, things are simplest when the area $\mathcal{S}_{\text{latt}}$ coincides with some planar surface of lattice plaquettes. Then one has (for an area dual to the 3-direction)

$$\mathcal{A}_i(\mathcal{S}_{\text{latt}}) = \sum_{n \in \mathcal{S}_{\text{latt}}} \mathcal{A}_i(n, \hat{3}) \xrightarrow{a \rightarrow 0} \sum_{\mathcal{S}_{\text{latt}}} a^2 \left(\sqrt{E_i^3 E^{3i}} + \mathcal{O}(a) \right). \tag{4.7}$$

Likewise, to obtain the length of a finite lattice curve $\mathcal{C}_{\text{latt}}$, one possibility is to add up the contributions from the initial points of the individual links of $\mathcal{C}_{\text{latt}}$. For the simple case of a curve along the 1-direction, this results in

$$\mathcal{L}_i(\mathcal{C}_{\text{latt}}) = \sum_{(n, \hat{1})} \mathcal{L}_i(n, \hat{2}, \hat{3}) \xrightarrow{a \rightarrow 0} \sum_{(n, \hat{1})} a \left(\sqrt{\frac{1}{\det E} (E_j^2 E^{2j} E_k^3 E^{3k} - (E^{2j} E_j^3)^2)} + \mathcal{O}(a) \right). \tag{4.8}$$

One observes that in all three cases the lowest power of a in the expansion is just the correct one to saturate the relevant continuum integral. Obviously these definitions are not complete without specifying how points along boundaries of spatial regions are to be counted, but for our present discussion we are not interested in spelling out the details of how this may be done.

Now let us demonstrate that on the lattice the commutator of a *finite* area with a *finite* volume also vanishes in the continuum limit. Consider, for example, the Poisson brackets between (4.5) and (4.7). Abbreviating $C(n, \hat{3}) := p_i(n, \hat{3}) p^i(n, \hat{3})$, one has

$$\left\{ \sum_{n \in \mathcal{S}_{\text{latt}}} \sqrt{C(n, \hat{3})}, \sum_{n \in \mathcal{R}_{\text{latt}}} \sqrt{\frac{1}{3!} D(n)} \right\} = \sum_{n \in \mathcal{S}_{\text{latt}} \cap \mathcal{R}_{\text{latt}}} C(n, \hat{3})^{-1/2} D(n)^{-1/2} \{C(n, \hat{3}), D(n)\} \xrightarrow{a \rightarrow 0} \sum_{n \in \mathcal{S}_{\text{latt}} \cap \mathcal{R}_{\text{latt}}} a^3 ((E^5)^{-1/2} E^4 + \mathcal{O}(a)). \tag{4.9}$$

Since the intersection $\mathcal{S}_{\text{latt}} \cap \mathcal{R}_{\text{latt}}$ describes (if it is non-vanishing) a two-dimensional surface in the continuum, we can substitute $\sum_n a^2 \rightarrow \int d^2x$, and the right-hand side of (4.9) becomes a two-dimensional integral over a function that vanishes as $a \rightarrow 0$, i.e. goes itself to zero in this limit. Hence the fact that we computed commutators of the local instead of the integrated geometric functions did not alter our main argument.

5. Summary and discussion

The conclusion of our computations in the previous sections is that geometric operators, as they are usually defined in the continuum loop representation for quantum gravity, do not in general commute, although they are taken as the quantum analogues of quantities that commute classically, being functions of half of the canonical variables only. We have shown this explicitly for the case of the volume and area operator because in this case the lattice calculations could be directly related to constructions already available in the continuum theory. However, our analysis suggests that this is a general feature of operators constructed along similar lines in the continuum representation and may therefore also spoil other commutation relations, such as those involving suitably regularized length or Hamiltonian operators.

We have argued that the non-preservation of classical commutators is not primarily a quantum effect, since the same algebraic relations can already be obtained in a classical, discretized version of the theory, where all basic variables are constrained to live on one-dimensional lattice links. In this case, non-commutativity comes about because of an implicit holonomy dependence of the non-local momentum variables (this can be seen explicitly if one writes them as functions of the basic variables of the Lagrangian formulation [18]). The usual canonical commutation relations are only obtained in the limit as the lattice spacing a is taken to zero. As we have shown, in this limit all the geometric functions (volumes, areas, lengths) become Poisson-commuting, as one would expect. At least for the case of volume and area, the same is also true for the corresponding quantum operators.

Presumably, non-commutativity in the continuum theory can be understood in a similar way, i.e. as arising through a particular way of regularizing. However, unlike in the lattice theory, the non-commutativity survives the removal of the regulator. This is at least unpleasant (as there *is* another regularization that does not have this property), but may have further physical implications, unless one can argue that the right-hand sides of commutators are small in some suitable sense. This is an issue that clearly deserves further investigation.

What could one do in the continuum loop representation to achieve commutativity, if one wanted to stick with this particular set of geometric operators? It does not seem that quotienting out by spatial diffeomorphisms would remedy the situation, since the action of the geometric operators is encoded in topological reroutings at graph intersections, which behave covariantly under spatial diffeomorphisms. One may try to select a subspace of the Hilbert space spanned by the spin-network states on which the geometric operators commute, in which case one would have to show that it is non-trivial (for example, that it contained states with non-vanishing volume).

Alternatively, one may interpret the result as an indication that it is after all necessary to take some sort of continuum limit, even if the ingredients of the quantum theory have been defined using a continuous ‘background’ differential manifold. In keeping with the general spirit of this ansatz, one would perhaps need some small ‘topological’ expansion parameter, presumably coming from some topological characteristics of the quantum states, for example, counting intersections of a certain type. (Further support for the need of a continuum limit comes from the following consideration. The construction of geometric (and other) operators in the continuum loop quantization is not universal. Since a small-distance expansion is always invoked as a motivation for the functional form of a regularized operator, in principle the same type of ambiguity as in the lattice theory appears. In the latter case, since the ambiguity is the result of a choice of the discretization (i.e. regularization), a necessary condition one has to impose on the continuum limit is that different discretizations

must lead to equivalent quantum theories. A similar criterion seems to be missing so far in the continuum construction.)

The lattice-regularized theory does not share this problem, although one may of course argue that it has not yet been shown that it leads to a sensible continuum quantum theory. The lattice approach has difficulties of its own, as there is no obvious way of implementing spatial diffeomorphism symmetry at the discretized level. This can, for example, be done ‘to lowest order in a ’, in which case one must show that the algebra of the diffeomorphism constraint does not acquire anomalous terms in the quantization. We are now in a state to tackle this rather involved computation and hope to be able to report soon on its outcome [23].

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