



EXISTENCE AND NON-EXISTENCE RESULTS FOR GLOBAL CONSTANT MEAN CURVATURE FOLIATIONS

ALAN D. RENDALL

Max-Planck-Institut für Gravitationsphysik, Schlaatzweg 1, 14473 Potsdam, Germany

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1. INTRODUCTION

If (M, g) is a Lorentz manifold and S a spacelike hypersurface, let h and k denote the induced metric and second fundamental form of S respectively. The mean curvature of S is the trace $\text{tr}_g k$. An interesting class of spacelike hypersurfaces are those whose mean curvature is constant (CMC hypersurfaces). The Lorentz manifolds of primary interest in the following are those which possess a compact Cauchy hypersurface and whose Ricci tensor satisfies $r(V, V) \geq 0$ for any timelike vector V . They will be referred to as cosmological spacetimes. The curvature condition is known as the strong energy condition and implies certain uniqueness statements for CMC hypersurfaces. Under very general conditions, if a region of a cosmological spacetime is foliated by compact CMC hypersurfaces, then each of these has a different value of the mean curvature and a time coordinate t can be defined by the condition that its value at a point be equal to the mean curvature of the leaf of the foliation passing through that point. A time coordinate of this type will be referred to in the following as a CMC time coordinate.

The questions, whether a cosmological spacetime contains a compact CMC hypersurface and which values the mean curvature can take on a hypersurface of this kind may seem purely geometrical in nature. However it turns out that the answers to these questions depend crucially on factors which have no obvious geometrical interpretation, but which have a physical meaning, when the Lorentz manifold is considered as a model for spacetime.

The Einstein equations for a Lorentz metric g take the form $G = 8\pi T$, where G is the Einstein tensor of the metric g and T is the energy-momentum tensor. To get a determined system of evolution equations for the geometry and the matter, it is necessary to say more about the nature of the matter model used. This means specifying some matter fields, denoted collectively by ϕ , a definition of T in terms of g and ϕ and the differential equations which describe the dynamics of the matter. Putting these things together with the Einstein equations gives a system of evolution equations, the Einstein-matter equations. It will be seen that the existence of global foliations by CMC hypersurfaces in a cosmological spacetime which is a solution of the Einstein-matter equations depends essentially on the matter model chosen.

Let s denote the scalar curvature of the metric g . The spacetime is said to satisfy the weak energy condition if $r(V, V) \geq (1/2)sg(V, V)$ for all timelike vectors V . (Note that, despite the terminology, the strong energy condition does not imply the weak one.) There is a topological obstruction to the existence of a compact spacelike hypersurface with vanishing mean curvature (maximal hypersurface) in a spacetime which satisfies the weak energy condition. In a cosmological spacetime this implies that if the Cauchy hypersurface is a manifold which admits no Riemannian metric with non-negative scalar curvature then the spacetime contains no compact maximal hypersurfaces. Moreover, there are strong restrictions in the case of a manifold which admits no Riemannian metric of positive scalar

curvature. The main questions are whether a cosmological spacetime satisfying the weak energy condition and containing at least one compact CMC hypersurface can be covered by a foliation of compact CMC hypersurfaces and whether the mean curvature of these hypersurfaces takes on all values not forbidden by the topological obstruction already mentioned. In other words, does it take all values in the interval $(-\infty, 0)$ or $(0, \infty)$ in the case where there is an obstruction and all real values in the case there is none? This statement concerning the values attained by the mean curvature is equivalent to a statement of whether a solution of the Einstein-matter equations exists globally in a CMC time coordinate. The theorem proved in Section 2 shows by example that the answer is negative if no restriction is put on the matter model used. In Section 3 results are reviewed which show that under certain symmetry assumptions there are matter models for which the answer is positive. Possible extensions of these results are also discussed. For general information on the points just mentioned, the reader is referred to [1], which is complementary to the treatment here.

These results are related to the concept of crushing singularities. A cosmological spacetime is said to have an initial crushing singularity if there is a compact spacelike hypersurface S_{t_0} and a foliation of the past of S_{t_0} by compact spacelike hypersurfaces S_t , such that as t tends to its limiting value towards the past, the mean curvature of S_t tends uniformly to $-\infty$. (There is a similar definition with 'initial' replaced by 'final', 'past' replaced by 'future' and $-\infty$ replaced by ∞ .) If the S_t form a CMC foliation and the parameter t labelling the leaves of the foliation is a CMC time coordinate, then a sufficient condition in order to have an initial crushing singularity is that t should take all values in the interval $(-\infty, t_0)$. The positive results of Section 3 prove that this is true under certain hypotheses on the matter model and the symmetry of the spacetimes considered, and so prove the existence of crushing singularities under certain circumstances. On the other hand, the results of Section 2 provide examples of spacetimes where an initial (or final) crushing singularity is not present. Informally, this can be expressed by saying that the initial (or final) singularity in these spacetimes is not crushing. It will now be indicated briefly how the statement about the absence of crushing singularities follows from the theorem proved in Section 2. If there is an initial crushing singularity, the hypersurfaces S_t provide barriers, as they are used in the well known existence theory for CMC hypersurfaces. Thus it can be assumed without loss of generality that the S_t are in fact CMC hypersurfaces and that t is a CMC time coordinate. Combining this with the uniqueness theorems for CMC hypersurfaces would show that the local CMC foliation which exists close to the initial hypersurface could be extended to arbitrarily negative values of t , contradicting the conclusions of the theorem.

The techniques which are used to obtain the positive and negative results are closely related. They will be explained in the case of spacetimes with $U(1) \times U(1)$ symmetry where the Cauchy hypersurface has the topology of a three-dimensional torus. Assume that a solution of the Einstein-matter equations satisfying the strong energy condition with a CMC Cauchy hypersurface of topology T^3 is invariant under the action of the group $U(1) \times U(1)$ consisting in rotating two of the three S^1 factors. Then (see [2]) in a neighbourhood of the initial hypersurface the metric can be written in the form:

$$-\alpha^2 dt^2 + A^2[(dx + \beta^1 dt)^2 + a^2 \tilde{g}_{AB}(dy^A + \beta^A dt)(dy^B + \beta^B dt)] \quad (1.1)$$

where t is a CMC time coordinate. Here the coordinates are t, x, y^2, y^3 . Upper case Roman indices take the values 2, 3 while lower case ones take the values 1, 2, 3, the value 1 corresponding to the coordinate x . The functions α, β^a, A and \tilde{g}_{AB} depend on t and x and \tilde{g}_{AB} has unit determinant. They are periodic in x . The quantity a depends only on t . Some of the field equations are:

$$\partial_x^2(A^{1/2}) = -\frac{1}{8}A^{5/2}[\frac{3}{2}(K_1 - \frac{1}{3}t)^2 - \frac{2}{3}t^2 + 2\eta_A\eta^A + \tilde{\kappa}^{AB}\tilde{\kappa}_{AB} + \tilde{\lambda}^{AB}\tilde{\lambda}_{AB} + 16\pi\rho] \quad (1.2)$$

$$\begin{aligned} \partial_x^2\alpha + A^{-1}\partial_x A\partial_x\alpha &= \alpha A^2[\frac{3}{2}(K_1 - \frac{1}{3}t)^2 + \frac{1}{3}t^2 \\ &+ 2\eta_A\eta^A + \tilde{\kappa}_{AB}\tilde{\kappa}^{AB} + 4\pi(\rho + \text{tr}S)] - A^2 \end{aligned} \quad (1.3)$$

$$\partial_x K_1 + 3A^{-1}\partial_x A K_1 - A^{-1}\partial_x A t - \tilde{\kappa}^{AB}\tilde{\lambda}_{AB} = 8\pi J A \quad (1.4)$$

$$\partial_t a = a[-\partial_x \beta^1 + \frac{1}{2}\alpha(3K_1 - t)] \tag{1.5}$$

$$\partial_t A = -\alpha K_1 A + \partial_x(\beta^1 A) \tag{1.6}$$

The quantity K_1 appearing in these equations is an eigenvalue of the second fundamental form. Alternatively it may be thought of as an auxiliary quantity defined in terms of the basic quantities contained in (1.1) by (1.5). The quantity η_A is given by

$$\eta_A = (1/2)\alpha^{-1} A a^2 \tilde{g}_{AB} \partial_x \beta^B \tag{1.7}$$

while $\tilde{\kappa}_{AB}$ and $\tilde{\lambda}_{AB}$ are the tracefree parts of the second fundamental forms of the group orbits in spacetime corresponding to the normal vector to the hypersurface $t=\text{const.}$ and the normal to the orbit in the hypersurface $t=\text{const.}$, respectively. The components \tilde{g}_{AB} can be parametrized as follows:

$$\tilde{g}_{22} = e^W \cosh V, \quad \tilde{g}_{33} = e^{-W} \cosh V, \quad \tilde{g}_{23} = \sinh V \tag{1.8}$$

In terms of W and V the squares of $\tilde{\kappa}_{AB}$ and $\tilde{\lambda}_{AB}$ have the explicit forms:

$$\tilde{\lambda}_{AB} \tilde{\lambda}^{AB} = \frac{1}{2} A^{-2} (\cosh^2 V W_x^2 + V_x^2) \tag{1.9}$$

$$\tilde{\kappa}_{AB} \tilde{\kappa}^{AB} = \frac{1}{2} \alpha^{-2} [\cosh^2 V (W_t - \beta^1 W_x)^2 + (V_t - \beta^1 V_x)^2] \tag{1.10}$$

The quantities ρ , J and $\text{tr}S$ denote the energy density, the matter current and three times the mean pressure, respectively.

In the proofs of the existence and non-existence theorems it is important to have estimates for a solution of these equations on a finite time interval (t_1, t_2) with $t_2 < 0$ in terms of the data it induces at some intermediate time t_0 . Consider a point where α attains its maximum on a hypersurface of constant time. The strong energy condition implies that $\rho + \text{tr}S \geq 0$ and so it follows from equation (1.3) that $\alpha \leq 3/t^2$. In fact this estimate is true, and can be proved in the same way, without any symmetry assumption on the spacetime. The next step is to use a generalization of an argument of Malec and Ó Murchadha [3] to show that if the dominant energy condition holds then $|K_1| \leq 5|t_1|$, $|A^{-2} \partial_x A| \leq 2|t_1|$. (For details of this argument see [2].) With these estimates (which depend very much on the symmetry assumption) in hand, equations (1.5) and (1.6) can be integrated in time to give the following bounds for a and A .

$$\begin{aligned} a(t_0) \exp(-C|t - t_0|) &\leq a(t) \leq a(t_0) \exp(C|t - t_0|) \\ \sup\{A(t, x), A^{-1}(t, x)\} &\leq \sup\{\|A(t_0)\|_\infty, \|A^{-1}(t_0)\|_\infty\} \exp(\exp(C|t - t_0|)) \end{aligned} \tag{1.11}$$

Along the way it also comes out that $|\partial_x \beta^1|$ satisfies the same kind of upper bound as a . For details the reader is once again referred to [2]. These bounds are probably far from optimal but they are sufficient for the present purposes. The general point is that many aspects of the geometry can be bounded in terms of t_1, t_2 and the maximum and minimum values of a and A on the initial hypersurface $t = t_0$. Using this information and integrating equation (1.2) in space at each fixed time gives a bound for the integral $\int_0^{2\pi} Q(t, x) dx$ where

$$Q = 16\pi\rho + 2\eta^A \eta_A + \tilde{\kappa}^{AB} \tilde{\kappa}_{AB} + \tilde{\lambda}^{AB} \tilde{\lambda}_{AB} \tag{1.12}$$

which is uniform in time on the given interval.

2. EXAMPLES OF SINGULARITY FORMATION

A matter model which is well known for its tendency to form singularities is dust. In this section a type of example will be presented which shows that global existence in a CMC time coordinate

does not hold for the Einstein-dust system. The main interest of this is the contrast it provides with the known positive results for some other matter models which will be reviewed in the next section. To put it another way, this example says something about the sharpness of the results previously obtained. The intuitive idea behind the example is very simple. If two dust particles start out close together and with velocities which are moderate in magnitude and opposite in direction then they should collide after finite time producing a shell-crossing singularity. To turn this intuition into a proof we can use the bounds on various geometric quantities presented in the last section. However these estimates alone do not suffice. The difficulty is connected with the phrase 'in finite time'. The problem is that a finite amount of proper time along the worldline of a particle might correspond to an infinite amount of coordinate time. This would correspond to the phenomenon known as 'collapse of the lapse', where the lapse function α becomes very small. To show that this does not happen it is necessary to have a positive lower bound on the lapse function on a finite interval of CMC time. This is the subject of the following lemma.

LEMMA (supporting the lapse) Consider a solution of the Einstein-matter equations with $U(1) \times U(1)$ symmetry on a finite interval (t_1, t_2) of CMC time with $t_2 < 0$ and let t_0 belong to this time interval. Suppose that the strong and dominant energy conditions hold. Then there is constant C , depending only on t_1, t_2 and the maximum values of a, a^{-1}, A and A^{-1} on the hypersurface $t = t_0$ such that $\alpha^{-1} \leq C$.

Proof. Equation (1.3) can be written in the form

$$\partial_x(A\partial_x\alpha) = \alpha A^3 P - A^3 \quad (2.1)$$

where P is non-negative. Under the given hypotheses $G = \int_0^{2\pi} P(x)dx$ is bounded. Consider now a fixed time t and an interval $I = [x_1, x_2]$ of values of x . Let $L = |x_2 - x_1|$ and denote by F and F' the maximum values of $A\alpha$ and $\partial_x(A\alpha)$ respectively on the interval I . Now for any $x \in I$

$$(A\alpha)(x) = (A\alpha)(x_1) + \int_{x_1}^x \partial_x(A\alpha)(y)dy$$

and the integral can be bounded by LF' . Hence:

$$F \leq (A\alpha)(x_1) + LF' \quad (2.2)$$

On the other hand

$$\begin{aligned} \partial_x(A\alpha)(x) &= (A\partial_x\alpha)(x) + (A^{-1}\partial_x A(A\alpha))(x) \\ &= (A\partial_x\alpha)(x_1) + \int_{x_1}^x \partial_x(A\partial_x\alpha)(y)dy + (A^{-1}\partial_x A(A\alpha))(x) \\ &= \partial_x(A\alpha)(x_1) - (A^{-1}\partial_x A(A\alpha))(x_1) + \int_{x_1}^x (\alpha A^3 P - A^3)(y)dy + (A^{-1}\partial_x A(A\alpha))(x) \end{aligned}$$

In the last line equation (2.1) has been used. Since $A^{-1}\partial_x A$ has already been bounded, the second and fourth terms in the final expression are bounded by CF for some positive constant C . Consider now the third term. The second term in the integrand makes a negative contribution, while the first can be bounded by CFP . Hence:

$$F' \leq \partial_x(A\alpha)(x_1) + C(1+G)F \quad (2.3)$$

It follows by substituting (2.3) into (2.2) that

$$F \leq (A\alpha)(x_1) + L[(\partial_x(A\alpha))(x_1) + C(1+G)F] \quad (2.4)$$

If $L \leq \frac{1}{2}[C(1 + G)]^{-1}$ then this implies that

$$F \leq 2[(A\alpha)(x_1) + L\partial_x(A\alpha)(x_1)] \tag{2.5}$$

Similarly, under the same restriction on L :

$$F' \leq 2[\partial_x(A\alpha)(x_1) + C(1 + G)(A\alpha)(x_1)] \tag{2.6}$$

Thus we have the inequalities

$$\begin{aligned} (A\alpha)(x + L) &\leq 2[(A\alpha)(x) + L\partial_x(A\alpha)(x)] \\ \partial_x(A\alpha)(x + L) &\leq 2[\partial_x(A\alpha)(x) + C(1 + G)(A\alpha)(x)] \end{aligned} \tag{2.7}$$

Now two cases will be considered. The first is where $C(1 + G) \leq (4\pi)^{-1}$. Then L can be chosen to be 2π and it follows that if $A\alpha$ takes its minimum at a given time at x_- then

$$(A\alpha)(x) \leq 2(A\alpha)(x_-) \tag{2.8}$$

If $C(1 + G) > (4\pi)^{-1}$ then it is possible to divide the circle into a number of equal intervals, starting at x_- whose length L satisfies the desired inequality and whose number does not exceed $C(1 + G)$. Let $z(x) = \max\{(A\alpha)(x), \partial_x(A\alpha)(x)\}$. Then

$$z(x + L) \leq 2C(1 + G)z(x) \tag{2.9}$$

Applying this on k successive intervals gives

$$z(x + kL) \leq 2^k C^k (1 + G)^k z(x) \tag{2.10}$$

On the other hand $A\alpha(x)$ can be bounded by a fixed constant times $z(x + kL)$ for some $k \leq C(1 + G)$. There results an estimate of the form

$$(A\alpha)(x) \leq [C(1 + G)]^{C(1+G)} (A\alpha)(x_-) \tag{2.11}$$

As discussed in the previous section, the assumptions of the lemma imply bounds on A and A^{-1} . Hence (2.8) and (2.11) imply an estimate of the form $\|\alpha\|_\infty \leq C\alpha_-$, where α_- is the minimum value of α at the given time. On the other hand, integrating equation (2.1) with respect to x shows that:

$$\int_0^{2\pi} A^3(x) dx = \int_0^{2\pi} (\alpha A^3 P)(x) dx \leq C\|\alpha\|_\infty$$

Since $\int_0^{2\pi} A^3(x) dx$ is bounded from below, putting these estimates together gives the desired lower bound for α_- .

The estimates (2.8) and (2.11) resemble the specialization of Harnack's inequality to one space dimension with the difference that the constant in the inequality only depends on the L^1 norm of P rather than its L^∞ norm.

Next the above lemma will be applied to the Einstein-dust system. The matter fields are a non-negative function μ , the proper energy density of the dust, and a unit vector u^α , the four-velocity of the dust particles. The energy-momentum tensor is given by $T_{\alpha\beta} = \mu u_\alpha u_\beta$ and the equations describing the dynamics of the matter fields are simply $\nabla_\alpha T^{\alpha\beta} = 0$, which is of course a necessary

compatibility condition for the Einstein equations. The dominant, strong and weak energy conditions are satisfied by this matter model. The theorem to be proved is the following:

THEOREM Let t_0 be a negative real number and $\epsilon > 0$. Then there exist initial data with constant mean curvature t_0 for the Einstein-dust system such that in any Cauchy development of these data the CMC foliation which exists in a neighbourhood of the initial hypersurface cannot be extended to values of the mean curvature greater than $t_0 + \epsilon$. Similarly there exist data for which the foliation cannot be extended to values of the mean curvature less than $t_0 - \epsilon$.

Proof. The initial data to be constructed will have $U(1) \times U(1)$ symmetry and so it is possible to use the form (1.1) of the metric. They will also be such that the initial velocity of the dust particles is in the x -direction. Then the velocity can be parametrized by the inner product v of the four-velocity u^α with the unit vector $A^{-1}\partial/\partial x$. The integral curves of u^α are geodesics and hence there are conservation laws corresponding to the Killing vectors $\partial/\partial y^A$. This means that if the velocity is initially in the x -direction, it remains so. Thus these dust solutions can be described completely by the two functions μ and v .

Initial data for dust spacetimes can be constructed using the conformal method. (A general discussion of this method can be found in [4].) However the situation here is sufficiently simple that we will not need all of the general theory. The data constructed will be such that $\tilde{\kappa}_{AB} = 0$, $\tilde{\lambda}_{AB} = 0$ and $\eta_A = 0$. The matter quantities occurring in the constraints are related to the matter fields μ and v in the special case of the type of data under consideration by:

$$\begin{aligned}\rho &= \mu(1 + v^2) \\ J &= \mu(1 + v^2)^{1/2}v\end{aligned}\tag{2.12}$$

To construct initial data, choose first the constant value $t = t_0$ for the mean curvature and a non-negative function $\tilde{\mu}$, a function v and a scalar function $\tilde{K}_1(x)$ on the circle. The solution of the constraints is sought in the form:

$$\begin{aligned}\tilde{K}_1 &= \frac{1}{3}t + A^{-3}(\tilde{K}_1 - \frac{1}{3}t) \\ \mu &= A^{-4}\tilde{\mu}\end{aligned}\tag{2.13}$$

and the Lichnerowicz equation, which is just (1.2) rewritten in terms of the rescaled quantities, reads:

$$\partial_x^2(A^{1/2}) = -\frac{1}{8}A^{5/2}[\frac{3}{2}A^{-3}(\tilde{K}_1 - \frac{1}{3}t)^2 - \frac{2}{3}t^2 + 16\pi A^{-4}\tilde{\mu}(1 + v^2)]$$

In terms of the rescaled quantities the equation (1.4) takes the simple form $\partial_x(\tilde{K}_1 - \frac{1}{3}t) = 8\pi\tilde{J}$, where $\tilde{J} = \tilde{\mu}(1 + v^2)^{1/2}v$. This equation can be solved if and only if \tilde{J} has integral zero. One way of ensuring that this condition is satisfied is to impose the symmetry conditions that $\tilde{\mu}(x) = \tilde{\mu}(\pi - x)$ and $v(x) = -v(\pi - x)$. When this equation is solvable the L^∞ norm of $\tilde{K}_1 - \frac{1}{3}t$ can be estimated in terms of the L^1 norm of \tilde{J} . The next step is to solve the Lichnerowicz equation and for that an idea will be borrowed from the general method, namely that of using sub- and supersolutions. In order for this to run as smoothly as possible, assume that $\tilde{\mu}$ is bounded below by a positive constant B . Then it is possible to find constant sub- and supersolutions, namely

$$\begin{aligned}A_- &= t_0^{-1}(24\pi B)^{1/2} \\ A_+ &= \max\{t_0^{-1}(48\pi\|\tilde{\mu}\|_\infty(1 + \|v\|_\infty^2))^{1/2}, t_0^{-2/3}(\frac{9}{2}\|\tilde{K}_1 - \frac{1}{3}t\|_\infty^2)^{2/3}\}\end{aligned}\tag{2.14}$$

These ensure the existence of a solution of the Lichnerowicz equation and give pointwise estimates for the solution. This allows A and A^{-1} to be bounded pointwise in terms of the L^∞ norms of $\tilde{K}_1 - \frac{1}{3}t$, $\tilde{\mu}$ and v and the constants t_0 and B .

The above provides a possibility of constructing a variety of initial data in such a way that the quantities entering the hypotheses of the lemma above can be easily controlled in terms of the free data. To obtain the desired example a specific subclass of this data will be considered. Let $x_1 < x_2$ and choose initial data for v such that $v_1 = v(x_1) = 1$ and $v_2 = v(x_2) = -1$. Consider two dust particles which start at time t_0 at the points x_1 and x_2 and with the velocities v_1 and v_2 respectively. Let the positions and velocities of these particles at time t be denoted by $x_1(t), x_2(t), v_1(t)$ and $v_2(t)$. These quantities satisfy the equations:

$$dx/dt = \alpha A^{-1} v / \sqrt{1 + v^2} - \beta^1 \tag{2.15}$$

$$dv/dt = -A^{-1} \alpha' \sqrt{1 + v^2} + \alpha K v \tag{2.16}$$

It is elementary to see that on a given finite interval of CMC time bounds for v_1 and v_2 can be obtained using Gronwall's inequality. Moreover these bounds do not depend on the distance $|x_2(t_0) - x_1(t_0)|$. The idea now is to assume that for a family of initial data of this type with $|x_2(t_0) - x_1(t_0)|$ tending to zero the corresponding solutions exist at least up to and including the time $t_0 + \epsilon$ and to show that this assumption leads to a contradiction. Note that if $t_0 + \epsilon \geq 0$ the statement of the theorem is an immediate consequence of the non-existence of maximal hypersurfaces. Thus it is assumed in the following that $t_0 < -\epsilon$.

Let C be a constant greater than $5 \sup_{t_0 < t < t_0 + \epsilon} \{(\|A^{-1} \alpha'\|_\infty + \|\alpha K\|_\infty)\}$. By what has been said above we know that the data can be chosen so that a single constant C works for all data in the family. It will now be shown that for the solution evolving from any one of these data $v_1(t) \geq \frac{1}{2}$ for all times $t \leq t_0 + \epsilon$ such that $t_0 \leq t \leq t_0 + C^{-1}$. Let t_* be the largest time in the interval $[t_0, t_0 + \epsilon]$ such that $v_1(t) \geq \frac{1}{2}$ for all t in the interval $[t_0, t_*]$. If $t_* < t_0 + C^{-1}$ and $t < t_0 + \epsilon$, let t' be the last time before t_* that $v_1(t)$ was equal to unity. Now $v(t_*) = \frac{1}{2}$. On the other hand it follows by integrating (2.16) from t' to t_* that $v(t_*) \geq \frac{5}{3}$. This contradiction shows that in fact either $t_* \geq t_0 + C^{-1}$ or $t_* = t_0 + \epsilon$. This gives the desired conclusion. In a similar way it can be shown that $v_2(t) \leq -\frac{1}{2}$ for all times $t \leq t_0 + \epsilon$ such that $t \leq t_0 + C^{-1}$. For convenience of notation, let $t_3 = \min\{t_0 + \epsilon, t_0 + C^{-1}\}$. From (2.15):

$$d/dt(x_1 - x_2) = \alpha A^{-1} v_1 / \sqrt{1 + v_1^2} - \alpha A^{-1} v_2 / \sqrt{1 + v_2^2} - \beta^1(x_1) + \beta^1(x_2) \tag{2.17}$$

On the interval $[0, t_3]$ we have a lower bound for v_1 , an upper bound for v_2 and a crude upper bound for $\sqrt{1 + v_1^2}$ and $\sqrt{1 + v_2^2}$. Moreover, we have a lower bound for αA^{-1} . (It is at this point that the lemma on supporting the lapse is used.) Putting all this together gives a negative upper bound for the sum of the first and second terms on the right hand side of (2.17). On the other hand

$$|\beta^1(x_1) - \beta^1(x_2)| \leq \|\partial_x \beta^1\|_\infty |x_1 - x_2| \tag{2.18}$$

Hence by choosing $|x_1(t_0) - x_2(t_0)|$ small it can be ensured that the sum of third and fourth terms on the right hand side of (2.17) is negligible in comparison with the sum of the first and second terms. Hence it can be arranged that $x_1(t) - x_2(t)$ tends to zero after a time smaller than t_3 . However this contradicts the existence of a regular solution of the Einstein-dust equations up to time $t_0 + \epsilon$, since in a regular solution the world lines of dust particles can never cross. Thus the first statement of the theorem has been proved. The proof of the second statement is strictly analogous.

3. EXAMPLES OF GLOBAL REGULARITY

There are matter models for which the situation is quite different from that presented in the last section, in that for spacetimes with $U(1) \times U(1)$ symmetry possessing a CMC Cauchy surface global

existence in CMC time is obtained. In this context 'global existence' means existence on the longest time interval consistent with the topological obstruction discussed in Section 1. A global existence theorem in this sense was proved in [2] for collisionless matter described by the Vlasov equation and for the massless scalar field (or more generally wave maps). The nature of these two matter models will now be recalled. In the case of collisionless matter, the matter field is a non-negative real-valued function f on the space of future-pointing unit timelike vectors in spacetime (the mass shell) and the equation which describes the dynamics of the matter is the Vlasov equation. This simply says that the function f is constant along the curves which are the natural lifts of timelike geodesics to the mass shell. The energy-momentum tensor at a given spacetime point is obtained by integrating the product of f with a suitable weight over the part of the mass shell over that point. Details can be found in [5]. In the case of the massless scalar field, the matter field is a real-valued function ϕ on spacetime which is supposed to satisfy the wave equation. The energy-momentum tensor is of the form $T = d\phi \otimes d\phi - \frac{1}{2}|d\phi|^2 g$. A wave map, which is a Lorentzian analogue of the harmonic maps familiar in Riemannian geometry, is a generalization of this, where the field, instead of taking values in the real numbers, takes values in an arbitrary complete Riemannian manifold, known as the target manifold. It is worth to note that the vacuum case, i. e. the case $T = 0$, is contained in these results as the special case $f = 0$ or $\phi = 0$. The global existence problem is hard even in the case of $U(1) \times U(1)$ symmetric vacuum spacetimes. The physics issue is whether gravitational waves propagate smoothly for arbitrarily long times or whether they could develop shocks. It is reasonable to expect that when a global existence theorem in CMC time holds, it should not be possible to extend the spacetime beyond the region covered by the CMC foliation, while maintaining the property that the initial hypersurface be a Cauchy hypersurface for the extended spacetime. Unfortunately, this statement has not been proved up to now.

The example of collisionless matter is particularly interesting due to the fact that dust solutions can be considered as distributional solutions of the Einstein-Vlasov system [6]. They have Dirac δ -function dependence on the velocity variables. The results already quoted show that approximating the δ -function by smooth functions in the initial data leads to a dramatic change in the long-time behaviour.

The basis of the global existence theorem is formed by the estimates mentioned in Section 1. One then proceeds by bounding higher and higher derivatives of the metric and the matter fields on the given finite interval (t_1, t_2) . When all derivatives have been bounded it follows that the solution can be extended to the closed interval. A local existence theorem (which is a consequence of standard results on the Cauchy problem and the existence of CMC foliations) then allows it to be extended to a longer time interval. Finally, consideration of the maximal interval of existence implies the desired global theorem.

The fact that higher derivatives can be bounded is connected to the fact that the given matter model does not form singularities in a given regular spacetime. This condition is violated by dust. The quantity $\int_0^{2\pi} \rho(t, x) dx$ is a bounded function of t on the given time interval, independent of the matter model. On the other hand, it is to be expected that ρ can blow up pointwise on a finite time interval in the case of dust. For other matter models it could happen that, although the energy density remains bounded, regularity breaks down at a higher level. This could result from formation of shocks by the matter. Coming back to the general case, the PDE problem to be studied is that of a semilinear equation of wave map type for W and V coupled to the matter equations. The target space of the wave map is the hyperbolic plane. This system is defined on a curved two-dimensional geometry defined by α , β^1 , a and A . What needs to be shown is that the finiteness of a certain C^k norm of the two-dimensional geometry implies that of a similar norm of W , V and the matter quantities. Conversely, the finiteness of the latter implies, via equations (1.2)-(1.6), the finiteness of a stronger norm of the two-dimensional geometry. This allows higher derivatives of all quantities of interest to be bounded inductively in favourable cases. The details of the argument for bounding low

order derivatives depend very much on the particular matter model.

There are similar results for spherically symmetric spacetimes with a Cauchy hypersurface of topology $S^2 \times S^1$. In that case there is no obstruction to the existence of a maximal hypersurface and all real values are attained by the mean curvature [7,8]. Moreover, it can be shown that the CMC foliation covers the entire spacetime. There are also some other cases where results are known but they do not involve larger classes of solutions than those discussed above (for details see [1]). All the known positive results are for spacetimes with at least two symmetries, so that the problem effectively reduces to studying a system in one (or less) space dimension. It seems that even the case with one symmetry is very difficult, not to mention the general case. However, investigations into generalizing the results reported here to those cases are being carried out. It should be noted that symmetries with fixed points also lead to difficulties so that, for instance, the case of spherical symmetry on S^3 remains open.

Another possible direction for generalizations is to keep the high symmetry but to relax the assumptions on the matter model. There are two basic types of matter model to be considered. There are the phenomenological matter models (e.g. dust, collisionless matter) and the field-theoretic matter models (e.g. massless scalar field, wave maps). The phenomenological matter models represent a macroscopic description of matter. They often have a tendency to form singularities in finite time in a given smooth spacetime. Dust is a good example. It would probably be possible to prove an analogue of the theorem of Section 2 for perfect fluids with pressure. The difference would be that, since the expected singularities are shocks, the energy density would probably remain finite at the time when the CMC foliation broke down. To try and get positive results for a fluid, it would be natural to introduce viscosity. Unfortunately, the concept of viscous fluids is known to be problematic even in special relativity. A case where it is difficult to make predictions is that of the Boltzmann equation. This is because, to the author's knowledge, there is up to now not even a global existence theorem for classical solutions of the special relativistic Boltzmann equation in the case of spatially homogeneous initial data.

Field theoretic matter models are supposed to describe matter at a more fundamental level. They seem less prone to forming singularities in a given smooth spacetime than phenomenological matter models. They do have a different kind of problem, which may purely be an incompatibility with the known techniques, rather than an essential difficulty. The problem is that in certain steps of the proofs one would like to have the non-negative pressures condition, i.e. the condition that $T(X, X) \geq 0$ for all spacelike vectors X . This condition is almost never satisfied by field-theoretic matter models. It is not even satisfied by the massless scalar field. However in the case of $U(1) \times U(1)$ symmetry (or spherical symmetry) it suffices that $T(X, X) \geq 0$ for spacelike vectors X orthogonal to the orbits of the symmetry group. This latter condition is satisfied by the massless scalar field and, more generally, by wave maps. It is not satisfied by an electromagnetic field, a Yang-Mills field or a massive scalar field. This also has the inconvenience that the positive results on existence of CMC hypersurfaces in the case of collisionless matter do not obviously extend to the case of charged collisionless matter, coupled to an electromagnetic field. An argument used in [2], and which does not require non-negative pressures might allow one to circumvent that difficulty in the $U(1) \times U(1)$ symmetric case. However, it does not, as it stands, apply to the spherically symmetric case. The massive scalar field is even worse, since it does not satisfy the strong energy condition, which is the standard condition used to ensure uniqueness of CMC hypersurfaces. Perhaps some use can be made of the fact that the condition is satisfied in an average sense ([9], p. 95). It would be desirable to have more flexible techniques and clearly a lot remains to be learned in this area.

To conclude, it is in order to make some remarks which put the question of the existence of global foliations by CMC hypersurfaces into a wider context. One of the most important mathematical open questions in general relativity is the cosmic censorship hypothesis of Penrose. It is closely related to the issue of the global behaviour of solutions of the Einstein equations corresponding to

general initial data. In [10] Eardley and Moncrief suggested that CMC hypersurfaces could be useful in trying to confirm this hypothesis. More generally, results of the kind discussed in this paper represent knowledge about long-time existence of solutions of the Einstein equations. This is not by itself enough to say something about cosmic censorship. For that one would not only need to know long time existence but also need detailed information on the asymptotic behaviour of the solutions.

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