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Missing Modules, the Gnome Lie Algebra, and E_{10}

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Abstract: We study the embedding of Kac–Moody algebras into Borcherds (or generalized Kac–Moody) algebras which can be explicitly realized as Lie algebras of physical states of some completely compactified bosonic string. The extra "missing states" can be decomposed into irreducible highest or lowest weight "missing modules" w.r.t. the relevant Kac–Moody subalgebra; the corresponding lowest weights are associated with imaginary simple roots whose multiplicities can be simply understood in terms of certain polarization states of the associated string. We analyse in detail two examples where the momentum lattice of the string is given by the unique even unimodular Lorentzian lattice $II_{1,1}$ or $II_{9,1}$, respectively. The former leads to the Borcherds algebra $\mathfrak{g}_{II_{1,1}}$, which we call "gnome Lie algebra", with maximal Kac–Moody subalgebra A_1. By the use of the denominator formula a complete set of imaginary simple roots can be exhibited, while the DDF construction provides an explicit Lie algebra basis in terms of purely longitudinal states of the compactified string in two dimensions. The second example is the Borcherds algebra $\mathfrak{g}_{II_{9,1}}$, whose maximal Kac–Moody subalgebra is the hyperbolic algebra E_{10}. The imaginary simple roots at level 1, which give rise to irreducible lowest weight modules for E_{10}, can be completely characterized; furthermore, our explicit analysis of two non-trivial level-2 root spaces leads us to conjecture that these are in fact the only imaginary simple roots for $\mathfrak{g}_{II_{9,1}}$.

Contents

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1. Introduction

The main focus of this paper is the interplay between Borcherds algebras[1] and their maximal Kac–Moody subalgebras. The potential importance of these infinite dimensional Lie algebras for string unification is widely recognized, but it is far from clear at this time what their ultimate role will be in the scheme of things (see e.g. [20, 31] for recent overviews and motivation). In addition to their uncertain status with regard to physical applications, these algebras are very incompletely understood and present numerous challenges from the purely mathematical point of view. Because recent advances in string theory have greatly contributed to clarifying some of their mathematical intricacies we believe that the best strategy for making progress is to exploit string technology as far as it can take us. This is the path we will follow in this paper.

As is well known, Kac–Moody and Borcherds algebras can both be defined recursively in terms of a Cartan matrix A (with matrix entries a_{ij}) and a set of generating elements $\{e_i, f_i, h_i | i \in I\}$ called Chevalley–Serre generators, which are subject to certain relations involving a_{ij} (see e.g. [24, 28]). For Kac–Moody algebras, the matrix A has to satisfy the properties listed on page one of [24]; the resulting Lie algebra is designated as $\mathfrak{g}(A)$.[2] For Borcherds algebras more general matrices A are possible [4]; in particular, imaginary (i.e., lightlike or timelike) simple roots are allowed, corresponding to zero or negative entries on the diagonal of the Cartan matrix, respectively. The root system of a Kac–Moody algebra is simple to describe, yet for any other but positive or positive semi-definite Cartan matrices (corresponding to finite and affine Lie algebras, resp.), the structure of the algebra itself is exceedingly complicated and not completely known even for a single example. By contrast, Borcherds algebras can sometimes be explicitly realized as Lie algebras of physical states of some compactified bosonic string. Famous examples are the fake monster Lie algebra $\mathfrak{g}_{II_{25,1}}$ and the (true) monster Lie algebra $\mathfrak{g}^\natural$, arising as the Lie algebra of transversal states of a bosonic string in 26 dimensions fully compactified on a torus or a $\mathbb{Z}_2$-orbifold thereof, respectively [5, 6]. Recently, such algebras were also discovered in vertex operator algebras associated with the compactified heterotic string [22]; likewise, the Borcherds superalgebras constructed in [21] may admit such explicit realizations. However, the root systems are now much more difficult to

[1] In the literature, these algebras are also referred to as "generalized Kac–Moody algebras."

[2] We use the labeling $i, j \in \{1, \ldots, d\}$ for $A > 0$ and $i, j \in \{-1, 0, 1, \ldots, d-2\}$ for Lorentzian A (which have Lorentzian signature), where d = rank(A). The affine case of positive semi-definite A which has a slightly different labeling will not concern us here.

characterize, because one is confronted with an (generically) infinite tower of imaginary simple roots; in fact, the full system of simple roots is known only in some special cases.

In this paper we exploit the complementarity of these difficulties. As shown some time ago, both Lorentzian Kac–Moody algebras and Borcherds algebras can be conveniently and explicitly represented in terms of a DDF construction [11, 8] adapted to the root lattice in question [17]. More precisely, any Lorentzian algebra $\mathfrak{g}(A)$ can be embedded into a possibly larger, but in some sense simpler Borcherds algebra of physical states $\mathfrak{g}_\Lambda$ associated with the root lattice Λ of $\mathfrak{g}(A)$. The DDF construction then provides a *complete basis* for $\mathfrak{g}_\Lambda$ and thereby also for $\mathfrak{g}(A)$, although the actual determination of the latter is very difficult. A distinctive feature of Lorentzian Kac–Moody algebras of "subcritical" rank (i.e., $d < 26$) is the occurrence of longitudinal states besides the transversal ones. This result applies in particular to the maximally extended hyperbolic algebra E_{10} which can be embedded into $\mathfrak{g}_{II_{9,1}}$, the Lie algebra of physical states of a subcritical bosonic string fully compactified on the unique 10-dimensional even unimodular Lorentzian lattice $II_{9,1}$. The problem of understanding E_{10} can thus be reduced to the problem of characterizing the "missing states" (alias "decoupled states"), i.e. those physical states in $\mathfrak{g}_{II_{9,1}}$ not belonging to E_{10}. The problem of counting these states, in turn, is equivalent to the one of identifying all the imaginary simple roots of $\mathfrak{g}_{II_{9,1}}$ with their multiplicities.

In general terms, our proposal is therefore to study the embedding

$$\mathfrak{g}(A) \subset \mathfrak{g}_\Lambda ,$$

and to group the missing states

$$\mathcal{M} \equiv \mathfrak{g}_\Lambda \ominus \mathfrak{g}(A)$$

into an infinite direct sum of "missing modules", that is, irreducible highest or lowest weight representations of the subalgebra $\mathfrak{g}(A)$. This idea of decomposing a Borcherds algebra with respect to its maximal Kac–Moody subalgebra was already used by Kang [26] for deriving formulas for the root multiplicites of Borcherds algebras and was treated in the axiomatic setup in great detail by Jurisich [23]. We present here an alternative approach exploiting special features of the string model. After exposing the general structure of the embedding, we will work out two examples in great detail. The first is $\mathfrak{g}_{II_{1,1}}$, the Lie algebra of physical states of a bosonic string compactified on $II_{1,1}$; because of its kinship with the monster Lie algebra $\mathfrak{g}^\natural$ which has the same root lattice, we will refer to it as the "gnome Lie algebra". Its maximal Kac–Moody subalgebra $\mathfrak{g}(A) \subset \mathfrak{g}_{II_{1,1}}$ is just the finite Lie algebra $A_1 \equiv \mathfrak{sl}_2$. The other example which we will investigate is $\mathfrak{g}_{II_{9,1}}$ with the maximal Kac–Moody subalgebra $E_{10} \subset \mathfrak{g}_{II_{9,1}}$. Very little is known about this hyperbolic Lie algebra, and even less is known about its representation theory (see, however, [13] for some recent results on the representations of hyperbolic Kac–Moody algebras). Our main point is that by combining the ill-understood Lie algebra with its representations into the Lie algebra $\mathfrak{g}_{II_{9,1}}$, we arrive at a structure which can be handled much more easily.

The gnome Lie algebra has not yet appeared in the literature so far, although it is possibly the simplest non-trivial example of a Borcherds algebra for which not only one has a satisfactory understanding of the imaginary simple roots, but also a completely explicit realization of the algebra itself in terms of physical string states. (Readers should keep in mind, that so far most investigations of such algebras are limited to counting dimensions of root spaces and studying the modular properties of the associated partition functions.) It is almost "purely Borcherds" since it has only two real roots (and hence only one real simple root), but infinitely many imaginary (in fact, timelike) simple roots. From the generalized denominator formula we shall derive a generating function for their multiplicities. Even better, the root spaces – and not just their dimensions – can be analyzed in a completely explicit manner using the DDF construction. If the fake monster Lie algebra is extremal in the sense that it contains only transversal, but no longitudinal states, the gnome Lie algebra $\mathfrak{g}_{II_{1,1}}$ is at the extreme opposite end of the classification in that it has only longitudinal but no transversal states. This is of course in accordance with expectations for a $d = 2$ subcritical string.

For the Borcherds algebra $\mathfrak{g}_{II_{9,1}}$ the analysis is not so straightforward. It has to be performed level by level where "level" refers to the $\mathbb{Z}$-grading of the Lie algebra induced by the eigenvalue of the central element of the affine subalgebra E_9 (which makes up the level-0 piece). At level 1, we exhibit a complete set of missing lowest weight vectors for the hyperbolic Lie algebra E_{10} obtainable from the tachyonic groundstate $|\mathbf{r}_{-1}\rangle$ (associated with the overextended real simple root $\mathbf{r}_{-1}$) by repeated application of the longitudinal DDF operators. To the best of our knowledge, the corresponding E_{10}-modules provide the first examples for explicit realizations of unitary irreducible highest weight representations of a hyperbolic Kac–Moody algebra. We also examine the non-trivial root spaces associated with the two level-2 roots (or fundamental weights w.r.t. the affine subalgebra) Λ_7 and Λ_1, which were recently worked out explicitly in [17, 1] and which exemplify the rapidly increasing complications at higher level. An important result of this paper is the explicit demonstration that the missing states for Λ_7 and Λ_1 can be completely reproduced by commuting missing level-1 states either with themselves or with other level-1 E_{10} elements. This calculation not only furnishes a non-trivial check on our previous results, which were obtained in a rather different manner; even more importantly, it shows that the simple multiplicity (i.e., the multiplicity as a simple root) of both Λ_7 and Λ_1 is zero. In view of this surprising conclusion and the fact that E_{10} is a "huge" subalgebra of $\mathfrak{g}_{II_{9,1}}$, we conjecture that *all* missing states of E_{10} should be obtainable in this way. In other words, the "easy" imaginary simple roots of $\mathfrak{g}_{II_{9,1}}$ at level-1 would in fact be the only ones. In spite of the formidable difficulties of verifying (or falsifying) this conjecture at arbitrary levels, we believe that its elucidation would take us a long way towards understanding E_{10} and what is so special about it.

2. The Lie Algebra of Physical States

We shall study one chiral sector of a closed bosonic string moving on a Minkowski torus as spacetime, i.e., with all target space coordinates compactified. Uniqueness of the quantum mechanical wave function then forces the center of mass momenta of the string to form a lattice λ with Minkowskian signature. Upon "old" covariant quantization this system turns out to realize a mathematical structure called vertex algebra [3]. In these models the physical string states form an infinite-dimensional Lie algebra $\mathfrak{g}_\Lambda$ which has the structure of a so-called Borcherds algebra. It is possible to identify a maximal Kac–Moody subalgebra $\mathfrak{g}(A)$ inside $\mathfrak{g}_\Lambda$ which is generically of Lorentzian indefinite type. The physical states not belonging to $\mathfrak{g}(A)$ are called missing states and can be grouped into irreducible highest or lowest weight representations of $\mathfrak{g}(A)$. In principle, the DDF construction allows us to identify the corresponding vacuum states.

2.1. The completely compactified bosonic string. For a detailed account of this topic the reader may wish to consult the review [16]. Here, we will follow closely [17], omitting most of the technical details.

Let Λ be an even Lorentzian lattice of rank $d < \infty$, representing the lattice of allowed center-of-mass momenta for the string. To each lattice point we assign a groundstate $|\mathbf{r}\rangle$ which plays the role of a highest weight vector for a d-fold Heisenberg algebra $\hat{\mathbf{h}}$ of string oscillators α_m^μ ($n \in \mathbb{Z}$, $0 \le \mu \le d-1$),

$$\alpha_0^\mu|\mathbf{r}\rangle = r^\mu|\mathbf{r}\rangle, \qquad \alpha_m^\mu|\mathbf{r}\rangle = 0 \quad \forall m > 0,$$

where

$$[\alpha_m^\mu, \alpha_n^\nu] = m\eta^{\mu\nu}\delta_{m+n,0}.$$

The Fock space is obtained by collecting the irreducible $\hat{\mathbf{h}}$-modules built on all possible groundstates, viz.

$$\mathcal{F} := \bigoplus_{\mathbf{r}\in\Lambda} \mathcal{F}^{(\mathbf{r})},$$

where

$$\mathcal{F}^{(\mathbf{r})} := \operatorname{span}\{\alpha_{-m_1}^{\mu_1}\cdots\alpha_{-m_M}^{\mu_M}|\mathbf{r}\rangle \mid 0 \le \mu_i \le d-1,\ m_i > 0\}.$$

To each state $\psi \in \mathcal{F}$, one assigns a vertex operator

$$\mathcal{V}(\psi, z) = \sum_{n\in\mathbb{Z}} \psi_n z^{-n-1},$$

which is an operator-valued ($\psi_n \in \operatorname{End}\mathcal{F}\ \forall n$) formal Laurent series. For notational convenience we put $\boldsymbol{\xi}(m) \equiv \boldsymbol{\xi}\cdot\boldsymbol{\alpha}_m$ for any $\boldsymbol{\xi} \in \mathbb{R}^{d-1,1}$, and we introduce the current

$$\boldsymbol{\xi}(z) := \sum_{m\in\mathbb{Z}} \boldsymbol{\xi}(m) z^{-m-1}.$$

The vertex operator associated with a single oscillator is defined as

$$\mathcal{V}\big(\boldsymbol{\xi}(-m)|\mathbf{0}\rangle, z\big) := \frac{1}{(m-1)!}\left(\frac{d}{dz}\right)^{m-1}\boldsymbol{\xi}(z), \tag{2.1}$$

whereas for a groundstate $|\mathbf{r}\rangle$ one puts

$$\mathcal{V}\big(|\mathbf{r}\rangle, z\big) := \mathrm{e}^{\int \mathbf{r}_-(z)dz}\mathrm{e}^{\mathrm{i}\mathbf{r}\cdot\mathbf{q}} z^{\mathbf{r}\cdot\mathbf{p}} \mathrm{e}^{\int \mathbf{r}_+(z)dz} c_{\mathbf{r}}, \tag{2.2}$$

with $c_{\mathbf{r}}$ denoting some cocycle factor, $\mathbf{r}_\pm(z) := \sum_{m>0}\mathbf{r}(\pm m)z^{\mp m-1}$, and q^μ being the position operators conjugate to the momentum operators $p^\mu \equiv \alpha_0^\mu$ ($[q^\mu, p^\nu] = \mathrm{i}\eta^{\mu\nu}$). For a general homogeneous element $\psi = \boldsymbol{\xi}_1(-m_1)\cdots\boldsymbol{\xi}_M(-m_M)|\mathbf{r}\rangle$, say, the associated vertex operator is then defined by the normal-ordered product

$$\mathcal{V}(\psi, z) := \ :\mathcal{V}\big(\boldsymbol{\xi}_1(-m_1)|\mathbf{0}\rangle, z\big)\cdots\mathcal{V}\big(\boldsymbol{\xi}_M(-m_M)|\mathbf{0}\rangle, z\big)\mathcal{V}\big(|\mathbf{r}\rangle, z\big):. \tag{2.3}$$

This definition can be extended by linearity to the whole of $\mathcal{F}$.

The above data indeed fulfill all the requirements of a vertex algebra [3, 15]. The two preferred elements in $\mathcal{F}$, namely the vacuum and the conformal vector, are given here by $\mathbf{1} := |\mathbf{0}\rangle$ and $\boldsymbol{\omega} := \frac{1}{2}\boldsymbol{\alpha}_{-1}\cdot\boldsymbol{\alpha}_{-1}|\mathbf{0}\rangle$, respectively. Note that the corresponding vertex operators are respectively given by the identity $\mathrm{id}_{\mathcal{F}}$ and the stress–energy tensor $\mathcal{V}(\boldsymbol{\omega}, z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}$, where the latter provides the generators L_n of the constraint Virasoro algebra Vir_L (with central charge $c = d$), such that the grading of $\mathcal{F}$ is obtained by the eigenvalues of L_0 and the role of a translation generator is played by L_{-1} satisfying

$\mathcal{V}(L_{-1}\psi, z) = \frac{d}{dz}\mathcal{V}(\psi, z)$. Finally, we mention that among the axioms of a vertex algebra there is a crucial identity relating products and iterates of vertex operators called the Cauchy–Jacobi identity.

We denote by $\mathcal{P}^h$ the space of (conformal) highest weight vectors or primary states of weight $h \in \mathbb{Z}$, satisfying

$$L_0\psi = h\psi, \tag{2.4}$$

$$L_n\psi = 0 \quad \forall n > 0. \tag{2.5}$$

We shall call the vectors in $\mathcal{P}^1$ physical states from now on. The vertex operators associated with physical states enjoy rather simple commutation relations with the generators of Vir_L. In terms of the mode operators we have $[L_n, \psi_m] = -m\psi_{m+n}$ for $\psi \in \mathcal{P}^1$. In particular, the zero modes ψ_0 of physical vertex operators commute with the Virasoro constraints and consequently map physical states into physical states. This observation leads to the following definition of a bilinear product on the space of physical states [3]:

$$[\psi, \varphi] := \psi_0\varphi \equiv \mathrm{Res}_z\left[\mathcal{V}(\psi, z)\varphi\right], \tag{2.6}$$

using an obvious formal residue notation. The Cauchy–Jacobi identity for the vertex algebra immediately ensures that the Jacobi identity $[\xi, [\psi, \varphi]] + [\psi, [\varphi, \xi]] + [\varphi, [\xi, \psi]] = 0$ always holds (even on $\mathcal{F}$). But the antisymmetry property turns out to be satisfied only modulo L_{-1} terms. Hence one is led to introduce the Lie algebra of observable physical states by

$$\mathfrak{g}_\Lambda := \mathcal{P}^1 \big/ L_{-1}\mathcal{P}^0, \tag{2.7}$$

where "observable" refers to the fact that the subspace $L_{-1}\mathcal{P}^0$ consists of (unobservable) null physical states, i.e., physical states orthogonal to all physical states including themselves (w.r.t. the usual string scalar product). Indeed, for $d \neq 26$, $L_{-1}\mathcal{P}^0$ accounts for all null physical states.

2.2. The DDF construction. For a detailed analysis of $\mathfrak{g}_\Lambda$ one requires an explicit basis. First, one observes that the natural $\mathfrak{g}_\Lambda$-gradation by momentum already provides a root space decomposition for $\mathfrak{g}_\Lambda$, viz.

$$\mathfrak{g}_\Lambda = \mathfrak{h}_\Lambda \oplus \bigoplus_{\mathbf{r}\in\Delta} \mathfrak{g}_\Lambda{}^{(\mathbf{r})},$$

where the root space $\mathfrak{g}_\Lambda{}^{(\mathbf{r})}$ consists of all observable physical states with momentum $\mathbf{r}$:

$$\mathfrak{g}_\Lambda{}^{(\mathbf{r})} := \{\psi \in \mathfrak{g}_\Lambda \,|\, p^\mu\psi = r^\mu\psi\}.$$

The set of roots, Δ, is determined by the requirement that the roots should represent physically allowed string momenta. Hence we have

$$\Delta \equiv \Delta_\Lambda := \{\mathbf{r} \in \Lambda \,|\, \mathbf{r}^2 \leq 2,\ \mathbf{r} \neq \mathbf{0}\} = \Delta^{\mathrm{re}} \cup \Delta^{\mathrm{im}},$$

where we have also split the set of roots into two subsets of real and imaginary roots which are respectively given by

$$\Delta^{\mathrm{re}} := \{\mathbf{r} \in \Delta \,|\, \mathbf{r}^2 > 0\}, \qquad \Delta^{\mathrm{im}} := \{\mathbf{r} \in \Delta \,|\, \mathbf{r}^2 \leq 0\}.$$

Zero momentum is by definition not a root but is incorporated into the d-dimensional Cartan subalgebra

$$\mathfrak{h}_\Lambda := \{\boldsymbol{\xi}(-1)|\mathbf{0}\rangle \mid \boldsymbol{\xi} \in \mathbb{R}^{d-1,1}\}.$$

Thus the task is to find a basis for each root space. This is achieved by the so-called DDF construction [11, 8] which we will sketch.

Given a root $\mathbf{r} \in \Delta$, it is always possible to find a DDF decomposition for it,

$$\mathbf{r} = \mathbf{a} - n\mathbf{k} \qquad \text{with } n := 1 - \tfrac{1}{2}\mathbf{r}^2,$$

where $\mathbf{a}, \mathbf{k} \in \mathbb{R}^{d-1,1}$ satisfy $\mathbf{a}^2 = 2$, $\mathbf{a}\cdot\mathbf{k} = 1$, and $\mathbf{k}^2 = 0$. Having fixed $\mathbf{a}$ and $\mathbf{k}$ we choose a set of orthonormal polarization vectors $\boldsymbol{\xi}^i \in \mathbb{R}^{d-1,1}$ $(1 \le i \le d-2)$ obeying $\boldsymbol{\xi}^i\cdot\mathbf{a} = \boldsymbol{\xi}^i\cdot\mathbf{k} = 0$. Then the transversal and longitudinal DDF operators are respectively defined by

$$A^i_m = A^i_m(\mathbf{a},\mathbf{k}) := \mathrm{Res}_z\left[\mathcal{V}\left(\boldsymbol{\xi}^i(-1)|m\mathbf{k}\rangle, z\right)\right], \tag{2.8}$$

$$\begin{aligned} A^-_m = A^-_m(\mathbf{a},\mathbf{k}) := \mathrm{Res}_z\Bigg[&-\mathcal{V}\big(\mathbf{a}(-1)|m\mathbf{k}\rangle, z\big) + \frac{m}{2}\frac{d}{dz}\log\big(\mathbf{k}(z)\big)\mathcal{V}\big(|m\mathbf{k}\rangle, z\big)\Bigg] \\ &-\frac{1}{2}\sum_{n\in\mathbb{Z}} {}^\times_\times A^i_n A^i_{m-n}{}^\times_\times + 2\delta_{m0}\mathbf{k}\cdot\mathbf{p}. \end{aligned} \tag{2.9}$$

We shall need to make use of the following important facts about the DDF operators (see e.g. [17]).

Theorem 1. *Let* $\mathbf{r} \in \Delta$. *The DDF operators associated with the DDF decomposition* $\mathbf{r} = \mathbf{a} - n\mathbf{k}$ *enjoy the following properties on the space of physical string states with momentum* $\mathbf{r}$, $\mathcal{P}^{1,(\mathbf{r})}$*:*

1. *(Physicality)* $[L_m, A^i_n] = [L_m, A^-_n] = 0$*;*
2. *(Transversal Heisenberg algebra)* $[A^i_m, A^j_n] = m\delta^{ij}\delta_{m+n,0}$*;*
3. *(Longitudinal Virasoro algebra)* $[A^-_m, A^-_n] = (m-n)A^-_{m+n} + \frac{26-d}{12}m(m^2-1)\delta_{m+n,0}$*;*
4. *(Null states)* $A^-_{-1}|\mathbf{a}\rangle \propto L_{-1}|\mathbf{a}-\mathbf{k}\rangle$*;*
5. *(Orthogonality)* $[A^-_m, A^i_n] = 0$*;*
6. *(Highest weight property)* $A^i_k|\mathbf{a}\rangle = A^-_k|\mathbf{a}\rangle = 0$ *for all* $k \ge 0$*;*
7. *(Spectrum generating)* $\mathcal{P}^{1,(\mathbf{r})} = \mathrm{span}\{A^{i_1}_{-m_1}\cdots A^{i_M}_{-m_M}A^-_{-n_1}\cdots A^-_{-n_N}|\mathbf{a}\rangle \mid m_1+\ldots+n_N = 1-\frac{1}{2}\mathbf{r}^2\}$*;*

for all $m, n \in \mathbb{Z}$ *and* $1 \le i \le d-2$.

As a simple consequence, we have the following explicit formula for the multiplicity of a root $\mathbf{r}$ in $\mathfrak{g}_\Lambda$:

$$\mathrm{mult}_{\mathfrak{g}_\Lambda}(\mathbf{r}) \equiv \dim \mathfrak{g}_\Lambda{}^{(\mathbf{r})} = \pi_{d-1}(n) := p_{d-1}(n) - p_{d-1}(n-1), \tag{2.10}$$

where $n = 1 - \frac{1}{2}\mathbf{r}^2$ and $\sum_{n\ge0} p_d(n)q^n = [\phi(q)]^{-d} \equiv \prod_{n\ge1}(1-q^n)^{-d}$, so that

$$\sum_{n=0}^{\infty}\pi_{d-1}(n)q^n = \frac{1-q}{[\phi(q)]^{d-1}} = 1 + (d-2)q + \tfrac{1}{2}(d-1)dq^2 + \cdots. \tag{2.11}$$

The above theorem is also useful for constructing a positive definite symmetric bilinear form on $\mathfrak{g}_\Lambda$ as follows:

$$\langle \mathbf{r}|\mathbf{s}\rangle := \delta_{\mathbf{r},\mathbf{s}} \quad \text{for } \mathbf{r},\mathbf{s} \in \Lambda, \qquad (\alpha^\mu_m)^\dagger := \alpha^\mu_{-m}.$$

For the DDF operators this yields

$$(A^i_m)^\dagger = A^i_{-m}, \qquad (A^-_m)^\dagger = A^-_{-m}.$$

In view of the above commutation relations it is then clear that $\langle\ |\ \rangle$ is positive definite on any root space $\mathfrak{g}_\Lambda{}^{(\mathbf{r})}$ if $d < 26$. For the critical dimension, $d = 26$, we redefine $\mathfrak{g}_\Lambda$ by dividing out the additional null states which correspond to the remaining longitudinal DDF states. Thus we have to replace π_{d-1} by p_{24} in the multiplicity formula. Note that the scalar product has Minkowskian signature on the Cartan subalgebra.

For our purposes we shall also need an invariant symmetric bilinear form on $\mathfrak{g}_\Lambda$ which is defined as

$$(\psi|\varphi) := -\langle\theta(\psi)|\varphi\rangle$$

for $\psi, \varphi \in \mathfrak{g}_\Lambda$, where the Chevalley involution is given by

$$\theta(|\mathbf{r}\rangle) := |-\mathbf{r}\rangle, \qquad \theta \circ \alpha^\mu_m \circ \theta := -\alpha^\mu_m.$$

Clearly, both bilinear forms are preserved by this involution and they enjoy the invariance and contravariance properties, respectively, viz.

$$([\psi,\chi]|\varphi) = (\psi|[\chi,\varphi]), \quad \langle[\psi,\chi]|\varphi\rangle = \langle\psi|[\theta(\chi),\varphi]\rangle \quad \forall \psi,\chi,\varphi \in \mathfrak{g}_\Lambda. \tag{2.12}$$

2.3. Borcherds algebras and Kac–Moody algebras. We now have all ingredients at hand to show that $\mathfrak{g}_\Lambda$ for any $d > 0$ belongs to a certain class of infinite-dimensional Lie algebras.

Definition 1. *Let J be a countable index set (identified with some subset of $\mathbb{Z}$). Let $B = (b_{ij})_{i,j\in J}$ be a real matrix, satisfying the following conditions:*

(C1) B is symmetric;
(C2) If $i \neq j$ then $b_{ij} \leq 0$;
(C3) If $b_{ii} > 0$ then $\frac{2b_{ij}}{b_{ii}} \in \mathbb{Z}$ for all $j \in J$.

Then the universal Borcherds algebra $\mathfrak{g}(B)$ associated with B is defined as the Lie algebra generated by elements e_i, f_i and h_{ij} for $i, j \in J$, with the following relations:

(R1) $[h_{ij}, e_k] = \delta_{ij} b_{ik} e_k, \quad [h_{ij}, f_k] = -\delta_{ij} b_{ik} f_k$;
(R2) $[e_i, f_j] = h_{ij}$;
(R3) If $b_{ii} > 0$ then $(\mathrm{ad} e_i)^{1-2b_{ij}/b_{ii}} e_j = (\mathrm{ad} f_i)^{1-2b_{ij}/b_{ii}} f_j = 0$;
(R4) If $b_{ij} = 0$ then $[e_i, e_j] = [f_i, f_j] = 0$.

The elements h_{ij} span an abelian subalgebra of $\mathfrak{g}(B)$ called the Cartan subalgebra. In fact, the elements h_{ij} with $i \neq j$ lie in the center of $\mathfrak{g}(B)$. It is easy to see that h_{ij} is zero unless the i^{th} and j^{th} columns of the matrix B are equal.[3] A Lie algebra is called a Borcherds algebra, if it can be obtained from a universal Borcherds algebra by

[3] Actually, the elements h_{ij} for $i \neq j$ do not play any role and in fact cannot appear in the present context, where $\mathfrak{g}(B) = \mathfrak{g}_\Lambda$ is based on a non-degenerate Lorenzian lattice Λ. Namely, for the i^{th} and j^{th} columns of B to be equal the corresponding roots must be equal, and therefore such h_{ij} are always of the form $\mathbf{v}_{ij}(-1)|\mathbf{0}\rangle$ with $\mathbf{v}_{ij} \in \Lambda$. Since furthermore the h_{ij} with $i \neq j$ are central elements, the lattice vectors $\mathbf{v}_{ij}$ must be orthogonal to all (real and imaginary) roots. Because Λ is non-degenerate, we conclude that $\mathbf{v}_{ij} = 0$, and hence $h_{ij} = 0$ for $i \neq j$.

dividing out a subspace of its center and adding an abelian algebra of outer derivations. An important property of (universal) Borcherds algebras is the existence of a triangular decomposition

$$\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+, \tag{2.13}$$

where $\mathfrak{n}_+$ and $\mathfrak{n}_-$ denote the subalgebras generated by the e_i's and the f_i's, respectively. This can be established by the usual methods for Kac–Moody algebras (see [23] for a careful proof).

Given the Lie algebra of physical string states, $\mathfrak{g}_\Lambda$, it is extremely difficult to decide whether it is a Borcherds algebra in the sense of the above definition. Luckily, however, there are alternative characterizations of Borcherds algebras which can be readily applied to the case of $\mathfrak{g}_\Lambda$. We start with the following one [4].

Theorem 2. *A Lie algebra $\mathfrak{g}$ is a Borcherds algebra if it has an almost positive definite contravariant form $\langle\ |\ \rangle$, which means that $\mathfrak{g}$ has the following properties:*

1. *(Grading)* $\mathfrak{g} = \bigoplus_{n\in\mathbb{Z}} \mathfrak{g}_n$ *with* $\dim \mathfrak{g}_n < \infty$ *for* $n \neq 0$*;*
2. *(Involution)* $\mathfrak{g}$ *has an involution θ which acts as -1 on $\mathfrak{g}_0$ and maps $\mathfrak{g}_n$ to $\mathfrak{g}_{-n}$;*
3. *(Invariance)* $\mathfrak{g}$ *carries a symmetric invariant bilinear form* $(\ |\ )$ *preserved by θ and such that* $(\mathfrak{g}_m \mid \mathfrak{g}_n) = 0$ *unless* $m + n = 0$*;*
4. *(Positivity) The contravariant form $\langle x|y\rangle := -(\theta(x)|y)$ is positive definite on $\mathfrak{g}_n$ if* $n \neq 0$.

The converse is almost true, which means that, apart from some pathological cases, a Borcherds algebra always satisfies the conditions in the above theorem (cf. [23]).

Hence $\mathfrak{g}_\Lambda$ for $d \leq 26$ is a Borcherds algebra if we can equip it with an appropriate $\mathbb{Z}$-grading. Note that the grading given by assigning degree $1 - \frac{1}{2}\mathbf{r}^2$ to a root space ${\mathfrak{g}_\Lambda}^{(\mathbf{r})}$ will not work since there are infinitely many lattice points lying on the hyperboloid $\mathbf{x}^2 = \text{const} \in 2\mathbb{Z}$. The solution is to slice the forward (resp. backward) light cone by a family of $(d-1)$-dimensional parallel hyperplanes whose common normal vector is timelike and has integer scalar product with all the roots of $\mathfrak{g}_\Lambda$(i.e., it is an element of the weight lattice Λ^*). There is one subtlety here, however. It might well happen that for a certain choice of the timelike normal vector $\mathbf{t} \in \Lambda^*$ there are some real roots $\mathbf{r} \in \Lambda$ which are orthogonal to $\mathbf{t}$ so that the associated root spaces would have degree zero.[4] But then we would run into trouble since the Chevalley involution does not act as -1 on a root space ${\mathfrak{g}_\Lambda}^{(\mathbf{r})}$ but rather maps it into ${\mathfrak{g}_\Lambda}^{(-\mathbf{r})}$. We call a timelike vector $\mathbf{t} \in \Lambda^*$ a grading vector if it is "in general position", which means that it has nonzero scalar product with all roots. So let us fix some grading vector[5] and define

$$\mathfrak{g}_n := \bigoplus_{\substack{\mathbf{r}\in\Delta \\ \mathbf{r}\cdot\mathbf{t}=n}} {\mathfrak{g}_\Lambda}^{(\mathbf{r})}, \qquad \mathfrak{g}_0 := \mathfrak{h}_\Lambda .$$

(The associated degree operator is just $\mathbf{t}{\cdot}\mathbf{p}$.) Then this yields the grading necessary for $\mathfrak{g}_\Lambda$ to be a Borcherds algebra. Note that the pairing property $(\mathfrak{g}_m \mid \mathfrak{g}_n) \propto \delta_{m+n,0}$ is fulfilled since θ is induced from the reflection symmetry of the lattice. Observe also that if the lattice admits a (timelike) Weyl vector $\boldsymbol{\rho}$ we can set $\mathbf{t} = \boldsymbol{\rho}$ since this vector has all the requisite properties. We conclude: if the lattice Λ has a grading vector and $d \leq 26$, then

[4] By choosing $\mathbf{t}$ to be timelike it is also assured that it has nonzero scalar product with all imaginary roots.

[5] Grading vectors always exist since the hyperplanes orthogonal to the real roots cannot exhaust all the points of Λ^* inside the lightcone.

the Lie algebra of physical states, $\mathfrak{g}_\Lambda$, is a Borcherds algebra. This result suggests that above the critical dimension the Lie algebra of physical string states somehow changes in type, as one would also naively expect from a string theoretical point of view. But this impression is wrong. It is an artefact caused by the special choice of the string scalar product. To see this, we recall another characterization of Borcherds algebras [7].

Theorem 3. *A Lie algebra $\mathfrak{g}$ satisfying the following conditions is a Borcherds algebra:*

(B1) $\mathfrak{g}$ *has a nonsingular invariant symmetric bilinear form* $(\ |\ )$*;*
(B2) $\mathfrak{g}$ *has a self-centralizing subalgebra* $\mathfrak{h}$ *such that* $\mathfrak{g}$ *is diagonalizable with respect to* $\mathfrak{h}$ *and all the eigenspaces are finite-dimensional;*
(B3) $\mathfrak{h}$ *has a regular element* $h^\times$*, i.e., the centralizer of* $h^\times$ *is* $\mathfrak{h}$ *and there are only a finite number of roots* $\mathbf{r} \in \mathfrak{h}^*$ *such that* $|\mathbf{r}(h^\times)| < R$ *for any* $R \in \mathbb{R}$*;*
(B4) The norms of roots of $\mathfrak{g}$ *are bounded above;*
(B5) Any two imaginary roots which are both positive or negative have inner product at most 0, and if they are orthogonal their root spaces commute.

Here, as usual, the nonzero eigenvalues of $\mathfrak{h}$ acting on $\mathfrak{g}$ are elements of the dual $\mathfrak{h}^*$ and are called roots of $\mathfrak{g}$. A root is called positive or negative depending on whether its value on the regular element is positive or negative, respectively; and a root is called real if its norm (naturally induced from $(\ |\ )$ on $\mathfrak{g}$) is positive, and imaginary otherwise. Note that the regular element provides a triangular decomposition (2.13) by gathering all root spaces associated with positive (resp. negative) roots into the subalgebra $\mathfrak{n}_+$ (resp. $\mathfrak{n}_-$).

For our purposes we shall need a special case of this theorem. Suppose that the bilinear form has Lorentzian signature on $\mathfrak{h}$ (and consequently also on $\mathfrak{h}^*$). For the regular element $h^\times$ we can take any $\mathbf{t}(-1)|\mathbf{0}\rangle$ associated with a timelike vector $\mathbf{t}$ in general position (cf. the above remark about grading vectors!). But the Lorentzian geometry implies more; namely, that two vectors inside or on the forward (or backward) lightcone have to have nonpositive inner product with each other, and they can be orthogonal only if they are multiples of the same lightlike vector. Therefore we have [7]

Corollary 1. *A Lie algebra* $\mathfrak{g}$ *satisfying the following properties conditions is a Borcherds algebra:*

(B1') $\mathfrak{g}$ *has a nonsingular invariant symmetric bilinear form* $(\ |\ )$*;*
(B2') $\mathfrak{g}$ *has a self-centralizing subalgebra* $\mathfrak{h}$ *such that* $\mathfrak{g}$ *is diagonalizable with respect to* $\mathfrak{h}$ *and all the eigenspaces are finite-dimensional;*
(B3') The bilinear form restricted to $\mathfrak{h}$ *is Lorentzian;*
(B4') The norms of roots of $\mathfrak{g}$ *are bounded above;*
(B5') If two roots are positive multiples of the same norm 0 vector then their root spaces commute.

Apparently, $\mathfrak{g}_\Lambda$ for any d fulfills the conditions *(B1')–(B4')*. A straightforward exercise in oscillator algebra also verifies *(B5')* (see formula (3.1) in [17]). We conclude that $\mathfrak{g}_\Lambda$ is indeed always a Borcherds algebra.

Although we do not know the Cartan matrix B associated to $\mathfrak{g}_\Lambda$ (and so the set of simple roots) we can determine the maximal Kac–Moody subalgebra of $\mathfrak{g}_\Lambda$ given by the submatrix A obtained from B by deleting all rows and columns $j \in J$ such that $b_{jj} \le 0$.

A special role is played by the lattice vectors of length 2 which are called the real roots of the lattice and which give rise to tachyonic physical string states. Lightlike or timelike roots are referred to as imaginary roots. We associate with every real root $\mathbf{r} \in \Lambda$ a reflection by $\mathfrak{w}_\mathbf{r}(\mathbf{x}) := \mathbf{x} - (\mathbf{x}\cdot\mathbf{r})\mathbf{r}$ for $\mathbf{x} \in \mathbb{R}^{d-1,1}$. The reflecting hyperplanes then divide the vector space $\mathbb{R}^{d-1,1}$ into regions called Weyl chambers. The reflections in the real roots of Λ generate a group called the Weyl group $\mathfrak{W}$ of Λ, which acts simply transitively on the Weyl chambers. Fixing one chamber to be the fundamental Weyl chamber $\mathcal{C}$ once and for all, we call the real roots perpendicular to the faces of $\mathcal{C}$ and with inner product at most 0 with elements of $\mathcal{C}$, the simple roots. We denote such a set of real simple roots by $\Pi^{\mathrm{re}} = \Pi^{\mathrm{re}}(\mathcal{C}) = \{\mathbf{r}_i | i \in I\}$ for a countable index set I.[6] Note that a priori there is no relation between the rank d of the lattice and the number of simple roots, $|I|$.[7]

The main new feature of Borcherds algebras in comparison with ordinary Kac–Moody algebras is the appearance of *imaginary simple roots*. An important property of Borcherds algebras is the existence of a character formula which generalizes the Weyl–Kac character formula for ordinary Kac–Moody algebras and which leads as a special case to the following Weyl–Kac–Borcherds denominator formula.

Theorem 4. *Let $\mathfrak{g}$ be a Borcherds algebra with Weyl vector $\boldsymbol{\rho}$ (i.e., $\boldsymbol{\rho}\cdot\mathbf{r} = -\frac{1}{2}\mathbf{r}^2$ for all simple roots) and Weyl group $\mathfrak{W}$ (generated by the reflections in the real simple roots). Then*

$$\prod_{\mathbf{r}\in\Delta_+}(1-\mathrm{e}^{\mathbf{r}})^{\mathrm{mult}(\mathbf{r})} = \sum_{\mathfrak{w}\in\mathfrak{W}}(-1)^{\mathfrak{w}}\,\mathrm{e}^{\mathfrak{w}(\boldsymbol{\rho})-\boldsymbol{\rho}}\sum_{\mathbf{s}}\epsilon(\mathbf{s})\mathrm{e}^{\mathfrak{w}(\mathbf{s})}, \tag{2.14}$$

where $\epsilon(\mathbf{s})$ is $(-1)^n$ if $\mathbf{s}$ is a sum of n distinct pairwise orthogonal imaginary simple roots and zero otherwise.

Note that the Weyl vector may be replaced by any other vector having inner product $-\frac{1}{2}\mathbf{r}^2$ with all *real* simple roots since $\mathrm{e}^{\mathfrak{w}(\boldsymbol{\rho})-\boldsymbol{\rho}}$ involves only inner products of $\boldsymbol{\rho}$ with real simple roots. This will be important for the gnome Lie algebra below where there is no true Weyl vector but the denominator formula nevertheless can be used to determine the multiplicities of the imaginary simple roots.

The physical states

$$e_i := |\mathbf{r}_i\rangle, \qquad f_i := -|-\mathbf{r}_i\rangle, \qquad h_i := \mathbf{r}_i(-1)|\mathbf{0}\rangle, \tag{2.15}$$

for $i \in I$, obey the following commutation relations (see [3]):

$$\begin{aligned} [h_i, h_j] &= 0, \\ [e_i, f_j] &= \delta_{ij} h_i, \\ [h_i, e_j] &= a_{ij} e_j, \qquad [h_i, f_j] = -a_{ij} f_j, \\ (\mathrm{ad}e_i)^{1-a_{ij}} e_j &= 0, \qquad (\mathrm{ad}f_i)^{1-a_{ij}} f_j = 0 \quad \forall i \neq j, \end{aligned} \tag{2.16}$$

which means that they generate via multiple commutators the Kac–Moody algebra $\mathfrak{g}(A)$ associated with the Cartan matrix $A = (a_{ij})_{i,j\in I}$, $a_{ij} := \mathbf{r}_i\cdot\mathbf{r}_j$. As usual, we have the triangular decomposition

$$\mathfrak{g}(A) = \mathfrak{n}_-(A) \oplus \mathfrak{h}(A) \oplus \mathfrak{n}_+(A), \tag{2.17}$$

[6] I may be identified with a subset of J. Note, however, that apart from some special examples, the matrix B for $\mathfrak{g}_\Lambda$ as a Borcherds algebra is not known.

[7] The extremal case occurs for the lattice $\Pi_{25,1}$ where $d = 26$ but $|I| = \infty$ [9]. We should mention here that in order to get the set of imaginary roots "well-behaved", one assumes that the semidirect product of the Weyl group with the group of graph automorphisms associated with the Coxeter–Dynkin diagram of Π^{re} has finite index in the automorphism group of the lattice Λ (see e.g. [29]).

where $\mathfrak{n}_+(A)$ (resp. $\mathfrak{n}_-(A)$) denotes the subalgebra generated by the e_i's (resp. f_i's) for $i \in I$. This corresponds to a choice of the grading vector $\mathbf{t}$ (and the regular element $h^\times := \mathbf{t}(-1)|\mathbf{0}\rangle$) satisfying $\mathbf{t}\cdot\mathbf{r}_i > 0\ \forall i \in I$. The Lie algebra $\mathfrak{g}(A)$ is a proper subalgebra of the Lie algebra of physical states $\mathfrak{g}_\Lambda$,

$$\mathfrak{g}(A) \subset \mathfrak{g}_\Lambda.$$

If we finally introduce the Kac–Moody root lattice

$$Q(A) := \sum_{i\in I} \mathbb{Z}\mathbf{r}_i,$$

then obviously $Q(A) \subseteq \Lambda$ and in particular $\operatorname{rank} Q(A) \leq d$, even though $|I|$ might be larger than d.

2.4. Missing modules. Having found the Kac–Moody algebra $\mathfrak{g}(A)$, the idea is now to analyze the "rest" of $\mathfrak{g}_\Lambda$ from the point of view of $\mathfrak{g}(A)$. It is clear that, via the adjoint action, $\mathfrak{g}_\Lambda$ is a representation of $\mathfrak{g}(A)$. Since the contravariant bilinear form is positive definite on the root spaces ${\mathfrak{g}_\Lambda}^{(\mathbf{r})}$, $\mathbf{r} \in \Delta$, it is sensible to consider the direct sum of orthogonal complements of $\mathfrak{g}(A) \cap {\mathfrak{g}_\Lambda}^{(\mathbf{r})}$ in ${\mathfrak{g}_\Lambda}^{(\mathbf{r})}$ with respect to $\langle\ |\ \rangle$ and explore its properties under the action of $\mathfrak{g}(A)$. We shall see that the resulting space of so-called missing states is a completely reducible $\mathfrak{g}(A)$-module, decomposable into irreducible highest or lowest weight representations. The issue of zero momentum, however, requires some care. If $Q(A) \not= \Lambda$, then there must be a set of $d - \operatorname{rank} Q(A)$ linearly independent imaginary simple roots, $\{\mathbf{r}_j | j \in H \subset J \setminus I\}$, linearly independent of the set of real simple roots, such that $\mathfrak{h}_\Lambda = \mathfrak{h}(A) \oplus \mathfrak{h}'$ with $\mathfrak{h}' := \operatorname{span}\{h_j | j \in H\}$. The latter subspace of the Cartan subalgebra is in general not a $\mathfrak{g}(A)$-module but rather an abelian algebra of outer derivations for $\mathfrak{g}(A)$ in view of the commutation relations *(R1)*. This observation suggests to consider an extension of $\mathfrak{g}(A)$ by these derivations. There is also another argument that this is a natural thing to do. Namely, extending $\mathfrak{h}(A)$ to $\mathfrak{h}_\Lambda$ ensures that any root $\mathbf{r}$ is a nonzero weight for the extended Lie algebra, while this is not guaranteed for $\mathfrak{g}(A)$ because there might exist roots in Δ orthogonal to all real simple roots. This procedure is in spirit the same for the general theory of affine Lie algebras where one extends the algebra by adjoining outer derivations to the Cartan subalgebra such that the standard invariant form becomes nondegenerate.

Definition 2. *The Lie algebra $\hat{\mathfrak{g}}(A) := \mathfrak{g}(A) + \mathfrak{h}_\Lambda = \mathfrak{n}_-(A) \oplus \mathfrak{h}_\Lambda \oplus \mathfrak{n}_+(A)$ is called the extended Kac–Moody algebra associated with Λ. The orthogonal complement of $\hat{\mathfrak{g}}(A)$ in $\mathfrak{g}_\Lambda$ with respect to the contravariant bilinear form $\langle\ |\ \rangle$ is called the space of missing (or decoupled) states, $\mathcal{M}$.*

It is clear that $\hat{\mathfrak{g}}(A)$ has the same root system and root space decomposition as $\mathfrak{g}(A)$. Note that $\mathcal{M}$ is the same as the orthogonal complement of $\hat{\mathfrak{g}}(A)$ in $\mathfrak{g}_\Lambda$ with respect to the invariant form $(\ |\ )$. Obviously, $\mathcal{M}$ has zero intersection with the Cartan subalgebra $\mathfrak{h}_\Lambda$ and with all the tachyonic root spaces ${\mathfrak{g}_\Lambda}^{(\mathbf{r})}$, $\mathbf{r} \in \Delta^{\mathrm{re}} = \Delta^{\mathrm{re}}(A)$. Hence we can write

$$\mathcal{M} = \mathcal{M}_- \oplus \mathcal{M}_+, \qquad \mathcal{M}_\pm := \bigoplus_{\mathbf{r}\in\Delta^{\mathrm{im}}_\pm} \mathcal{M}^{(\mathbf{r})}, \tag{2.18}$$

where $\Delta_{\pm}^{\text{im}}$ denotes the set of imaginary roots inside the forward or the backward lightcone, respectively,[8] and $\mathcal{M}^{(\mathbf{r})}$ is given as the orthogonal complement of the root space for $\mathfrak{g}(A)$ in $\mathfrak{g}_\Lambda$, viz.

$$\mathfrak{g}_\Lambda{}^{(\mathbf{r})} = \mathfrak{g}(A)^{(\mathbf{r})} \oplus \mathcal{M}^{(\mathbf{r})} \qquad \forall \mathbf{r} \in \Delta^{\text{im}}. \tag{2.19}$$

Note that it might (and in some examples does) happen that $\mathfrak{g}(A)^{(\mathbf{r})}$ is empty for some $\mathbf{r} \in \Delta^{\text{im}}$, namely when $\mathbf{r}$ is not a root for $\mathfrak{g}(A)$.

Generically, $\mathfrak{g}(A)$ is a (infinite-dimensional) Lorentzian Kac–Moody algebra about which not much is known. On the other hand we are in the lucky situation of having a root space decomposition with known multiplicities for $\mathfrak{g}_\Lambda$. So the main problem in this string realization of $\mathfrak{g}(A)$ is to understand the space of missing states. The starting point for the analysis presented below is the following theorem [23].

Theorem 5. 1. *$\mathcal{M}$ is completely reducible under the adjoint action of $\mathfrak{g}(A)$. It decomposes into an orthogonal (w.r.t. $\langle\ |\ \rangle$) direct sum of irreducible lowest or highest weight modules for $\mathfrak{g}(A)$:*

$$\mathcal{M}_\pm = \bigoplus_{\mathbf{r}\in\mathcal{B}} m_{\mathbf{r}} L(\mp\mathbf{r}), \tag{2.20}$$

where $\mathcal{B} \subset \Lambda \cap (-\mathcal{C})$ denotes some appropriate set of dominant integral weights for $\mathfrak{h}(A)$, $L(\mathbf{r})$ (resp. $L(-\mathbf{r})$) denotes an irreducible highest (resp. lowest) weight module for $\mathfrak{g}(A)$ with highest weight $\mathbf{r}$ (resp. lowest weight $-\mathbf{r}$), which occurs with multiplicity $m_{\mathbf{r}}$ (= $m_{-\mathbf{r}}$) inside $\mathcal{M}_-$ (resp. $\mathcal{M}_+$).

2. *Let $\mathcal{H}_\pm \subset \mathcal{M}_\pm$ denote the space of missing lowest and highest weight vectors, respectively. Equipped with the bracket in $\mathfrak{g}_\Lambda$, $\mathcal{H}_+$ and $\mathcal{H}_-$ are (isomorphic) Lie algebras. If there are no pairwise orthogonal imaginary simple roots in $\mathfrak{g}_\Lambda$, then they are free Lie algebras.*

Proof. Let $x \in \hat{\mathfrak{g}}(A)$, $m \in \mathcal{M}$. Then we can write $[x, m] = x' + m'$ for some $x' \in \hat{\mathfrak{g}}(A)$, $m' \in \mathcal{M}$. It follows that $(y|x') = (y|[x, m]) = ([y, x]|m) = 0$ for all $y \in \hat{\mathfrak{g}}(A)$ using invariance. Since the radical of the invariant form has been divided out we conclude that $x' = 0$. Thus $[\hat{\mathfrak{g}}(A), \mathcal{M}] \subseteq \mathcal{M}$ and the homomorphism property of $\rho : \mathfrak{g}(A) \to \text{End}\mathcal{M}$, $\rho(x)m := [x, m]$, follows from the Jacobi identity in $\mathfrak{g}_\Lambda$. But $\mathcal{M}_\pm$ are already $\hat{\mathfrak{g}}(A)$-modules by themselves. To see this, we exploit the $\mathbb{Z}$-grading of $\mathfrak{g}_\Lambda$ induced by the grading vector $\mathbf{t}$. An element of $\mathfrak{g}_\Lambda$ with momentum $\mathbf{r}$ is said to have height $\mathbf{r}\cdot\mathbf{t}$. Then $\mathcal{M}_+$ and $\mathcal{M}_-$ consist of elements of positive and negative height, respectively. Going from positive to negative weight with the action of $\hat{\mathfrak{g}}(A)$ requires missing states of height zero, which cannot exist since $\mathfrak{h}_\Lambda \subset \hat{\mathfrak{g}}(A)$.

By applying the Chevalley involution θ, it is sufficient to consider $\mathcal{M}_-$. Let $\mathcal{N} \subset \mathcal{M}_-$ be a $\hat{\mathfrak{g}}(A)$-submodule. Then

$$\mathcal{N} = \bigoplus_{\mathbf{r}\in\Delta_-^{\text{im}}} \mathcal{N}^{(\mathbf{r})}, \qquad \mathcal{N}^{(\mathbf{r})} := \mathcal{M}_-^{(\mathbf{r})} \cap \mathcal{N}.$$

Since $\dim \mathcal{M}_-^{(\mathbf{r})} \le \dim \mathfrak{g}_\Lambda{}_-^{(\mathbf{r})} < \infty$ and $\langle\ |\ \rangle$ is positive definite on $\mathcal{M}_-^{(\mathbf{r})}$ for all $\mathbf{r} \in \Delta$, it follows that we have the decomposition

$$\mathcal{M}_-^{(\mathbf{r})} = \mathcal{N}^{(\mathbf{r})} \oplus \mathcal{N}^{(\mathbf{r})\perp} \qquad \forall \mathbf{r} \in \Delta_-^{\text{im}}.$$

[8] This means that we choose the grading vector to lie inside the backward lightcone.

If we define

$$\mathcal{N}^{\perp} := \bigoplus_{\mathbf{r}\in\Delta_-^{\mathrm{im}}} \mathcal{N}^{(\mathbf{r})\perp},$$

then

$$\mathcal{M}_- = \mathcal{N} \oplus \mathcal{N}^{\perp},$$

and

$$\langle \mathcal{N}|x(m)\rangle = \langle \theta(x)(\mathcal{N})|m\rangle = 0$$

for all $x \in \hat{\mathfrak{g}}(A)$, $m \in \mathcal{N}^{\perp}$, since $\mathcal{N}$ is a submodule by assumption. Hence $\mathcal{N}^{\perp}$ is also a $\hat{\mathfrak{g}}(A)$-submodule and $\mathcal{M}_-$ is indeed completely reducible.

Finally, it is easy to see that each irreducible $\hat{\mathfrak{g}}(A)$-submodule $\mathcal{N} \subset \mathcal{M}_-$ is of highest-weight type. Indeed, $\mathcal{N}$ inherits the grading of $\mathcal{M}_-$ by height which is bounded from above by zero, whereas the Chevalley generators e_i $(i \in I)$ associated with real simple roots increase the height when applied to elements of $\mathcal{N}$.

Now we want to show that each irreducible $\hat{\mathfrak{g}}(A)$-module $\mathcal{N} \subset \mathcal{M}_-$ is also irreducible under the action of $\mathfrak{g}(A)$. We shall use an argument similar to the proof of Prop. 11.8 in [24]. Recall that we have the decomposition $\mathfrak{h}_\Lambda = \mathfrak{h}(A) \oplus \mathfrak{h}'$, where $\mathfrak{h}'$ is spanned by suitable elements $h_i = \mathbf{r}_i(-1)|\mathbf{0}\rangle$ $(i \in H)$ associated with imaginary simple roots $\mathbf{r}_i$. Obviously, any imaginary simple root $\mathbf{r}_i$ satisfies $\mathbf{r}_i\!\cdot\!\mathbf{r} \geq 0$ for all $\mathbf{r} \in \Delta_-^{\mathrm{im}}$ and $\mathbf{r}_i\!\cdot\!\mathbf{r}_j \leq 0$ for all $\mathbf{r}_j \in \Pi^{\mathrm{re}}$. Let us introduce a restricted grading vector by $\mathbf{t}' := \sum_{i\in H} \mathbf{r}_i$. We shall call the inner product of $\mathbf{t}'$ with any root $\mathbf{r}$ the restricted height of that root. The subspaces of $\mathcal{N}$ of constant restricted height are then given by

$$\mathcal{N}_h := \bigoplus_{\substack{\mathbf{r}\in\Delta_-^{\mathrm{im}} \\ \mathbf{t}'\cdot\mathbf{r}=h}} \mathcal{N}^{(\mathbf{r})}.$$

Since $\mathbf{t}'\!\cdot\!\mathbf{r} \geq 0$ for all $\mathbf{r} \in \Delta_-^{\mathrm{im}}$, there exists some minimal $h_{\min}$ such that $\mathcal{N}_{h_{\min}} \neq 0$ and $\mathcal{N}_h = 0$ for $h < h_{\min}$. We have a decomposition of $\mathfrak{g}(A)$ w.r.t. to the restricted height as well, viz.

$$\mathfrak{g}(A) = \mathfrak{g}_- \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_+, \qquad \mathfrak{g}_\pm := \bigoplus_{h \gtrless 0} \mathfrak{g}(A)_h.$$

Note that this triangular decomposition is different from the previous one encountered in (2.17). In general, they are related by $\mathfrak{n}_\pm(A) \subseteq \mathfrak{g}_\pm \oplus \mathfrak{g}_0$ and $\mathfrak{h}(A) \subseteq \mathfrak{g}_0$. Now, apparently each $\mathcal{N}_h$ is a $\mathfrak{g}_0$ module. In particular, $\mathcal{N}_{h_{\min}}$ must be irreducible, since any $\mathfrak{g}_0$-invariant proper subspace would generate a proper $\hat{\mathfrak{g}}(A)$ submodule of $\mathcal{N}$ contradicting its irreducibility. By the same argument, $\{v \in \mathcal{N}_h|\mathfrak{g}_-(v) = 0\} = 0$ for $h > h_{\min}$. Hence

$$\mathcal{N} = \mathfrak{U}(\mathfrak{g}_+)\mathcal{N}_{\mathrm{vac}},$$

where

$$\mathcal{N}_{\mathrm{vac}} := \{v \in \mathcal{N}|\mathfrak{g}_-(v) = 0\} = \mathcal{N}_{h_{\min}}$$

is an irreducible $\mathfrak{g}_0$ module. From this we conclude that $\mathcal{N}$ is indeed an irreducible $\mathfrak{g}(A)$ module.

So $\mathcal{M}_-$ decomposes into an othogonal direct sum

$$\mathcal{M}_- = \bigoplus_{\alpha\in\mathcal{B}} m_\alpha L_\alpha,$$

where $\mathcal{B}$ denotes some appropriate index set and each L_α is an irreducible $\mathfrak{g}(A)$-module occurring with multiplicity $m_\alpha > 0$. Finally, it is easy to see that each irreducible $\mathfrak{g}(A)$-submodule $L_\alpha \subset \mathcal{M}_-$ is of highest-weight type. Indeed, L_α inherits the grading of $\mathcal{M}_-$ by height which is bounded from above by zero, whereas the Chevalley generators e_i ($i \in I$) associated with real simple roots increase the height when applied to vectors in L_α. So there exists an element $v_\mathbf{r} \in L_\alpha$ associated with a dominant integral weight $\mathbf{r} \in \Lambda \cap (-\mathcal{C})$ such that $e_i(v_\mathbf{r}) = 0$ for all $i \in I$ and $L_\alpha \equiv L(\mathbf{r}) = \mathfrak{U}\big(\mathfrak{n}_-(A)\big)v_\mathbf{r}$.

To prove the second part of the theorem, let $v_1, v_2 \in \mathcal{H}_-$. It follows that $x\big([v_1, v_2]\big) := [x,[v_1,v_2]] = [x(v_1), v_2] + [v_1, x(v_2)]$. If we choose $x = e_i$ or $x = h_i$, respectively, it is clear that $[v_1, v_2]$ is again a highest weight vector. To see that it is missing we note that $\langle x|[v_1,v_2]\rangle = \langle x\big(\theta(v_1)\big)|v_2\rangle$ for all $x \in \mathfrak{g}(A)$ by contravariance. But since $x\big(\theta(v_1)\big) \in \mathcal{M}_+$ and $v_2 \in \mathcal{M}_- \perp \mathcal{M}_+$ we see that indeed $[v_1, v_2] \in \mathcal{H}_-$. Finally, since $\mathfrak{g}_\Lambda$ is a Borcherds algebra we know that extra Lie algebra relations (in addition to those for $\mathfrak{g}(A)$) can occur only if there are pairwise orthogonal imaginary simple roots in $\mathfrak{g}_\Lambda$. If this is not the case $\mathcal{H}_\pm$ must be free. $\square$

So the space of missing states decomposes into an orthogonal direct sum of irreducible $\mathfrak{g}(A)$-multiplets each of which is obtained by repeated application of the raising operators e_i (resp. f_i) to some lowest (resp. highest) weight vector. This beautiful structure, however, looks rather messy from the point of view of a single missing root space, $\mathcal{M}^{(\mathbf{r})}$, say. Generically, it decomposes into an orthogonal direct sum of three subspaces with special properties, viz.

$$\mathcal{M}^{(\mathbf{r})} = \mathcal{R}^{(\mathbf{r})} \oplus \mathcal{H}^{(\mathbf{r})} \oplus \mathcal{J}^{(\mathbf{r})}, \qquad \text{for } \mathbf{r} \in \Delta_+^{\text{im}}, \tag{2.21}$$

where $\mathcal{R}^{(\mathbf{r})}$ consists of states belonging to lower-height $\mathfrak{g}(A)$-multiplets and $\mathcal{H}^{(\mathbf{r})} := [\mathcal{H}_+, \mathcal{H}_+] \cap \mathcal{M}^{(\mathbf{r})}$ is spanned by multiple commutators of appropriate lower-height vacuum vectors. What can we say about the remaining piece, $\mathcal{J}^{(\mathbf{r})}$? Its states are vacuum vectors for $\mathfrak{g}(A)$, which cannot be reached by multiple commutators inside the space of missing lowest weight vectors, $\mathcal{H}_+$. So a basis for $\mathcal{J}^{(\mathbf{r})}$ is part of a basis for $\mathcal{H}_+$. At the level of the Borcherds algebra $\mathfrak{g}_\Lambda$, this just means that the root $\mathbf{r}$ is an imaginary simple root of multiplicity dim $\mathcal{J}^{(\mathbf{r})}$. For this reason we introduce the so-called simple multiplicity $\mu(\mathbf{r})$ of a root $\mathbf{r}$ in the fundamental Weyl chamber as

$$\mu(\mathbf{r}) := \dim \mathcal{J}^{(\mathbf{r})}. \tag{2.22}$$

Obviously we have $\mu(\mathbf{r}) \leq \text{mult}(\mathbf{r})$. Once we know the simple multiplicity of a fundamental root, it is clear how to proceed. Recursively by height, we adjoin to $\mathfrak{g}(A)$ for each fundamental root $\mathbf{r}$ a set of $\mu(\mathbf{r})$ generators $\{e_j, f_j, h_j\}$. This also explains why it is sufficient to concentrate on fundamental roots. Indeed, by the action of the Weyl group we conclude that the simple multiplicity of any non-fundamental positive imaginary root is zero, while the Chevalley involution tells us that $\mu(\mathbf{r}) = \mu(-\mathbf{r})$ – this just reflects that the fact that we adjoin the Chevalley generators e_j and f_j always in pairs.

Let us point out that for ordinary (i.e. not generalized in the sense of Borcherds) Kac–Moody algebras, for which *all* elements of any root space are obtained as multiple commutators of the Chevalley–Serre generators (by the very definition of a Kac–Moody algebra!), we have $\mu(\mathbf{r}) = 0$, and therefore the notion of simple multiplicity is superfluous.

3. The Gnome Lie Algebra

The gnome Lie algebra $\mathfrak{g}_{II_{1,1}}$, which we will investigate in this section, is the simplest example of a Borcherds algebra that can be explicitly described as the Lie algebra of physical states of a compactified string. It is based on the lattice $II_{1,1}$ as the momentum lattice of a fully compactified bosonic string in two space-time dimensions. Since there are no transversal degrees of freedom in $d = 2$ and only longitudinal string excitations occur, the Lie algebra of physical states may be regarded as the precise opposite of the fake monster Lie algebra in 26 dimensions which has only transversal and no longitudinal physical states. It constitutes an example of a generalized Kac–Moody algebra which is almost "purely Borcherds" in that with one exception, all its simple roots are imaginary (timelike). The gnome Lie algebra is also a cousin of the true monster Lie algebra because they both have the same root lattice, $II_{1,1}$. In fact, we shall see that the gnome Lie algebra is a Borcherds subalgebra not only of the fake monster Lie algebra but also of any Lie algebra of physical states associated with a momentum lattice that can be decomposed in such a way that it contains $II_{1,1}$ as a sublattice.

3.1. The lattice $II_{1,1}$. We start by summarizing some properties of the unique two-dimensional even unimodular Lorentzian lattice $II_{1,1}$. It can be realized as

$$II_{1,1} := \mathbb{Z}(\tfrac{1}{2}; \tfrac{1}{2}) \oplus \mathbb{Z}(-1; 1) = \{(\ell/2 - n; \ell/2 + n) \,|\, \ell, n \in \mathbb{Z}\},$$

where for the (Minkowskian) product of two vectors our convention is

$$(x^1; x^0) \cdot (y^1; y^0) := x^1 y^1 - x^0 y^0.$$

Alternatively, we will represent the elements of $II_{1,1}$ in a light cone basis, i.e., in terms of pairs $\langle \ell, n\rangle \in \mathbb{Z} \oplus \mathbb{Z}$ with inner product matrix $\left(\begin{smallmatrix} 0 & -1 \\ -1 & 0 \end{smallmatrix}\right)$, so that $\langle \ell, n\rangle^2 = -2\ell n$. The lattice points are shown in Fig. 1 below. The main importance of this lattice for us derives from the fact that it is the root lattice of the Lie algebra $\mathfrak{g}_{II_{1,1}}$ we are about to construct. As already explained in the last section, allowed physical string momenta have norm squared at most two and consequently any root Λ for $\mathfrak{g}_{II_{1,1}}$ must obey $\Lambda^2 \leq 2$. There are no lightlike roots here: the corresponding root spaces are empty owing to the absence of transversal polarizations in two dimensions. Therefore, imaginary roots for $\mathfrak{g}_{II_{1,1}}$ are all lattice vectors lying in the interior of the lightcone. Real roots satisfy $\Lambda^2 = 2$, and the lattice $II_{1,1}$ possesses only two such roots $\Lambda = \pm\mathbf{r}_{-1}$, where

$$\mathbf{r}_{-1} := (\tfrac{3}{2}; -\tfrac{1}{2}) = \langle 1, -1\rangle.$$

Our notation has been chosen so as to make explicit the analogy with E_{10}, where $\mathbf{r}_{-1}$ is the over-extended root. In addition we need the lightlike vector

$$\boldsymbol{\delta} := (-1; 1) = \langle 0, 1\rangle,$$

obeying $\mathbf{r}_{-1} \cdot \boldsymbol{\delta} = -1$. Hence it serves as a lightlike Weyl vector for $\mathfrak{g}_{II_{1,1}}$.[9] It is analogous to the null root of the affine subalgebra $E_9 \subset E_{10}$, but the crucial difference is that for

[9] It is, however, only a "real" Weyl vector since it has scalar product -1 with all real simple roots, whereas it will not have the correct scalar products with all imaginary simple roots. In fact, there is no true Weyl vector for $\mathfrak{g}_{II\,1,1}$.

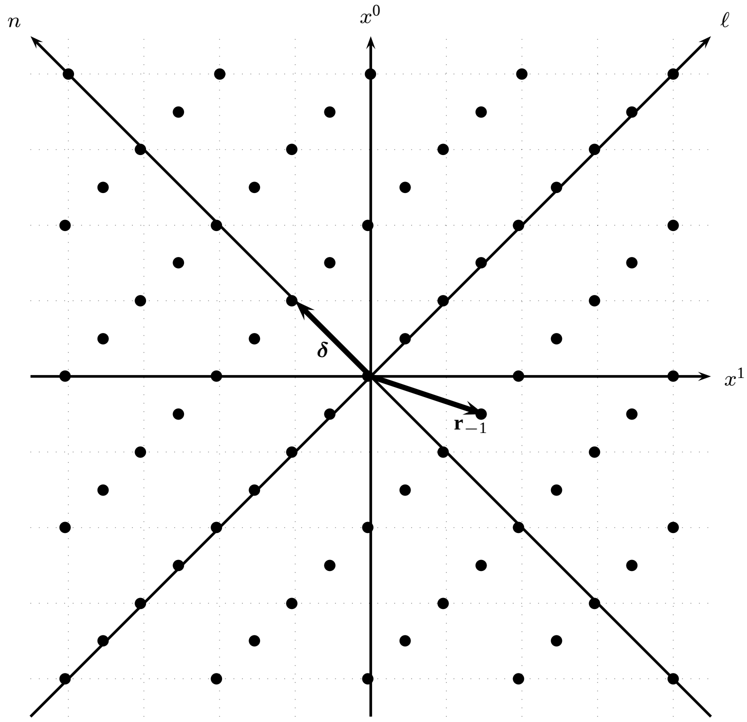


Fig. 1. The Lorentzian lattice $\mathit{II}_{1,1}$

$\mathit{II}_{1,1}$ it is *not* a root (see the above remark). Nonetheless, we can use $\boldsymbol{\delta}$ to introduce the notion of level (again by analogy with E_{10}), namely, by assigning to a root Λ the integer

$$\ell := -\boldsymbol{\delta}\cdot\Lambda .$$

This gives us a $\mathbb{Z}$-grading of the set of roots. The reflection symmetry of the lattice, which gives rise to the Chevalley involution of $\mathfrak{g}_{\mathit{II}_{1,1}}$ and which introduces the splitting of the set of roots into positive and negative roots, apparently changes the level into its negative. Consequently, the sign of the level of a root determines whether it is positive or negative, and for an analysis of $\mathfrak{g}_{\mathit{II}_{1,1}}$ it is sufficient to consider positive roots only. We conclude that the set of positive roots for $\mathfrak{g}_{\mathit{II}_{1,1}}$ consists of the level-1 root $\mathbf{r}_{-1}$ and the infinitely many lattice vectors lying inside the forward lightcone.

The Weyl group of $\mathit{II}_{1,1}$ is very simple: since we can only reflect with respect to the single root $\mathbf{r}_{-1}$, it has only two elements and is thus isomorphic to $\mathbb{Z}_2$ just like the Weyl group of the monster Lie algebra [5]. On any vector $\mathbf{x} \in \mathbb{R}^{1,1}$ it acts as $\mathfrak{w}_{-1}(\mathbf{x}) := \mathbf{x} - (\mathbf{x}\cdot\mathbf{r}_{-1})\mathbf{r}_{-1}$; in light cone coordinates we have the simple formula

$$\mathfrak{w}_{-1}\big(\langle \ell, n\rangle\big) = \langle n, \ell\rangle .$$

Hence the forward lightcone is the union of only two Weyl chambers; the fundamental Weyl chamber leading to our choice of the real simple root has been shaded in Fig. 2. It is given by

$$\mathcal{C} = \{\mathbf{x} \in \mathbb{R}^{1,1} \,|\, \mathbf{x}^2 \le 0, \mathbf{x}\cdot\mathbf{r}_{-1} \le 0, \mathbf{x}\cdot\boldsymbol{\delta} \le 0\}.$$

The imaginary positive roots inside $\mathcal{C}$ will be called fundamental roots. Combining the action of the Weyl group with the reflection symmetry of the lattice, the whole analysis of $\mathfrak{g}_{II_{1,1}}$ is thereby reduced to understanding the root spaces associated with fundamental roots.

Obviously, $\mathbf{r}_{-1}$ and $\boldsymbol{\delta}$ span $II_{1,1}$, and thus any positive level-ℓ root can be written as

$$\Lambda = \ell\mathbf{r}_{-1} + n\boldsymbol{\delta} = \langle \ell, n - \ell\rangle,$$

where $n > \ell > 0$ because of $\Lambda^2 = 2\ell(\ell - n)$.

As explained in [17], the DDF construction necessitates the introduction of fractional momenta which do not belong to the lattice. We define

$$\mathbf{a}_\ell := \ell\mathbf{r}_{-1} + \left(\ell - \frac{1}{\ell}\right)\boldsymbol{\delta}, \qquad \mathbf{k}_\ell := -\frac{1}{\ell}\boldsymbol{\delta},$$

such that we can write down the so-called DDF decomposition

$$\Lambda = \mathbf{a}_\ell - \left(1 - \frac{1}{2}\Lambda^2\right)\mathbf{k}_\ell \tag{3.1}$$

for any positive level-ℓ root Λ. The tachyonic momenta $\mathbf{a}_\ell$ lie on a mass shell hyperbola $\mathbf{a}_\ell^2 = 2$ which has been depicted in Fig. 2 below. This figure also displays the intermediate points (as small circles) "between the lattice" required by the DDF construction, and allows us to visualize how the lattice becomes more and more "fractionalized" with increasing level. We call vectors $\mathbf{a}_\ell - m\mathbf{k}_\ell$, $0 \le m \le -\frac{1}{2}\Lambda^2$, which are not lattice points fractional roots. Note that fractional roots can only occur for $\ell > 1$. We stress that the physical states associated with these intermediate points are *not* elements of the Lie algebra $\mathfrak{g}_{II_{1,1}}$, as their operator product expansions will contain fractional powers.

3.2. Basic structure of the gnome Lie algebra. The gnome Lie algebra is by definition the Borcherds algebra $\mathfrak{g}_{II_{1,1}}$ of physical states of a bosonic string fully compactified on the lattice $II_{1,1}$. We would first like to describe its root space decomposition. To do so, we assign the grading $\langle \ell, n\rangle$ to any string state with momentum $\langle \ell, n\rangle = \ell\mathbf{r}_{-1} + (n - \ell)\boldsymbol{\delta} \in II_{1,1}$. The no-ghost theorem in the guise of Thm. 1 then implies that the contravariant form $\langle\ |\ \rangle$ is positive definite on the piece of nonzero degree of the gnome Lie algebra $\mathfrak{g}_{II_{1,1}}$. The degree $\langle 0, 0\rangle$ piece of $\mathfrak{g}_{II_{1,1}}$ is isomorphic to $\mathbb{R}^2$, while the tachyonic states $|\pm\mathbf{r}_{-1}\rangle$ yield two one-dimensional subspaces of degrees $\langle -1, 1\rangle$ and $\langle 1, -1\rangle$, respectively. With these conventions, the gnome Lie algebra looks schematically like the monster Lie algebra (see Fig. 3 and [6]). Here we have indexed the subspace associated with the root $\Lambda = \langle \ell, n\rangle$ by $[\ell n]$ because the dimension of this root space depends only on the product ℓn. Indeed, since $1 - \frac{1}{2}\langle \ell, n\rangle^2 = 1 + \ell n$ we have, according to (2.10),

$$\mathrm{mult}_{\mathfrak{g}_{II_{1,1}}}(\Lambda) \equiv \dim {\mathfrak{g}_{II_{1,1}}}^{(\Lambda)} = \pi_1(1 + \ell n),$$

where the partition function $\pi_1(n)$ was already defined in (2.11).

While this description of $\mathfrak{g}_{II_{1,1}}$ is rather abstract, we can give a much more concrete realization of this Lie algebra by means of the discrete DDF construction developed in [17]. In fact, the DDF construction provides us with a *complete basis* for the gnome Lie algebra.

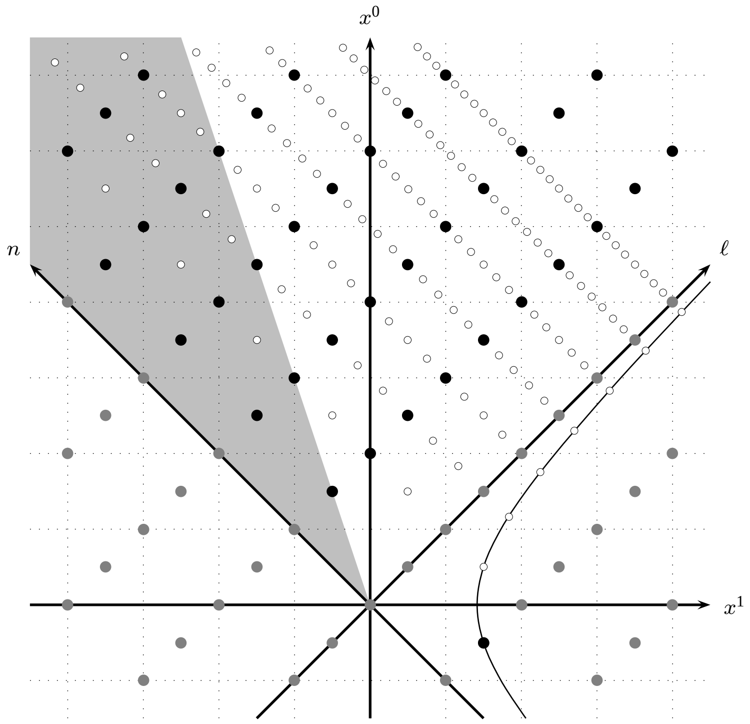


Fig. 2. Fundamental Weyl chamber, positive and fractional roots for $\mathfrak{g}_{II\,1,1}$

The single real simple root $\mathbf{r}_{-1}$ of $II_{1,1}$ gives rise to Lie algebra elements (cf. Eq. (2.15))

$$h_{-1} := \mathbf{r}_{-1}(-1)|\mathbf{0}\rangle, \qquad e_{-1} := |\mathbf{r}_{-1}\rangle, \qquad f_{-1} := -|-\mathbf{r}_{-1}\rangle, \tag{3.2}$$

which generate the finite Kac–Moody subalgebra $\mathfrak{g}(A) = \mathfrak{sl}_2 \equiv A_1 \subset \mathfrak{g}_{II_{1,1}}$. On the other hand, there are infinitely many imaginary (timelike) roots inside the lightcone. We shall see that out of these all fundamental roots (except for one) will be simple roots as well.

We notice that the one-dimensional Cartan subalgebra $\mathfrak{h}(A)$ spanned by h_{-1} does not coincide with the two-dimensional Cartan subalgebra $\mathfrak{h}_{II_{1,1}}$. Hence we need to introduce the Lie algebra

$$\hat{\mathfrak{g}}(A) := \mathfrak{sl}_2 + \mathfrak{h}_{II_{1,1}} = \mathfrak{sl}_2 \oplus \mathbb{R}\Lambda_0$$

by adjoining to $\mathfrak{sl}_2$ the element

$$\Lambda_0 := (\mathbf{r}_{-1} + 2\delta)(-1)|\mathbf{0}\rangle,$$

which commutes with $\mathfrak{sl}_2$ and therefore behaves like a central charge (but notice that the affine extension of $\mathfrak{sl}_2$ is *not* a subalgebra of $\mathfrak{g}_{II_{1,1}}$). It may be regarded as a remnant of the Cartan subalgebra of the hyperbolic extension of a zero-dimensional (virtual) Lie algebra.

We see that in this example the Lie algebra $\mathfrak{g}(A)$ is too small to yield a lot of information (the "smallness" of $\mathfrak{g}(A)$ is due to the absence of transversal physical string

$$
\begin{array}{ccccccccccccc}
n & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ell \\
 & \mathfrak{g}^{[6]} & & \mathfrak{g}^{[10]} & & \mathfrak{g}^{[12]} & & \mathfrak{g}^{[12]} & & \mathfrak{g}^{[10]} & & \mathfrak{g}^{[6]} & \\
\cdots & & \mathfrak{g}^{[5]} & & \mathfrak{g}^{[8]} & & \mathfrak{g}^{[9]} & & \mathfrak{g}^{[8]} & & \mathfrak{g}^{[5]} & & \cdots \\
\cdots & 0 & & \mathfrak{g}^{[4]} & & \mathfrak{g}^{[6]} & & \mathfrak{g}^{[6]} & & \mathfrak{g}^{[4]} & & 0 & \cdots \\
\cdots & & 0 & & \mathfrak{g}^{[3]} & & \mathfrak{g}^{[4]} & & \mathfrak{g}^{[3]} & & 0 & & \cdots \\
\cdots & 0 & & 0 & & \mathfrak{g}^{[2]} & & \mathfrak{g}^{[2]} & & 0 & & 0 & \cdots \\
\cdots & & 0 & & 0 & & \mathfrak{g}^{[1]} & & 0 & & 0 & & \cdots \\
\cdots & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & \cdots \\
\cdots & & 0 & & \mathfrak{g}^{[-1]} & & \mathbb{R}^2 & & \mathfrak{g}^{[-1]} & & 0 & & \cdots \\
\cdots & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & \cdots \\
\cdots & & 0 & & 0 & & \mathfrak{g}^{[1]} & & 0 & & 0 & & \cdots \\
\cdots & 0 & & 0 & & \mathfrak{g}^{[2]} & & \mathfrak{g}^{[2]} & & 0 & & 0 & \cdots \\
\cdots & & 0 & & \mathfrak{g}^{[3]} & & \mathfrak{g}^{[4]} & & \mathfrak{g}^{[3]} & & 0 & & \cdots \\
\cdots & 0 & & \mathfrak{g}^{[4]} & & \mathfrak{g}^{[6]} & & \mathfrak{g}^{[6]} & & \mathfrak{g}^{[4]} & & 0 & \cdots \\
\cdots & & \mathfrak{g}^{[5]} & & \mathfrak{g}^{[8]} & & \mathfrak{g}^{[9]} & & \mathfrak{g}^{[8]} & & \mathfrak{g}^{[5]} & & \cdots \\
 & \mathfrak{g}^{[6]} & & \mathfrak{g}^{[10]} & & \mathfrak{g}^{[12]} & & \mathfrak{g}^{[12]} & & \mathfrak{g}^{[10]} & & \mathfrak{g}^{[6]} & \\
 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots &
\end{array}
$$

Fig. 3. Root space decomposition of the gnome Lie algebra

states in two dimensions). Nonetheless, there are infinitely many purely longitudinal physical states present which are of the form

$$A^-_{-n_1}(\mathbf{a}_\ell) \cdots A^-_{-n_N}(\mathbf{a}_\ell)|\mathbf{a}_\ell\rangle, \tag{3.3}$$

where $n_1 \geq n_2 \geq \ldots \geq n_N \geq 2$ and the longitudinal DDF operators $A^-_{-n_a}$ are associated with a tachyon momentum $\mathbf{a}_\ell$ and a lightlike vector $\mathbf{k}_\ell$ satisfying $\mathbf{a}_\ell \cdot \mathbf{k}_\ell = 1$. Of course, not all of these string states belong to $\mathfrak{g}_{II_{1,1}}$; in addition, we must require that (cf. Eq. (3.1))

$$\Lambda := \mathbf{a}_\ell - M\mathbf{k}_\ell$$

is a root, i.e. $\Lambda \in II_{1,1}$ with $\Lambda^2 \leq 2$, so that

$$M := \sum_{j=1}^{N} n_j = 1 - \tfrac{1}{2}\Lambda^2 \geq 0.$$

In other words, given a root $\Lambda = \ell\mathbf{r}_{-1} + n\boldsymbol{\delta}$, a basis of the associated root space $\mathfrak{g}_{II_{1,1}}{}^{(\Lambda)}$ is provided by longitudinal DDF states of the above form with total excitation number $M = \ell(n-\ell)+1$. For momenta of the form $\mathbf{a}_\ell - m\mathbf{k}_\ell$, $0 \leq m < M$, such that $m-1$ is not a multiple of ℓ, i.e., for fractional roots "between the lattice points" (cf. Fig. 2), we obtain "intermediate (physical) states" which are not elements of the Lie algebra $\mathfrak{g}_{II_{1,1}}$. In fact, they are not full-fledged states of the string model under consideration but rather states of the uncompactified string model.

It is clear that, apart from the subalgebra $\hat{\mathfrak{g}}(A)$, all elements of the gnome Lie algebra are associated with imaginary roots. And since none of the longitudinal states can be obtained by multiple commutation of elements of $\mathfrak{sl}_2$, all of them are missing states. Thus

$$\begin{aligned}\mathcal{M}_+^{(\Lambda)} &= {\mathfrak{g}_{II_{1,1}}}^{(\Lambda)} \\ &= \text{span}\{A^-_{-n_1}(\mathbf{a}_\ell)\cdots A^-_{-n_N}(\mathbf{a}_\ell)|\mathbf{a}_\ell\rangle \mid n_j > 1, n_1 + \ldots + n_N = 1 - \tfrac{1}{2}\Lambda^2\},\end{aligned} \tag{3.4}$$

for all $\Lambda \in \Delta_+^{\text{im}}$ and similarly for $\mathcal{M}_-$. From the point of view of $\mathfrak{sl}_2$, all these states must be added "by hand" to fill up $\mathfrak{sl}_2$ to $\mathfrak{g}_{II_{1,1}}$. Having a complete basis for the space of missing states the task is now to determine the complete set of imaginary simple roots. In principle, this can be achieved in two steps. First, we have to identify all the missing lowest weight vectors in $\mathcal{M}_+$. Then we have to determine a basis for the Lie algebra of lowest weight vectors. This provides us with the complete information about the imaginary simple roots and their multiplicities. In the next subsection, this strategy is discussed in more detail and is illustrated by some examples.

For the gnome Lie algebra, the information about the imaginary simple roots and their multiplicities can be determined by means of the Weyl–Kac–Borcherds denominator formula. One reason for this is the simplicity of the Weyl group of $\mathfrak{sl}_2$ which simplifies the denominator formula enormously. It reads

$$\begin{aligned}&\left(x^{-1} - y^{-1}\right)\prod_{\ell,n>0}(1 - x^\ell y^n)^{\pi_1(1+\ell n)} \\ &\qquad = \left(x^{-1} - y^{-1}\right) + \sum_{n\geq\ell>0}\mu_{\ell,n}\left(x^n y^{\ell-1} - x^{\ell-1}y^n\right),\end{aligned} \tag{3.5}$$

where we write $x \equiv \mathrm{e}^{\langle 1,0\rangle}$ and $y \equiv \mathrm{e}^{\langle 0,1\rangle}$ for the generators of the group algebra of $II_{1,1}$ and we put $\mu_{\ell,n} \equiv \mu\big(\langle \ell, n\rangle\big)$. Recall that the action of the Weyl group simply interchanges x and y. Also note that the fundamental roots have nonzero inner product with each other so that there is no extra contribution of pairwise orthogonal imaginary simple roots on the right-hand side. Therefore we are in the fortunate situation that the sum on the right-hand side runs only once over the imaginary simple roots and that the relevant coefficients are just the simple multiplicities. Furthermore, the associated Lie algebra of lowest weight vectors, $\mathcal{H}_+$, is a free Lie algebra, which follows from Thm. 5 due to the fact that there are no lightlike roots (cf. [23]).

We summarize: a set of imaginary simple roots for the gnome Lie algebra $\mathfrak{g}_{II_{1,1}}$ is given by the vectors $\{\langle \ell, n\rangle \mid n \geq \ell \geq 1\}$, each with multiplicity $\mu_{\ell,n}$ which is the coefficent of $x^n y^{\ell-1}$ in the left-hand side of Eq. (3.5) as generating function.

Expanding the latter, one readily obtains the results (see Fig. 4)

$$\begin{aligned}\mu_{1,n} &= \pi_1(1+n) \qquad \text{for } n \geq 1, \\ \mu_{2,n} &= \pi_1(1+2n) - \pi_1(2+n) - \tfrac{1}{2}\pi_1(1+\tfrac{n}{2})\big[\pi_1(1+\tfrac{n}{2}) - 1\big] \\ &\quad - \sum_{k=1}^{\left[\frac{n-1}{2}\right]}\pi_1(1+k)\pi_1(1+n-k) \qquad \text{for } n \geq 2,\end{aligned} \tag{3.6}$$

where we have defined $\pi_1(1+\frac{n}{2}) := 0$ for any odd integer n. The first formula tells us that all level-1 longitudinal states are missing states associated with imaginary simple roots; from the second we learn that this is no longer true at higher level since $\mu_{2,n} < \pi_1(1+2n)$ and consequently some of the associated states can be generated by commutation of level-1 states. In fact, one easily sees that not only does $\mu(\Lambda)$ not vanish in general, and hence all higher-level roots are simple with a certain multiplicity, but also that $\mu(\Lambda) < \text{mult}(\Lambda)$ at

$\boxed{\mu_{\ell,n}}$

$\ell\backslash n$	1	2	3	4	5	6
1	1	1	2	2	4	4
2		0	1	2	6	10
3			3	6	20	40
4				5	36	101
5					63	239
6						331

$\boxed{\mathrm{mult}(\langle \ell, n\rangle)}$

$\ell\backslash n$	1	2	3	4	5	6
1	1	1	2	2	4	4
2	1	2	4	8	14	24
3	2	4	12	24	55	105
4	2	8	24	66	165	383
5	4	14	55	165	478	1238
6	4	24	105	383	1238	3660

Fig. 4. Multiplicity of imaginary simple roots vs. dimension of root spaces

higher level. This illustrates the point we have already made in the introduction and in the past [17]: while generalized Kac–Moody algebras such as the gnome may have a rather simple structure in terms of the DDF construction, they are usually quite complicated to analyze from the point of view of their root space decompositions. For hyperbolic Kac–Moody algebras, the situation is precisely the reverse: the simple roots can be read off from the Coxeter–Dynkin diagram, but the detailed structure of the root spaces is exceedingly complicated.

Due to the complicated pattern of the imaginary simple roots and their multiplicities, the approach of decomposing $\mathfrak{g}_{II_{1,1}}$ into multiplets of $\mathfrak{sl}_2$ seems to be not very fruitful. One reason for this is that $\mathfrak{sl}_2$ is just "too small" to yield non-trivial information about the full Lie algebra – in stark contrast to the algebra $\mathfrak{g}_{II_{9,1}}$ whose corresponding subalgebra $\mathfrak{g}(A) = E_{10}$ is much bigger. Another reason, which is not so obvious, comes from the observation that for increasing level the dimensions of the root spaces grow much faster than the simple multiplicities. This explains why additional imaginary simple roots are needed at every level. There is a beautiful example where this situation is rectified. The true monster Lie algebra [6] is a Borcherds algebra which is based on the same lattice $II_{1,1}$ as the root lattice; but the multiplicity of a root $\langle \ell, n\rangle$ is given by $c(\ell n)$ (replacing $\pi_1(1 + \ell n)$) which is the coefficient of $q^{\ell n}$ in the elliptic modular function $j(q) - 744 = \sum_{n\geq -1} c_n q^n = q^{-1} + 196884q + \ldots$. In [6], Borcherds was able to determine a set of imaginary simple roots and their simple multiplicities by establishing an identity for the elliptic modular function which turned out to be precisely the above denominator formula. In that example, the imaginary simple roots are all level-1 vectors $\langle 1, n\rangle$ $(n \geq 1)$, each with multiplicity $c(n)$. Thus the simple multiplicities are large enough so that the level-1 $\mathfrak{sl}_2$ vacuum vectors can generate by multiple commutators the full Lie algebra of missing lowest weight vectors.

Even though the infinite Cartan matrix looks rather messy, the gnome Lie algebra $\mathfrak{g}_{II_{1,1}}$ has now been cast into the form of a Borcherds algebra in the sense of Def. 1. The next step in the analysis would be the calculation of the structure constants. Since we have exhibited an explicit basis of the algebra in terms of the DDF states, this can be done in principle. Practically, however, the calculations still have to be performed by use of the humble oscillator basis $\{\alpha_m^\mu\}$, whereas we would prefer to be able to calculate the commutators of DDF states in a manifestly physical way, i.e., in a formalism based on the DDF operators only. For the transversal DDF operators this problem was solved recently [18]. However, since we are dealing with purely longitudinal excitations here, one would certainly have to consider exponentials of longitudinal DDF operators. This is technically much more delicate, since the operators do not form a Heisenberg algebra but a Virasoro algebra. Let us also point out the evident relation between the gnome Lie algebra and Liouville theory, which remains to be understood in more detail.

3.3. DDF states and examples. We will now perform some explicit checks and for some examples exhibit the split of the root spaces into parts that can be generated by commutation of low-level elements and the remaining states which must be adjoined by hand, and whose number equals the simple multiplicity of the root in question. Since the actual calculations are quite cumbersome it is helpful to use a computer. We would like to emphasize that these examples not only provide completely explicit realizations of the Lie algebra elements, but also enable us to determine the "structure constants", whereas for other Borcherds algebras (such as the true or the fake monster Lie algebra), investigations so far have been limited to the determination of root space multiplicities and the modular properties of the associated partition functions.

It is natural to investigate the subspace $\mathcal{M}_+$ of missing states of the gnome Lie algebra recursively level by level:

$$\mathcal{M}_+ = \bigoplus_{\ell>0} \mathcal{M}^{[\ell]}, \qquad \mathcal{M}^{[\ell]} := \bigoplus_{\substack{\Lambda\in\Delta_+^{\mathrm{im}} \\ \Lambda\cdot\boldsymbol{\delta}=-\ell}} \mathcal{M}_+^{(\Lambda)}. \tag{3.7}$$

We observe that, already at level 1, we have an infinite tower of missing states; indeed, the states

$$A^-_{-n_1}(\mathbf{r}_{-1})\cdots A^-_{-n_N}(\mathbf{r}_{-1})|\mathbf{r}_{-1}\rangle \tag{3.8}$$

span $\mathcal{M}^{[1]}$. Adjoining these states to the algebra is therefore tantamount to adjoining infinitely many imaginary simple level-1 roots $\mathbf{r}_{-1} + n\boldsymbol{\delta} = \langle 1, n-1\rangle$ $(n > 1)$ with multiplicity $\pi_1(n)$.[10] Although this statement is evident, we would like to demonstrate explicitly that these states are indeed lowest weight vectors for irreducible $\mathfrak{sl}_2$-modules. So let us consider the state $v_\Lambda := A^-_{-n_1}(\mathbf{r}_{-1})\cdots A^-_{-n_N}(\mathbf{r}_{-1})|\mathbf{r}_{-1}\rangle$, where $\Lambda := \mathbf{r}_{-1}+n\boldsymbol{\delta}$, $n := \sum_{j=1}^N n_j > 1$. Using the adjoint action in $\mathfrak{g}_{II_{1,1}}$ and the formulas for $\mathfrak{sl}_2$ given in (3.2), we infer that

$$\begin{aligned} h_{-1}(v_\Lambda) &= (2-n)v_\Lambda, \\ f_{-1}(v_\Lambda) &\propto L_{-1}|n\boldsymbol{\delta}\rangle \equiv 0, \\ (e_{-1})^{1-\mathbf{r}_{-1}\cdot\Lambda}(v_\Lambda) &\propto L_{-1}|n(\mathbf{r}_{-1}+\boldsymbol{\delta})\rangle \equiv 0. \end{aligned}$$

Note that the last two relations (the lowest weight and the null vector condition, respectively) follow from momentum conservation (cf. Eq. (2.2)) and the fact that physical string states in two dimensions are bound to be null states. Hence v_Λ is indeed a vacuum vector for an irreducible $\mathfrak{sl}_2$-module with spin $\frac{1}{2}(n-2)$. These multiplets can be constructed by repeated application of the raising operator e_{-1} which each time increases the level by one. Clearly, the higher-level states belong to irreducible $\mathfrak{sl}_2$-multiplets, but the structure quickly becomes rather messy. As already mentioned, we have to decompose each missing root space $\mathcal{M}_+^{(\Lambda)}$ into an orthogonal direct sum of three subspaces with special properties: one consists of states belonging to lower-level $\mathfrak{sl}_2$-multiplets, the other is made up of appropriate multiple commutators of lower-level vacuum vectors,

[10] As already mentioned, there are no proper physical states on the lightcone, i.e., with momenta proportional to the lightlike vectors $\boldsymbol{\delta} = \langle 0, 1\rangle$ and $\mathbf{r}_{-1} + \boldsymbol{\delta} = \langle 1, 0\rangle$, since these would require transversal polarizations.

and the rest comes from states corresponding to imaginary simple roots. We will now illustrate this pattern by a few examples.

So the question is which of the higher-level states can be generated by multiple commutators of the missing level-1 states. As it turns out we will have to add new states at each higher-level root, apart from an exceptional level-2 root which we will exhibit below.

We have calculated the following commutators (by means of `MAPLE V`)

$$\big[|\mathbf{r}_{-1}\rangle, A^-_{-3}|\mathbf{r}_{-1}\rangle\big] = A^-_{-3}|\mathbf{a}_2\rangle, \tag{3.9}$$

$$\big[|\mathbf{r}_{-1}\rangle, A^-_{-4}|\mathbf{r}_{-1}\rangle\big] = \Big(-\tfrac{3}{8}A^-_{-3}A^-_{-2} - \tfrac{5}{8}A^-_{-5}\Big)|\mathbf{a}_2\rangle, \tag{3.10}$$

$$\big[|\mathbf{r}_{-1}\rangle, A^-_{-2}A^-_{-2}|\mathbf{r}_{-1}\rangle\big] = \Big(-A^-_{-3}A^-_{-2} + A^-_{-5}\Big)|\mathbf{a}_2\rangle, \tag{3.11}$$

$$\big[|\mathbf{r}_{-1}\rangle, A^-_{-5}|\mathbf{r}_{-1}\rangle\big] = \Big(\tfrac{35}{64}A^-_{-7} + \tfrac{7}{32}A^-_{-5}A^-_{-2} + \tfrac{5}{64}A^-_{-3}A^-_{-2}A^-_{-2}\Big)|\mathbf{a}_2\rangle, \tag{3.12}$$

$$\begin{aligned}\big[|\mathbf{r}_{-1}\rangle, A^-_{-3}A^-_{-2}|\mathbf{r}_{-1}\rangle\big] = \Big(&-\tfrac{61}{128}A^-_{-7} + \tfrac{1}{4}A^-_{-4}A^-_{-3} + \tfrac{7}{64}A^-_{-5}A^-_{-2} \\ &+ \tfrac{37}{128}A^-_{-3}A^-_{-2}A^-_{-2}\Big)|\mathbf{a}_2\rangle,\end{aligned} \tag{3.13}$$

$$\begin{aligned}\big[A^-_{-2}|\mathbf{r}_{-1}\rangle, A^-_{-3}|\mathbf{r}_{-1}\rangle\big] = \Big(&-\tfrac{83}{128}A^-_{-7} + \tfrac{1}{4}A^-_{-4}A^-_{-3} + \tfrac{41}{64}A^-_{-5}A^-_{-2} \\ &- \tfrac{21}{128}A^-_{-3}A^-_{-2}A^-_{-2}\Big)|\mathbf{a}_2\rangle,\end{aligned} \tag{3.14}$$

where $\mathbf{a}_2 = 2\mathbf{r}_{-1} + \frac{3}{2}\boldsymbol{\delta}$ is the tachyonic level-2 root. Furthermore, we have adopted the convention from [17] according to which the DDF operators are always understood to be the ones appropriate for the states on which they act (i.e. $A^-_m(\mathbf{r}_{-1})$ on the l.h.s. and $A^-_m(\mathbf{a}_2)$ on the r.h.s.).

The first commutator generates an element of the root space associated with $\Lambda = 2\mathbf{r}_{-1} + 3\boldsymbol{\delta}$. But since this space is one-dimensional, $\mathrm{mult}(2\mathbf{r}_{-1} + 3\boldsymbol{\delta}) = \pi_1(3) = 1$, we infer that we do not need an additional imaginary simple root here (recall that $\mathrm{mult}(2\mathbf{r}_{-1} + n\boldsymbol{\delta}) = \mathrm{mult}\langle 2, n-2\rangle = \pi_1(2n-3)$). This is, of course, a rather trivial observation because $\langle 2, 1\rangle$ is not a fundamental root anyhow.

The next two commutators leading to states in the root space associated with $\Lambda = 2\mathbf{r}_{-1} + 4\boldsymbol{\delta}$ are already more involved. By taking suitable linear combinations we obtain $A^-_{-3}A^-_{-2}|\mathbf{a}_2\rangle$ and $A^-_{-5}|\mathbf{a}_2\rangle$, which, as one can easily convince oneself, already span the full two-dimensional root space, $\mathrm{mult}(2\mathbf{r}_{-1} + 4\boldsymbol{\delta}) = \pi_1(5) = 2$. Consequently, this root space can be entirely generated by commutators of level-1 missing states, which means that $\mu_{2,2} = 0$. This is the only root in the fundamental Weyl chamber which is not simple.

Let us finally consider a generic example. The commutators (3.12)–(3.14) give states with momentum $\Lambda = 2\mathbf{r}_{-1} + 5\boldsymbol{\delta}$. Note that the commutators (3.12) and (3.13) are states of spin 3/2 $\mathfrak{sl}_2$-modules built on the vacuum vectors $A^-_{-5}|\mathbf{r}_{-1}\rangle$ and $A^-_{-3}A^-_{-2}|\mathbf{r}_{-1}\rangle$, respectively. In the notations of the last section (see Eq. (2.21)), they span the two-dimensional space $\mathcal{R}^{(\Lambda)}$, whereas $\mathcal{H}^{(\Lambda)}$ is one-dimensional with basis element given by the commutator (3.14) of two level-1 vacuum vectors. By building suitable linear combinations these states can be simplified somewhat; in this way, we get the three linearly independent states

$$\big(A^-_{-7} + \tfrac{3}{5}A^-_{-5}A^-_{-2}\big)\,|\mathbf{a}_2\rangle\ , \tag{3.15}$$

$$\big(A^-_{-3}A^-_{-2}A^-_{-2} - \tfrac{7}{5}A^-_{-5}A^-_{-2}\big)\,|\mathbf{a}_2\rangle\ , \tag{3.16}$$

$$\big(A^-_{-4}A^-_{-3} + \tfrac{16}{5}A^-_{-5}A^-_{-2}\big)\,|\mathbf{a}_2\rangle\ . \tag{3.17}$$

However, we know that the full root space has dimension $\pi_1(7) = 4$, generated by the longitudinal DDF operators $A^-_{-3}A^-_{-2}A^-_{-2}, A^-_{-4}A^-_{-3}, A^-_{-5}A^-_{-2}, A^-_{-7}$. Hence $\mathcal{J}^{(\Lambda)}$ must be one-dimensional. Indeed, the physical state

$$\left(-2457413A^-_{-7} + 1354090A^-_{-5}A^-_{-2} - 1613422A^-_{-4}A^-_{-3} + 157593A^-_{-3}A^-_{-2}A^-_{-2}\right)|\mathbf{a}_2\rangle$$

is orthogonal to the above three states and cannot be generated by commutation. Hence it is a missing state which must be added by hand to arrive at the total count of four. We conclude that $2\mathbf{r}_{-1} + 5\boldsymbol{\delta}$ is an imaginary simple root with simple multiplicity $\mu_{2,3} = 1$.

Of course, these explicit results are in complete agreement with the Weyl–Kac–Borcherds formula predicting $\mu_{2,2} = 0$ and $\mu_{2,3} = 1$ (cf. Fig. 4).

3.4. Direct sums of lattices. We conclude this section with a remark about direct sums of lattices and how this translates into the associated Lie algebras of physical states.

Suppose we have two lattices Λ_1 and Λ_2. Then the direct sum

$$\Lambda := \Lambda_1 \oplus \Lambda_2$$

enjoys the following properties (see e.g. [28]):

(i) $\operatorname{rank}\Lambda = \operatorname{rank}\Lambda_1 + \operatorname{rank}\Lambda_2$;
(ii) $\operatorname{sgn}\Lambda = \operatorname{sgn}\Lambda_1 + \operatorname{sgn}\Lambda_2$;
(iii) $\det \Lambda = (\det \Lambda_1)(\det \Lambda_2)$;
(iv) Λ is even iff both Λ_1 and Λ_2 are even ;

where sgn denotes the signature of a lattice. For Λ to be even Lorentzian we shall therefore assume that Λ_1 is even Lorentzian and Λ_2 is even Euclidean. For example, the root lattice of E_{10} can be decomposed into a direct sum of the unique even selfdual Lorentzian lattice $I\!I_{1,1}$ in two dimensions and the E_8 root lattice. More generally, we have

$$I\!I_{8n+1,1} = I\!I_{1,1} \oplus \Gamma_{8n},$$

where Γ_{8n} denotes an even selfdual Euclidean lattice of rank $8n$.[11]

We would like to answer the question how the Lie algebra of physical states in $\mathcal{F}_\Lambda := \mathcal{F}_{\Lambda_1} \otimes \mathcal{F}_{\Lambda_2}$ is built up from the states in $\mathcal{F}_{\Lambda_1}$ and $\mathcal{F}_{\Lambda_2}$. This amounts to rewriting both $\mathcal{P}^1_\Lambda$ and $L_{-1}\mathcal{P}^0_\Lambda$ as direct sums of tensor products of subspaces of $\mathcal{F}_{\Lambda_i}$. Using the facts about tensor products of vertex algebras [14] and that $\mathcal{F}^h_{\Lambda_2} = 0$ for $h < 0$, we deduce that any state in $\psi \in \mathcal{P}^1_\Lambda$ is a finite linear combination of the form

$$\psi = \sum_{h=0}^{H} \psi_1^{1-h} \otimes \psi_2^h,$$

with $\psi_i^h \in \mathcal{F}^h_{\Lambda_i}$ and satisfying

[11] As is well known (see e.g. [10]), there exists only one such lattice for $n = 1$ (associated with E_8), two for $n = 2$ (associated with $E_8 \oplus E_8$ and $\mathrm{Spin}(32)/\mathbb{Z}_2$, resp.), and 24 for $n = 3$ (the 24 Niemeier lattices with the famous Leech lattice as one of them). For higher rank, an explicit classification seems impossible. This is due to the explosive growth of the number of even selfdual Euclidean lattices according to the Minkowski–Siegel mass formula which, for example, gives us 8×10^7 as a lower limit on the number of such lattices with rank 32.

$$\begin{aligned} \psi_1^{1-h} \otimes L_{2,n}\psi_2^h &= 0 \qquad \text{for } 0 \le h < n, \\ L_{1,n}\psi_1^{1-h} \otimes \psi_2^h + \psi_1^{1-h-n} \otimes L_{2,n}\psi_2^{h+n} &= 0 \qquad \text{for } 0 \le h \le H-n, \\ L_{1,n}\psi_1^{1-h} \otimes \psi_2^h &= 0 \qquad \text{for } H-n < h \le H, \end{aligned} \tag{3.18}$$

for all $n > 0$. We immediately see that $\psi_2^0 \in \mathcal{P}^0_{\Lambda_2} = \mathbb{R}|\mathbf{0}\rangle_2$ and $\psi_1^{1-H} \in \mathcal{P}^{1-H}_{\Lambda_1}$, but it is difficult to extract from the above relations similar information about the other states. Nonetheless, the last two observations are sufficient to pinpoint the gnome Lie algebra inside $\mathfrak{g}_\Lambda$. Namely, by considering the special case $\psi = \psi_1^1 \otimes |\mathbf{0}\rangle_2$, we can immediately infer that $\mathfrak{g}_{II_{1,1}} \cong \mathfrak{g}_{II_{1,1}} \otimes |\mathbf{0}\rangle_2 \subset \mathfrak{g}_\Lambda$. So the gnome Lie algebra is a Borcherds subalgebra of any Lie algebra of physical states for which the root lattice can be decomposed into a direct sum in such a way that $II_{1,1}$ arises as a sublattice. This in particular holds for the Lie algebras based on the lattices $II_{9,1}$, $II_{17,1}$, and $II_{25,1}$, respectively, the latter being the celebrated fake monster Lie algebra [5].

We can explore the decomposition of $\mathcal{P}^1_\Lambda$ further by the use of the DDF construction. Let us suppose that Λ_1 is the lattice $II_{1,1}$ and that Λ_2 has rank $d-2$ (> 0). We shall write vectors in Λ as $(\mathbf{r}, \mathbf{v})$, where $\mathbf{r} \in II_{1,1}$ and $\mathbf{v} \in \Lambda_2$, respectively, so that $(\mathbf{r}, \mathbf{v})^2 = \mathbf{r}^2 + \mathbf{v}^2$. We wish to find a tensor product decomposition of the subspace of $\mathcal{P}^1_{\Lambda_1}$ which has fixed momentum component $\mathbf{r} \in \Lambda_1$, i.e., of the space

$$\mathcal{P}^{1,\mathbf{r}}_\Lambda := \mathcal{P}^1_\Lambda \cap \bigoplus_{\mathbf{v}\in\Lambda_2} \mathcal{F}^{(\mathbf{r},\mathbf{v})}_\Lambda .$$

The idea is to perform the DDF construction in a clever way such that the $d-2$ transversal directions all belong to the Euclidean lattice Λ_2 and thus the transversal DDF operators can be identified with the string oscillators in $\mathcal{F}_{\Lambda_2}$. We start from the DDF decomposition $\mathbf{r} = \mathbf{a}_\ell - \left(1 - \frac{1}{2}\mathbf{r}^2\right)\mathbf{k}_\ell$ (see Eq. (3.1)), which gives rise to the decomposition

$$(\mathbf{r}, \mathbf{v}) = \left(\mathbf{a}_\ell - \tfrac{1}{2}\mathbf{v}^2\mathbf{k}_\ell, \mathbf{v}\right) - \left(1 - \tfrac{1}{2}(\mathbf{r}, \mathbf{v})^2\right)(\mathbf{k}_\ell, \mathbf{0})$$

within Λ. A suitable set of polarization vectors is obtained from any orthonormal basis $\{\boldsymbol{\xi}^i | 1 \le i \le d-2\}$ of $\mathbb{R} \otimes_\mathbb{Z} \Lambda_2$ by putting $\boldsymbol{\xi}^i \equiv (\mathbf{0}, \boldsymbol{\xi}^i)$. From Thm. 1 it follows that

$$\begin{aligned} \mathcal{P}^{1,\mathbf{r}}_\Lambda = \text{span}\{ & A^{i_1}_{-m_1} \cdots A^{i_M}_{-m_M} A^-_{-n_1} \cdots A^-_{-n_N} |\mathbf{a}_\ell - \tfrac{1}{2}\mathbf{v}^2\mathbf{k}_\ell, \mathbf{v}\rangle \\ & |\mathbf{v} \in \Lambda_2, m_1 + \ldots + n_N = 1 - \tfrac{1}{2}(\mathbf{r}, \mathbf{v})^2 \}. \end{aligned}$$

For fixed $h := \frac{1}{2}\mathbf{v}^2 + \sum_a m_a$, we may identify

$$A^{i_1}_{-m_1} \cdots A^{i_M}_{-m_M} |\mathbf{a}_\ell - h\mathbf{k}_\ell, \mathbf{v}\rangle \cong |\mathbf{a}_\ell - h\mathbf{k}_\ell\rangle_1 \otimes \alpha^{i_1}_{-m_1} \cdots \alpha^{i_M}_{-m_M} |\mathbf{v}\rangle_2,$$

or

$$\text{span}\{A^{i_1}_{-m_1} \cdots A^{i_M}_{-m_M} |\mathbf{a}_\ell - h\mathbf{k}_\ell, \mathbf{v}\rangle\} \cong |\mathbf{a}_\ell - h\mathbf{k}_\ell\rangle_1 \otimes \mathcal{F}^h_{\Lambda_2}.$$

If we finally use the fact that $\mathcal{P}^h_{\Lambda_1}$ for any integer h is generated by longitudinal operators we conclude that

$$\mathcal{P}^{1,\mathbf{r}}_\Lambda \cong \bigoplus_{h=0}^{1-\frac{1}{2}\mathbf{r}^2} \mathcal{P}^{1-h,(\mathbf{r})}_{\Lambda_1} \otimes \mathcal{F}^h_{\Lambda_2}$$

for any $\mathbf{r} \in \Lambda_1$. There is one subtlety here concerning the central charge. The longitudinal Virasoro algebra occurring on the right-hand side as spectrum-generating algebra for any $\mathcal{P}^h_{\Lambda_1}$ does not have the naive central charge $c = 24$ (like for the gnome Lie algebra) but rather $c = 26 - d$, the extra contribution coming from the transversal space Λ_2. So for $d = 26$ we get modulo null states the trivial representation of the longitudinal Virasoro algebra and hence $\mathfrak{g}_\Lambda{}^{\mathbf{r}} \cong \mathcal{F}_{\Lambda_2}^{1-\frac{1}{2}\mathbf{r}^2}$ in agreement with the literature [6].

4. Missing Modules for E_{10}

We now turn to the hyperbolic Kac–Moody algebra $\mathfrak{g}(A) = E_{10}$, which arises as the maximal Kac–Moody subalgebra of the Borcherds algebra $\mathfrak{g}_{II_{9,1}}$ of physical states associated with a subcritical open bosonic string moving in 10-dimensional space-time fully compactified on a torus, so that the momenta lie on the lattice $II_{9,1}$. As such, it plays the same role for $\mathfrak{g}_{II_{9,1}}$ as $\mathfrak{sl}_2$ did for the gnome Lie algebra, but is incomparably more complicated. Again, the central idea to split the larger algebra $\mathfrak{g}_{II_{9,1}}$ into E_{10} and its orthogonal complement which can be decomposed into a direct sum of E_{10} lowest and highest weight modules, respectively. Since the root lattice of E_{10} is identical with the momentum lattice $II_{9,1}$, there is no need to extend E_{10} by outer derivations. Thus we start from

$$\mathfrak{g}_{II_{9,1}} = E_{10} \oplus \mathcal{M},$$

where the space of missing states $\mathcal{M}$ decomposes as

$$\mathcal{M} = \mathcal{M}_+ \oplus \mathcal{M}_-, \qquad \mathcal{M}_\pm = \bigoplus_{v\in\mathcal{H}_\pm} \mathfrak{U}(E_{10})v;$$

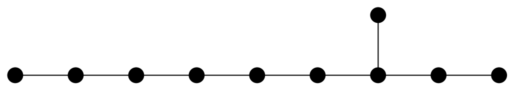
each of the (irreducible) E_{10} modules $\mathfrak{U}(E_{10})v$ is referred to as a “missing module”. To be sure, this decomposition still does not provide us with an explicit realization of the E_{10} algebra since we know as little about the E_{10} modules as about the E_{10} itself (see [13] for some recent progress). On the other hand, we do gain insight by combining the unknown algebra and its unknown modules into something which we understand very well, namely the Lie algebra of physical states $\mathfrak{g}_{II_{9,1}}$ for which a basis is explicitly given in terms of the DDF construction. Moreover, we will formulate a conjecture according to which all higher-level missing states can be obtained by commuting the missing states at level 1 whose structure is completely known. Our explicit tests of this conjecture for the root spaces of Λ_7 and Λ_1 constitute highly non-trivial checks, but of course major new insights are required to settle the question for higher levels. We should mention that the results of the previous section immediately show that the conjecture fails for the gnome Lie algebra $\mathfrak{g}_{II_{1,1}}$. As we have already pointed out, the $\mathfrak{sl}_2$ module structure of the missing states for $\mathfrak{g}_{II_{1,1}}$ is not especially enlightening due to the “smallness” of $\mathfrak{sl}_2$. Here the situation is completely different, because E_{10} and its representations are “huge” (even in comparison with irreducible representations of the affine E_9 subalgebra!). If our conjecture were true it would not only take us a long way towards a complete understanding of E_{10} but also provide another hint that E_{10} is very special indeed. Conversely, it would also allow us to understand the Borcherds algebra $\mathfrak{g}_{II_{9,1}}$ by exhibiting its complete set of imaginary simple roots. In addition to the fake monster, the true monster, and the gnome Lie algebra, this would be the fourth example of an explicit realization of a Borcherds algebra.

4.1. Basics of E_{10}. As the momentum lattice for the completely compactified string we shall take the unique 10-dimensional even unimodular Lorentzian lattice $\mathit{II}_{9,1}$. It can be defined as the lattice of all points $\mathbf{x} = (x_1, \ldots, x_9|x_0)$ for which the x_μ's are all in $\mathbb{Z}$ or all in $\mathbb{Z} + \frac{1}{2}$ and which have integer inner product with the vector $\mathbf{l} = (\frac{1}{2}, \ldots, \frac{1}{2} \mid \frac{1}{2})$, all norms and inner products being evaluated in the Minkowskian metric $\mathbf{x}^2 = x_1^2 + \ldots + x_9^2 - x_0^2$ (cf. [32]).

To identify the maximal Kac–Moody subalgebra of the Borcherds algebra $\mathfrak{g}_{\mathit{II}_{9,1}}$ of physical string states we have to determine a set of real simple roots for the lattice. According to [9], such a set is given by the ten vectors $\mathbf{r}_{-1}, \mathbf{r}_0, \mathbf{r}_1, \ldots, \mathbf{r}_8$ in $\mathit{II}_{9,1}$ for which $\mathbf{r}_i^2 = 2$ and $\mathbf{r}_i \cdot \boldsymbol{\rho} = -1$, where the Weyl vector is $\boldsymbol{\rho} = (0, 1, 2, \ldots, 8|38)$ with $\mathbf{r}^2 = -1240$.[12] Explicitly,

$$
\begin{array}{lcl}
\mathbf{r}_{-1} & = & (\;0,\; 0,\; 0,\; 0,\; 0,\; 0,\; 0,\; 1, -1 \mid 0), \\
\mathbf{r}_{0} & = & (\;0,\; 0,\; 0,\; 0,\; 0,\; 0,\; 1, -1,\; 0 \mid 0), \\
\mathbf{r}_{1} & = & (\;0,\; 0,\; 0,\; 0,\; 0,\; 1, -1,\; 0,\; 0 \mid 0), \\
\mathbf{r}_{2} & = & (\;0,\; 0,\; 0,\; 0,\; 1, -1,\; 0,\; 0,\; 0 \mid 0), \\
\mathbf{r}_{3} & = & (\;0,\; 0,\; 0,\; 1, -1,\; 0,\; 0,\; 0,\; 0 \mid 0), \\
\mathbf{r}_{4} & = & (\;0,\; 0,\; 1, -1,\; 0,\; 0,\; 0,\; 0,\; 0 \mid 0), \\
\mathbf{r}_{5} & = & (\;0,\; 1, -1,\; 0,\; 0,\; 0,\; 0,\; 0,\; 0 \mid 0), \\
\mathbf{r}_{6} & = & (-1, -1,\; 0,\; 0,\; 0,\; 0,\; 0,\; 0,\; 0 \mid 0), \\
\mathbf{r}_{7} & = & (\;\tfrac{1}{2},\; \tfrac{1}{2},\; \tfrac{1}{2},\; \tfrac{1}{2},\; \tfrac{1}{2},\; \tfrac{1}{2},\; \tfrac{1}{2},\; \tfrac{1}{2},\; \tfrac{1}{2} \mid \tfrac{1}{2}), \\
\mathbf{r}_{8} & = & (\;1, -1,\; 0,\; 0,\; 0,\; 0,\; 0,\; 0,\; 0 \mid 0).
\end{array}
$$

These simple roots indeed generate the reflection group of $\mathit{II}_{9,1}$. The corresponding Coxeter–Dynkin diagram associated with the Cartan matrix $a_{ij} := \mathbf{r}_i \cdot \mathbf{r}_j$ looks as follows:



The algebra $\mathfrak{g}(A)$ is the hyperbolic Kac–Moody algebra E_{10}, defined in terms of generators and relations (2.16). Moreover, from $|\det A| = 1$ we infer that the root lattice $Q(E_{10})$ indeed coincides with $\mathit{II}_{9,1}$, and hence $\hat{\mathfrak{g}}(A) \equiv \mathfrak{g}(A)$ here.

The E_9 null root is

$$\boldsymbol{\delta} = \sum_{i=0}^{8} n_i \mathbf{r}_i = (0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 1 \mid 1),$$

where the marks n_i can be read off from

$$\left[\begin{array}{ccccccccc} & & & & & & 3 & & \\ 0 & 1 & 2 & 3 & 4 & 5 & 6 & 4 & 2 \end{array}\right] \quad .$$

The fundamental Weyl chamber $\mathcal{C}$ of E_{10} is the convex cone generated by the fundamental weights Λ_i,[13]

$$\Lambda_i = -\sum_{j=-1}^{8} (A^{-1})_{ij} \mathbf{r}_j \text{for } i = -1, 0, 1, \ldots 8,$$

[12] Note that $\boldsymbol{\rho}$ fulfills all the requirements of a grading vector for $\mathfrak{g}_{\mathit{II}9,1}$.

[13] Notice that our convention is opposite to the one adopted in [25]. The fundamental weights here are *positive* and satisfy $\Lambda_i \cdot \mathbf{r}_j = -\delta_{ij}$.

where A^{-1} is the inverse Cartan matrix. Thus,

$$\Lambda \in \mathcal{C} \quad \Longleftrightarrow \quad \Lambda = \sum_{i=-1}^{8} k_i \Lambda_i,$$

for $k_i \in \mathbb{Z}_+$. A special feature of E_{10} is that we need not distinguish between root and weight lattice, since these are the same for self-dual root lattices.[14] Note also that the null root plays a special role: the first fundamental weight is just $\Lambda_{-1} = \boldsymbol{\delta}$, and all null-vectors in $\mathcal{C}$ must be multiples of Λ_{-1} since $\Lambda_i^2 < 0$ for all other fundamental weights.

We can employ the affine null root to introduce a $\mathbb{Z}$-grading of E_{10}. If we introduce the so-called level ℓ of a root $\Lambda \in \Delta(E_{10})$ by

$$\ell := -\Lambda \cdot \boldsymbol{\delta},$$

then we may decompose the algebra into a direct sum of subspaces of fixed level, viz.

$$E_{10} = \bigoplus_{\ell \in \mathbb{Z}} {E_{10}}^{[\ell]},$$

where

$${E_{10}}^{[0]} \cong E_9, \qquad {E_{10}}^{[\ell]} := \bigoplus_{\substack{\Lambda \in \Delta(E_{10}) \\ -\Lambda \cdot \boldsymbol{\delta} = \ell}} {E_{10}}^{(\Lambda)} \quad \text{for } \ell \neq 0.$$

Besides the obvious fact that ℓ counts the number of e_{-1} (resp. f_{-1}) generators in multiple commutators, the level derives its importance from the fact that it grades the algebra E_{10} with respect to its affine subalgebra E_9 [12]. The subspaces belonging to a fixed level can be decomposed into irreducible representations of E_9, the level being equal to the eigenvalue of the central term of the E_9 algebra on this representation (hence the full E_{10} algebra contains E_9 representations of *all* integer levels!). Let us emphasize that for general hyperbolic algebras there would be a separate grading associated with every regular affine subalgebra, and therefore the graded structure would no longer be unique.

Using the Jacobi identity it is possible to represent any subspace of fixed level in the form

$${E_{10}}^{[\ell]} = \underbrace{\big[{E_{10}}^{[1]}, \big[{E_{10}}^{[1]}, \ldots \big[{E_{10}}^{[1]}, {E_{10}}^{[1]}\big]}_{\ell \text{ times}}\big] \ldots\big],$$

for $\ell > 0$, and in an analogous form for $\ell < 0$. This simple fact turns out to be extremely useful in connection with the DDF construction, as soon as one wishes to effectively construct higher-level elements of E_{10}.

Little is known about the general structure of this algebra. Partial progress has been made in determining the multiplicity of certain roots. Although the general form of the multiplicity formulas for arbitrary levels appears to be beyond reach for the moment, the following results for levels $\ell \leq 2$ have been established. For $\ell = 0$ and $\ell = 1$, we have $\text{mult}_{E_{10}}(\Lambda) = p_8(1 - \frac{1}{2}\Lambda^2)$ (see [24]), i.e., the multiplicities are just given by the number of transversal states; as was demonstrated in [17] the corresponding states are indeed transversal. For $\ell = 2$, it was shown in [25] that $\text{mult}_{E_{10}}(\Lambda) = \xi(3 - \frac{1}{2}\Lambda^2)$, where $\sum_n \xi(n) q^n = \big[1 - \phi(q^2)\big/\phi(q^4)\big]\big/\phi(q)^8$, $\phi(q)$ denoting the Euler function as before.

[14] In the remainder, we will consequently denote arbitrary roots by Λ and reserve the letter $\mathbf{r}$ for real roots.

Beyond $\ell = 2$, no general formula seems to be known although for $\ell = 3$ the multiplicity problem was recently solved [2]. However, the resulting formulas are somewhat implicit and certainly more cumbersome than the above results. Of course, if one is only interested in a particular root, the relevant multiplicity can always be determined by means of the Peterson recursion formula (see e.g. [27]).

4.2. Lowest and highest weight modules of E_{10}. We know from Thm. 5 that $\mathcal{M}_+$ (resp. $\mathcal{M}_-$) decomposes into a direct sum of lowest (resp. highest) weight modules for E_{10}. As before, $\mathcal{H}_\pm$ denotes the subspace spanned by the corresponding lowest and highest weight states, respectively. Clearly, $\mathcal{H}_\pm$ inherits from $\mathfrak{g}_{II_{9,1}}$ the grading by the level,

$$\mathcal{H}_\pm = \bigoplus_{\ell \gtrless 0} \mathcal{H}^{[\ell]}, \qquad \mathcal{H}^{[\ell]} := \mathcal{H} \cap {\mathfrak{g}_{II_{9,1}}}^{[\ell]}.$$

Since the Chevalley involution provides an isomorphism between $\mathcal{H}^{[\ell]}$ and $\mathcal{H}^{[-\ell]}$ and since we are ultimately interested in identifying the imaginary simple roots and their multiplicities, it is sufficient to restrict the explicit analysis to $\mathcal{H}_+$. We will first study the structure of the space $\mathcal{H}^{[1]}$ and will explicitly demonstrate how it is made up of purely longitudinal DDF states. Intuitively, this is what one should expect. Recall that the level-1 sector of E_{10} is isomorphic to the basic representation of E_9 (cf. [12]); in terms of the DDF construction, it is generated by the transversal states built on $|\mathbf{r}_{-1}\rangle$, i.e., it is spanned by all states of the form $A^{j_1}_{-m_1} \cdots A^{j_k}_{-m_k}|\mathbf{r}_{-1}\rangle$ and their orbits under the action of the E_9 affine Weyl group [17]. Thus the longitudinal states at level 1 do not belong to E_{10} and must be counted as missing states. Furthermore, the level-1 transversal DDF operators can be identified with the adjoint action of appropriate E_9 elements (corresponding to multiples of the affine null root). Hence the purely longitudinal DDF states built on the level-1 roots of E_{10} are candidates for missing lowest weight vectors. But apparently this set can be further reduced, because each (real) level-1 root of E_{10} is conjugated to some root of the form $\mathbf{r}_{-1} + M\boldsymbol{\delta}$ ($M \geq 0$) under the action of the affine Weyl group. So we end up with purely longitudinal states built on $|\mathbf{r}_{-1}\rangle$ – the same set we already encountered in Sect. 3.3 for the case of the gnome Lie algebra! And indeed, we have

Proposition 1. *The space of missing level-1 lowest weight vectors consists of purely longitudinal DDF states built on $|\mathbf{r}_{-1}\rangle$,*

$$\mathcal{H}^{[1]} = \operatorname{span}\Big\{ A^-_{-n_1} \cdots A^-_{-n_N} |\mathbf{r}_{-1}\rangle \,\Big|\, n_1 \geq n_2 \geq \ldots \geq n_N \geq 2 \Big\},$$

i.e., it is (modulo null states) the longitudinal Virasoro–Verma module with $|\mathbf{r}_{-1}\rangle$ as highest weight vector. In particular, $\mathbf{r}_{-1} + n\boldsymbol{\delta}$ for any $n \geq 2$ is an imaginary simple root for $\mathfrak{g}_{II_{9,1}}$ with multiplicity $\mu(\mathbf{r}_{-1} + n\boldsymbol{\delta}) = \pi_1(1 + n)$.

Proof. Let us consider the state

$$v_\Lambda := A^-_{-n_1}(\mathbf{r}_{-1}) \cdots A^-_{-n_N}(\mathbf{r}_{-1})|\mathbf{r}_{-1}\rangle$$

with momentum $\Lambda := \mathbf{r}_{-1} + M\boldsymbol{\delta}$, $M := \sum_j n_j > 1$. We first check that, under the adjoint action in $\mathfrak{g}_{II_{9,1}}$, it is a lowest weight vector for the basic representation of E_9. Acting with either of the affine Chevalley generators $e_i = |\mathbf{r}_i\rangle$ and $f_i = -|-\mathbf{r}_i\rangle$ ($i = 0, 1, \ldots, 8$) on v_Λ, we can move it through the longitudinal DDF operators by the use of the general "intertwining relation" [18]

$$E_n^{\mathbf{r}} A_m^-(\mathbf{r}_{-1}) = A_m^-(\mathbf{a}') E_n^{\mathbf{r}},$$

where $\mathbf{a}' := \mathbf{r}_{-1} + \mathbf{r} + \boldsymbol{\delta}$ and $E_n^{\mathbf{r}}$ denotes the step operator associated with the real affine root $\mathbf{r} + n\boldsymbol{\delta}$. Thereby we end up with the same state but where the Chevalley generator now acts on $|\mathbf{r}_{-1}\rangle$. The latter, however, is just a lowest weight vector for the basic representation of E_9, viz.

$$f_i|\mathbf{r}_{-1}\rangle = 0, \qquad e_i^{1-\mathbf{r}_i\cdot\mathbf{r}_{-1}}|\mathbf{r}_{-1}\rangle = 0, \qquad \text{for } i = 0, 1, \dots, 8\,,$$

which is readily seen by inspection of the momenta. Indeed, $(\mathbf{r}_{-1} - \mathbf{r}_i)^2 = \big(\mathbf{r}_{-1} + (1 - \mathbf{r}_i\cdot\mathbf{r}_{-1})\mathbf{r}_i\big)^2 = 2(2 - \mathbf{r}_i\cdot\mathbf{r}_{-1}) \geq 4$, contradicting the mass shell condition (2.4). Since $h_i(v_\Lambda) = \mathbf{r}_i\cdot\Lambda v_\Lambda = -\delta_{i0}v_\Lambda$, we conclude that v_Λ is a vacuum vector for the adjoint action of E_9 generating the basic representation. According to [17] it is given by the transversal states built on v_Λ, i.e., $\mathfrak{U}(E_9)v_\Lambda$ is spanned by the states $A^{j_1}_{-m_1}\cdots A^{j_k}_{-m_k}v_\Lambda$, where $A^j_m \equiv A^j_m(\mathbf{r}_{-1})$.

To show that the state v_Λ is a lowest weight vector for the full E_{10} algebra, we have to check the remaining two Chevalley generators. Again by momentum conservation, the state $f_{-1}(v_\Lambda) = -\big[|-\mathbf{r}_{-1}\rangle, v_\Lambda\big]$ has momentum $M\boldsymbol{\delta}$. But since the physical states associated with lightlike momentum are purely transversal and are elements of E_{10}, the resulting missing state must be a null state (or vanishes identically). Within $\mathfrak{g}_{II_{9,1}}$, we therefore have $f_{-1}(v_\Lambda) = 0$. On the other hand, acting with the Chevalley generator e_{-1} on v_Λ repeatedly, say k times, we obtain a state of momentum $\boldsymbol{\lambda} = (1+k)\mathbf{r}_{-1} + M\boldsymbol{\delta}$. By the mass shell condition, this state identically vanishes for $\boldsymbol{\lambda}^2 = 2(1+k)(1+k-M) > 2$, i.e., $k > M - 1 = 1 - \mathbf{r}_{-1}\cdot\Lambda$. For $k = M-1$, the momentum vector $\boldsymbol{\lambda}$ is lightlike, and by the same reasoning as before we conclude that the state is null also for this value of k.

Altogether, we have shown that

$$f_i(v_\Lambda) = (e_i)^{1-\mathbf{r}_i\cdot\Lambda}(v_\Lambda) = 0 \qquad \text{for } i = -1, 0, 1, \dots, 8\,.$$

These are the defining conditions for v_Λ to be a lowest weight state for E_{10}. Since $\mathrm{ad}h_i = \mathbf{r}_i\cdot\mathbf{p}$, it is clear that the lowest weight is just Λ. The fact that f_{-1} annihilates the state v_Λ in particular implies that we can "only go up" in level (for positive level lowest weight states) and that it is not possible to cross the line $\ell = 0$ by the action of E_{10}. $\square$

In the context of representation theory of hyperbolic Kac–Moody algebras (see [13]), the above result provides the first examples of explicit realizations of unitary irreducible lowest weight representations of the hyperbolic algebra E_{10}. More specifically, they are associated with lowest weights $\Lambda_0 + m\Lambda_{-1}$ for any $m \geq 0$. By commutation we even obtain an infinite set of missing lowest-weight vectors with lowest weights $\ell\Lambda_0 + m\Lambda_{-1}$ for any $\ell \geq 1$ and $m \geq 0$, on which we can build irreducible E_{10} modules. Analogous statements can be also made for other hyperbolic algebras when we replace $II_{9,1}$ by the root lattice of the hyperbolic algebra. Due to the string realization this lattice should be even and Lorentzian, conditions which rule out some hyperbolic algebras (see e.g. [30] for a list of them).

The next question is now whether $\mathfrak{g}_{II_{9,1}}$ also provides realizations of other lowest weight representations of E_{10}. The results of the following section suggest that this may not be the case. More specifically, we are led to make the following

Conjecture 1. There are no imaginary simple roots for $\mathfrak{g}_{II_{9,1}}$ at level 2 or higher, i.e., the Lie algebra of missing lowest weight states, $\mathcal{H}_+$, is a free algebra generated by the states given in Prop. 1.

Note that for the true monster Lie algebra the analogous claim is actually valid: the imaginary simple roots are all of level 1. On the other hand, the conjecture obviously fails for the gnome Lie algebra. The reason for this is that the root spaces in the former example are much bigger (due to the "hidden" extra 24 dimensions of the moonshine module), even though the maximal Kac–Moody algebra in both examples is the same, namely $\mathfrak{sl}_2$. This appears to suggest that E_{10} has just the right size so that the missing modules built on elements of the free Lie algebra over $\mathcal{H}^{[1]}$ precisely fill up E_{10} to the full Lie algebra of physical states.

At present, we are not aware of any convincing general argument in favour of the above conjecture. In the next subsection, however, we will verify it for two explicitly constructed non-trivial root spaces. More specifically, we will consider a $201 = 192 + 9$ dimensional and a $780 = 727+53$ dimensional level-2 root space, respectively, where the first contribution in each sum equals the dimension of the E_{10} root space and the second term is the dimension of the space of missing states. We will show for both examples that all the missing states are contained in E_{10} modules built on level-1 missing lowest weight vectors or on commutators of them. Of course, these two zeros could be accidental like in the case of the gnome Lie algebra where we also found a zero at level 2 (see Fig. 4). In the latter example, this was not unexpected since the root multiplicities in this region of the fundamental cone are very low, anyway. For the E_{10} algebra, by contrast, there is no apparent reason why all missing states in certain level-2 root spaces should belong to E_{10} modules of the conjectured type. The fact that they do in the cases we have studied constitutes our primary motivation for the above conjecture.

4.3. Examples: Λ_7 and Λ_1. We use the same system of polarization vectors and DDF decomposition as in [17], which we recall here for convenience: Explicitly, Λ_7 is given by

$$\Lambda_7 = \begin{bmatrix} & & & & & & 7 & & \\ 2 & 4 & 6 & 8 & 10 & 12 & 14 & 9 & 4 \end{bmatrix} = (0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0 \,|\, 2),$$

so $\Lambda_7^2 = -4$. Its decomposition into two level-1 tachyonic roots is $\Lambda_7 = \mathbf{r} + \mathbf{s} + 2\boldsymbol{\delta}$, where

$$\mathbf{r} := \mathbf{r}_{-1} = \begin{bmatrix} & & & & & & 0 & & \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} = (0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 1, -1 \,|\, 0),$$
$$\mathbf{s} = \begin{bmatrix} & & & & & & 1 & & \\ 1 & 2 & 2 & 2 & 2 & 2 & 2 & 1 & 0 \end{bmatrix} = (0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0, -1, -1 \,|\, 0).$$

Since $n = 1 - \frac{1}{2}\Lambda_7^2 = 3$, we have the DDF decomposition $\Lambda_7 = \mathbf{a} - 3\mathbf{k}$, where $\mathbf{k} := -\frac{1}{2}\boldsymbol{\delta}$ and

$$\mathbf{a} := \mathbf{r} + \mathbf{s} - \mathbf{k} = (0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0, -\tfrac{3}{2} \,|\, \tfrac{1}{2}).$$

As for the three sets of polarization vectors associated with the tachyon momenta $|\mathbf{r}\rangle$, $|\mathbf{s}\rangle$ and $|\mathbf{a}\rangle$, respectively, a convenient choice is

$$\begin{aligned}
\boldsymbol{\xi}_\alpha \equiv \boldsymbol{\xi}_\alpha(\mathbf{r}) &= \boldsymbol{\xi}_\alpha(\mathbf{s}) = \boldsymbol{\xi}_\alpha(\mathbf{a}) \text{ for } \alpha = 1, \ldots, 7\ ,\\
\boldsymbol{\xi}_1 &:= (1,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0\,|\,0),\\
&\ \ \vdots\\
\boldsymbol{\xi}_7 &:= (0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 1,\ 0,\ 0\,|\,0);\\
\boldsymbol{\xi}_8(\mathbf{r}) &:= (0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 1,\ 1\,|\,1),\\
\boldsymbol{\xi}_8(\mathbf{s}) &:= (0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,-1,\ 1\,|\,1),\\
\boldsymbol{\xi}_8 \equiv \boldsymbol{\xi}_8(\mathbf{a}) &:= (0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 1,\ 0\,|\,0).
\end{aligned}$$

The little group is $\mathfrak{W}(\Lambda_7, \boldsymbol{\delta}) = \mathfrak{W}(D_8) = S_8 \ltimes (\mathbb{Z}_2)^7$ of order $2^{14}3^1 5^1 7^1$. We only have to evaluate the following commutator, where ϵ denotes a cocycle-factor:

$$\big[|\mathbf{s}\rangle, A^-_{-2}|\mathbf{r}\rangle\big] = \epsilon\bigg(-\frac{1}{2}A^-_{-3} - \frac{5}{6}A^8_{-1}A^8_{-1}A^8_{-1} + \frac{1}{3}A^8_{-3} + \frac{1}{2}\sum_{\mu=1}^{7} A^\mu_{-1}A^\mu_{-1}A^8_{-1}\bigg)|\mathbf{a}\rangle.$$

To identify the remaining missing states, we act on this state with the little Weyl group (which leaves the longitudinal contribution invariant): S_8 permutes all transversal polarizations, and hence generates another seven states. To see that the longitudinal state can be separated from the transversal ones, we act with $\mathfrak{w}_0 \cdots \mathfrak{w}_5\mathfrak{w}_8\mathfrak{w}_6\mathfrak{w}_5 \cdots \mathfrak{w}_0$ on the above state; this operation switches the relative sign between the transversal and the longitudinal terms. Altogether we can thus isolate the following nine states:

$$\begin{aligned}
A^-_{-3}|\mathbf{a}\rangle &\qquad 1 \ \text{ state},\\
\{2A^i_{-3} - 8A^i_{-1}A^i_{-1}A^i_{-1} + 3A^i_{-1}\textstyle\sum_{j=1}^{8} A^j_{-1}A^j_{-1}\}|\mathbf{a}\rangle &\qquad 8 \ \text{ states}.
\end{aligned}$$

We use Roman letters i, j running from 1 to 8 to label the transversal indices. These nine states indeed span the orthogonal complement of the 192-dimensional root space $E_{10}{}^{(\Lambda_7)}$ in $\mathfrak{g}_{II_{9,1}}{}^{(\Lambda_7)}$ as was already noticed in [19] where the result was derived by a completely different approach based on multistring vertices and overlap identities.

Our second (more involved) example is the fundamental root Λ_1 given by

$$\Lambda_1 = \begin{bmatrix} & & & & & & 9 & & \\ 2 & 4 & 6 & 9 & 12 & 15 & 18 & 12 & 6 \end{bmatrix} = (0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 1,\ 1,\ 1\,|\,3),$$

hence $\Lambda_1^2 = -6$ (our conventions used here are the same as in [1]). We have the DDF decomposition $\Lambda_1 = \mathbf{a} - 4\mathbf{k}$, where $\mathbf{k} = -\frac{1}{2}\boldsymbol{\delta}$ and

$$\mathbf{a} := \Lambda_1 + 4\mathbf{k} = \big(0, 0, 0, 0, 0, 0, 1, 1, -1|1\big)\ .$$

We will need two different decompositions of Λ_1 into level-1 roots, namely:

1. $\Lambda_1 = \mathbf{r} + \mathbf{s} + 3\boldsymbol{\delta}$ with

$$\mathbf{r} := \begin{bmatrix} & & & & & & 0 & & \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} = (0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 1,\ -1\,|\,0),$$

$$\mathbf{s} := \begin{bmatrix} & & & & & & 0 & & \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} = (0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 1,\ 0,\ -1\,|\,0);$$

2. $\Lambda_1 = \mathbf{r}' + \mathbf{s}' + 2\boldsymbol{\delta}$ with

$$\mathbf{r}' := \left[\begin{matrix} & & & & & & 0 & & \\ 1&1&1&0&0&0&0&0&0\end{matrix}\right] = (0,\ 0,\ 0,\ 0,\ 0,\ 1,\ 0,\ 0,\ -1\,|\,0),$$

$$\mathbf{s}' := \left[\begin{matrix} & & & & & & 3 & & \\ 1&1&1&3&4&5&6&4&2\end{matrix}\right] = (0,\ 0,\ 0,\ 0,\ 0,\ -1,\ 1,\ 1,\ 0\,|\,1).$$

Although we will need several sets of polarization vectors adjusted to these different decompositions, we will present the basis using the following set, which is adjusted to the first decomposition:

$$\begin{aligned}
\boldsymbol{\xi}_\alpha &\equiv \boldsymbol{\xi}_\alpha(\mathbf{r}) = \boldsymbol{\xi}_\alpha(\mathbf{s}) = \boldsymbol{\xi}_\alpha(\mathbf{a}) \qquad \text{for } \alpha = 1, \dots, 7,\\
\boldsymbol{\xi}_1 &= (1,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0\,|\,0),\\
&\ \vdots\\
\boldsymbol{\xi}_6 &= (0,\ 0,\ 0,\ 0,\ 0,\ 1,\ 0,\ 0,\ 0\,|\,0),\\
\boldsymbol{\xi}_7 &= \tfrac{1}{2}\sqrt{2}(0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 1,\ 1,\ 1\,|\,1),\\
\boldsymbol{\xi}_8(\mathbf{a}) &= \tfrac{1}{2}\sqrt{2}(0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 1,\ -1,\ 0\,|\,0),\\
\boldsymbol{\xi}_8(\mathbf{r}) &= \tfrac{1}{2}\sqrt{2}(0,\ 0,\ 0,\ 0,\ 0,\ 0,\ -1,\ 1,\ 1\,|\,1),\\
\boldsymbol{\xi}_8(\mathbf{s}) &= \tfrac{1}{2}\sqrt{2}(0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 1,\ -1,\ 1\,|\,1).
\end{aligned}$$

The little Weyl group, $\mathfrak{W}(\Lambda_1, \boldsymbol{\delta})$, which is isomorphic to $\mathbb{Z}_2 \times \mathfrak{W}(E_7)$ in this case, acts on this set by permuting $\boldsymbol{\xi}_1, \dots \boldsymbol{\xi}_6$, as a $\mathbb{Z}_2$ on $\boldsymbol{\xi}_8$ and by a more complicated transformation on $\boldsymbol{\xi}_7$. We worked out the following commutator equations, ϵ denoting some irrelevant cocycle factor:

$$\begin{aligned}
\big[|\mathbf{s}\rangle, A^-_{-3}|\mathbf{r}\rangle\big] = \epsilon\Big\{ & \tfrac{1}{8}\textstyle\sum_{\mu,\nu=1}^7 A^\mu_{-1}A^\mu_{-1}A^\nu_{-1}A^\nu_{-1} - \tfrac{1}{2}\sqrt{2}A^8_{-1}A^-_{-3} - \tfrac{3}{4}A^8_{-1}A^8_{-1}A^-_{-2}\\
&-\tfrac{3}{4}\textstyle\sum_{\mu=1}^7 A^\mu_{-1}A^\mu_{-1}A^8_{-1}A^8_{-1} - \tfrac{5}{24}A^8_{-1}A^8_{-1}A^8_{-1}A^8_{-1}\\
&-\tfrac{5}{6}A^8_{-1}A^8_{-3} + \tfrac{1}{4}\textstyle\sum_{\mu=1}^7 A^\mu_{-1}A^\mu_{-1}A^-_{-2} + \tfrac{1}{2}\sum_{\mu=1}^7 A^\mu_{-1}A^\mu_{-3}\Big\}|\mathbf{a}\rangle,
\end{aligned}$$

$$\begin{aligned}
\big[|\mathbf{s}\rangle, A^8_{-1}A^-_{-2}|\mathbf{r}\rangle\big] = \epsilon\Big\{ & \tfrac{7}{64}\sqrt{2}A^8_{-1}A^8_{-1}A^8_{-1}A^8_{-1} - \tfrac{5}{32}\sqrt{2}\textstyle\sum_{\mu=1}^7 A^\mu_{-1}A^\mu_{-1}A^8_{-1}A^8_{-1}\\
&+\tfrac{7}{16}\sqrt{2}A^8_{-1}A^8_{-3} - \tfrac{1}{16}\sqrt{2}\textstyle\sum_{\mu=1}^7 A^\mu_{-1}A^\mu_{-3}\\
&+\tfrac{1}{16}\sqrt{2}A^-_{-2}A^-_{-2} - \tfrac{1}{64}\sqrt{2}\textstyle\sum_{\mu,\nu=1}^7 A^\mu_{-1}A^\mu_{-1}A^\nu_{-1}A^\nu_{-1}\\
&-\tfrac{3}{4}A^8_{-1}A^8_{-1}A^8_{-2} + \tfrac{1}{4}\textstyle\sum_{\mu=1}^7 A^\mu_{-1}A^\mu_{-1}A^8_{-2} - \tfrac{1}{4}\sqrt{2}A^-_{-4}\Big\}|\mathbf{a}\rangle,
\end{aligned}$$

$$\begin{aligned}
\big[A^8_{-1}|\mathbf{s}\rangle, A^-_{-2}|\mathbf{r}\rangle\big] = \epsilon\Big\{ & -\tfrac{3}{4}A^8_{-1}A^8_{-1}A^8_{-2} + \tfrac{1}{4}\textstyle\sum_{\mu=1}^7 A^\mu_{-1}A^\mu_{-1}A^8_{-2} - \tfrac{1}{16}\sqrt{2}A^-_{-2}A^-_{-2}\\
&-\tfrac{7}{16}\sqrt{2}A^8_{-1}A^8_{-3} - \tfrac{7}{64}\sqrt{2}A^8_{-1}A^8_{-1}A^8_{-1}A^8_{-1}\\
&+\tfrac{1}{64}\sqrt{2}\textstyle\sum_{\mu,\nu=1}^7 A^\mu_{-1}A^\mu_{-1}A^\nu_{-1}A^\nu_{-1} + \tfrac{1}{16}\sqrt{2}\sum_{\mu=1}^7 A^\mu_{-1}A^\mu_{-3}\\
&+\tfrac{5}{32}\sqrt{2}\textstyle\sum_{\mu=1}^7 A^\mu_{-1}A^\mu_{-1}A^8_{-1}A^8_{-1} + \tfrac{1}{4}\sqrt{2}A^-_{-4}\Big\}|\mathbf{a}\rangle,
\end{aligned}$$

$$\begin{aligned}
\big[A^\mu_{-1}|\mathbf{s}\rangle, A^-_{-2}|\mathbf{r}\rangle\big] = \epsilon\Big\{ & -\tfrac{1}{4}\textstyle\sum_{\nu=1}^7 A^\nu_{-1}A^\nu_{-1}A^\mu_{-2} + \tfrac{1}{2}\sqrt{2}A^8_{-1}A^\mu_{-3} - \tfrac{1}{6}\sqrt{2}A^\mu_{-1}A^8_{-3}\\
&-\tfrac{1}{2}A^\mu_{-1}A^-_{-3} + \tfrac{3}{4}A^8_{-1}A^8_{-1}A^\mu_{-2} + \tfrac{1}{12}\sqrt{2}A^\mu_{-1}A^8_{-1}A^8_{-1}A^8_{-1}\\
&-\tfrac{1}{4}\sqrt{2}\textstyle\sum_{\nu=1}^7 A^\nu_{-1}A^\nu_{-1}A^\mu_{-1}A^8_{-1}\Big\}|\mathbf{a}\rangle,
\end{aligned}$$

$$\begin{aligned}\big[|\mathbf{s}\rangle, A^{\mu}_{-1}A^{-}_{-2}|\mathbf{r}\rangle\big] = \epsilon\Big\{ & -\tfrac{3}{4}A^{8}_{-1}A^{8}_{-1}A^{\mu}_{-2} + \tfrac{1}{4}\textstyle\sum_{\nu=1}^{7} A^{\nu}_{-1}A^{\nu}_{-1}A^{\mu}_{-2} + \tfrac{1}{2}\sqrt{2}A^{8}_{-1}A^{\mu}_{-3} \\ & -\tfrac{1}{6}\sqrt{2}A^{\mu}_{-1}A^{8}_{-3} - \tfrac{1}{2}A^{\mu}_{-1}A^{-}_{-3} + \tfrac{1}{12}\sqrt{2}A^{\mu}_{-1}A^{8}_{-1}A^{8}_{-1}A^{8}_{-1} \\ & -\tfrac{1}{4}\sqrt{2}\textstyle\sum_{\nu=1}^{7} A^{\nu}_{-1}A^{\nu}_{-1}A^{\mu}_{-1}A^{8}_{-1}\Big\}|\mathbf{a}\rangle.\end{aligned}$$

We need one more commutator, associated with a second DDF decomposition. Namely,

$$\begin{aligned}\big[|\mathbf{s}'\rangle, A^{-}_{-2}|\mathbf{r}'\rangle\big] = \epsilon\Big\{ & \tfrac{1}{32}\textstyle\sum_{\mu=1}^{7} A^{\mu}_{-1}A^{\mu}_{-1}A^{8}_{-1}A^{8}_{-1} + \tfrac{1}{64}A^{8}_{-1}A^{8}_{-1}A^{8}_{-1}A^{8}_{-1} \\ & +\tfrac{1}{16}\textstyle\sum_{\mu=1}^{7} A^{\mu}_{-1}A^{\mu}_{-3} + \tfrac{1}{64}\textstyle\sum_{\mu,\nu=1}^{7} A^{\mu}_{-1}A^{\mu}_{-1}A^{\nu}_{-1}A^{\nu}_{-1} \\ & -\tfrac{1}{16}A^{-}_{-2}A^{-}_{-2} - \tfrac{1}{3}\sqrt{3}A^{7}_{-1}A^{-}_{-3} + \tfrac{1}{3}A^{6}_{-1}A^{-}_{-3}\sqrt{6} + \tfrac{1}{6}\sqrt{2}A^{6}_{-3}A^{7}_{-1} \\ & +\tfrac{1}{4}\sqrt{2}A^{6}_{-1}A^{7}_{-1}A^{8}_{-1}A^{8}_{-1} - \tfrac{1}{3}\sqrt{2}A^{6}_{-1}A^{7}_{-1}A^{7}_{-1}A^{7}_{-1} \\ & +\tfrac{1}{6}\sqrt{2}A^{6}_{-1}A^{7}_{-3} + \tfrac{1}{16}A^{8}_{-3}A^{8}_{-1} - \tfrac{1}{3}A^{6}_{-3}A^{6}_{-1} - \tfrac{1}{4}A^{6}_{-1}A^{6}_{-1}A^{8}_{-1}A^{8}_{-1} \\ & -\tfrac{1}{6}A^{7}_{-3}A^{7}_{-1} - \tfrac{1}{8}A^{7}_{-1}A^{7}_{-1}A^{8}_{-1}A^{8}_{-1} + \tfrac{1}{3}A^{6}_{-1}A^{6}_{-1}A^{6}_{-1}A^{6}_{-1} \\ & +\tfrac{1}{12}A^{7}_{-1}A^{7}_{-1}A^{7}_{-1}A^{7}_{-1} - \tfrac{1}{4}\textstyle\sum_{\mu=1}^{7} A^{\mu}_{-1}A^{\mu}_{-1}A^{6}_{-1}A^{6}_{-1} \\ & -\tfrac{1}{8}\textstyle\sum_{\mu=1}^{7} A^{\mu}_{-1}A^{\mu}_{-1}A^{7}_{-1}A^{7}_{-1} + A^{6}_{-1}A^{6}_{-1}A^{7}_{-1}A^{7}_{-1} + \tfrac{1}{4}A^{-}_{-4} \\ & -\tfrac{2}{3}\sqrt{2}A^{6}_{-1}A^{6}_{-1}A^{6}_{-1}A^{7}_{-1} + \tfrac{1}{4}\sqrt{2}\textstyle\sum_{\mu=1}^{7} A^{\mu}_{-1}A^{\mu}_{-1}A^{6}_{-1}A^{7}_{-1}\Big\}|\mathbf{a}\rangle.\end{aligned}$$

We displayed this result using the basis of polarization associated with the first decomposition. Appropriate linear combinations and the little Weyl group action lead to the following 53 states, spanning the orthogonal complement of the 727-dimensional root space ${E_{10}}^{(\Lambda_1)}$ in ${\mathfrak{g}_{II_{9,1}}}^{(\Lambda_1)}$. We use the following conventions to label the transversal indices: Roman letters $i, j, \ldots$ run from 1 to 8, Greek letters from the middle of the alphabet $\mu, \nu, \ldots$ run from 1 to 7 and Greek letters from the beginning of the alphabet $\alpha, \beta, \ldots$ run from 1 to 6.

$$\begin{aligned} A^{i}_{-1}A^{-}_{-3}|\mathbf{a}\rangle & \qquad \text{8 states,} \\ \{A^{8}_{-1}A^{8}_{-1}A^{i}_{-2} - \tfrac{1}{3}\textstyle\sum_{\mu=1}^{7} A^{\mu}_{-1}A^{\mu}_{-1}A^{i}_{-2}\}|\mathbf{a}\rangle & \qquad \text{8 states,} \\ \{A^{-}_{-2}A^{-}_{-2} - 4A^{-}_{-4}\}|\mathbf{a}\rangle & \qquad \text{1 state,} \\ \{A^{\mu}_{-1}A^{8}_{-1}A^{8}_{-1}A^{8}_{-1} - 3\textstyle\sum_{\nu=1}^{7} A^{\nu}_{-1}A^{\nu}_{-1}A^{\mu}_{-1}A^{8}_{-1} - 2A^{\mu}_{-1}A^{8}_{-3} & \\ +6A^{8}_{-1}A^{\mu}_{-3}\}|\mathbf{a}\rangle & \qquad \text{7 states,} \\ \{A^{\alpha}_{-1}A^{7}_{-3} + A^{7}_{-1}A^{\alpha}_{-3} - 2A^{\alpha}_{-1}A^{7}_{-1}A^{7}_{-1}A^{7}_{-1} - 4A^{\alpha}_{-1}A^{\alpha}_{-1}A^{\alpha}_{-1}A^{7}_{-1} & \\ +\tfrac{3}{2}\textstyle\sum_{i=1}^{8} A^{i}_{-1}A^{i}_{-1}A^{\alpha}_{-1}A^{7}_{-1}\}|\mathbf{a}\rangle & \qquad \text{6 states,} \\ \{A^{\alpha}_{-3}A^{\alpha}_{-1} - \tfrac{3}{2}A^{8}_{-3}A^{8}_{-1} + \tfrac{1}{2}A^{7}_{-3}A^{7}_{-1} - A^{\alpha}_{-1}A^{\alpha}_{-1}A^{\alpha}_{-1}A^{\alpha}_{-1} & \\ -\tfrac{1}{4}A^{7}_{-1}A^{7}_{-1}A^{7}_{-1}A^{7}_{-1} + \tfrac{3}{4}\textstyle\sum_{i=1}^{8} A^{i}_{-1}A^{i}_{-1}A^{\alpha}_{-1}A^{\alpha}_{-1} & \\ +\tfrac{3}{8}\textstyle\sum_{i=1}^{8} A^{i}_{-1}A^{i}_{-1}A^{7}_{-1}A^{7}_{-1} - A^{\alpha}_{-1}A^{\alpha}_{-1}A^{7}_{-1}A^{7}_{-1} & \\ +\tfrac{3}{8}\textstyle\sum_{\mu=1}^{7} A^{\mu}_{-1}A^{\mu}_{-1}A^{8}_{-1}A^{8}_{-1} - \tfrac{3}{8}A^{8}_{-1}A^{8}_{-1}A^{8}_{-1}A^{8}_{-1}\}|\mathbf{a}\rangle & \qquad \text{6 states,} \\ \{A^{\alpha}_{-1}A^{\beta}_{-3} + A^{\beta}_{-1}A^{\alpha}_{-3} + \tfrac{1}{2}A^{\alpha}_{-1}A^{\beta}_{-1}A^{\beta}_{-1}A^{\beta}_{-1} + \tfrac{1}{2}A^{\alpha}_{-1}A^{\alpha}_{-1}A^{\alpha}_{-1}A^{\beta}_{-1} & \\ -\tfrac{3}{2}\textstyle\sum_{\substack{\gamma=1 \\ \gamma\neq\alpha,\beta}}^{6} A^{\alpha}_{-1}A^{\beta}_{-1}A^{\gamma}_{-1}A^{\gamma}_{-1} + \tfrac{3}{2}A^{\alpha}_{-1}A^{\beta}_{-1}A^{8}_{-1}A^{8}_{-1} & \\ +\tfrac{3}{2}A^{\alpha}_{-1}A^{\beta}_{-1}A^{7}_{-1}A^{7}_{-1} + 4A^{\gamma}_{-1}A^{\delta}_{-1}A^{\epsilon}_{-1}A^{\eta}_{-1}\}|\mathbf{a}\rangle & \qquad \text{15 states,} \\ \{\tfrac{4}{3}A^{8}_{-1}A^{8}_{-3} - \tfrac{1}{8}\textstyle\sum_{\mu=1}^{7} A^{\mu}_{-1}A^{\mu}_{-1}A^{-}_{-2} + \tfrac{3}{8}A^{8}_{-1}A^{8}_{-1}A^{-}_{-2} & \\ +\tfrac{1}{3}A^{8}_{-1}A^{8}_{-1}A^{8}_{-1}A^{8}_{-1} + \tfrac{1}{4}\textstyle\sum_{\mu=1}^{7} A^{\mu}_{-1}A^{\mu}_{-1}A^{8}_{-1}A^{8}_{-1}\}|\mathbf{a}\rangle & \qquad \text{1 state,} \\ \{7A^{8}_{-1}A^{8}_{-3} + \tfrac{7}{4}A^{8}_{-1}A^{8}_{-1}A^{8}_{-1}A^{8}_{-1} - \tfrac{5}{2}\textstyle\sum_{\mu=1}^{7} A^{\mu}_{-1}A^{\mu}_{-1}A^{8}_{-1}A^{8}_{-1} & \\ -\tfrac{1}{4}\textstyle\sum_{\mu,\nu=1}^{7} A^{\mu}_{-1}A^{\mu}_{-1}A^{\nu}_{-1}A^{\nu}_{-1} - \textstyle\sum_{\mu=1}^{7} A^{\mu}_{-1}A^{\mu}_{-3}\}|\mathbf{a}\rangle & \qquad \text{1 state.} \end{aligned}$$

These are precisely the missing states found in [1].

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Note added in proof

We have meanwhile performed an independent test of the Conjecture 1 by means of a modified denominator formula, establishing its validity for all roots of norm ≥ -8. However, the conjecture fails for roots of norm ≤ -10. See O. Bärwald, R.W. Gebert and J. Niocolai, "On the Imaginary Simple Roots of the Borcherds Algebra $\mathfrak{g}_{II_{9,1}}$". Nuclear Physics **B510**, 721–738 (1998).

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