Master Partitions for Large N Matrix Field Theories

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Abstract: We introduce a systematic approach for treating the large N limit of matrix *field* theories.

1. Introduction

It has been known for thirty years that quantum field theory simplifies enormously if the number N of internal field components tends to infinity. In the case where the N components form a *vector* this leads to exact solutions in any dimension of space-time. For physical applications, ranging from solid state physics to gauge theories and quantum gravity, a different situation is much more pertinent: The case of N^2 internal components that form a *matrix*. Here exact solutions have only been produced for very low dimensionalities. It is one of the outstanding problems of theoretical physics to extend large N technology to physically interesting dimensions.

In the present article we will be concerned with matrix "spin systems", that is D-dimensional Euclidean lattice field theories whose internal degrees of freedom are hermitian, complex or unitary $N \times N$ matrices. The idea is to treat the problem by a three step procedure:

(1) Eguchi–Kawai reduction: Replace the $N = \infty$ field theory by a one-matrix model coupled to appropriate constant external field matrices.
(2) Character expansion: Express the partition function of the one-matrix model of (1) as a sum over polynomial representations – labelled by Young diagrams – of $U(N)$.
(3) Saddle point analysis: Find an effective Young diagram that dominates the partition sum of (2) in the large N limit.

The insight that step (1) is possible is due to Eguchi and Kawai [1]. Intuitively it says that, if a saddle point configuration exists at $N = \infty$, it should be given by a single translationally invariant matrix (the so-called master field). In practice the reduction is rather subtle, and we will be using the twisted EK reduction [2] which results in a one-matrix model in external constant fields encoding the original (discrete) space-time.

Step (2) is novel in this context and is the main focus of the present work. The one-matrix model of (1) still has N^2 degrees of freedom, and it is well known that a saddle point for matrix models can only be found once the degrees of freedom are reduced as $N^2 \to N$. The external fields encoding space-time prevent any naive reduction to the N eigenvalues of the matrix, which is the route of choice for simpler models without external fields. But is it possible to replace the matrix integral by a sum over *partitions* corresponding to a sum over all polynomial representations of $U(N)$. The crucial point is then that one ends up with a kind of one-dimensional spin model in Young diagram space with only N variables: the possible lengths of the N rows of the diagram.

Step (3) might appear to be an exotic idea: we claim that the $N = \infty$ "master field" can be described by a "master partition". However, it has already been recently demonstrated in a series of papers [3,4] that certain infinite sums over partitions are dominated by a saddle point configuration. This led to the solution of matrix models in external fields not treatable by any other method. The present models are more complicated, but not fundamentally different.

The character expansions we find lead to a very interesting and apparently novel combinatorial problem in Young pattern space (see Sect. 4). More insight into this problem will be needed in order to proceed with the final step (3) of our program, the saddle point analysis. We introduce what we call "lattice polynomials" Ξ_h, Υ_h which are polynomials in $\frac{1}{N}$. They depend on the Young diagram h and the precise nature of the space-time lattice.

It might be objected that the present approach is futile unless one can demonstrate that the lattice polynomials Ξ_h, Υ_h can be explicitly computed or at least bootstraped at $N = \infty$. But there is one important argument against this pessimistic assessment: The lattice polynomials Ξ_h, Υ_h only depend on the nature of the lattice but *not* on the local measure of the minimally coupled (matrix) spins of the model[1]. Therefore, solving interacting field theory in our language is of the same degree of complexity as solving the free field case.

Finally we should mention that our program is very general since it applies in principle to *any* large N matrix spin system. It would be interesting to extend the method to matrix field theories with a *gauge symmetry* such as Yang-Mills theory. Indeed the EK reduction was initially designed for lattice gauge theory [1]. Recently it was demonstrated by Monte Carlo methods that even the path integral of continuum gauge theory may be EK reduced to a *convergent* ordinary multiple matrix integral [5]. A rigorous mathematical proof, as well as an investigation on whether the reduced model reinduces the field theory as $N \to \infty$, are still lacking. At any rate, reducing a D-dimensional gauge theory, one so far ends up with a nonlinearly coupled D-matrix model, which is not yet tractable by the present machinery unless it is understood how to perform a further reduction $DN^2 \to N^2$.

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[1] Except for the global symmetry of the matrix spins. In this paper we develop the theory in parallel for the case of U(N) global symmetry (hermitian matrices) and U(N) × U(N) symmetry (complex matrices). The other classical groups could presumably be treated as well, but it is well known that they do not lead to different large N limits.

2. Reduced Matrix Spin Systems

Consider a spin model on a periodic lattice. In order to be specific we will sketch the method for a two-dimensional lattice, but higher (or lower) dimensions can be treated as well. We will not dwell on details since they are well explained elsewhere. The variables are $N \times N$ hermitian matrices $M(x)$ defined on the lattice sites x

$$\mathcal{Z}_{\mathrm{H}} = \int \prod_x \mathcal{D}M(x) \, e^{-\mathcal{S}_{\mathrm{H}}},$$

$$\mathcal{S}_{\mathrm{H}} = N \operatorname{Tr} \sum_x \left[\frac{1}{2} M(x)^2 + V\big(M(x)\big) \right.$$
$$\left. - \frac{\beta}{2} \sum_{\mu=1,2} [M(x)M(x+\hat{\mu}) + M(x)M(x-\hat{\mu})] \right], \tag{1}$$

where $\hat{\mu}$ denotes the unit vector in the μ-direction. It is equally natural to consider general complex matrices $\Phi(x) \in \mathrm{GL}(N, \mathbb{C})$, in which case

$$\mathcal{Z}_{\mathrm{GL}} = \int \prod_x \mathcal{D}\Phi(x) \, e^{-\mathcal{S}_{\mathrm{GL}}},$$

$$\mathcal{S}_{\mathrm{GL}} = N \operatorname{Tr} \sum_x \left[\Phi(x)\Phi^{\dagger}(x) + V\big(\Phi(x)\Phi^{\dagger}(x)\big) \right.$$
$$\left. - \beta \sum_{\mu=1,2} [\Phi(x)\Phi^{\dagger}(x+\hat{\mu}) + \Phi(x)\Phi^{\dagger}(x-\hat{\mu})] \right]. \tag{2}$$

If $V = 0$ in Eqs. (1),(2) the model is free. The integration measures in Eqs. (1),(2) are the flat measures for hermitian and complex matrices:

$$\mathcal{D}M = \prod_{i=1}^{N} \frac{dM_{ii}}{\sqrt{2\pi N^{-1}}} \prod_{i<j}^{N} \frac{d\operatorname{Re}M_{ij} d\operatorname{Im}M_{ij}}{\pi N^{-1}}, \qquad \mathcal{D}\Phi = \prod_{i,j=1}^{N} \frac{d\operatorname{Re}\Phi_{ij} d\operatorname{Im}\Phi_{ij}}{\pi N^{-1}}. \tag{3}$$

A third, very important type of spin model is the so-called *chiral field*, which looks like the free complex model Eq. (2)

$$\mathcal{Z}_{\mathrm{U}} = \int \prod_x \mathcal{D}U(x) \, e^{-\mathcal{S}_{\mathrm{U}}},$$

$$\mathcal{S}_{\mathrm{U}} = -\beta N \operatorname{Tr} \left[\sum_x \sum_{\mu=1,2} [U(x)U^{\dagger}(x+\hat{\mu}) + U(x)U^{\dagger}(x-\hat{\mu})] \right], \tag{4}$$

but the matrices $U(x) \in \mathrm{U}(N)$ are unitary. In this case the measure $\mathcal{D}U(x)$ is the Haar measure on the group. The model is therefore not free.

The Eguchi–Kawai reduction [1,2] states that the above lattice models can be replaced at $N = \infty$ by, respectively, the following one-matrix models coupled to constant external field matrices P and Q:

$$Z_{\mathrm{H}} = \int \mathcal{D}M \exp N \operatorname{Tr}\left[-\frac{1}{2}M^2 - V(M) + \beta\left(MPMP^{\dagger} + MQMQ^{\dagger} \right) \right], \quad (5)$$

$$Z_{\mathrm{GL}} = \int \mathcal{D}\Phi \, \exp N \operatorname{Tr}\left[-\Phi\Phi^{\dagger} - V(\Phi\Phi^{\dagger}) \right] \times$$
$$\times \exp \beta N \operatorname{Tr}\left(\Phi P\Phi^{\dagger} P^{\dagger} + \Phi P^{\dagger}\Phi^{\dagger} P + \Phi Q\Phi^{\dagger} Q^{\dagger} + \Phi Q^{\dagger}\Phi^{\dagger} Q \right), \quad (6)$$

$$Z_{\mathrm{U}} = \int \mathcal{D}U \exp \beta N \operatorname{Tr}\left(UPU^{\dagger} P^{\dagger} + UP^{\dagger}U^{\dagger} P + UQU^{\dagger} Q^{\dagger} + UQ^{\dagger}U^{\dagger} Q \right). \quad (7)$$

Here $P = P_N$ and $Q = Q_N$ are the famous $N \times N$ unitary "shift and clock" matrices

$$P_N = \begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & 0 & 1 \\ 1 & & & & 0 \end{pmatrix}, \quad Q_N = \begin{pmatrix} 1 & & & & \\ & \omega_N & & & \\ & & \ddots & & \\ & & & \omega_N^{N-2} & \\ & & & & \omega_N^{N-1} \end{pmatrix}, \quad (8)$$

where $\omega_N = \exp \frac{2\pi i}{N}$ and $P_N Q_N = \omega_N Q_N P_N$. To be more precise, the free energies as well as appropriate correlation functions (see [2]) are identical to leading order in $\frac{1}{N}$ in the lattice field theory and the corresponding one-matrix model. The thermodynamic limit, that is a lattice of infinite extent, is approached when $N \to \infty$. We see that the structure of the lattice has been "hidden" in index space! It is natural to generalize the situation to a toroidal $K \times L$ lattice:

$$P = P_K \otimes \mathbb{1}_{\frac{N}{K}}, \qquad Q = Q_L \otimes \mathbb{1}_{\frac{N}{L}}, \quad (9)$$

where N is chosen to be divisible by K and L. This allows to take the thermodynamic limit and the large N limit independently. If we put $L = 1$ (we can then equivalently omit Q altogether) the target space becomes a closed one-dimensional chain.

We suspect that matrix models on arbitrary discrete target spaces can be EK reduced by appropriate external matrices, but this has not been worked out yet.

3. Character Expansions

Now we turn to step (2) and rewrite the reduced hermitian, complex and unitary matrix integrals Eqs. (5), (6), (7) as sums over representations of U(N). To this end introduce the following source integrals:

$$Z_{\mathrm{H}}[J] = \int \mathcal{D}M \, \exp N \operatorname{Tr}\left[-\frac{1}{2}M^2 - V(M) + JM \right], \quad (10)$$

$$Z_{\mathrm{GL}}[J\bar{J}] = \int \mathcal{D}\Phi \, \exp N \operatorname{Tr}\left[-\Phi\Phi^{\dagger} - V(\Phi\Phi^{\dagger}) + J\Phi + \Phi^{\dagger}\bar{J} \right], \quad (11)$$

$$Z_U[J\bar{J}] = \int \mathcal{D}U \ \exp \ N \operatorname{Tr}\left[JU + U^\dagger \bar{J}\right]. \tag{12}$$

The two different ways of introducing a source are due to the $U(N)$ symmetry of hermitian matrices on the one hand and the $U(N) \times U(N)$ symmetry of complex (and complex unitary) matrices on the other. The reduced models are easily obtained from the source integrals by applying an operator:

$$Z_H = \exp \frac{\beta}{N} \operatorname{Tr}\left(\partial P \partial P^\dagger + \partial Q \partial Q^\dagger\right) \cdot Z_H[J]\Big|_{J=0}, \tag{13}$$

$$Z_{GL,U} = \exp \frac{\beta}{N} \operatorname{Tr}\left(\partial P \bar{\partial} P^\dagger + \partial P^\dagger \bar{\partial} P + \partial Q \bar{\partial} Q^\dagger + \partial Q^\dagger \bar{\partial} Q\right) \cdot Z_{GL,U}[J\bar{J}]\Big|_{J=\bar{J}=0}. \tag{14}$$

Here $\partial, \bar{\partial}$ denote $N \times N$ matrix differential operators whose matrix elements are $\partial_{ji} = \frac{\partial}{\partial J_{ij}}$ and $\bar{\partial}_{ji} = \frac{\partial}{\partial \bar{J}_{ij}}$. It is clear that the source integrals are *class* functions of, respectively, J and $J\bar{J}$. Therefore they can be expressed as character expansions, with known (see [3, 4,6]) expansion coefficients. If $V = 0$, they read for the hermitian and complex source integrals, respectively,

$$Z_H[J] = \exp \frac{1}{2} N \operatorname{Tr} J^2 = \sum_h \chi_h(A_2)\chi_h(J), \tag{15}$$

$$Z_{GL}[J\bar{J}] = \exp \ N \operatorname{Tr} J\bar{J} = \sum_h \chi_h(A_1)\chi_h(J\bar{J}), \tag{16}$$

while for the unitary source integral one has [6]

$$Z_U[J\bar{J}] = \sum_h \frac{\chi_h(A_1)\chi_h(A_1)}{\chi_h(\mathbb{1})}\chi_h(J\bar{J}). \tag{17}$$

Here the sum runs over all partitions h labeled by the shifted weights $h_i = N - i + m_i$, where $m_i \geq 0, i = 1, \ldots, N$, is the number of boxes in the ith row of the Young pattern associated to h. $\chi_h(J)$ is the Schur function, dependent on J, on the diagram h. It is identical to the Weyl character of the matrix J corresponding to the representation labeled by h. A_1 and A_2 are defined through $\operatorname{Tr} A_1^k = N(\delta_{k,0}+\delta_{k,1})$ and $\operatorname{Tr} A_2^k = N(\delta_{k,0}+\delta_{k,2})$, and $\chi_h(\mathbb{1})$ is the dimension of the representation. For more details on the notation, and for explicit formulas for the characters $\chi_h(A_1)$, $\chi_h(A_2)$ and $\chi_h(\mathbb{1})$ see [3,4]. For a non-zero potential V, the hermitian and complex character expansions become a bit more complicated, but are still available:

$$Z_H[J] = \sum_h \Theta_h \chi_h(J), \tag{18}$$

$$Z_{GL}[J\bar{J}] = \sum_h \Omega_h \chi_h(J\bar{J}), \tag{19}$$

where Θ_h is given by

$$\Theta_h = \frac{\chi_h(A_1)}{\chi_h(\mathbb{1})} \int \mathcal{D}M \, \exp \, N \operatorname{Tr} \left[-\frac{1}{2}M^2 - V(M) \right] \chi_h(M), \qquad (20)$$

and Ω_h by

$$\Omega_h = \left(\frac{\chi_h(A_1)}{\chi_h(\mathbb{1})} \right)^2 \int \mathcal{D}\Phi \, \exp \, N \operatorname{Tr} \left[-\Phi\Phi^\dagger - V(\Phi\Phi^\dagger) \right] \chi_h(\Phi\Phi^\dagger). \qquad (21)$$

The integrals appearing in Eqs. (20), (21) are ordinary one-matrix integrals which may be computed rather explicitly as $N \times N$ determinants. Their analysis in the $N \to \infty$ limit proceeds by employing standard techniques, supplemented by the methods of [3].

Now we apply the operators in Eqs. (13), (14) in order to generate the space-time lattice; this results in character expansions for the reduced matrix field theories. In the hermitian case one has (here $|h| = \sum_i m_i =$ number of boxes in the Young diagram)

$$Z_{\mathrm{H}} = \sum_h \chi_h(A_2) \, \Xi_h \, \beta^{\frac{|h|}{2}} \quad \text{for } V = 0, \qquad (22)$$

$$Z_{\mathrm{H}} = \sum_h \Theta_h \, \Xi_h \, \beta^{\frac{|h|}{2}} \qquad \text{for } V \neq 0, \qquad (23)$$

with

$$\Xi_h = \exp \frac{1}{N} \operatorname{Tr} \left(\partial P \partial P^\dagger + \partial Q \partial Q^\dagger \right) \cdot \chi_h(J) \Big|_{J=0}. \qquad (24)$$

The free complex, interacting complex, and the unitary case become

$$Z_{\mathrm{GL}} = \sum_h \chi_h(A_1) \, \Upsilon_h \, \beta^{|h|} \qquad \text{for } V = 0, \qquad (25)$$

$$Z_{\mathrm{GL}} = \sum_h \Omega_h \, \Upsilon_h \, \beta^{|h|} \qquad \text{for } V \neq 0, \qquad (26)$$

$$Z_{\mathrm{U}} = \sum_h \frac{\chi_h(A_1)\chi_h(A_1)}{\chi_h(\mathbb{1})} \, \Upsilon_h \, \beta^{|h|}, \qquad (27)$$

with

$$\Upsilon_h = \exp \frac{1}{N} \operatorname{Tr} \left(\partial P \bar{\partial} P^\dagger + \partial P^\dagger \bar{\partial} P + \partial Q \bar{\partial} Q^\dagger + \partial Q^\dagger \bar{\partial} Q \right) \cdot \chi_h(J\bar{J}) \Big|_{J=\bar{J}=0}. \qquad (28)$$

The character expansions Eqs. (22), (23), (25), (26), (27) are at the heart of our proposal. It is seen that they neatly *separate* the nature of the local spin weight ($\chi_h(A_2), \Theta_h, \chi_h(A_1), \Omega_h,$ $(\chi_h(A_1))^2(\chi_h(\mathbb{1}))^{-1}$) and the nature of the embedding space (Ξ_h, Υ_h). As a striking example, note that from the point of view of our character expansion method the difference between the free Gaussian model on a toroidal lattice Eq. (25) and the non-trivial chiral model Eq. (27) is a simple, explicitly known factor

$$\frac{\chi_h(A_1)}{\chi_h(\mathbb{1})} = N^{|h|} \prod_{i=1}^N \frac{(N-i)!}{h_i!}.$$

The character expansions involve sums over N variables only and we can write down a saddle point equation for the effective density of the master partition. In order to complete the program, we need a second bootstrap equation for the novel quantities Ξ_h and Υ_h, which contain the connectivity information of the lattice.

4. Lattice Polynomials

Inspection of the quantities Ξ_h and Υ_h in eqs.(24),(28) shows that they are polynomials in the variable $\frac{1}{N}$ of degree not higher than, respectively, $\frac{1}{2}|h| - 1$ and $|h| - 2$. They are zero if the number $|h|$ of boxes in the Young pattern is odd. Conjugating the diagram gives the same polynomial except for the replacement $\frac{1}{N} \rightarrow -\frac{1}{N}$. The first few can be computed by brute force calculation directly from the definitions Eqs. (24), (28), see Table 1.

Table 1. The first few $D = 2$ lattice polynomials

h	Ξ_h	Υ_h
2	2	2
1^2	2	2
4	$3 + 12\frac{1}{N}$	$3 + 24\frac{1}{N} + 54\frac{1}{N^2}$
31	$5 + 4\frac{1}{N}$	$5 + 8\frac{1}{N} + 18\frac{1}{N^2}$
2^2	6	6
21^2	$5 - 4\frac{1}{N}$	$5 - 8\frac{1}{N} + 18\frac{1}{N^2}$
1^4	$3 - 12\frac{1}{N}$	$3 - 24\frac{1}{N} + 54\frac{1}{N^2}$

Here we used $\mathrm{Tr}(P^k Q^l) = N\delta_{k,0}\delta_{l,0}$, which is true as long as $|k| < N$, $|l| < N$. We also replaced $\omega_N \rightarrow 1$, $\omega_N^* \rightarrow 1$ (remember $\omega_N = \exp\frac{2\pi i}{N}$): in other words, we assumed P and Q to commute at large N. Both assumptions are innocent at least in the strong coupling (small β) phase. If the model possesses a weak coupling phase (like e.g. the chiral field Eq. (7)), these assumptions may have to be reconsidered, if we want the character expansion to describe this second phase as well. This is because in the present approach we expect large N phase transitions to correspond to the situation where the number of rows of the master partition is of $\mathcal{O}(N)$ ("touching transition"). Note that we *cannot* drop the other terms of $\mathcal{O}(\frac{1}{N})$ in Ξ_h, Υ_h since the character expansions are for the partition function and not for the free energy.

The direct calculation of the lattice polynomials quickly gets very tedious. The combinatorics involved seems to be of a novel type. While we have not yet found an efficient calculational scheme or recursive method, let us give some interesting representations for Ξ_h and Υ_h that may prove useful later. Introduce the following Gaussian measure on the space of $M \times N$ ($M \le N$) complex matrices Λ:

$$[\mathcal{D}\Lambda] = \prod_{i=1}^{M}\prod_{j=1}^{N}\left(\frac{d\mathrm{Re}\Lambda_{ij}\, d\mathrm{Im}\Lambda_{ij}}{\pi N^{-1}}\right)\exp N\,\mathrm{Tr}\left[-\Lambda\Lambda^\dagger\right]. \tag{29}$$

This measure is invariant under $\mathrm{U}(M) \times \mathrm{U}(N)$. It is then fairly easy to prove (cf. [4]) the following representation for the character of the source:

$$\chi_h(J) = \int \mathcal{D}U\, \chi_h(U^\dagger) \int [\mathcal{D}\Lambda]\, \exp N\,\mathrm{Tr}\, U\Lambda J\Lambda^\dagger, \tag{30}$$

where $U \in \mathrm{U}(M)$ is unitary and $\mathcal{D}U$ is the Haar measure on $\mathrm{U}(M)$. This formula is valid for diagrams h with at most M rows. Therefore Ξ_h becomes, cf. Eq. (24)

$$\Xi_h = \int \mathcal{D}U \chi_h(U^\dagger) \int [\mathcal{D}\Lambda] \exp N \operatorname{Tr} \left(\Lambda^\dagger U \Lambda P \Lambda^\dagger U \Lambda P^\dagger + \Lambda^\dagger U \Lambda Q \Lambda^\dagger U \Lambda Q^\dagger \right).$$

(31)

After a Hubbard-Stratanovich transformation decoupling the quartic terms by Gaussian $M \times M$ complex matrices S and T (with measure as in Eq. (29) with $N \to M$), and integration over Λ, we obtain the representation

$$\Xi_h = \int \mathcal{D}U \chi_h(U^\dagger) \int [\mathcal{D}S][\mathcal{D}T]$$
$$\times \exp \left[\operatorname{Tr}_{M \otimes N} \sum_{k=1}^{\infty} \frac{1}{k} \left(SU \otimes P + S^\dagger U \otimes P^\dagger + TU \otimes Q + T^\dagger U \otimes Q^\dagger \right)^k \right].$$

(32)

The combinatorial interpretation of the exponential in Eq. (32) is the following: we have a generating function for a non-commutative random walk on a two-dimensional lattice with variable U. The representation is useful for getting some exact results on the Ξ_h, but we have not yet been able to compute the integral Eq. (32) exactly except for $M = 1$ (characters with just one row). E.g. we can find a generating function (with z_i being the eigenvalues of U) for the large N limit of Ξ_h

$$\prod_{i,j}^{M} \frac{1}{(1 - z_i z_j)^2} = \sum_h \Xi_h^{N=\infty} \chi_h(z)$$

(33)

giving the constant terms of the lattice polynomials. This is however not sufficient for the large N limit of the field theory, as already mentioned. A curious feature of Eq. (32) is that we can take $N \to \infty$ while keeping M in the range $1 \ll M \ll N$. That is, it should be possible to find a saddle point for the situation where the row lengths are large compared to the number of rows, corresponding to the extreme strong coupling limit. Furthermore, it should be investigated whether the $M \times M$ matrices can be taken to commute as $N \to \infty$.

Similar, if slightly more complicated representations are possible for Υ_h; here the starting point is the expression

$$\chi_h(J\bar{J}) = \int \mathcal{D}U \chi_h(U^\dagger) \int [\mathcal{D}\Lambda_1][\mathcal{D}\Lambda_2] \exp N \operatorname{Tr} \left(U^{\frac{1}{2}} \Lambda_1 J \Lambda_2^\dagger + \Lambda_2 \bar{J} \Lambda_1^\dagger U^{\frac{1}{2}} \right),$$

(34)

which means the lattice polynomials become

$$\Upsilon_h = \int \mathcal{D}U \chi_h(U^\dagger) \int [\mathcal{D}\Lambda_1][\mathcal{D}\Lambda_2] \times$$
$$\times \exp N \operatorname{Tr} \left(\Lambda_2^\dagger U^{\frac{1}{2}} \Lambda_1 P \Lambda_1^\dagger U^{\frac{1}{2}} \Lambda_2 P^\dagger + \Lambda_2^\dagger U^{\frac{1}{2}} \Lambda_1 P^\dagger \Lambda_1^\dagger U^{\frac{1}{2}} \Lambda_2 P \right) \times$$
$$\times \exp N \operatorname{Tr} \left(\Lambda_2^\dagger U^{\frac{1}{2}} \Lambda_1 Q \Lambda_1^\dagger U^{\frac{1}{2}} \Lambda_2 Q^\dagger + \Lambda_2^\dagger U^{\frac{1}{2}} \Lambda_1 Q^\dagger \Lambda_1^\dagger U^{\frac{1}{2}} \Lambda_2 Q \right), \quad (35)$$

and the non-commutative random walk representation is

$$
\Upsilon_h = \int \mathcal{D}U \, \chi_h(U^\dagger) \int [\mathcal{D}S][\mathcal{D}\bar{S}][\mathcal{D}T][\mathcal{D}\bar{T}]
$$

$$
\times \exp\left[\text{Tr}_{M \otimes N} \sum_{k=1}^{\infty} \frac{1}{k} \left(SU^{\frac{1}{2}} \otimes P + \bar{S}U^{\frac{1}{2}} \otimes P^\dagger + TU^{\frac{1}{2}} \otimes Q + \bar{T}U^{\frac{1}{2}} \otimes Q^\dagger \right)^k \right]
$$

$$
\times \exp\left[\text{Tr}_{M \otimes N} \sum_{k=1}^{\infty} \frac{1}{k} \left(\bar{S}^\dagger U^{\frac{1}{2}} \otimes P + S^\dagger U^{\frac{1}{2}} \otimes P^\dagger + \bar{T}^\dagger U^{\frac{1}{2}} \otimes Q + T^\dagger U^{\frac{1}{2}} \otimes Q^\dagger \right)^k \right],
$$

$$\tag{36}$$

from which we find that $\Upsilon_h^{N=\infty} = \Xi_h^{N=\infty}$, cf. Eq. (33), but $\frac{1}{N}$ corrections are different (see Table 1). Again, for arbitrary one-row representations ($M = 1$) it is possible to obtain Υ_h rather explicitly.

Another potentially useful representation[2] of the lattice polynomials is given by the following *dual* equations: Eq. (24) becomes

$$
\Xi_h = \chi_h(\partial) \cdot \exp \frac{1}{N} \text{Tr} \left(J P J P^\dagger + J Q J Q^\dagger \right) \Big|_{J=0},
\tag{37}
$$

and Eq. (28) is dual to

$$
\Upsilon_h = \chi_h(\partial \bar{\partial}) \cdot \exp \frac{1}{N} \text{Tr} \left(J P \bar{J} P^\dagger + J P^\dagger \bar{J} P + J Q \bar{J} Q^\dagger + J Q^\dagger \bar{J} Q \right) \Big|_{J=\bar{J}=0}.
\tag{38}
$$

We could go on and discuss correlation functions which are naturally included into the present formalism. In particular, it is straightforward to give expressions for their character expansions in terms of modified lattice polynomials, and it remains true that the combinatorics is independent on whether the reduced field theory is free or interacting. This is however beyond the scope of the present article.

While it is unclear whether the $D \geq 2$ lattice polynomials can be computed exactly for a general partition, it should be stressed once more that this is unnecessary; all we need is an indirect method in order to extract the large N behavior.

5. Conclusions

This solution to the problem of the large N limit of (non-gauge) matrix field theories is not yet complete since the structure of the lattice polynomials we introduced still needs to be further analyzed in order to be able to write the full set of saddle point equations. However we feel that we are definitely closing in on the large N problem, and that we have brought it into the simplest form to date. The proposed approach is concrete, systematic and rather general: we demonstrated that the reduction from N^2 to N variables is possible once one changes variables from matrices to partitions. In this language the *master field* becomes a *master partition*. Presumably one should first (re)derive in the current framework the exact solutions for some lower dimensional target spaces before dealing with the two (and higher) dimensional field theories.

[2] We thank D.-N. Verma for pointing this out to us.

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