On M-Theory

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Abstract. This contribution gives a personal view on recent attempts to find a unified framework for non-perturbative string theories, with special emphasis on the hidden symmetries of supergravity and their possible role in this endeavor. A reformulation of $d = 11$ supergravity with enlarged tangent space symmetry SO(1, 2) × SO(16) is discussed from this perspective, as well as an ansatz to construct yet further versions with SO(1, 1) × SO(16)$^\infty$ and possibly even SO(1, 1)$_+$ × ISO(16)$^\infty$ tangent space symmetry. It is suggested that upon "third quantization", dimensionally reduced maximal supergravity may have an equally important role to play in this unification as the dimensionally reduced maximally supersymmetric $SU(\infty)$ Yang Mills theory.

Key words. Supergravity—hidden symmetries—superstrings.

1. Introduction

Many theorists now believe that there is a unified framework for all string theories, which also accomodates $d = 11$ supergravity (Cremmer, Julia & Scherk 1978). Much of the evidence for this elusive theory, called "M-Theory" (Witten 1995; Townsend 1995), is based on recent work on duality symmetries in string theory which suggests that all string theories are connected through a web of non-perturbative dualities (Font *et al.* 1990; Rey 1991; Sen 1993,94; Schwarz & Sen 1994; Duff & Khuri 1994; Giveon *et al.* 1994; Hull & Townsend 1995; Witten 1995; Kachru & Vafa 1995; Schwarz 1995, 96; Duff 1996; Horava 1996). Although it is unknown what M-theory really is, we can probably assert with some confidence

(i) that it will be a pregeometrical theory, in which space-time as we know it will emerge as a secondary concept (which also means that it makes little sense to claim that the theory "lives" in either ten or eleven dimensions), and
 (ii) that it should possess a huge symmetry involving new and unexplored types of Lie algebras (such as hyperbolic Kac Moody algebras), and perhaps other exotic structures such as quantum groups. In particular, the theory should be background independent and should be logically deducible from a vast generalization of the principles underlying general relativity.

According to a widely acclaimed recent proposal (Banks *et al.* 1997) M-Theory "is" the $N \to \infty$ limit of the maximally supersymmetric quantum mechanical $SU(N)$ matrix model (Claudson & Halpern 1985; Flume 1985; Baake, Reinicke & Rittenberg 1985) (see deWit (1997), Banks (1997) and Bigatti & Susskind (1997) for recent reviews, points of view and comprehensive lists of references). This model had already

appeared in an earlier investigation of the $d = 11$ supermembrane (Bergshoeff, Sezgin & Townsend 1987; 1988) in a flat background in the light cone gauge (deWit, Hoppe & Nicolai 1988). Crucial steps in the developments leading up to this proposal were the discovery of Dirichlet p-branes and their role in the description of non-perturbative string states (Polchinski 1995) and the realization that the dynamics of an ensemble of such objects is described by dimensionally reduced supersymmetric Yang Mills theories (Witten 1996; Polchinski 1996). Although there are a host of unsolved problems in matrix theory, two central ones can perhaps be singled out: one is the question whether the matrix model admits massless normalizable states for any N (see Fröhlich & Hoppe 1997; Yi 1997; Sethi & Stern 1997; Porrati & Rozenberg 1997; Hoppe 1997; Green & Gutperle 1997; Halpern & Schwarz 1997) for recent work in this direction); the other is related to the still unproven existence of the $N \to \infty$ limit. This would have to be a weak limit in the sense of quantum field theory, requiring the existence of a universal function $g = g(N)$ (the coupling constant of the $SU(N)$ matrix model) such that the limit $N \to \infty$ exists for all correlators. The existence of this limit would be equivalent to the renormalizability of the supermembrane (deWit, Hoppe & Nicolai 1988). However, even if these problems can be solved eventually, important questions remain with regard to the assertions made above: while matrix theory is pregeometrical in the sense that the target space coordinates are replaced by matrices, thus implying a kind of non-commutative geometry, the hidden exceptional symmetries of dimensionally reduced supergravities discovered long ago (Julia 1979) are hard to come by (see Elitzur *et al.* (1997) and references therein).

In the first part of this contribution, I will report on work (Melosch & Nicolai 1997), which was motivated by recent advances in string theory as well as the possible existence of an Ashtekar-type canonical formulation of $d = 11$ supergravity. Although at first sight our results, which build on the earlier work of (deWit & Nicolai 1985; Nicolai 1987), may seem to be of little import for the issues raised above, I will argue that they could actually be relevant, assuming (as we do) that the success of the search for M-Theory will crucially depend on the identification of its underlying symmetries, and that the hidden exceptional symmetries of maximal super-gravity theories may provide important clues as to where we should be looking. Namely, as shown in (deWit & Nicolai 1985; Nicolai 1987) the local symmetries of the dimensionally reduced theories can be partially "lifted" to eleven dimensions, indicating that these symmetries may have a role to play also in a wider context than that of dimensionally reduced supergravity. The existence of alternative versions of $d = 11$ supergravity, which, though equivalent on-shell to the original version of (Cremmer, Julia & Scherk 1978), differ from it off-shell, suggests the existence of a novel kind of "exceptional geometry" for $d = 11$ supergravity and the bigger theory containing it. This new geometry would be intimately tied to the special properties of the exceptional groups, and would be characterized by relations such as (3)–(5) below, which have no analog in ordinary Riemannian geometry. The hope is, of course, that one may in this way gain valuable insights into what the (surely exceptional) geometry of M-Theory might look like, and that our construction may provide a simplified model for it. After all, we do not even know what the basic physical concepts and mathematical "objects" (matrices, BRST string functionals, spin networks,...?) of such a theory should be, especially if it is to be a truly pregeometrical theory of quantum gravity.

The second part of this paper discusses the infinite dimensional symmetries of $d = 2$ supergravities (Julia 1981; Julia 1982, 1984; Breitenlohner & Maison 1987; Breitenlohner, Maison & Gibbons 1988; Nicolai 1991; Julia & Nicolai 1996; Bernard & Julia 1997) and an ansatz that would incorporate them into the construction of (Melosch & Nicolai 1997; deWit & Nicolai 1985; Nicolai 1987). The point of view adopted here is that the fundamental object of M-Theory could well be a kind of "Unendlichbein" belonging to an infinite dimensional coset space (Ashtekar 1986), which would generalize the space $GL(4, \mathbf{R})$ /SO(1, 3) of general relativity. This bein would be acted upon from the right side by a huge extension of the Lorentz group, containing not only space-time, but also internal symmetries, and perhaps even local supersymmetries. For the left action, one would have to appeal to some kind of generalized covariance principle. An intriguing, but also puzzling, feature of the alternative formulations of $d = 11$ supergravity is the apparent loss of manifest general covariance, as well as the precise significance of the global E_{11-d} symmetries of the dimensionally reduced theories. This could mean that in the final formulation, general covariance will have to be replaced by something else.

The approach taken here is thus different from and arguably even more speculative than current ideas based on matrix theory, exploiting the observation that instead of dimensionally reducing the maximally extended *rigidly* supersymmetric theory to one dimension, one might equally well contemplate reducing the maximally extended *locally* supersymmetric theory to one (light-like ≡ null) dimension. While matrix theory acquires an infinite number of degrees of freedom only in the $N \to \infty$ limit, the chirally reduced supergravity would have an infinite number from the outset, being one half of a field theory in two dimensions. The basic idea is then that upon quantization the latter might undergo a similarly far-reaching metamorphosis as the quantum mechanical matrix model, its physical states being transmuted into "target space" degrees of freedom as in string theory (Nicolai 1987). This proposal would amount to a third quantization of maximal ($N = 16$) supergravity in two dimensions, where by "third quantization" I mean that the quantum treatment should take into account the gravitational degrees of freedom on the worldsheet, i.e. its (super)moduli for arbitrary genus. The model can be viewed as a very special example of $d = 2$ quantum cosmology; with the appropriate vertex operator insertions the resulting multiply connected $d = 2$ "universes" can be alternatively interpreted as multistring scattering diagrams (Mandelstam 1973; Giddings & Wolpert 1987). One attractive feature of this proposal is that it might naturally bring in E_{10} as a kind of non-perturbative spectrum generating (rigid) symmetry acting on the third quantized Hilbert space, which would mix the worldsheet moduli with the propagating degrees of freedom. A drawback is that these theories are even harder to quantize than the matrix model (see, however, Nicolai, Korotkin & Samtleben (1997) and references therein).

2. SO(1, 2) × SO(16) invariant supergravity in eleven dimensions

In (deWit & Nicolai 1985; Nicolai 1987), new versions of $d = 11$ supergravity (Cremmer, Julia & Scherk 1985) with local SO(1, 3) × SU(8) and SO(1, 2) × SO(16) tangent space symmetries, respectively, have been constructed. Melosch & Nicolai (1997) develop these results further (for the SO(1, 2) × SO(16) invariant version of

(Nicolai 1987) and also discusses a hamiltonian formulation in terms of the new variables. In both versions the supersymmetry variations acquire a polynomial form from which the corresponding formulas for the maximal supergravities in four and three dimensions can be read off directly and without the need for complicated duality redefinitions. This reformulation can thus be regarded as a step towards the complete fusion of the bosonic degrees of freedom of $d = 11$ supergravity (i.e. the elfbein E_M^A and the antisymmetric tensor A_{MNP}) in a way which is in harmony with the hidden symmetries of the dimensionally reduced theories.

For lack of space, and to exhibit the salient features as clearly as possible I will restrict the discussion to the bosonic sector. To derive the SO(1, 2) × SO(16) invariant version of (Nicolai 1987; Melosch & Nicolai 1997) from the original formulation of $d = 11$ supergravity, one first breaks the original tangent space symmetry SO(1,10) to its subgroup SO(1, 2) × SO(8) through a partial choice of gauge for the elfbein, and subsequently enlarges it again to SO(1, 2) × SO(16) by introducing new gauge degrees of freedom. The symmetry enhancement of the transverse (helicity) group SO(9) ⊂ SO(1,10) to SO(16) requires suitable redefinitions of the bosonic and fermionic fields, or, more succinctly, their combination into tensors w.r.t. the new tangent space symmetry. The construction thus requires a 3 + 8 split of the $d = 11$ coordinates and indices, implying a similar split for all tensors of the theory. It is important, however, that the dependence on all eleven coordinates is retained throughout.

The elfbein and the three-index photon are thus combined into new objects covariant w.r.t. to the new tangent space symmetry. In the special Lorentz gauge preserving SO(1, 2) × SO(8) the elfbein takes the form

$$E_M^A = \begin{pmatrix} \Delta^{-1} e_\mu^a & B_\mu^m e_m^a \\ 0 & e_m^a \end{pmatrix} \tag{1}$$

where curved $d = 11$ indices are decomposed as $M = (\mu, m)$ with $\mu = 0, 1, 2$ and $m = 3, \ldots, 10$ (with a similar decomposition of the flat indices), and $\Delta := \det e_m^a$. In this gauge, the elfbein contains the (Weyl resealed) dreibein and the Kaluza Klein vector $B_\mu^{\ m}$ both of which will be kept in the new formulation. By contrast, the internal achtbein is replaced by a rectangular 248-bein (e_{IJ}^m, e_A^m) containing the remaining "matter-like" degrees of freedom, where $([IJ], A)$ label the 248-dimensional adjoint representation of E_8 in the SO(16) decomposition. This 248-bein, which in the reduction to three dimensions contains all the propagating bosonic matter degrees of freedom of $d = 3$, $N = 16$ supergravity, is defined in a special SO(16) gauge by

$$(e_{IJ}^m, e_A^m) := \begin{cases} \Delta^{-1} e_a^m \Gamma_{\alpha\dot\beta}^a & \text{if } [IJ] \text{ or } A = (\alpha\dot\beta) \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

where the SO(16) indices IJ or A are decomposed w.r.t. the diagonal subgroup SO(8) ≡ (SO(8) × SO(8))$_{\text{diag}}$ of SO(16) (see Nicolai (1987) for details). Being the inverse densitized internal achtbein contracted with an SO(8) Γ-matrix, this object is very much analogous to the inverse densitized triad in the framework of Ashtekar's reformulation of Einstein's theory (Ashtekar 1986). Note that, due to its rectangularity, there does not exist an inverse for the 248-bein (nor is one needed for the supersymmetry variations and the equations of motion!). In addition we need the composite fields (Q_μ^{IJ}, P_μ^A) and (Q_m^{IJ}, P_m^A), which together make up an E_8 connection in

eleven dimensions and whose explicit expressions in terms of the $d = 11$ coefficients of anholonomy and the four-index field strength F_{MNPQ} can be found in (Nicolai 1987).

The new geometry is encoded into algebraic constraints between the vielbein components, which are without analog in ordinary Riemannian geometry because they rely in an essential way on special properties of the exceptional group E_8. We have

$$e_A^m e_A^n - \tfrac{1}{2} e_{IJ}^m e_{IJ}^n = 0 \tag{3}$$

and

$$\Gamma_{AB}^{IJ}(e_B^m e_{IJ}^n - e_B^n e_{IJ}^m) = 0 \qquad \Gamma_{AB}^{IJ} e_A^m e_B^n + 4 e_{K[I}^m e_{J]K}^n = 0 \tag{4}$$

where $\Gamma_{A\dot{A}}^I$ are the standard SO(16) Γ-matrices and $\Gamma_{AB}^{IJ} \equiv (\Gamma^{[I} \Gamma^{J]})_{AB}$, the minus sign in (3) reflects the fact that we are dealing with the maximally non-compact form $E_{8(+8)}$. While the SO(16) covariance of these equations is manifest, it turns out, remarkably, that they are also covariant under E_8. Obviously, (3) and (4) correspond to the singlet and the adjoint representations of E_8. More complicated are the following relations transforming in the 3875 representation of E_8

$$e_{IK}^{(m} e_{JK}^{n)} - \tfrac{1}{16} \delta_{IJ} e_{KL}^m e_{KL}^n = 0,$$

$$\Gamma_{AB}^K e_B^{(m} e_{IK}^{n)} - \tfrac{1}{14} \Gamma_{\dot{A}B}^{IKL} e_B^{(m} e_{KL}^{n)} = 0,$$

$$e_{[IJ}^{(m} e_{KL]}^{n)} + \tfrac{1}{24} e_A^m \Gamma_{AB}^{IJKL} e_B^n = 0. \tag{5}$$

The 248-bein and the new connection fields are subject to a "vielbein postulate" similar to the usual vielbein postulate stating the covariant constancy of the vielbein w.r.t. to generally covariant and Lorentz covariant derivative:

$$(\partial_\mu - B_\mu^n \partial_n)e_{IJ}^m + \partial_n B_\mu{}^n e_{IJ}^m + \partial_n B_\mu{}^m e_{IJ}^n + 2 Q_\mu{}^K{}_{[I} e_{J]K}^m + P_\mu^A \Gamma_{AB}^{IJ} e_m^B = 0,$$

$$(\partial_\mu - B_\mu^n \partial_n)e_A^m + \partial_n B_\mu{}^m e_A^n + \partial_n B_\mu{}^n e_A^m + \tfrac{1}{4} Q_\mu^{IJ} \Gamma_{AB}^{IJ} e_B^m - \tfrac{1}{2} \Gamma_{AB}^{IJ} P_\mu^B e_{IJ}^m = 0,$$

$$\partial_m e_{IJ}^n + 2 Q_m{}^K{}_{[I} e_{J]K}^n + P_m^A \Gamma_{AB}^{IJ} e_B^n = 0,$$

$$\partial_m e_A^n + \tfrac{1}{4} Q_m^{IJ} \Gamma_{AB}^{IJ} e_B^n - \tfrac{1}{2} \Gamma_{AB}^{IJ} P_m^B e_{IJ}^n = 0. \tag{6}$$

Like (3)–(5), these relations are E_8 covariant. It must be stressed, however, that the full theory of course does not respect E_8 invariance. A puzzling feature of (6) is that the covariantization w.r.t. an affine connection is "missing" in these equations, even though the theory is still invariant under $d = 11$ coordinate transformations. One can now show that the supersymmetry variations of $d = 11$ supergravity can be entirely expressed in terms of these new variables (and their fermionic partners).

The reduction of $d = 11$ supergravity to three dimensions yields $d = 3$, $N = 16$ supergravity (Marcus & Schwarz 1983), and is accomplished rather easily, since no duality redefinitions are needed any more, unlike in (Cremmer & Julia 1979). The propagating bosonic degrees of freedom in three dimensions are all scalar, and combine into a matrix $\mathcal{V}(x)$, which is an element of a non-compact $E_{8(+8)}/SO(16)$ coset space, and whose dynamics is governed by a non-linear σ-model coupled to $d = 3$ gravity. The identification of the 248-bein with the a-model field $\mathcal{V} \in E_8$ is

given by

$$e_{IJ}^m = \tfrac{1}{60}\,\mathrm{Tr}(Z^m \mathcal{V} X^{IJ} \mathcal{V}^{-1}) \qquad e_A^m = \tfrac{1}{60}\,\mathrm{Tr}(Z^m \mathcal{V} Y^A \mathcal{V}^{-1}) \tag{7}$$

where X^{IJ} and Y^A are the compact and non-compact generators of E_8, respectively, and where the Z_m for $m = 3, \ldots, 10$ are eight non-compact commuting generators obeying $\mathrm{Tr}(Z^m Z^n) = 0$ for all m and n (the existence of eight such generators is a consequence of the fact that the coset space $E_{8(+8)}/S0(16)$ has real rank 8 and therefore admits an eight-dimensional maximal flat and totally geodesic submanifold (Helgason 1962). This reduction provides a "model" for the exceptional geometry, where the relations (3)–(6) can be tested by means of completeness relations for the E_8 Lie algebra generators in the adjoint representation. Of course, this is not much of a test since all dependence on the internal coordinates is dropped in (7), and the terms involving B_μ^m disappear altogether. It would be desirable to find other "models" with non-trivial dependence on the internal coordinates. The only example of this type so far is provided by the S^7 truncation of $d = 11$ supergravity for the SO(1, 3) × SU(8) invariant version of $d = 11$. supergravity (deWit & Nicolai 1987).

3. More symmetries

The emergence of hidden symmetries of the exceptional type in extended super-gravities (Cremmer & Julia 1978) was a remarkable and, at the time, quite unexpected discovery. It took some effort to show that the general pattern continues when one descends to $d = 2$ and that the hidden symmetries become infinite dimensional (Julia 1981; Julia 1982, 1984; Breitenlohner & Maison 1987; Breitenlohner Maison & Gibbons 1988; Nicolai 1991; Julia & Nicolai 1996; Bernard & Julia 1997) generaliz-ing the Geroch group of general relativity (Geroch 1972; Kinnersley & Chitre 1977). As we will see, even the coset structure remains, although the mathematical objects one deals with become a lot more delicate. The fact that the construction described above works with a 4 + 7 and 3 + 8 split of the indices suggests that we should be able to go even further and to construct versions of $d = 11$ supergravity with infinite dimensional tangent space symmetries, which would be based on a 2 + 9 or even a 1 + 10 split of the indices. This would also be desirable in view of the fact that the new versions are "simple" only in their internal sectors. The general strategy is thus to further enlarge the internal sector by absorbing more and more degrees of freedom into it, such that in the final step corresponding to a 1 + 10 split, only an einbein is left in the low dimensional sector. Although the actual elaboration of these ideas has to be left to future work, I will try to give at least a flavor of some anticipated key features.

3.1 *Reduction to two dimensions*

Let us first recall some facts about dimensional reduction of maximal supergravity to two dimensions. Following the empirical rules of dimensional reduction one is led to predict $E_9 = E_8^{(1)}$ as a symmetry for the dimensional reduction of $d = 11$ super-gravity to two dimensions (Julia 1981). This expectation is borne out by the existence of a linear system for maximal $N = 16$ supergravity in two dimensions (Nicolai

1987; Nicolai & Warner 1989) (see Maison (1978); Belinski & Zakharov (1978); Breitenlohner & Maison (1987) for the bosonic theory). The linear system requires the introduction of an extra "spectral" parameter t, and the extension of the σ-model matrix $\mathcal{V}(x)$ to a matrix $\widehat{\mathcal{V}}(x;t)$ depending on this extra parameter t, as is generally the case for integrable systems in two dimensions. An unusual feature is that, due to the presence of gravitational degrees of freedom, this parameter becomes coordinate dependent, i.e. we have $t = t(x; w)$, where w is an integration constant, sometimes referred to as the "constant spectral parameter" whereas t itself is called the "variable spectral parameter".

Here, we are mainly concerned with the symmetry aspects of this system, and with what they can teach us about the $d = 11$ theory itself. The coset structure of the higher dimensional theories has a natural continuation in two dimensions, with the only difference that the symmetry groups are infinite dimensional. This property is manifest from the transformation properties of the linear system matrix $\widehat{\mathcal{V}}$, with a global affine symmetry acting from the left, and a local symmetry corresponding to some "maximal compact" subgroup acting from the right:

$$\widehat{\mathcal{V}}(x;t) \longrightarrow g(w)\widehat{\mathcal{V}}(x;t)h(x;t). \tag{8}$$

Here $g(w) \in E_9$ with affine parameter w, and the subgroup to which $h(x; t)$ belongs is characterized as follows (Julia 1982; Breitenlohner & Maison 1987). Let τ be the involution characterizing the coset space $E_{8(+8)}/SO(16)$: then $h(t) \in SO(16)^{\infty}_{\varepsilon}$ is defined to consist of all τ^{∞} invariant elements of E_9, where the extended involution τ^{∞} is defined by $\tau^{\infty}(h(t)) = \tau h(\varepsilon t^{-1})$, with $\varepsilon = +1$ (or -1) for a Lorentzian (Euclidean) worldsheet. For $\varepsilon = 1$, which is the case we are mainly interested in, we will write $SO(16)^{\infty} \equiv SO(16)^{\infty}_{\varepsilon}$. We also note that $SO(16)^{\infty}_{\varepsilon}$ is different from the affine extension of $SO(16)$ for either choice of sign.

What has been achieved by the coset space description is the following: by representing the "moduli space of solutions" $\mathcal{M}$ (of the bosonic equations of motion of $d = 11$ supergravity with nine commuting space-like Killing vectors) as

$$\mathcal{M} = \frac{\text{solutions of field equations}}{\text{diffeomorphisms}} = \frac{E_9}{SO(16)^{\infty}} \tag{9}$$

we have managed to endow this space, which a priori is very complicated, with a group theoretic structure, that makes it much easier to handle. In particular, the integrability of the system is directly linked to the fact that $\mathcal{M}$ possesses an infinite dimensional "isometry group" E_9. The introduction of infinitely many gauge degrees of freedom embodied in the subgroup $SO(16)^{\infty}$ linearizes and localizes the action of this isometry group on the space of solutions. Of course, in making such statements, one should keep in mind that a mathematically rigorous construction of such spaces is a thorny problem. This is likewise true for the infinite dimensional groups* and their associated Lie algebras; the latter being infinite dimensional vector spaces, there are myriad ways of equiping them with a topology. We here take the liberty of ignoring

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* For instance, the Geroch group can be defined rigorously to consist of all maps from the complex w plane to $SL(2, \mathbf{R})$ with meromorphic entries. With this definition, one obtains all multisoliton solutions of Einstein's equations, and on this solution space the group acts transitively by construction.

these subleties, not least because these spaces ultimately will have to be "quantized" anyway.

There is a second way of defining the Lie algebra of SO $(16)_\varepsilon^\infty$ which relies on the Chevalley-Serre presentation. Given a finite dimensional non-compact Lie group G with maximal compact subgroup H, a necessary condition for this prescription to work is that dim $H = \frac{1}{2}$ (dim G – rank G), and we will subsequently extend this prescription to the infinite Lie group. Let us first recall that any (finite or infinite dimensional) Kac Moody algebra is recursively defined in terms of multiple com-mutators of the Chevalley generators subject to certain relations (Bourbaki 1968; Kac 1930). More specifically, given a Cartan matrix Au and the associated Dynkin diagram, one starts from a set of $sl(2, \mathbf{R})$ generators $\{e_i, f_i, h_1\}$, one for each node of the Dynkin diagram, which in addition to the standard $sl(2, \mathbf{R})$ commutation relations

$$[h_i, h_j] = 0 \qquad [e_i, f_j] = \delta_{ij} h_j,$$
$$[h_i, e_j] = A_{ij} e_j \quad [h_i, f_j] = -A_{ij} f_j \tag{10}$$

Are subject to the multilinear Serre relations

$$[e_i, [e_i, \ldots [e_i, e_j] \ldots]] = 0 \qquad [f_i, [f_i, \ldots [f_i, f_j] \ldots]] = 0 \tag{11}$$

where the commutators are $(1 - A_{ij})$-fold ones. The Lie algebra is then by definition the linear span of all multiple commutators which do not vanish by virtue of these relations.

To define the subalgebra SO$(16)_\varepsilon^\infty$, we first recall that the Chevalley involution θ is defined by

$$\theta(e_i) = -f_i \qquad \theta(f_i) = -e_i \qquad \theta(h_i) = -h_i. \tag{12}$$

This involution, like the ones to be introduced below, leaves invariant the defining relations (18) and (19) of the Kac Moody algebra, and extends to the whole Lie algebra via the formula $\theta([x, y]) = [\theta(x), \theta(y)]$. It is not difficult to see that, for E_8 (and also for $sl(n, \mathbf{R})$), we have $\tau = \theta$, and the maximal compact subalgebras defined above correspond to the subalgebras generated by the multiple commutators of the θ invariant elements $(e_i - f_i)$ in both cases. The trick is now to carry over this definition to the affine extension, whose associated Cartan matrix has a zero eigenvalue. To do this, however, we need a slight generalization of the above definition; for this purpose, we consider involutions ω that can be represented as products of the form

$$\omega = \theta \cdot s \tag{13}$$

where the involution s acts as

$$s(e_i) = s_i e_i \qquad s(f_i) = s_i f_i \qquad s(h_i) = h_i \tag{14}$$

with $s_i = \pm 1$. It is important that different choices of s_i do not necessarily lead to inequivalent involutions (the general problem of classifying the involutive automorphisms of infinite dimensional Kac Moody algebras has so far not been completely solved, see e.g. (Levstein 1988; Bausch & Rousseau 1989)[†]). In particular

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[†] I am very grateful to C. Daboul for helpful discussions on this topic.

for E_9, which is obtained from E_8 by adjoining another set $\{e_0, f_0, h_0\}$ of Chevalley generators, we take $s_i = 1$ for all $i \geq 1$, whereas $s_0 = \varepsilon$, with ε as before, i.e. $\epsilon = +1$ (or -1) for Lorentzian (Euclidean) worldsheet. Thus, on the extended Chevalley generators,

$$\omega(e_0) = -\varepsilon f_0 \qquad \omega(f_0) = -\varepsilon e_0 \qquad \omega(h_0) = -h_0. \qquad (15)$$

With this choice, the involution ω coincides with the involutions defined before for the respective choices of ε, i.e. $\omega = \tau^\infty$, and therefore the invariant subgroups are the same, too. For $\varepsilon = 1$, the involution ω defines an infinite dimensional "maximal compact" subalgebra consisting of all the negative norm elements w.r.t. to the standard bilinear form

$$\langle e_i | f_j \rangle = \delta_{ij} \qquad \langle h_i | h_j \rangle = A_{ij} \qquad (16)$$

(the norm of any given multiple commutator can be determined recursively from the fundamental relation $\langle [x, y] | z \rangle = \langle x | [y, z] \rangle$). The notion of "compactness" here is thus algebraic, not topological: the subgroup $SO(16)^\infty$ will not be compact in the topological sense (recall the well known example of the unit ball in an infinite dimensional Hilbert space, which is bounded but not compact in the norm topology). On the other hand, for $\varepsilon = -1$, the group $SO(16)^\infty_\varepsilon$ is not even compact in the algebraic sense, as $e_0 + f_0$ has positive norm. However, this is in accord with the expectation that $SO(16)^\infty_\varepsilon$ should contain the (non-compact) group $SO(1,8)$ rather than $SO(9)$ if one of the compactified dimensions is time-like.

3.2 2 + 9 *split*

Let us now consider the extension of the results described in section 2 to the situation corresponding to a 2 + 9 split of the indices. Elevating the local symmetries of $N = 16$ supergravity from two to eleven dimensions would require the existence of yet another extension of the theory, for which the Lorentz group $SO(1,10)$ is replaced by $SO(1, 1) \times SO(16)^\infty$; the subgroup $SO(16)^\infty$ can be interpreted as an extension of the transverse group $SO(9)$ in eleven dimensions. Taking the hints from (1), we would now decompose the elfbein into a zweibein and nine Kaluza Klein vectors B_μ^m (with $m = 2, \ldots, 10$). The remaining internal neunbein would have to be replaced by an "Unendlichbein" $(e_U^m(x; t),\ e_A^m(x; t))$, depending on a spectral parameter t, necessary to parametrize the infinite dimensional extension of the symmetry group. However, in eleven dimensions, there is no anolog of the dualization mechanism, which would ensure that despite the existence of infinitely many dual potentials, there are only finitely many physical degrees of freedom. This indicates that if the construction works it will take us beyond $d = 11$ supergravity.

  Some constraints on the geometry can be deduced from the requirement that in the dimensional reduction to $d = 2$, there should exist a formula analogous to (7), but with $\mathcal{V}$ replaced by the linear system matrix $\widehat{\mathcal{V}}$, or possibly even the enlarged linear system of (Julia & Nicolai 1996). Evidently, we would need a ninth nilpotent generator to complement the Z^m's of (7); an obvious candidate is the central charge generator c, since it obeys $\langle c | c \rangle = \langle c | Z^m \rangle = 0$ for all $m = 3, \ldots, 10$. The parameter t, introduced somewhat ad hoc for the parametrization of the unendlichbein, must obviously coincide with the spectral parameter of the d 2 theory, and

the generalized "unendlichbein postulate" should evidently reduce to the linear system of $d = 2$ supergravity in this reduction. To write it down, we need to generalize the connection coefficients appearing in the linear system. The latter are given by

$$\mathcal{Q}_\mu^{IJ} = Q_\mu^{IJ} + \cdots \qquad \mathcal{P}_\mu^A = \frac{1 + t^2}{1 - t^2} P_\mu^A + \frac{2t}{1 - t^2} \varepsilon_{\mu\nu} P^{\nu A} + \cdots \qquad (17)$$

with Q_μ^{IJ} and $P_{A\mu}$ as before; the dots indicate t dependent fermionic contributions which we omit. A very important difference with section 2, where the tangent space symmetry was still finite dimensional, is that the Lie algebra of $SO(16)^\infty$ also involves the P's, and not only the Q's. More specifically, from the t dependence of the dimensionally reduced connections in (17) we infer that the connections $(\mathcal{Q}_M^{IJ}(x;\ t),$ $\mathcal{P}_M^A(x;\ t))$ constitute an $SO(16)^\infty$ (and not an E_9) gauge connection. This means that the covariantizations in the generalized vielbein postulate are now in precise correspondence with the local symmetries, in contrast with the relations (6) which look E_8 covariant, whereas the full theory is invariant only under local $SO(16)$.

To write down an ansatz, we put

$$\mathcal{D}_\mu := \partial_\mu - B_\mu^n \partial_n + \cdots \qquad (18)$$

where the dots stand for terms involving derivatives of the Kaluza Klein vector fields. Then the generalization of (6) should read

$$\mathcal{D}_\mu e_{IJ}^m(t) + 2\,\mathcal{Q}_\mu^K{}_{[I}(t) e_{J]K}^m(t) + \mathcal{P}_\mu^A(t) \Gamma_{AB}^{IJ} e_B^m(t) = 0,$$
$$\mathcal{D}_\mu e_A^m(t) + \tfrac{1}{4}\mathcal{Q}_\mu^{IJ}(t)\Gamma_{AB}^{IJ} e_B^m(t) - \tfrac{1}{2}\Gamma_{AB}^{IJ}\mathcal{P}_\mu^B(t) e_{IJ}^m(t) = 0,$$
$$\partial_m e_{IJ}^n(t) + 2\,\mathcal{Q}_m^K{}_{[I}(t) e_{J]K}^n(t) + \mathcal{P}_m^A(t) \Gamma_{AB}^{IJ} e_B^n(t) = 0,$$
$$\partial_m e_A^n(t) + \tfrac{1}{4}\mathcal{Q}_m^{IJ}(t)\Gamma_{AB}^{IJ} e_B^n(t) - \tfrac{1}{2}\Gamma_{AB}^{IJ}\mathcal{P}_m^B(t) e_{IJ}^n(t) = 0. \qquad (19)$$

Of course, the challenge is now to find explicit expressions for the internal components $\mathcal{Q}_m^{IJ}(x;\ t)$ and $\mathcal{P}_m^A(x;\ t)$, such that (19) can be interpreted as a $d = 11$ generalization of the linear system of dimensionally reduced supergravity. Another obvious question concerns the fermionic partners of the unendlichbein: in two dimensions, the linear system matrix contains all degrees of freedom, including the fermionic ones, and the local $N = 16$ supersymmetry can be bosonized into a local $SO(16)^\infty$ gauge transformation (Nicolai & Warner 1989). Could this mean that there is a kind of bosonization in eleven dimensions or M-Theory? This idea may not be as outlandish as it sounds because a truly pregeometrical theory might be subject to a kind of "pre-statistics", such that the distinction between bosons and fermions arises only through a process of spontaneous symmetry breaking.

4. Yet more symmetries?

In 1982, B. Julia conjectured that the dimensional reduction of maximal supergravity to one dimension should be invariant under a further extension of the *E*-series, namely (a non-compact form of) the hyperbolic Kac Moody algebra E_{10} obtained by adjoining another set $\{e_{-1}, f_{-1}, h_{-1}\}$ of Chevalley generators to those of E_9 (Julia

1985)‡. As shown in Nicolai (1992), the last step of the reduction requires a null reduction if the affine symmetry of the $d = 2$ theory is not to be lost. The reason is that the infinite dimensional affine symmetries of the $d = 2$ theories always involve dualizations of the type

$$\partial_\mu \varphi = \varepsilon_{\mu\nu} \partial^\nu \tilde\varphi \tag{20}$$

(in actual fact, there are more scalar fields, and the duality relation becomes non-linear, which is why one ends up with infinitely many dual potentials for each scalar degree of freedom). Dimensional reduction w.r.t. to a Killing vector ξ^μ amounts to imposing the condition $\xi^\mu \partial_\mu \equiv 0$ on *all* fields, including dual potentials. Hence,

$$\xi^\mu \partial_\mu \varphi = 0, \quad \xi^\mu \partial_\mu \tilde\varphi \equiv \eta^\mu \partial_\mu \varphi = 0 \tag{21}$$

where $\eta^\mu \equiv \varepsilon^{\mu\nu} \xi_\nu$. If ξ^μ and η^μ are linearly independent, this constraint would force all fields to be constant, which is clearly too strong a requirement. Hence we must demand that ξ^μ and η^μ are collinear, which implies

$$\xi^\mu \xi_\mu = 0, \tag{22}$$

i.e. the Killing vector must be null. Starting from this observation, it was shown in Nicolai (1992) that the Matzner Misner $sl(2, \mathbf{R})$ symmetry of pure gravity can be formally extended to an $sl(3, \mathbf{R})$ algebra in the reduction of the vierbein from four to one dimensions. Combining this $sl(3, \mathbf{R})$ with the Ehlers $sl(2, \mathbf{R})$ of ordinary gravity, or with the E_8 symmetry of maximal supergravity in three dimensions, one is led to the hyperbolic algebra $\mathcal{F}_3$ (Feingold & Frenkel 1993) for ordinary gravity, and to E_{10} for maximal supergravity. The transformations realizing the action of the Chevalley generators on the vierbein components can be worked out explicitly, and the Serre relations can be formally verified (Nicolai 1992) (for E_{10}, this was shown more recently in Mizoguchi (1970).

There is thus some evidence for the emergence of hyperbolic Kac Moody algebras in the reduction to one null dimension, but the difficult open question that remains is what the configuration space is on which this huge symmetry acts. This space is expected to be much bigger than the coset space (9). Now, already for the $d = 2$ reduction there are extra degrees of freedom that must be taken into account in addition to the propagating degrees of freedom. Namely, the full moduli space involve-ing all bosonic degrees of freedom should also include the moduli of the zweibein, which are not contained in (9). For each point on the worldsheet, the zweibein is an element of the coset space GL(2, $\mathbf{R}$)/SO(1, 1); although it has no local degrees of freedom any more, it still contains the global information about the conformal structure of the world sheet Σ. Consequently, we should consider the Teichmüller space

$$\mathcal{T} = \frac{\{e_\mu^\alpha(x) | x \in \Sigma\}}{\mathrm{SO}(1,1) \times \mathrm{Weyl}(\Sigma) \times \mathrm{Diff}_0(\Sigma)} \tag{23}$$

as part of the configuration space of the theory (see Verlinde (1990) for a detailed description of $\mathcal{T}$). In fact, we should even allow for arbitrary genus of the worldsheet,

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‡The existence of a maximal dimension for supergravity (Nahm 1978) would thus be correlated with the existence of a "maximally extended" hyperbolic Kac Moody algebra, which might thus explain the occurrence of maximum spin 2 for massless gauge particles in nature.

and replace $\mathcal{T}$ by the "universal Teichmüller space" $\widetilde{\mathcal{T}}$. This infinite dimensional space can be viewed as the configuration space of non-perturbative string theory (Friedan & Shenker 1987). For the models under consideration here, however, even $\widetilde{\mathcal{T}}$ is not big enough, as we must also take into account the dilaton ρ and the non-propagating Kaluza Klein vector fields in two dimensions. For the former, a coset space description was proposed in Julia & Nicolai (1996). On the other hand, the Kaluza Klein vectors and the cosmological constant they could generate in two dimensions have been largely ignored in the literature. Even if one sets their field strengths equal to zero (there are arguments that the Geroch group, and hence infinite duality symmetries, are incompatible with a nonzero cosmological constant in two dimensions), there still remain topological degrees of freedom for higher genus world sheets.

The existence of inequivalent conformal structures is evidently important for the null reductions, as the former are in one-to-one correspondence with the latter. Put differently, the inequivalent null reductions are precisely parametrized by the space (23). The extended symmetries should thus not only act on one special null reduction (set of plane wave solutions of Einstein's equations), but relate different reductions. Indeed, it was argued in Mizoguchi (1997) that, for a toroidal worldsheet, the new $sl(2, \mathbf{R})$ transformations associated with the over-extended Chevalley generators change the conformal structure, but only for non-vanishing holonomies of the Kaluza Klein vector fields on the worldsheet. This indicates that the non-trivial realization of the hyperbolic symmetry requires the consideration of non-trivial worldsheet topologies. The dimensionally reduced theory thereby retains a memory of its two-dimensional ancestor. It is therefore remarkable that, at least for isomonodromic solutions of Einstein's theory, the $d = 2$ theory exhibits a factorization of the equations of motion akin to, but more subtle than the holomorphic factorization of conformal field theories (Korotkin & Nicolai 1995). In other words, there may be a way to think of the $d = 2$ theory as being composed of two chiral halves just as for the closed string. Consequently, a truncation to one null dimension may not be necessary after all if the theory factorizes all by itself.

In summary, what we are after here is a group theoretic unification of all these moduli spaces that would be analogous to (9) above, and fuse the matter and the topological degrees of freedom. No such description seems to be available for (23) (or $\widetilde{\mathcal{T}}$), and it is conceivable that only the total moduli space $\widetilde{\mathcal{M}}$ containing both $\mathcal{M}$ and $\widetilde{\mathcal{T}}$ as well as the dilaton and the Kaluza Klein, and perhaps even the fermionic, degrees of freedom is amenable to such an interpretation. Extrapolating the previous results, we are thus led to consider coset spaces E_{10}/H with $SO(16)^\infty \subset H \subset E_{10}$. As before, the introduction of the infinitely many spurious degrees of freedom associated with the gauge group H would be necessary in order to "linearize" the action of E_{10}.

What are the choices for H? One possibility would be to follow the procedure of the foregoing section, and to define $H = SO(16)^{\infty\infty} \subset E_{10}$ in analogy with $SO(16)^\infty \subset E_9$ by taking its associated Lie algebra to be the linear span of all ω invariant combinations of E_{10} Lie algebra elements. To extend the affine involution to the full hyperbolic algebra, we would again invoke (13), setting $\varepsilon = +1$ in (15) (since we now assume the worldsheet to be Lorentzian), which leaves us with the two choices $s_{-1} = \pm 1$. For $s_{-1} = +1$ we would get the "maximal compact" subalgebra of E_{10}, corresponding to the compactification of ten spacelike dimensions. A subtlety here is that a definition in terms of the standard bilinear form is no longer possible,

unlike for affine and finite algebras, as this would now also include part of the Cartan subalgebra of E_{10}: due to the existence of a negative eigenvalue of the E_{10} Cartan matrix, there exists a negative norm element $\sum_i n_i h_i$ of the Cartan subalgebra, which would have to be excluded from the definition of H (cf. the footnote on p. 438 of (Julia & Nicolai 1996). The alternative choice $s_{-1} = -1$ would correspond to reduction on a 9 + 1 torus.

However, for the null reduction advocated here, physical reasoning motivates us to propose yet another choice for H. Namely, in this case, H should contain the group ISO(9) $\subset$ SO(1, 10) leaving invariant a null vector in eleven dimensions (Julia & Nivolai 1995). To identify the relevant parabolic subgroup of E_{10}, which we denote by ISO(16)$^\infty$, we recall (Nicolai 1992) that the over-extended Chevalley generators correspond to the matrices

$$e_{-1} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad f_{-1} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix} \quad h_{-1} = \frac{1}{2} \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \qquad (24)$$

in a notation where we only write out the components acting on the 0, 1, 2 components of the elfbein, with all other entries vanishing. Evidently, we have $h_{-1} = d - c_{-}$ with

$$d = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad c_{-} = -\frac{1}{2} \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad (25)$$

where d is the scaling operator on the dilaton ρ, and c_{-} is the central charge, alias the "level counting operator" of E_{10}, obeying $[c_{-}, e_{-1}] = -e_{-1}$ and $[c_{-}, f_{-1}] = + f_{-1}$ (and having vanishing commutators with all other Chevalley generators). Writing

$$c_{\pm} := -\frac{1}{2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \pm \frac{1}{2} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad (26)$$

we see that the first matrix on the right scales the conformal factor, generating Weyl transformations (called Weyl(Σ) in (23)) on the zweibein, while the second generates the local SO(1,1) Lorentz transformations. In a lightcone basis, these symmetries factorize on the zweibein, which decomposes into two chiral einbeine. Consequently, Weyl transformations and local SO(1, 1) can be combined into two groups SO(1, 1)$_\pm$ with respective generators $c_\pm$, and which act separately on the chiral einbeine. One of these, SO(1, 1)$_{-}$ (generated by c_{-}), becomes part of E_{10}. The other, SO(1, 1)$_{+}$, acts on the residual einbein and can be used to eliminate it by gauging it to one. Since $c_\pm$ acts in the same way on the conformal factor, we also recover the result of Julia (1982).

We wish to include both ISO(9) and SO(1, 1)$_{-}$ into the enlarged local symmetry H = ISO(16)$^\infty$, and thereby unify the longitudinal symmetries with the "transversal" group SO(16)$^\infty$ discussed before. Accordingly, we define ISO(16)$^\infty$ to be the algebra generated by the SO (16)$^\infty$ Lie algebra together with c_{-} and e_{-1}, as well as all their nonvanishing multiple commutators. The "classical" configuration space of M-Theory should then be identified with the coset space

$$\widetilde{\mathcal{M}} = \frac{E_{10}}{\text{ISO}(16)^{\infty}}.$$

(27)

Of course, we will have to worry about the fate of these symmetries in the quantum theory. Indeed, some quantum version of the symmetry groups appearing in (27) must be realized on the Hilbert space of third quantized $N = 16$ supergravity, such that E_{10} becomes a kind of spectrum generating (rigid) symmetry on the physical states, while the gauge group $\text{ISO}(16)^{\infty}$ gives rise to the constraints defining them. Because "third quantization" here is analogous to the transition from first quantized string theory to string field theory, the latter would have to be interpreted as multi-string states in some sense (cf. Witten (1986)) for earlier suggestions in this direction; note also that the coset space (27) is essentially generated by half of E_{10}, so there would be no "anti-string states". According to Font *et al.* (1990); Rey (1991); Sen (1993, 94); Schwarz *&* Sen (1994); Duff & Khuri (1994); Giveon *et al.* (1994); Hull & Townsend (1995); Witten (1995); Kachru & Vafa (1995); Schwarz (1995, 96); Duff (1996); Horava (1996) the continuous duality symmetries are broken to certain discrete subgroups over the integers in the quantum theory. Consequently, the quantum configuration space would be the left coset

$$\widetilde{\mathcal{F}} = E_{10}(\mathbf{Z}) \backslash \widetilde{\mathcal{M}}$$

and the relevant partition functions would have to be new kinds of modular forms defined on $\widetilde{\mathcal{F}}$. However, despite recent advances (Bakas 1996; Sen 1995), the precise significance of the (discrete) "string Geroch group" remains a mystery, and it is far from obvious how to extend the known results and conjectures for finite dimensional duality symmetries to the infinite dimensional case (these statements apply even more to possible discrete hyperbolic extensions; see, however, (Mizoguchi 1997; Gebert *&* Mizoguchi 1997). Moreover, recent work (Korotkin & Samtleben 1997) confirms the possible relevance of quantum groups in this context (in the form of "Yangian doubles").

Returning to our opening theme, more should be said about the $1 + 10$ split, which would lift up the $\text{SO}(1, 1)_{+} \times \text{ISO}(16)^{\infty}$ symmetry, and the "bein" which would realize the exceptional geometry alluded to in the introduction, and on which $\text{ISO}(16)^{\infty}$ would act as a generalized tangent space symmetry. However, as long as the $2 + 9$ split has not been shown to work, and a manageable realization is not known for either E_{10} or $\text{ISO}(16)^{\infty}$, we must leave the elaboration of these ideas to the future. It could well prove worth the effort.

Acknowledgements

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