Differentially Rotating Disks of Dust

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We present a three-parameter family of solutions to the stationary axisymmetric Einstein equations that describe differentially rotating disks of dust. They have been constructed by generalizing the Neugebauer-Meinel solution of the problem of a rigidly rotating disk of dust. The solutions correspond to disks with angular velocities depending monotonically on the radial coordinate; both decreasing and increasing behaviour is exhibited. In general, the solutions are related mathematically to Jacobi's inversion problem and can be expressed in terms of Riemann theta functions. A particularly interesting two-parameter subfamily represents Bäcklund transformations to appropriate seed solutions of the Weyl class.

1. INTRODUCTION

Although many rigorous solutions to Einstein's field equations have been constructed, only few of them can be applied to physically relevant situations. In the particular field of stationary axisymmetric solutions so far only the Kerr black holes [1] and the rigidly rotating disks of dust [2–4] are known to be physically relevant. On the other hand, stationary axisymmetric solutions are of special interest to astrophysics to model equilibrium configurations like stars and galaxies.

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The complicated structure of the Einstein equations describing the interior of a rotating body gives little hope for a rigorous global solution in the near future. However, if one restricts oneself to considering only dust configurations of finite extension, the body flattens in an extreme manner (it becomes a disk) and the interior equations turn into boundary conditions for the exterior vacuum equations. Now these equations can be expressed by a single nonlinear equation – the so-called Ernst equation [5,6]. For treating the Ernst equation, there are powerful analytic methods available which come from soliton theory [7–12]. In particular, Korotkin [13,14] and Meinel and Neugebauer [15] (see also Refs. 16,17), were able to construct a class of solutions containing a finite number of complex parameters and one arbitrary real solution to the axisymmetric three dimensional Laplace equation. In this paper we present a three-parameter subclass describing disks of dust revolving with a non uniform angular velocity. These solutions are analytic in the sense that they belong to the class of solutions under discussion and therefore strictly satisfy the Ernst equation. On the other hand, these solutions are numerical solutions since the real-valued potential function mentioned above has been determined numerically in order to satisfy the boundary conditions. The accuracy that has been obtained was very high (generally 12 digits) such that for any practical use the solutions are just as good as purely analytic ones. Moreover, the accuracy may in principle be increased arbitrarily.

The paper is organized as follows. In the first section, the boundary value problem for differentially rotating disks of dust is introduced and the class of solutions in question is reviewed. The numerical methods by which we were able to obtain our subclass of differentially rotating disks of dust will be discussed in the first part of the second section. This will be followed by a thorough illustration of the parameter space of these solutions. The subsequent subsections contain detailed discussions about particular limits.

In what follows, units are used in which the velocity of light as well as Newton's constant of gravitation are equal to 1.

1.1. Metric tensor, Ernst equation, and boundary conditions

The metric tensor for axisymmetric stationary and asymptotically flat space-times reads as follows in Weyl–Papapetrou-coordinates $(\rho, \zeta, \varphi, t)$:

$$ds^2 = e^{-2U}[e^{2k}(d\rho^2 + d\zeta^2) + \rho^2 d\varphi^2] - e^{2U}(dt + a\,d\varphi)^2.$$

For this line element, the vacuum field equations are equivalent to a single

complex equation — the so-called Ernst equation[3]

$$(\mathrm{Re}\, f)\, \triangle f = (\nabla f)^2\,, \tag{1}$$

$$\triangle = \frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial}{\partial \rho} + \frac{\partial^2}{\partial \zeta^2}\,, \qquad \nabla = \left(\frac{\partial}{\partial \rho}, \frac{\partial}{\partial \zeta}\right),$$

where the Ernst potential f is given by

$$f = e^{2U} + \mathrm{i}\, b \quad \text{with} \quad b_{,\zeta} = \frac{e^{4U}}{\rho}\, a_{,\rho}, \qquad b_{,\rho} = -\frac{e^{4U}}{\rho}\, a_{,\zeta}\,. \tag{2}$$

To obtain the boundary conditions for differentially rotating disks of dust, one has to consider the field equations for an energy-momentum-tensor

$$T^{ik} = \epsilon u^i u^k = \sigma_p(\rho) e^{U-k}\delta(\zeta) u^i u^k,$$

where ϵ and σ_p stand for the energy-density and the invariant (proper) surface mass-density, respectively, δ is the usual Dirac delta-distribution, and u^i denotes the four-velocity of the dust material.[4]

Integration of the corresponding field equations from the lower to the upper side of the disk (with coordinate radius ρ_0) yields for $\zeta = 0^+$ and $0 \le \rho \le \rho_0$ the conditions (see Ref. 18, p. 81–83)

$$2\pi\sigma_p = e^{U-k}(U_{,\zeta} + \tfrac{1}{2}Q) \tag{3}$$

$$e^{4U}Q^2 + Q(e^{4U})_{,\zeta} + (b_{,\rho})^2 = 0 \tag{4}$$

with

$$Q = -\rho e^{-4U}[b_{,\rho}b_{,\zeta} + (e^{2U})_{,\rho}(e^{2U})_{,\zeta}]. \tag{5}$$

Note that boundary condition (4) for the Ernst potential f does not involve the surface mass-density σ_p. This condition comes from the nature of the material the disk is made of. Therefore, eq. (4) will be referred to as the *dust-condition*.

The angular velocity $\Omega = u^\varphi/u^t$ of the disk can be calculated from

$$\Omega = \frac{Q}{a_{,\zeta} - a\, Q}\,. \tag{6}$$

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[3] The remaining function k can be calculated from the Ernst potential f by a line integration.

[4] u^i has only φ- and t- components.

The following requirements resulting from symmetry conditions and asymptotical flatness complete our set of boundary conditions:

- Regularity at the rotation axis is guaranteed by

$$\frac{\partial f}{\partial \rho}(0, \zeta) = 0\,.$$

- At infinity asymptotical flatness is realized by $U \to 0$ and $a \to 0$. For the potential b this has the consequence $b \to b_\infty = \text{const}$. Without loss of generality, this constant can be set to 0, i.e. $f \to 1$ at infinity.
- Finally, we assume reflectional symmetry with respect to the plane $\zeta = 0$, i.e. $f(\rho, -\zeta) = \overline{f(\rho, \zeta)}$ (with a bar denoting complex conjugation).

1.2. Solutions related to Jacobi's inversion problem

Meinel and Neugebauer [15] showed that for an arbitrary integer p the function f defined by

$$f = \exp\left(\sum_{\nu=1}^{p} \int_{K_\nu}^{K^{(\nu)}} \frac{K^p dK}{W(K)} - v_p\right)$$

with

$$W(K) = \sqrt{(K + \mathrm{i}z)(K - \mathrm{i}\bar{z})\prod_{\nu=1}^{p}(K - K_\nu)(K - \bar{K}_\nu)}$$

satisfies the Ernst equation.[5] Hereby, the K_ν are arbitrary complex parameters. The variable $z = \rho + \mathrm{i}\zeta$ is the complex combination of the coordinates ρ and ζ. The (z-dependent) values for the $K^{(\nu)}$ as well as the integration paths on a two-sheeted Riemann surface have to be taken from the solution to the following Jacobian inversion problem:

$$\sum_{\nu=1}^{p} \int_{K_\nu}^{K^{(\nu)}} \frac{K^j dK}{W(K)} = v_j\,, \quad 0 \le j < p.$$

The potential functions v_j $(0 \le j \le p)$ may be any real solutions to the axisymmetric Laplace equation $\triangle v_j = 0$ satisfying the recursion conditions

$$\mathrm{i}v_{j,z} = \tfrac{1}{2}v_{j-1} + zv_{j-1,z}\,.$$

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[5]   Korotkin [13,14] (see also Refs. 16,17) found solutions to the Ernst equation which are closely related to the solutions considered here.

These recursion conditions are automatically satisfied by the *ansatz*

$$v_j = \frac{1}{2\pi i} \int_\Sigma \frac{K^j H(K)}{\sqrt{(K-\zeta)^2 + \rho^2}} \, dK, \tag{8}$$

where Σ is some curve (or even some set of curves) in the complex plane. With (8), the regularity of the resulting solution at $(\rho, \zeta) = (|\Im[K_\nu]|,$ $\mathrm{Re}\,[K_\nu])$ is guaranteed. However, there are discontinuities along the curve $\Sigma' = \{(\rho, \zeta) : \zeta \pm i\rho \in \Sigma\}$. Moreover, a free function H defined on Σ enters the class of solutions. Hence, this ansatz allows us to consider a restricted class of boundary value problems[6] in which the curve Σ results from the shape of the boundary Σ'.

Ernst potentials with reflectional symmetry $f(\rho, -\zeta) = \overline{f(\rho, \zeta)}$ are obtained if the following properties are all satisfied:
- For each parameter K_ν there is a different parameter K_μ with $K_\mu = -\overline{K_\nu}$.
- $K \in \Sigma \Leftrightarrow 1 \pm \bar{K} \in \Sigma$
- $H(1 \pm \bar{K}) = \overline{H(K)}$

For differentially rotating disks, we can set $\Sigma = \{K : K = i\rho_0 x, -1 \leq x \leq 1\}$. Thus we get

$$v_j = \rho_0^{j-p} \int_{-1}^{1} \frac{(ix)^j h(x^2)}{Z_D} \, dx,$$

$$Z_D = \sqrt{(ix - \zeta/\rho_0)^2 + (\rho/\rho_0)^2} \qquad (\mathrm{Re}\,(Z_D) < 0).$$

In this expression we require the real-valued function h to be analytic on the interval $[0, 1]$. This is necessary for an analytic behaviour of the angular velocity Ω for all $\rho \in [0, \rho_0]$.

The additional requirement

$$h(1) = 0 \tag{9}$$

leads to a surface mass-density σ_p of the form

$$\sigma_p(\rho) = \psi(\rho)\sqrt{\rho_0^2 - \rho^2} \qquad \text{(with } \psi \text{ analytic on } [0, \rho_0])$$

and therefore ensures vanishing σ_p at the rim of the disk. The resulting Ernst potential f depends on the normalized coordinates $(\rho/\rho_0, \zeta/\rho_0)$, on the parameters $X_\nu = K_\nu/\rho_0$ and functionally on h.

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[6] The restriction comes from the requirement that the potential functions v_j be *real*.

One obtains the special case of rigid rotation [3] when the following equations are all satisfied:

- $p = 2$

- $X_1^2 = -1 + \dfrac{\mathrm{i}}{\mu}, \qquad X_2 = -\overline{X_1}$

- $h(x^2) = \mu \dfrac{\operatorname{arsinh}[\mu(1 - x^2)]}{\pi\sqrt{1 + \mu^2(1 - x^2)^2}} \cdot$ \hfill (10)

The parameter μ is related to the angular velocity

$$\mu = 2\Omega^2 \rho_0^2 e^{-2V_0}, \qquad V_0 = U(\rho = 0, \zeta = 0)$$

and runs on the interval $(0, \mu_0)$ with $\mu_0 = 4.62966184\ldots$ For $\mu \ll 1$ one obtains the Newtonian limit of the Maclaurin disk. On the other hand, $\mu \to \mu_0$ and $\rho_0 \to 0$ yields the ultrarelativistic limit of the extreme Kerr black hole.

In this article we explore the subclass $p = 2$ of the solutions introduced above. It will be shown how, for a given value of the complex parameter X_1 ($K_1 = \rho_0 X_1$), the freedom of the choice of the function h with the property (9) has been used to find a solution satisfying the dust-condition (4). It turns out that for each X_1 within a certain region (a more precise description follows) there is a function h such that the resulting Ernst potential can be interpreted as having been created by a differentially rotating disk of dust. The accompanying surface mass-density and angular velocity may afterwards be calculated according to eqs. (3) and (6).

2. DIFFERENTIALLY ROTATING DISKS

2.1. The numerical scheme

As already mentioned above, in this paper we consider the class of solutions introduced in the previous chapter for the particular case $p = 2$. Here we can prescribe

(i) the coordinate radius ρ_0

(ii) the complex parameter X_1 with $\operatorname{Re}(X_1) \leq 0$ and $\Im(X_1) \leq 0$ (without loss of generality). Then K_1 and K_2 follow from $K_1 = \rho_0 X_1$, $K_2 = -\overline{K_1}$.

(iii) a real-valued function $h : [0, 1] \longrightarrow \mathbb{R}$ which is analytic everywhere in $[0, 1]$ (i.e. in particular at the boundaries of the interval) and vanishes at the upper boundary: $h(1) = 0$.

For such a choice all the requirements stated in subsection 1.1 are satisfied except the dust-condition (4)[7] . Now, this condition yields a complicated nonlinear integral equation for h:

$$D\left(x^2 = \frac{\rho^2}{\rho_0^2}\,;X_1;h\right) := \rho_0^2[e^{4U}Q^2 + Q(e^{4U})_{,\zeta} + (b_{,\rho})^2] \doteq 0$$

$$(\zeta = 0^+, 0 \leq \rho \leq \rho_0) \tag{11}$$

$$\{Q = -\rho e^{-4U}[b_{,\rho}b_{,\zeta} + (e^{2U})_{,\rho}(e^{2U})_{,\zeta}],\ \ f = e^{2U} + \mathrm{i}b = f(\rho/\rho_0, \zeta/\rho_0; X_1; h)\,\}.$$

Note that the resulting function h depends parametrically on X_1 (but not on ρ_0).

With expansions of the functions h and D in Chebyshev-polynomials (this can be done since both of them are analytic in $[0, 1]$) we try to discretize eq. (11):

- $$h(x^2) \approx \sum_{j=1}^{N} h_j T_{j-1}(2x^2 - 1) - \frac{1}{2}h_1\,, \quad T_j(\tau) = \cos[j\arccos(\tau)],$$

  $$h(1) \doteq 0 \Rightarrow h_1 = -2\sum_{j=2}^{N} h_j$$

- $$D(x^2; X_1; h) \approx \sum_{j=1}^{N-1} D_j(X_1; h_k)T_{j-1}(2x^2 - 1) - \frac{1}{2}D_1(X_1; h_k).$$

In this manner, the nonlinear integral equation (11) is approximated by a finite system of nonlinear equations

$$D_j(X_1; h_k) = 0 \qquad (1 \leq j < N, \quad 2 \leq k \leq N). \tag{12}$$

The system (12) has been solved numerically by a Newton–Raphson method. For this technique one needs a good initial guess for the solution. Fortunately, the values h_k are given exactly for the rigidly rotating disks. Therefore, we start with an X_1 that differs only slightly from those for the rigidly rotating disks [say $X_1^2 = (-1 + \varepsilon) + \mathrm{i}/\mu$] and take the h_k's for the rigidly rotating disks as initial values. The newly calculated nearby

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[7] Additionally one has to ensure that the surface mass-density σ_p given by eq. (3) is positive and finite within $[0, \rho_0]$. Furthermore, the global regularity of the Ernst potential has to be checked. Fortunately, our solutions possess these properties.

solution serves then as an initial guess for another solution further away from the rigidly rotating disks. Thus we can gradually explore the whole parameter region of X_1.

The numerical code written to implement this scheme produces results with excellent convergence. For almost all values X_1 inside the available parameter region[8] the cancellation of the terms in eq. (11) up to the 12th digit and even beyond has been achieved within the whole range $x^2 \in [0, 1]$. The resulting Chebyshev-coefficients fall off rapidly (generally $N = 20$ suffices to achieve the previously mentioned accuracy of 12 digits) and the resulting function h indeed has the desired smooth analytic behaviour. As already mentioned in footnote 7, the accompanying surface mass-density turns out to be positive and finite and the Ernst potential is regular outside the disk.
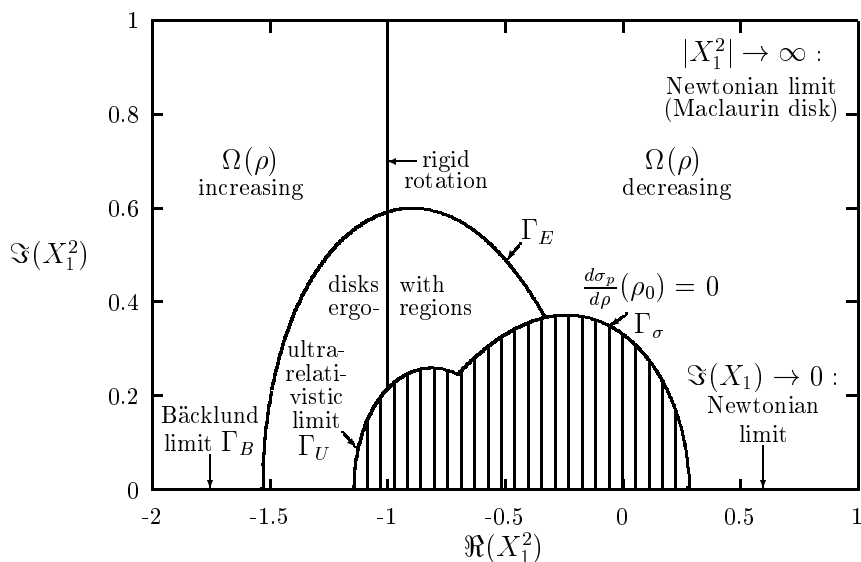

Figure 1. Outside the hatched region of the parameter space of X_1^2 regular solutions have been found. Physical characteristics of solutions corresponding to values inside the allowed region are indicated.

2.2. The parameter region of the solutions and examples

In the following we discuss the solutions obtained, as a function of the parameter X_1^2. For each value outside the hatched region in Figure 1 we have found a corresponding solution to the Ernst equation satisfying the dust-condition. What follows is a discussion of the details of Figure 1:

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[8] The exceptions are to be found in narrow stripes along the curve Γ_σ (see next section).

(i) *Differential rotation*: For $\mathrm{Re}\,(X_1^2) = -1$ we find the rigidly rotating disks. On the left-hand side of this line (i.e. for $\mathrm{Re}\,(X_1^2) < -1$) the solutions turned out to possess an angular velocity Ω which increases with the radial coordinate ρ. On the other side, for $\mathrm{Re}\,(X_1^2) > -1$, the function Ω decreases as ρ increases.

(ii) *Ergoregions*: For values X_1^2 inside the area encompassed by the curve Γ_E, the curve Γ_U and parts of the curves Γ_B and Γ_σ, the corresponding disks possess an ergoregion, i.e. a portion of the (ρ, ζ)-space within which the function e^{2U} is negative.

(iii) *Ultrarelativistic limit*: As will be shown in subsection 2.3, any simultaneous limit
- ρ_0 tends to 0
- X_1^2 tends to a value on Γ_U

turns out to be an ultrarelativistic limit. In the case of non vanishing values for $\rho^2 + \zeta^2$, the resulting f tends to the Ernst potential of an extreme Kerr black hole. If, on the contrary, finite values for $\sqrt{\rho^2 + \zeta^2}/\rho_0$ are maintained, non asymptotically flat solutions can be obtained. These results are in agreement with a conjecture by Bardeen and Wagoner [19].

(iv) *The Newtonian limit* $|X_1^2| \to \infty$: Here the Ernst potential tends to 1, i.e. it describes a Minkowski space. In a given neighbourhood (for large values of $|X_1^2|$) a post-Minkowskian expansion (with the first coefficient being a Newtonian potential) of the Ernst potential can be carried out. One finds that the resulting Newtonian coefficient is the gravitational field of a rigidly rotating Maclaurin-disk.

(v) *The Newtonian limit* $\Im(X_1) \to 0$: For real and positive values of X_1^2 (hence $X_1 = -X_2$, both real) again the corresponding Ernst potential is equal to 1. Here, the Newtonian coefficient of the post-Minkowskian expansion describes a disk with decreasing $\Omega(\rho)$.

(vi) *Bäcklund limit*: For real and negative values of X_1^2 we get $X_1 = X_2$. Then the complicated structure of the Ernst potential f simplifies considerably. One finds that these solutions can be interpreted as Bäcklund transforms of appropriate seed solutions of the Weyl class. A detailed discussion on this subclass is given in subsection 2.4.

(vii) *The hatched region and the curve* Γ_σ: Inside the hatched region no solutions have been found satisfying both of the requirements
- f is regular everywhere outside the disk.
- The function h is analytic for all $x^2 \in [0, 1]$.

Apart from the ultrarelativistic curve Γ_U the hatched region is encompassed by the curve Γ_σ. Starting from the Newtonian solutions described in (v) and ending at Γ_U, the corresponding Ernst potentials
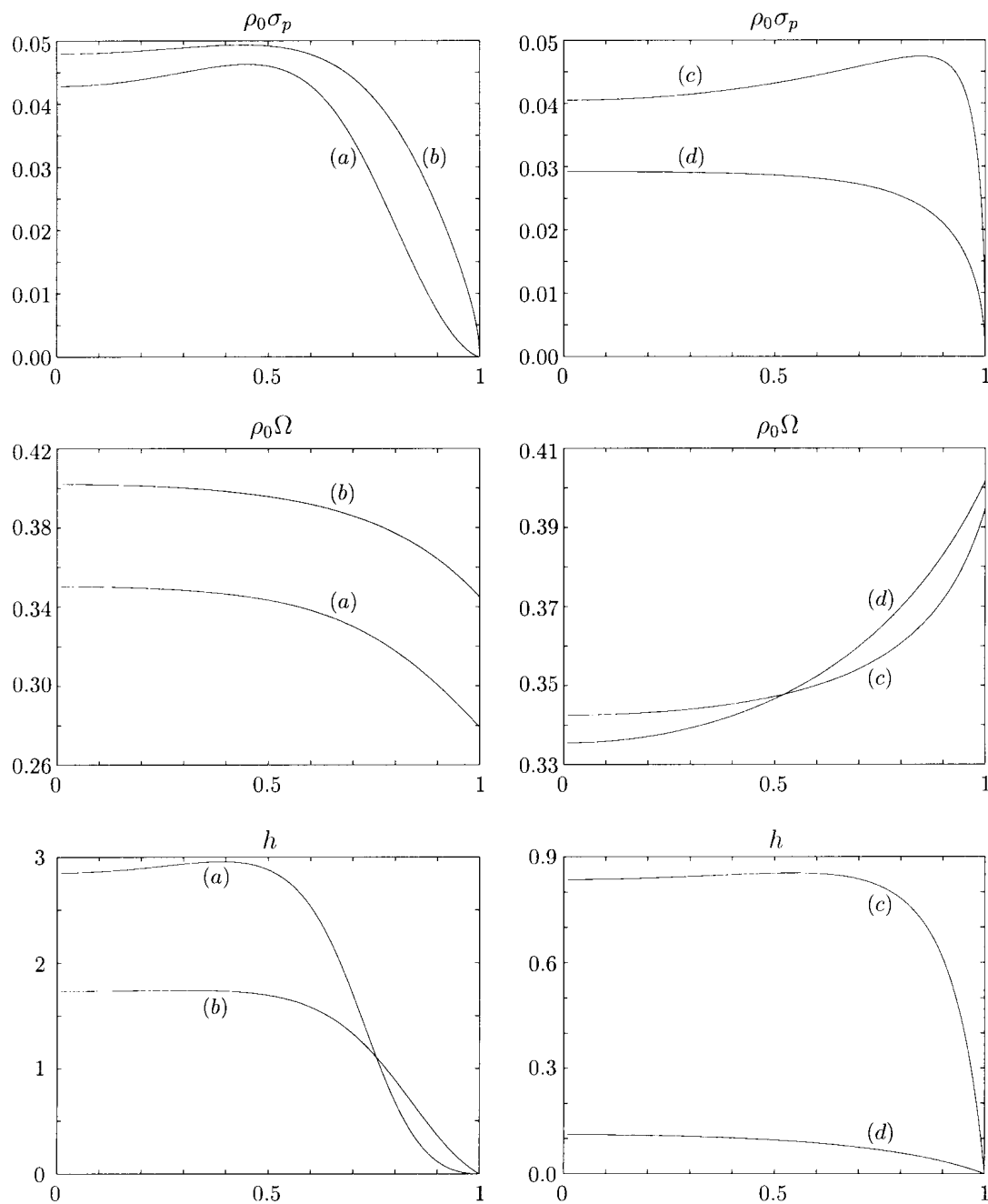
Figure 2. Examples for differentially rotating disks. The dimensionless quantities $\rho_0\sigma_p$, $\rho_0\Omega$ and the function h are plotted against the normalized radial coordinate ρ/ρ_0 and x, respectively, for (a) $X_1^2 \approx -1/2 + i/3$ (here $X_1^2 \in \Gamma_\sigma$), (b) $X_1^2 = -2/3 + i/2$, (c) $X_1^2 = -3/2 + i/5$, (d) $X_1^2 = -4$.

describe a transition from Minkowski space to the ultrarelativistic limit, just as the rigidly rotating disks and the Bäcklund solutions (at Γ_B) do. Now, all solutions along Γ_σ possess the property that the derivative of the surface mass-density vanishes at the rim of the disk, i.e. $\sigma_p(\rho) = (\rho_0^2 - \rho^2)^{3/2}\, \tilde{\psi}(\rho)$ (with $\tilde{\psi}$ analytic in $[0, \rho_0]$). Surprisingly, this physical property coincides with the failure of our numerical method at Γ_σ. Further investigations are necessary to clarify this coincidence.
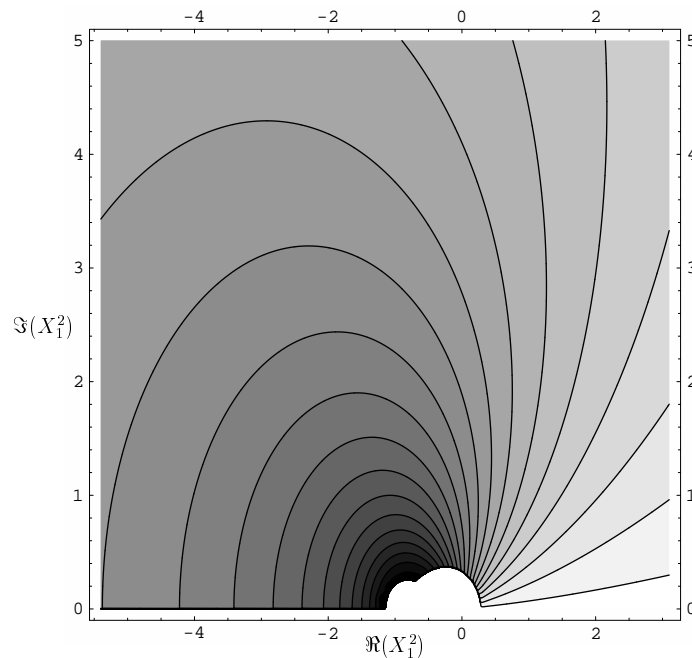

Figure 3. Contourplot of M^2/J. The stripes correspond to intervals of length $1/20$; they are white for Newtonian disks ($M^2/J \to 0$) and black for highly relativistic disks ($M^2/J \to 1$).

Figure 2 shows representative examples of solutions. From Figure 3 we can get an impression of how "relativistic" our solutions are. Here, the physical quantity M^2/J is plotted over the available parameter region of X_1^2 with the gravitational mass M and the total angular momentum J given by

$$M = 2\int_S (T_{ab} - \tfrac{1}{2}Tg_{ab})n^a\xi^b dV \qquad (T = g_{ab}T^{ab}), \quad J = -\int_S T_{ab}n^a\eta^b dV.$$

(S is the spacelike hypersurface $t = $ constant with the unit future-pointing normal vector n^a; the Killing vectors ξ^a and η^a correspond to stationarity

and axisymmetry, respectively.) Note that M and J can also be calculated from the behaviour of the Ernst potential at infinity:

$$U = -\frac{M}{r} + \mathcal{O}(r^{-2}), \qquad b = -2J\frac{\cos\theta}{r^2} + \mathcal{O}(r^{-3}) \quad (\rho = r\sin\theta, \zeta = r\cos\theta).$$

The agreement of the two expressions yields an excellent confirmation of the regularity of the solutions. For Figure 3, not only the values M^2/J have been determined but also this agreement was checked.

2.3. Ultrarelativistic limits

In this subsection we show how ultrarelativistic limits of our solutions can be obtained if simultaneously

- X_1^2 tends to a value on Γ_U.
- the coordinate radius ρ_0 tends to zero.

To this end we first consider the limit $\rho_0 \to 0$ for a fixed value $X_1^2 \notin \Gamma_U$ and finite values of $\rho^2 + \zeta^2 \neq 0$. Here, f tends to 1. In a second step, we investigate the above simultaneous limit again for $\rho^2 + \zeta^2 \neq 0$ and find that f tends to the Ernst potential of an extreme Kerr black hole. Finally, we indicate how further non asymptotically flat solutions can be obtained by fixing finite values of $\sqrt{\rho^2 + \zeta^2}/\rho_0$.

(i) *The limit $\rho_0 \to 0$ for $X_1^2 \notin \Gamma_U$:* Similar to the treatment in [20] we rewrite the expression for f and Jacobi's inversion problem in the equivalent form[9]

$$f = \exp\left(\int_{K^{(2)}}^{K^{(1)}} \frac{K^2 dK}{W} - \tilde{v}_2\right),$$

$$\int_{K^{(2)}}^{K^{(1)}} \frac{dK}{W} = \tilde{v}_0, \qquad \int_{K^{(2)}}^{K^{(1)}} \frac{K\,dK}{W} = \tilde{v}_1 \qquad (13)$$

with

$$\tilde{v}_j = v_j - w_j = v_j - \int_{K_1}^{K_2} \frac{K^j\,dK}{W} \qquad (j = 0, 1, 2).$$

In the limit $\rho_0 \to 0$, $\rho^2 + \zeta^2 \neq 0$ the potential function v_0 as well as the integral w_0 diverge whilst these values remain finite for $j > 0$:

$$v_0 = -\frac{1}{\rho_0\sqrt{\rho^2 + \zeta^2}} \int_{-1}^{1} h(x^2; X_1)\,dx + \mathcal{O}(\rho_0), \quad v_1 = \mathcal{O}(\rho_0), \quad v_2 = \mathcal{O}(\rho_0),$$

$$w_0 = \frac{2}{\rho_0\sqrt{\rho^2 + \zeta^2}} \mathrm{Re}\left[\frac{1}{X_1} K\left(\frac{X_2}{X_1}\right)\right] + \frac{\pi\mathrm{i}\zeta}{2(\rho^2 + \zeta^2)^{3/2}} + \mathcal{O}(\rho_0)$$

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[9] $K^{(2)}$ is now on the other sheet of the Riemann surface.

with K being the Jacobian elliptic function

$$K(m) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - m^2 \sin^2 \theta}} ,$$

$$w_1 = \frac{\pi \mathrm{i}}{2\sqrt{\rho^2 + \zeta^2}} + \mathcal{O}(\rho_0), \qquad w_2 = \mathcal{O}(\rho_0).$$

Now, we define the curve Γ_U ($X_2 = -\bar{X}_1$):

$$X_1^2 \in \Gamma_U$$

$$:\Leftrightarrow \ C(X_1) := - \int_{-1}^{1} h(x^2; X_1)\, dx - 2\mathrm{Re}\left[\frac{1}{X_1} K\left(\frac{X_2}{X_1} \right) \right] \doteq 0 . \quad (14)$$

For all values $X_1^2 \notin \Gamma_U$, $\tilde{v}_0$ diverges as well:

$$\tilde{v}_0 = \frac{C(X_1)}{\rho_0 \sqrt{\rho^2 + \zeta^2}} - \frac{\pi \mathrm{i} \zeta}{2(\rho^2 + \zeta^2)^{3/2}} + \mathcal{O}(\rho_0) \qquad \text{with } C(X_1) \neq 0.$$

In the limit $\rho_0 \to 0$, this leads to finite solutions $X^{(j)} = K^{(j)}/\rho_0$ ($j = 1, 2$) of Jacobi's inversion problem and eventually to

$$f = \exp\left(\frac{\rho_0}{\sqrt{\rho^2 + \zeta^2}} \int_{X^{(2)}}^{X^{(1)}} \frac{X^2 dX}{\sqrt{(X^2 - X_1^2)(X^2 - X_2^2)}} + \mathcal{O}(\rho_0) \right) \to 1.$$

(ii) *The black hole limit*: The above simultaneous limit is performed such that

$$\Omega_U := \frac{C(X_1)}{2\rho_0}$$

remains finite. We then get for $\rho_0 \to 0$ (with $\rho = r \sin\theta$, $\zeta = r\cos\theta$):

$$\tilde{v}_0 = \frac{2\Omega_U}{r} - \frac{\pi\mathrm{i}\cos\theta}{2r^2} , \qquad \tilde{v}_1 = -\frac{\pi\mathrm{i}}{2r} , \qquad \tilde{v}_2 = 0 .$$

Again we can follow the procedure of [20]. Since $K_j \to 0$ ($j = 1, 2$), the integrals in Jacobi's inversion problem (13) become elementary, and thus for f one obtains

$$f = \frac{2r\Omega_U - 1 - \mathrm{i}\cos\theta}{2r\Omega_U + 1 - \mathrm{i}\cos\theta} ,$$

i.e. the ($r > 0$ part of the) extreme Kerr solution with $J = 1/(4\Omega_U^2) = M^2$. The constant Ω_U plays the role of the 'angular velocity of the horizon'.

(iii) *The non asymptotically flat ultrarelativistic limit*: As shown in [20] for the rigidly rotating disks, our more general solutions also allow for a different, not asymptotically flat limit. To achieve this, one has to consider finite values of r/ρ_0. A coordinate transformation

$$\tilde{r} = \frac{r}{C(X_1)}, \quad \tilde{\varphi} = \varphi - \Omega_U t, \quad \tilde{\theta} = \theta,$$
$$\tilde{t} = C(X_1)t \quad (\text{hence} \quad r/\rho_0 = 2\tilde{r}\Omega_U),$$

yields a transformation to a corotating system (with angular velocity Ω_U) combined with a rescaling of r and t. In the limit $\rho_0 \to 0$ and $C(X_1) \to 0$, the resulting Ernst potential $\tilde{f}$ [which is related to the Ernst potential f'_U within the above corotating system by $\tilde{f} = f'_U/C^2(X_1)$] still describes a disk and is regular everywhere outside the disk. However, it is not asymptotically flat.

2.4. Bäcklund limit

In the limit of real and negative values of X_1^2 we obtain purely imaginary values for K_1 and K_2 with $K_1 = K_2$. Then the reformulation (13) of the expressions for f and Jacobi's inversion problem, leads to [with the abbreviation $r_1 = \sqrt{(K_1 - i\bar{z})(K_1 + iz)}$]:

$$f = \exp\left(\int_{K^{(2)}}^{K^{(1)}} \frac{dK}{\sqrt{(K - i\bar{z})(K + iz)}} - [v_2 - K_1^2 v_0] \right),$$

$$\int_{K^{(2)}}^{K^{(1)}} \frac{dK}{(K - K_1)\sqrt{(K - i\bar{z})(K + iz)}} = v_1 + K_1 v_0 - \frac{i\pi}{r_1},$$

$$\int_{K^{(2)}}^{K^{(1)}} \frac{dK}{(K + K_1)\sqrt{(K - i\bar{z})(K + iz)}} = v_1 - K_1 v_0.$$

Applying the substitution $\lambda(K) = \sqrt{(K - i\bar{z})/(K + iz)}$, we get the following system of equations for f, $\lambda^{(1)}\lambda^{(2)}$, and $(\lambda^{(2)} - \lambda^{(1)})$:

$$f = \frac{\lambda^{(1)}\lambda^{(2)} + (\lambda^{(2)} - \lambda^{(1)}) - 1}{\lambda^{(1)}\lambda^{(2)} - (\lambda^{(2)} - \lambda^{(1)}) - 1} \, e^{-(v_2 - K_1^2 v_0)},$$

$$\frac{\lambda^{(1)}\lambda^{(2)} - \lambda_1(\lambda^{(2)} - \lambda^{(1)}) - \lambda_1^2}{\lambda^{(1)}\lambda^{(2)} + \lambda_1(\lambda^{(2)} - \lambda^{(1)}) - \lambda_1^2} = -e^{r_1(v_1 + K_1 v_0)},$$

$$\frac{\lambda^{(1)}\lambda^{(2)} - \lambda_1^*(\lambda^{(2)} - \lambda^{(1)}) - \lambda_1^{*2}}{\lambda^{(1)}\lambda^{(2)} + \lambda_1^*(\lambda^{(2)} - \lambda^{(1)}) - \lambda_1^{*2}} = e^{r_1^*(v_1 - K_1 v_0)}$$

$$[\text{with} \quad \lambda^{(j)} = \lambda(K^{(j)}), \ \lambda_1 = \lambda(K_1), \ \lambda_1^* = 1/\bar{\lambda}_1, \ r_1^* = \bar{r}_1].$$

The solution for f is given by

$$f = f_0 \begin{vmatrix} 1 & 1 & 1 \\ -1 & \lambda_1\alpha_1 & \lambda_1^*\alpha_1^* \\ 1 & \lambda_1^2 & \lambda_1^{*2} \end{vmatrix} \Bigg/ \begin{vmatrix} 1 & 1 & 1 \\ 1 & \lambda_1\alpha_1 & \lambda_1^*\alpha_1^* \\ 1 & \lambda_1^2 & \lambda_1^{*2} \end{vmatrix} \tag{15}$$

where

$$f_0 = e^{-(v_2 - K_1^2 v_0)}, \qquad \alpha_1 = \frac{1 - \exp[r_1(v_1 + K_1 v_0)]}{1 + \exp[r_1(v_1 + K_1 v_0)]}, \qquad \alpha_1^* = \frac{1}{\bar{\alpha}_1}.$$

Equation (15) represents a Bäcklund transformation of the real seed solution f_0 (see Refs. 10,11). As a consequence of (7), α_1 satisfies the Riccati equations[10]

$$\alpha_{1,z} = \lambda_1(\alpha_1^2 - 1)\frac{f_{0,z}}{2f_0}, \qquad \alpha_{1,\bar{z}} = \frac{1}{\lambda_1}(\alpha_1^2 - 1)\frac{f_{0,\bar{z}}}{2f_0}.$$

Hence, in this limit, our solutions are physically interesting Bäcklund transforms of nontrivial seed solutions of the Weyl class.

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REFERENCES

1. Kerr, R. (1963). *Phys. Rev. Lett.* **11**, 237.
2. Neugebauer, G., and Meinel, R. (1993). *Astrophys. J.* **414**, L97.
3. Neugebauer, G., and Meinel, R. (1995). *Phys. Rev. Lett.* **75**, 3046.
4. Neugebauer, G., Kleinwächter, A., and Meinel, R. (1996). *Helv. Phys. Acta* **69**, 472.
5. Ernst, F. J. (1968). *Phys. Rev.* **167**, 1175.
6. Kramer, D., and Neugebauer, G. (1968). *Commun. Math. Phys.* **7**, 173.
7. Maison, D. (1978). *Phys. Rev. Lett.* **41**, 521.
8. Belinski, V. A., and Zakharov, V. E. (1978). *Zh. Eksper. Teoret. Fiz.* **75**, 195.

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[10] It is interesting to note that the special choice of the integration constant arising from (8) guarantees the absence of a singular behaviour of the final solution at $\rho = -\Im[K_1]$.

9. Harrison, B. K. (1978). *Phys. Rev. Lett.* **41**, 119.

10. Neugebauer, G. (1979). *J. Phys. A* **12**, L67.

11. Neugebauer, G. (1980). *J. Phys. A* **13**, L19.

12. Neugebauer, G. (1980). *J. Phys. A* **13**, 1737.

13. Korotkin, D. A. (1989). *Theor. Math. Phys.* **77**, 1018.

14. Korotkin, D. A. (1993). *Class. Quantum Grav.* **10**, 2587.

15. Meinel, R., and Neugebauer, G. (1996). *Phys. Lett. A* **210**, 160.

16. Korotkin, D. A. (1997). *Phys. Lett. A* **229**, 195.

17. Meinel, R., and Neugebauer, G. (1997). *Phys. Lett. A***229**, 200.

18. Kleinwächter, A. (1995). "Untersuchungen zu rotierenden Scheiben in der Allgemeinen Relativitätstheorie." Ph.D. Dissertation, Friedrich-Schiller-Universität Jena.

19. Bardeen, J. M., and Wagoner, R. V. (1971). *Astrophys. J.* **167**, 359.

20. Meinel, R. (1998). In *Recent Developments in Gravitation and Mathematical Physics*, A. García, C. Lämmerzahl, A. Macías, T. Matos, and D. Nuñez, eds. (Science Network Publishing, Konstanz); gr-qc/9703077.