On the duality between perturbative heterotic orbifolds and M-theory on T^4/Z_N ☆

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Abstract

The heterotic $E_8 \times E_8$ string compactified on an orbifold $T^4/\mathbf{Z}_N$ has gauge group $G \times G'$ with (massless) states in its twisted sectors which are charged under both gauge group factors. In the dual M-theory on $(T^4/\mathbf{Z}_N) \otimes (S^1/\mathbf{Z}_2)$ the two group factors are separated in the eleventh direction and the G and G' gauge fields are confined to the two boundary planes, respectively. We present a scenario which allows for a resolution of this apparent paradox and assigns all massless matter multiplets locally to the different six-dimensional boundary fixed planes. The resolution consists of diagonal mixing between the gauge groups which live on the connecting seven-planes (6d and the eleventh dimension) and one of the gauge group factors. We present evidence supporting this mixing by considering gauge couplings and verify local anomaly cancellation. We also discuss open problems which arise in the presence of U_1 factors. © 2000 Elsevier Science B.V. All rights reserved.

1. Introduction

The rôle of M-theory for string duality is undisputed. Nevertheless, we are far from understanding this 11-dimensional theory at a fundamental level. We know that at particular points in its moduli space all known string theories are recovered. Moreover, the low energy physics is captured by 11-dimensional supergravity. Compactification and duality symmetries provide data which allow us to gain some insight into the structure of M-theory (see, e.g., Refs. [1–3] for reviews).

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In this paper we concentrate on 10-dimensional $E_8 \times E_8$ heterotic string theory compactified on $T^4/\mathbf{Z}_N$ orbifolds and its dual M-theory description. The $E_8 \times E_8$ heterotic string is dual to M-theory on $S^1/\mathbf{Z}_2 \simeq I$. Hořava and Witten have shown [4,5] that one E_8 factor lives on each of the two boundary ten-planes [1] and that the corresponding gauge multiplets are confined to it. There is no a priori M-theoretical explanation of the appearance of the E_8 gauge group on each of the two ten-planes, but they are needed for local anomaly cancellation. Thus, anomaly cancellation provides important constraints which teach us about the structure of M-theory.

Considering the proposed duality between the heterotic string on $\mathbb{R}^{5,1} \otimes (T^4/\mathbf{Z}_N)$ and M-theory on $\mathbb{R}^{5,1} \otimes (T^4/\mathbf{Z}_N) \otimes (S^1/\mathbf{Z}_2)$ one immediately encounters the following puzzle. Suppose both ten-plane E_8 groups are broken in the six-dimensional theory; let $G \times G'$ $\subset E_8 \times E_8$ denote the surviving subgroup. The twisted sectors of heterotic orbifolds generally contain massless states which are charged under *both G and G'*. In the Hořava–Witten theory however, G is G and G' is G', they live on different ten-plains and nowhere the twain shall meet, so there does not seem to be any place where a massless state can be simultaneously charged with respect to both G and G'. Indeed, how would a state residing at one end of the eleventh dimension know about the gauge group acting on the other side? Somehow, in the effective seven-dimensional gauge theory on $\mathbb{R}^{5,1} \otimes (S^1/\mathbf{Z}_2)$ gauge quantum numbers ought to 'flow' from one end of the x^{11} to the other end; the main objective of our work is to understand how this works.

The same problem also arises in the phenomenologically more interesting examples of orbifold compactifications to four dimensions. In fact, our initial motivation was to understand how the gauge quantum numbers work in the four-dimensional orbifolds, but the situation in six dimensions turned out to be easier. In particular, we got very useful hints from the requirement of local anomaly cancellation, which are much stronger in $d = 6$ than in $d = 4$. Consequently, in this article we restrict ourselves to the discussion of the six-dimensional theories with the intention to come back to the four-dimensional case in the future.

In M-theory, an A_{n-1} singularity of the K3 compactification — such as a $\mathbf{Z}_n$ fixed point of an orbifold — supports an SU_n gauge theory on the corresponding seven-plane $\mathbb{R}^{5,1} \otimes$ {f.p.} $\otimes (S^1/\mathbf{Z}_2)$. The Cartan subalgebra arises from the zero modes of the three-form potential of 11d SUGRA and the completion to SU_n is achieved by M2-branes wrapped on a vanishing 2-cycle in the orbifold limit of the smooth K3 compactification manifold. These seven-plane gauge groups play a vital rôle in our resolution of the paradoxical $G \times G'$ charges of the twisted states: at one end of the eleventh dimension, say at $x^{11} = 0$, the seven-plane SU_n *mixes* with a similar factor of G — the unbroken subgroup of the E_8 living on the ten-plane boundary of the entire 11d spacetime. The mixing happens along the six-planes where the fixed seven-planes intersect the $x^{11} = 0$ ten-plane, but it has global consequences for the resulting effective theory: the *diagonal* ten-plane/seven-plane SU_n gauge group *appears* to be a subgroup of the ten-plane G, but actually reaches along the fixed seven-planes to the other end of the x^{11}. Consequently, along the six-planes where the

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[1] Throughout we refer to an extended object as a *plane*. E.g., a ten-plane has ten spacetime dimensions.

fixed seven-planes carrying the SU_n intersect the second ten-plane carrying G', we have both SU_n and G' gauge fields at the same location in space — and hence the twisted states living on those six-planes may have both the SU_n^{diag} and the G' charges in a perfectly local fashion.

The bottom line is, the twisted states have local $G' \times SU_n^{7P}$ charges but from the global point of view, they have simultaneous charges under the G' and the diagonal SU_n gauge groups. The apparent paradox of simultaneous G' and G charges is due to mis-identification of the diagonal ten-plane/seven-plane SU_n as a subgroup of the ten-plane $G \subset E_8$. This mis-identification is natural in the perturbative heterotic string theory where the entire eleventh dimension is invisible and everything lives in ten dimensions. In the M-theory, one needs to be more careful.

In this article, we shall marshal three lines of evidence for the mixing of ten-plane and seven-plane gauge groups. First, this is the only way to reconcile the massless spectra of heterotic orbifolds with locality of the dual M-theory description. Second, the heterotic gauge couplings (which can be computed exactly in six dimensions) will show that some SU_n gauge groups cannot be of purely perturbative origin but must be diagonally mixed with several non-perturbative factors,

$$SU_n^{d=6} = \text{diag}\big[SU_n^{\text{pert}} \times \big(SU_n^{\text{non-pert}}\big)^\nu\big], \tag{1.1}$$

and the number ν of the non-perturbative SU_n factors will always turn out to be equal to the number of fixed seven-planes in M-theory that carry SU_n gauge groups. Finally, each six-plane in M-theory suffers from local anomalies which are sensitive to spectra of massless particles living on the six-planes themselves, on the seven-planes, on the ten-planes and in the eleven-dimensional bulk as well as inflow and intersection anomalies due to the M-theory's Chern–Simons terms. In six dimensions, it is very difficult to cancel the local anomalies unless one has correct *local* spectra of all the fields — and we shall see that the ten-plane/seven-plane gauge group mixing indeed provides for cancellation of all the local anomalies.

The rest of this paper is organized as follows: we begin with an illustrative example of a $\mathbf{Z}_2$ orbifold. In Section 2 we take a close look at the seven-plane and the ten-plane SU_2 gauge groups of this orbifold and discuss their mixing from both heterotic and M-theory points of view. In particular, we show how the mixing explains the SU_2 charges of the twisted states as well as the exact value of the SU_2 gauge coupling. In Section 3 we confirm our solution by verifying local anomaly cancellation. In these two sections we try to be as explicit as possible. Section 4 generalizes our approach to other $\mathbf{Z}_N$ orbifolds. The subtleties that arise in non-prime orbifolds are treated in detail.

Unfortunately, our proposed solution works for some $\mathbf{Z}_N$ orbifolds but has difficulties with others. Section 5 describes two common types of complications, both associated with broken seven-plane SU_n groups. In some orbifolds (discussed in Section 5.1), the perturbative ten-plane gauge groups don't mix with the non-perturbative seven-plane groups. As far as the effective six-dimensional effective theory is concerned, the seven-plane groups are completely invisible, but the local anomalies on the six-planes are not so blind. To cancel the anomalies, we have to assume that the seven-plane gauge groups

are not SU_n's but rather their Cartan subgroups $U_1^{(n-1)}$. Alas, from the seven-dimensional point of view, all $\mathbf{Z}_n$ fixed planes are created equal and we have no idea how or why does the M-theory decide that such fixed planes carry full SU_n gauge groups in some orbifolds but only the Cartan $U_1^{(n-1)}$ subgroups in others. Worse problems plague orbifold models where the ten-plane and the seven-plane gauge groups do mix but the mixing involves abelian factors. In Section 5.2 we show that in such models local anomaly cancellation does not seem to work and we speculate how the seven-dimensional Chern–Simons terms might remedy this problem.

Section 6 summarizes our results. Appendices A, B and C contain some useful data about the anomalies in six dimensions and as well as some related group theory.

2. Compactification on $\mathbf{Z}_2$

2.1. Heterotic vs. M-theory point of view

Consider the perturbative compactification of the $E_8 \times E_8$ heterotic string on the $T^4/\mathbf{Z}_2$ orbifold limit of K3. $\mathbf{Z}_2$ acts as $(z^1, z^2) \rightarrow -(z^1, z^2)$ on the two complex coordinates of T^4. Under this transformation all four moduli of the torus are invariant and are thus also moduli of the orbifold. There are sixteen orbifold fixed points. This compactification has $\mathcal{N} = 1$ supersymmetry in $d = 6$ (eight supercharges). We represent the discrete $\mathbf{Z}_2$ transformation in the $E_8 \times E_8$ gauge lattice via the shift vector $\delta = (\frac{1}{2}, \frac{1}{2}, 0, 0, 0, 0, 0, 0; 1, 0, 0, 0, 0, 0, 0, 0)$. This results in the gauge group $G \times G' = [E_7 \times SU_2] \times SO_{16} \subset E_8 \times E_8$. The massless matter in the untwisted sector consists of hyper-multiplets transforming as $(\underline{56}, \underline{2}; \underline{1})$ and $(\underline{1}, \underline{1}; \underline{128})$. They only carry charges of G or G'. The untwisted sector also includes the four moduli which are gauge singlets. The massless matter in the twisted sector consists of sixteen half-hyper-multiplets, one localized at each fixed point, transforming as $(\underline{1}, \underline{2}; \underline{16})$. They carry quantum numbers under *both*, G *and* G'. This is the complete massless matter spectrum of this compactification. The rules to determine the massless states of heterotic K3 orbifold compactifications have recently been reviewed in [6]. The spectra of some of the models considered in this paper were constructed in [7]. We note that the difference of the number of hyper-multiplets (n_H) and of vector-multiplets (n_V) satisfies $n_H - n_V = 244$, as required for a consistent perturbative heterotic compactification.

We want to study this compactification within the context of the conjectured duality between the heterotic string on K3 and M-theory on K3 $\otimes (S^1/\mathbf{Z}_2) \simeq$ K3 $\otimes I$. In the latter description, the gauge fields of $G \times G'$ are confined to one of the two ten-planes at the ends of the x^{11} interval. We will denote them by 10P and 10P′, respectively. Since none of the perturbative gauge fields live in the bulk, it is therefore not a priori clear how the twisted matter fields, which are charged under G and G', can be accommodated in the M-theory picture. From the six-dimensional point of view each of the sixteen fixed points of the heterotic compactification $T^4/\mathbf{Z}_2$ is a fixed six-plane. In the M-theory picture there are sixteen seven-planes, denoted by 7P, of infinite extent in six spacetime directions and

of finite extent in the x^{11} direction, $x^{11} \in I = [0, \pi R_{11}]$. Their boundaries are six-planes which are the intersection of the 7P with the two ten-planes at $x^{11} = 0$ and $x^{11} = \pi R_{11}$. It is here where the perturbative heterotic gauge groups are located, G say at $x^{11} = 0$ and G' at $x^{11} = \pi R_{11}$. The sixteen intersection six-planes on the $E_7 \times SU_2$ side will be denoted as I6 and those on the SO_{16} side by I6'. All the I6 are of course completely equivalent, as are the I6'.

From this geometric picture it is reasonable to expect that the 7P's do play a central role in the resolution of the puzzle with the massless twisted heterotic states in the M-theory context. They are the only objects which connect both sides of the x^{11} interval.[2] On their worldvolume we have a seven-dimensional supersymmetric gauge theory.

For each $\mathbf{Z}_2$ fixed point of the orbifold there is an associated harmonic two-form, the Kähler form of the S^2 which has shrunk to zero size at the orbifold singularity. In M-theory on $T^4/\mathbf{Z}_2$ the three-form potential C of eleven-dimensional supergravity with two internal indices and one spacetime index has thus a zero mode associated with each orbifold fixed-point. In other words, there is a U_1 vector, and by supersymmetry, a complete seven-dimensional vector-multiplet for each fixed point.[3] Wrapping M2-branes around the S^2 gives rise to additional massless states in the limit of zero volume of the S^2. Taking into account two possible orientations gives SU_2 as the maximal gauge group. We now compactify further to six dimensions on $S^1/\mathbf{Z}_2$. This breaks half the supersymmetry and each seven-dimensional vector-multiplet decomposes into a six-dimensional vector-multiplet V_7 and a hyper-multiplet H_7 (the subscript reminds of their seven-dimensional origin). If the five-dimensional compact manifold is the direct product $(T^4/\mathbf{Z}_2) \otimes (S^1/\mathbf{Z}_2)$, the vector components which arise from C are projected out. This is because C is odd under $\mathbf{Z}_2$: $x^{11} \to -x^{11}$ [4,5]. In particular, the vector components must vanish on the intersection six-planes. There is no such restriction on the hyper-multiplet components.

The analysis of this model which we present in Sections 2 and 3 as well as the analysis of other models in Section 4 requires some modification of the set-up such that we retain vectors of the non-perturbative gauge group. In fact we will argue that the vector-components of the full non-perturbative SU_2 survives on I6' and the hyper component on I6. This clearly requires a departure from the direct product geometry assumed above to a 'twisted product'. We must admit that we do not know how this works in detail. We believe that the proposed field content on any of the planes involved, in particular on the 7P and the I6 and I6', is correct. We give several pieces of strong evidence. Rather than being able to specify how exactly the twist acts, we can only state the effect it has on the boundary conditions of the H_7 and V_7 fields. To understand what is meant here, recall that the presence of the ten-planes, or in other words, dividing by $\mathbf{Z}_2$, breaks half of the

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[2] Our discussion is restricted to massless states. Their masslessness is protected by their chirality. There are massive states which are charged under both E_7 and SO_{16}. Since the $\mathcal{N} = 1$ SUSY algebra in $d = 6$ has no central charge, they are not BPS and their masses are not protected against perturbative and non-perturbative corrections. One may speculate that these states originate from open M2-branes stretched between the two ten-planes [4,5]. From now on we only consider massless states.

[3] Since $H^2(\text{K3}) = 22$, there are six additional vector-multiplets which are not attached to an orbifold singularity. Upon compactification, their components arrange themselves, together with components of the metric, to the supergravity multiplet, a tensor multiplet and four moduli hyper-multiplets.

supersymmetries, 8 of the 16 supercharges which would be present in M-theory on K3 are even and 8 are odd under the $\mathbf{Z}_2$. This entails that under the 7d $\to$ 6d decomposition the vector and the hyper components of the seven-dimensional vector-multiplet have opposite (free vs. fixed) boundary conditions at each end of the interval. The choice of the boundary conditions on both sides will be crucial below. As we shall see, we have to impose Dirichlet boundary conditions for the SU_2 vector components on the I6$'$ and Neumann conditions on the I6. This is to be compared with the U_1 vectors which are not associated to orbifold fixed points (see the previous footnote). Here the boundary conditions are such that the zero mode for their hyper components are retained. The vector components are projected out by the $\mathbf{Z}_2$ twist. We want to stress once more that we are not able to derive these boundary conditions from first principles but we will present compelling arguments in favour of them.

Now that we have introduced the main ingredients of the model from the M-theory point of view, we can give a qualitative description of how they are assembled into a picture that is consistent with the heterotic description. This will involve the non-perturbative gauge groups in an essential way and we will in fact establish that the SU_2 visible in the heterotic description is the diagonal subgroup $SU_2^{\text{het}} = \text{diag}[SU_2^{\text{pert}} \times (SU_2^{\text{non-pert}})^{16}]$.

By looking at the heterotic spectrum of the model, it is clear that the charged states in the untwisted sector live on the ten-planes, ($\underline{56}, \underline{2}; \underline{1}$) on 10P and ($\underline{1}, \underline{1}; \underline{128}$) on 10P$'$. The major new ingredient in the M-theory description of the model is the presence of additional gauge groups, the non-perturbative SU_2's, one on each 7P, i.e., one for each orbifold fixed-point. This means that we have to reconsider the SU_2 charge assignments of the fields in the twisted sector. The E_7 and the SO_{16} gauge factors are unaffected by the presence of the non-perturbative SU_2's. The twisted matter fields necessarily live either on the I6 or I6$'$ intersection planes. As they are charged under the SO_{16}, which is confined to 10P$'$, they are located on the I6$''$s, one half-hyper-multiplet on each. It then also follows that we have to attribute their SU_2 quantum numbers to the non-perturbative SU_2 which lives on the 7P which is bounded by the I6$'$. The situation is illustrated in the Fig. 1.

We conclude that the ith twisted matter multiplet transforms as ($\underline{16}, \underline{2}$) of $SO_{16}^{(\text{pert})}|_{10\text{P}'} \times SU_2^{(\text{non-pert})}|_{7\text{P}_i}$, where $i = 1, \ldots, 16$. In particular it does not couple to the perturbative SU_2. For this picture to be consistent, we have to impose adequate boundary conditions for H_7 and V_7. The twisted matter multiplet can couple to the gauge field only if we impose free (Neumann) boundary condition on V_7 at the I6$'$ end of the 7P. As explained before, this implies fixed boundary conditions for H_7, which is thus invisible at I6$'$. We now have to cope with the fact that in the heterotic picture there is only one SU_2 gauge factor. This will be consistent with the M-theory description if the perturbative SU_2 which is confined to 10P mixes with the sixteen non-perturbative SU_2's such that the heterotic SU_2 is the diagonal subgroup $SU_2^{\text{het}} = \text{diag}[SU_2^{\text{pert}} \times (SU_2^{\text{non-pert}})^{16}]$. This requires that the SU_2 vector-multiplets V_7 are locked to the perturbative SU_2 on 10P. That is to say that we have to impose

$$A_\mu^{\text{non-pert}}(x^1, \ldots, x^6, x^{11} = 0) = A_\mu^{\text{pert}}(x^1, \ldots, x^6, x^7 = x^8 = x^9 = x^{10} = 0),$$
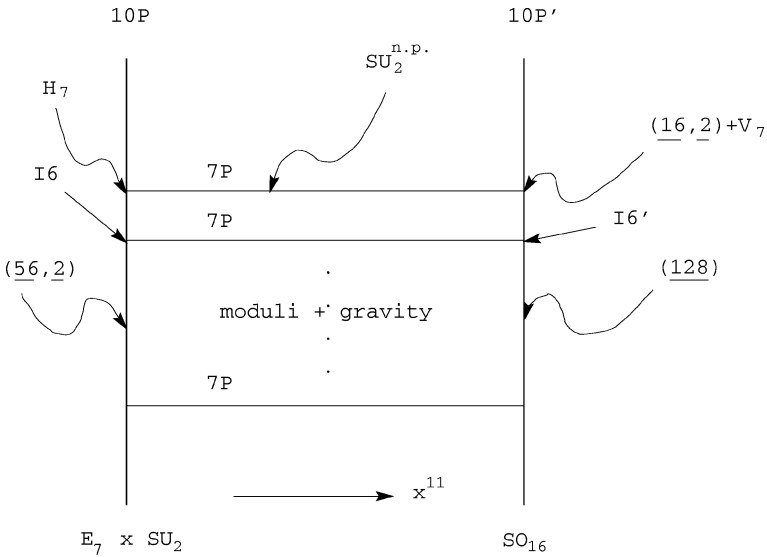$$\text{for } \mu = 1, \ldots, 6, \tag{2.1}$$

Fig. 1.

and likewise for the gauginos and at all other fixed points. Imposing fixed boundary conditions for V_7 on 10P requires free boundary conditions for the adjoint hyper-multiplet H_7. Thus the latter is visible on the I6. This is also indicated in the figure.

In the following we will substantiate this picture in two independent ways. In Section 2.2 we consider the heterotic gauge couplings and we will find that indeed the SU_2 coupling has both a perturbative contribution, which is due to $SU_2|_{10P}$ and a non-perturbative contribution which we interpret to arise from the sixteen $SU_2|_{7P}$'s. In Section 2.3 we then show that the local anomalies on all intersection six-planes cancel. In particular this includes quantum contributions from either V_7 or H_7 and also inflow contributions from the bulk and the seven-planes. This is another check on the assignment of the fields to the different planes, as summarized in the figure.

2.2. Consideration of the gauge couplings

The gauge kinetic energy of the six-dimensional low-energy effective $\mathcal{N} = 1$ SYM theory is, in string frame, up to a numerical constant [8]

$$\mathcal{L} \sim \frac{1}{\alpha'} \sum_\alpha \left(v_\alpha e^{-\phi} + \tilde{v}_\alpha \right) \operatorname{tr} F_\alpha^2. \tag{2.2}$$

Here ϕ is the unique dilaton of the perturbative heterotic string. Additional dilatons arise if we allow for additional tensor-multiplets, but this we will not do in this paper. $e^{-\phi} \sim \text{Vol}(K3)/\lambda_H^2 \alpha'^2$, where λ_H is the heterotic coupling constant. The sum is over all gauge group factors. v and $\tilde{v}$ are dimensionless constants. For perturbative gauge groups, $v = 1$ — it is in fact the level of the Kac–Moody algebra — and $\tilde{v}$ arises at one loop. For non-perturbative gauge groups, on the other hand, $v = 0$ and $\tilde{v}$ is fixed at tree level.

The coefficients v and $\tilde{v}$ are related, via supersymmetry, to the coefficients of the anomaly polynomial which must factorize to allow a Green–Schwarz mechanism to cancel the anomaly. For further explanation we refer to Appendix B. Factorizability of the anomaly polynomial imposes the constraint

$$b_\alpha = 6(v_\alpha + \tilde{v}_\alpha), \qquad (2.3)$$

where b_α is the coefficient of the one-loop beta-function of the $d = 4$ SYM theory that one obtains upon further compactification on T^2. Given the matter content of the theory we can thus compute $\tilde{v}_\alpha$.

The Hořava–Witten theory, at least when applied to compactifications on a *smooth* K3, relates $\tilde{v}$ to the net magnetic charge k on the ten-plane on which it lives. The field strength G of the three-from potential C satisfies [8]

$$dG = \frac{1}{16\pi^2}\Big\{\delta(x^{11})\big(\tfrac{1}{2}\operatorname{tr}R^2 - \operatorname{tr}F_1 \wedge F_1\big)$$
$$+ \delta(x^{11} - \pi R_{11})\big(\tfrac{1}{2}\operatorname{tr}R^2 - \operatorname{tr}F_2 \wedge F_2\big)\Big\}dx^{11} \qquad (2.4)$$

from which the magnetic charge of the ten-planes is determined as $k_{1,2} = n_{1,2} - 12$. $n_{1,2} = \frac{1}{16\pi^2}\int_{K3}\operatorname{tr}F_{1,2}^2$ are the instanton numbers on the two sides of the interval. In perturbative compactifications, i.e., in the absence of freely floating M5-branes, $n_1 + n_2 = 24$ and therefore $k_1 + k_2 = 0$. Also, integrating the anomaly polynomial of the ten-dimensional heterotic theory over a smooth K3 one derives $\tilde{v}_{1,2} = \frac{1}{2}k_{1,2}$ and thus $\tilde{v}_1 + \tilde{v}_2 = 0$. In particular, if two group factors arise from the same E_8, they should have the same $\tilde{v}$. For the model at hand, $n_1 = 8$ and $n_2 = 16$. Therefore, if the above analysis applied, we would find $\tilde{v} = +2$ for the SO_{16} factor and $\tilde{v} = -2$ for both the E_7 and the SU_2 factor. However, using (2.3) and the field content of the theory, we find

$$b(E_7) = -6 \quad \rightarrow \quad \tilde{v}(E_7) = -2,$$
$$b(SU_2) = 90 \quad \rightarrow \quad \tilde{v}(SU_2) = 14,$$
$$b(SO_{16}) = 18 \quad \rightarrow \quad \tilde{v}(SO_{16}) = 2. \qquad (2.5)$$

We thus realize that for the orbifold compactification the Hořava–Witten formulae work for the E_7 and the SO_{16} coupling, but they do not give the correct SU_2 coupling $\tilde{v}_{SU_2} = k/2 + 16 = 14$. As we have argued before, the origin of the additional contribution $+16$ has to do with the fact that the SU_2 that we actually observe is the diagonal subgroup of the perturbative SU_2 and 16 non-perturbative SU_2's on the 7P's.

In the heterotic theory the non-perturbative gauge fields do not contribute additional degrees of freedom but they do show up in the value for $\tilde{v}$ of the gauge factor with which they mix. The gauge coupling constant of SU_2^{het} is thus

$$\frac{1}{g_{\text{het}}^2} = \frac{1}{g_{10P}^2} + \sum_i \frac{1}{g_{7P}^2}, \qquad (2.6)$$

where the sum is over all those non-perturbative gauge groups which mix with the perturbative gauge group on the ten-plane. All coupling constants in Eq. (2.6) are in the

six-dimensional theory. The subscripts on the right-hand side refer to their origin. For the model considered in this section, the sum is over all sixteen non-perturbative SU_2 factors. Also, $1/g_{10P}^2 = 1/\alpha'(\text{Vol(K3)}/\lambda_H^2 \alpha'^2 + \tilde{v}_{\text{pert}})$ with $\tilde{v}_{\text{pert}} = k/2$. The seven-dimensional gauge couplings are $1/g^2|_{7P} \sim R_{11}/l_{11}^3 \sim 1/\alpha'$ where l_{11} is the eleven-dimensional Planck length. We thus find

$$\tilde{v}_{SU_2} = \frac{k_1}{2} + \#(\text{7P groups that mix}). \tag{2.7}$$

The non-perturbative contribution comes with an overall coefficient one. This coefficient was fixed using supersymmetry, which relates $\tilde{v}$ to the anomaly polynomial which gave $\tilde{v}_{SU_2} = -2 + 16$. We give further supporting evidence for this interpretation, based on the structure of the anomaly polynomial, in Appendix B.

It is now also straightforward to check that the above discussion supports the distribution of fields that we have put forward.[4] The non-perturbative gauge groups mix with the perturbative SU_2 and thus must be locked to it on each I6 intersection plane. For it not to produce extra massless degrees of freedom in the heterotic limit $R_{11} \to 0$, we must impose boundary conditions such that there are no zero modes, neither for V_7 nor for H_7. Since, as discussed above, they have opposite boundary conditions (free vs. fixed) we must impose for both, the vector-multiplet and the hyper-multiplet, Neumann conditions on one end of the x^{11} interval and 'Dirichlet' on the other.[5] In particular V_7 then has free boundary conditions on I6. The situation for H_7 is reversed.

We have already pointed out one difference between the compactification of the heterotic string on a smooth K3 and on a singular K3. In the former case we always have $\tilde{v} = k/2$ whereas this is not true in the latter case for those gauge group factors which mix with the non-perturbative gauge groups on the fixed seven-planes. Another difference between the smooth and the singular geometry is the fact that whereas in the former case the rank of the gauge group is reduced, this is not so in the latter.[6] The rank reduction in the smooth case is due to the presence of a non-trivial gauge bundle with instanton numbers $n_{1,2}$. On the orbifold a $\mathbf{Z}_n$ singularity can support a singular $\mathbf{Z}_n$ instanton with instanton number $1/n$, which breaks the gauge group without reducing its rank. E.g., for $\mathbf{Z}_2$, the gauge group is reduced to $[E_7 \times SU_2]/\mathbf{Z}_2$ or $\text{Spin}(16)/\mathbf{Z}_2$, depending on the choice of the $\mathbf{Z}_2 \subset E_8$ [9]. For the model considered in this section the instanton number on the $E_7 \times SU_2$ side is eight, i.e., there is one instanton of instanton number $1/2$ at each fixed point. On the SO_{16} side, we have instead one instanton of instanton number one at each fixed point.

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[4] We should however stress that we do not have a truly M-theoretic derivation of these results.

[5] Note that the locking condition (2.1) is not exactly Dirichlet, but has the same effect on mode counting for the seven-dimensional fields. Both, hyper and vector components, have half-integer modes only, i.e., no zero-modes.

[6] This presumes a $\mathbf{Z}_N$ symmetric torus of generic size and no $E_8 \times E_8$ Wilson lines on T^4. In the completely generic case of asymmetric orbifolds of a Narain $\Gamma_{4,20}$ compactification of the heterotic string the resulting gauge group can have rank as high as 20 or can be lower than 16.

3. Considerations of local anomaly cancellation

One important consistency condition that the low-energy effective field theory has to satisfy is anomaly freedom. This requirement is particularly powerful in six dimensions since in addition to pure gauge anomalies there are potential gravitational and mixed anomalies [10]. They all have to cancel. It is straightforward to verify that the anomaly of the six-dimensional field theory factorizes and the presence of one antisymmetric tensor in the massless spectrum guarantees that this anomaly can be cancelled via the Green–Schwarz mechanism.

In the M-theory description of the heterotic orbifold we have allocated all massless fields (perturbative and non-perturbative) to the bulk (gravity and moduli) and the various types of planes which are present. We can now consider the field theory on any one of these planes and since anomalies are a UV phenomenon, we need to require that they cancel locally, i.e., on any plane separately. In the bulk and on the seven-planes this is automatic, they are odd-dimensional. On each of the two ten-planes, away from the intersection six-planes, there are 16 supercharges and an entire E_8 gauge group. Anomaly cancellation works in exactly the same way as in the Hořava–Witten theory. The situation on the intersection six-planes, however, involves new features: here supersymmetry is broken further to eight super-charges and the gauge group is broken to a subgroup. The issue of anomaly cancellation on the six-planes has to be addressed and in fact we will find that it provides a non-trivial check on the scenario advocated in Section 2.

We now turn to the evaluation of the anomaly on the intersection six-planes. It gets contributions from two sources. (i) Quantum contributions: they arise from the massless states which are charged under the gauge group operating at the particular I6 or I6$'$ fixed plane we are considering. Fields residing in the bulk, on the ten-plane into which the six-plane is embedded, on the seven-plane which is bounded by the six-plane and the fields confined to the six-plane do contribute. (ii) Inflow and intersection contributions: they arise from gauge variance of the 11d SUGRA action. There is a contribution from a modified Bianchi identity and contributions arising from Chern–Simons (CS) terms. We first discuss the quantum contributions. Some basic results which we will be using are collected in Appendix A.

(1) Bulk fields: gravity multiplet, selfdual tensor multiplet and four moduli hypermultiplets. In this work we analyze only compactifications with one tensor multiplet in the bulk, thus restricting the discussion to the perturbative heterotic string. Recall that in the Hořava–Witten theory the gravitational anomaly is distributed evenly over the two end-of-the-world ten-planes. By the same logic we distribute the contribution of the bulk fields between all 2×16 fixed I6 and I6$'$ planes and obtain as their contribution to the anomaly on each intersection six-plane [7]

$$\mathcal{A}(\text{bulk}) = \frac{1}{2 \cdot 16} \left[-\frac{244}{240} \operatorname{tr} R^4 + \frac{44}{192} (\operatorname{tr} R^2)^2 \right] + \frac{4}{2 \cdot 16} \left[\frac{1}{240} \operatorname{tr} R^4 + \frac{1}{192} (\operatorname{tr} R^2)^2 \right]$$

$$= -\frac{1}{32} \operatorname{tr} R^4 + \frac{1}{128} (\operatorname{tr} R^2)^2. \tag{3.1}$$

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[7] The correctly normalized anomaly polynomial in $d = 6$ is $(-i/192\pi^3)\mathcal{A}$.

The first term in the first line is the contribution from the gravity multiplet and the tensor multiplet whereas the second term is the contribution from the four moduli fields.

(2) Ten-plane fields: these are vector and hyper multiplets from the untwisted sector. We have to distinguish between I6 and I6′ planes, as they are embedded in 10P and 10P′, respectively, which carry different perturbative gauge groups and untwisted matter, charged under G (for I6) and G' (for I6′). $\mathcal{N} = 1$ supersymmetry demands that the fermions in a vector-multiplet have opposite chirality than the fermions in a hyper-multiplet and consequently they contribute to the anomaly with opposite sign. Thus generically

$$
\mathcal{A} = (n_H - n_V)\left(\frac{1}{240} \operatorname{tr} R^4 + \frac{1}{192}\left(\operatorname{tr} R^2\right)^2\right)
$$
$$
- \frac{1}{4} \operatorname{tr} R^2 \left(\operatorname{Tr}_H F^2 - \operatorname{Tr}_V F^2\right) + \left(\operatorname{Tr}_H F^4 - \operatorname{Tr}_V F^4\right), \tag{3.2}
$$

where n_H and n_V is the number of hyper-multiplets and vector-multiplets, respectively. Distributing the anomaly of the untwisted fields charged under $E_7 \times SU_2$ evenly over the 16 I6 planes, we have $H = \frac{1}{16}(\underline{56}, \underline{2})$ and $V = \frac{1}{16}[(\underline{133}, \underline{1}) + (\underline{1}, \underline{3})]$. Taking into account the gauge quantum numbers we arrive, after a little algebra at [8]

$$
\mathcal{A}(10P) = -\frac{1}{160} \operatorname{tr} R^4 - \frac{1}{128}\left(\operatorname{tr} R^2\right)^2 + \operatorname{tr} R^2 \left(\frac{3}{32} \operatorname{tr} F_{E_7}^2 - \frac{13}{32} \operatorname{tr} F_{SU_2}^2\right)
$$
$$
- \frac{3}{16}\left(\operatorname{tr} F_{E_7}^2\right)^2 + \frac{5}{16}\left(\operatorname{tr} F_{SU_2}^2\right)^2 + \frac{9}{8} \operatorname{tr} F_{SU_2}^2 \operatorname{tr} F_{E_7}^2. \tag{3.3}
$$

Similarly, at each I6′ plane with $G' = SO_{16}$ we have $H = \frac{1}{16}(\underline{128})$ and $V = \frac{1}{16}(\underline{120})$, leading to

$$
\mathcal{A}(10P') = \frac{1}{480} \operatorname{tr} R^4 + \frac{1}{384}\left(\operatorname{tr} R^2\right)^2 - \frac{1}{32} \operatorname{tr} R^2 \cdot \operatorname{tr} F_{SO_{16}}^2
$$
$$
+ \frac{3}{16}\left(\operatorname{tr} F_{SO_{16}}^2\right)^2 - \operatorname{tr} F_{SO_{16}}^4. \tag{3.4}
$$

(3) Seven-plane fields: each of the sixteen 7P's connects (across the x^{11} direction) an I6 plane to an I6′ plane. As explained in the previous section, each 7P carries an $SU_2^{\text{non-pert}}$ with a $d = 7$ vector-multiplet whose reduction to $d = 6$ gives a vector-multiplet and an adjoint hyper-multiplet. With the assignment of boundary conditions as specified before, the hyper-multiplet component contributes to the anomaly on I6 whereas the vector-multiplet component contributes on I6′. There is however one subtlety concerning the precise contribution. A priori the contribution from both the vector and the hyper components is distributed evenly over the two bounding six-planes. The local boundary conditions however determine whether they do in fact give rise to an anomaly. This is only the case if the fields satisfy free boundary conditions. This is a manifestation of the local consistency assumption: what happens at a given boundary is sensitive only to the boundary conditions imposed there and is blind to what happens at another, distant boundary. In other words, the fact that each multiplet, due to the chosen boundary conditions, contributes to

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[8] See Appendix C for the group theory involved in this derivation.

the anomaly only on one of the two ends does not affect the amount by which it contributes. In summary, the contribution to the local anomaly from the 7P fields is *half* of that of a SU_2 vector-multiplet on I6$'$ and *half* of that of an an SU_2 adjoint hyper-multiplet on I6. We then find

$$\mathcal{A}(7P) = \frac{1}{160} \operatorname{tr} R^4 + \frac{1}{128} \left(\operatorname{tr} R^2 \right)^2 - \frac{1}{4} \operatorname{tr} R^2 \cdot \operatorname{tr} F_{SU_2}^2 + \left(\operatorname{tr} F_{SU_2}^2 \right)^2$$

on the I6 and

$$\mathcal{A}(7P) = -\frac{1}{160} \operatorname{tr} R^4 - \frac{1}{128} \left(\operatorname{tr} R^2 \right)^2 + \frac{1}{4} \operatorname{tr} R^2 \cdot \operatorname{tr} F_{SU_2}^2 - \left(\operatorname{tr} F_{SU_2} \right)^2 \tag{3.5}$$

on the I6$'$.

(4) Six-plane fields: these are the massless fields which are entirely confined to the I6 (I6$'$) planes. In the models which we investigate there are no such tensor or vector states and the only contribution arises from hyper-multiplets which appear in the twisted sectors of the heterotic theory. For the $\mathbf{Z}_2$ model considered here the twisted sector contains sixteen half-hyper-multiplets, one localized at each fixed point and transforming as $(\underline{1}, \underline{2}; \underline{16})$. It should be assigned to the I6$'$ side since it is charged under the gauge groups residing here, namely $(SU_2 \times SO_{16})$. There are no six-plane fields on I6. Note that the SU_2 quantum numbers should be understood as those pertaining to the diagonal SU_2 group. We thus find [9]

$$\mathcal{A}(6P) = 0,$$
$$\mathcal{A}(6P') = \frac{1}{15} \operatorname{tr} R^4 + \frac{1}{12} \left(\operatorname{tr} R^2 \right)^2 - \operatorname{tr} R^2 \left(\operatorname{tr} F_{SU_2}^2 - \tfrac{1}{4} \operatorname{tr} F_{SO_{16}}^2 \right)$$
$$+ \operatorname{tr} F_{SO_{16}}^4 + \left(\operatorname{tr} F_{SU_2} \right)^2 + \frac{3}{2} \operatorname{tr} F_{SU_2}^2 \cdot \operatorname{tr} F_{SO_{16}}^2. \tag{3.6}$$

This completes the enumeration of the quantum anomalies.

In addition to the quantum anomalies there are additional contributions which have their origin in the gauge-variance of the classical low-energy M-theory effective action [11]. More precisely, these 'inflow' contributions arise from the CS couplings $C \wedge G \wedge G$ and $C \wedge (\operatorname{tr} R^4 - \frac{1}{4} (\operatorname{tr} R^2)^2)$ in the eleven-dimensional bulk action. Here C is the three-form potential and G its field strength with (magnetic) sources on the ten-planes and the six-planes. This produces an anomaly in the gauge theory on the six-planes [11]

$$\mathcal{A}(\text{inflow}) = -g \left(\frac{1}{8} \operatorname{tr} R^4 - \frac{1}{32} \left(\operatorname{tr} R^2 \right)^2 \right) - \frac{3g}{4} \left(\operatorname{tr} F_{10}^2 - \frac{1}{2} \operatorname{tr} R^2 \right)^2. \tag{3.7}$$

F_{10} are the gauge fields on the ten planes only, i.e., G or G' and g is the magnetic charge of the six-plane. To the best of our knowledge the coefficients of these terms have not been reliably determined. We have fixed the normalizations such as to make the anomaly cancellation work for this model. Once fixed we use them for all other models which we

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[9] If there were vector-multiplets confined to the six-planes, they would contribute in an obvious way. If there were n_{T_6} tensor multiplets, there would be an additional contribution of $n_{T_6} \left(\frac{29}{240} \operatorname{tr} R^4 - \frac{7}{192} (\operatorname{tr} R^2)^2 \right)$.

consider.[10] Below we will give independent arguments in support of the normalizations in (3.7). The magnetic charges are easy to determine. We have found in Section 2.2 that the magnetic charge of the ten-planes is $k_{1,2}$. In the orbifold limit the geometric and the gauge curvatures are restricted to the orbifold singularities. Hence this is also where the magnetic charge is sitting. We thus get the following sum rule for the magnetic charges of the six-planes

$$\sum_i g_{i\alpha} = k_\alpha. \tag{3.8}$$

The sum extends over all six-planes on a given side of the x^{11} interval and $\alpha = 1, 2$. For the problem at hand $k_{1,2} = \pm 4$ and since all I6 are related by symmetry (and likewise the I6') we find $g_{I6} = 1/16 \cdot (-4) = -1/4$ and $g_{I6'} = 1/16 \cdot 4 = +1/4$. We thus obtain for the inflow contributions to the anomalies on the six-planes:

$$\mathcal{A}(\text{inflow}) = \frac{1}{32} \operatorname{tr} R^4 - \frac{1}{128} \left(\operatorname{tr} R^2 \right)^2 + \frac{3}{16} \left(\operatorname{tr} F_{E_7}^2 + \operatorname{tr} F_{SU_2}^2 - \frac{1}{2} \operatorname{tr} R^2 \right)^2$$

on each I6 plane and

$$\mathcal{A}(\text{inflow}) = -\frac{1}{32} \operatorname{tr} R^4 + \frac{1}{128} \left(\operatorname{tr} R^2 \right)^2 - \frac{3}{16} \left(\operatorname{tr} F_{SO_{16}^2} - \frac{1}{2} \operatorname{tr} R^2 \right)^2 \tag{3.9}$$

on each I6' plane. Adding the quantum and the inflow anomalies, we obtain

$$\mathcal{A}(\text{quantum} + \text{inflow}) = \frac{3}{2} \left(\operatorname{tr} F_{SU_2}^2 - \frac{1}{16} \operatorname{tr} R^2 \right) \cdot \left(\operatorname{tr} F_{E_7}^2 + \operatorname{tr} F_{SU_2}^2 - \frac{1}{2} \operatorname{tr} R^2 \right)$$

and

$$\mathcal{A}(\text{quantum} + \text{inflow}) = \frac{3}{2} \left(\operatorname{tr} F_{SU_2}^2 - \frac{1}{16} \operatorname{tr} R^2 \right) \cdot \left(\operatorname{tr} F_{SO_{16}}^2 - \frac{1}{2} \operatorname{tr} R^2 \right). \tag{3.10}$$

Note that these expressions factorize into two terms, the first of a seven-plane origin and the second of a ten-plane origin.

Consistency of the theory requires, in the absence of tensor multiplets on the I6 and I6' planes, that the anomaly vanishes identically.[11] Thus, were it only for these contributions to the local anomaly, the theory would be inconsistent. The factorization pattern of (3.10) does suggest yet another contribution to the anomaly. Indeed, we still have to take into account the intersection anomaly [11]. It arises from the electric coupling of the 7P to C. This gives rise to a CS term on each 7P worldvolume of the form $C \wedge Y_4$ which leads to a contribution to the anomaly on the intersection six-planes

$$\mathcal{A}(\text{intersection}) = \left(\operatorname{tr} F_{10}^2 - \frac{1}{2} \operatorname{tr} R^2 \right) \times Y_4 \tag{3.11}$$

with

---

[10] We would like to point out that our relative normalization differs by a crucial factor three from that of Ref. [11].

[11] In the presence of extra tensor multiplets one could invoke the GS mechanism provided the anomaly factorizes appropriately.

$$Y_4 = \frac{3}{2}\left(\eta \, \mathrm{tr}\, R^2 - \rho \, \mathrm{tr}\, F_7^2\right). \tag{3.12}$$

Here F_7 refers to the 7P gauge fields and F_{10} to the ten-plane gauge fields. η and ρ are as yet free parameters. They should follow from seven-dimensional physics and should in particular not depend on the details of the boundary conditions imposed at the ends of the interval. We will come back to this issue in Section 4. However, comparing Eqs. (3.10) and (3.11) requires that for the $\mathbf{Z}_2$ orbifold model considered here,

$$\eta = \frac{1}{16} \quad \text{and} \quad \rho = 1. \tag{3.13}$$

So far we have presented a consistent scenario for the M-theory description of the heterotic $T^4/\mathbf{Z}_2$ orbifold with gauge group $[E_7 \times SU_2] \times SO_{16}$. In the remaining sections we will generalize the analysis to other $T^4/\mathbf{Z}_N$ orbifolds.

4. Examples of $\mathbf{Z}_3$, $\mathbf{Z}_4$ and $\mathbf{Z}_6$ orbifolds

In this section we generalize the discussion of the two preceding sections to orbifolds with higher order fixed points. In Section 4.1 we recall some basic facts about $T^4/\mathbf{Z}_N$ orbifolds. We then discuss one model for $N = 3, 4, 6$ each. We will stress the new issues which arise for non-prime orbifolds, i.e., for $N = 4, 6$. We first analyse the gauge couplings (Section 4.2) and then turn to the analysis of local anomalies (Section 4.3) of these models. In Section 4.4 we 'derive' the values for the parameters ρ and η in (3.12). We give two independent arguments. The first is based on a comparison between the heterotic anomaly and the M-theory anomaly. The second argument does not rely on anomaly considerations but uses the duality between M-theory on K3 and heterotic theory on T^3. To help the reader through the discussion of the various models, we have collected the data concerning their spectra, gauge couplings and magnetic charges in three tables. They can be found in Section 4.2.

4.1. Some facts about $T^4/\mathbf{Z}_N$ orbifolds

To ensure $d = 6$, $\mathcal{N} = 1$ supersymmetry (eight unbroken supercharges) in $T^4/\mathbf{Z}_N$ compactifications of the heterotic string, N has to be restricted to $N \in \{2, 3, 4, 6\}$. Whereas for the $\mathbf{Z}_2$ orbifold the number of moduli is four, for the remaining cases there are only two moduli. This follows by considering which of the $(1, 1)$ forms on the torus are invariant under the $\mathbf{Z}_N$ twist. Recall that if T^4 is parametrized by two complex coordinates z^1 and z^2, the discrete $\mathbf{Z}_N$ transformation acts as $z^1 \to \alpha z^1$, $z^2 \to \alpha^{N-1} z^2$, where $\alpha^N = 1$. The action is such that $dz^1 \wedge dz^2$ survives, since this is the $(2, 0)$ form which we need for a K3 compactification or the orbifold limits thereof. Its presence also guarantees at least eight unbroken supercharges. There are various ways to embed the geometrical twists as shifts in the gauge lattice, leading to different spectra. In the absence of Wilson lines there are 2 (5, 12, 59) different embeddings for $\mathbf{Z}_2$ ($\mathbf{Z}_3$, $\mathbf{Z}_4$, $\mathbf{Z}_6$) with different gauge groups $G \times G' \subset E_8 \times E_8$ and matter content [12,13]. The number of fixed points is given by the Lefshetz fixed-point formula as

$$(1 - \alpha)^2 (1 - \alpha^*)^2 = \begin{cases} 4^2, & \text{for } \mathbf{Z}_2, \\ 3^2, & \text{for } \mathbf{Z}_3, \\ 2^2, & \text{for } \mathbf{Z}_4, \\ 1, & \text{for } \mathbf{Z}_6. \end{cases} \tag{4.1}$$

For the other possible lattice automorphisms, $N \in \{5, 8, 10, 12\}$, the fixed point formula (4.1) does not give an integer. The precise fixed-point structure for N non-prime will be discussed when we consider the $\mathbf{Z}_4$ and $\mathbf{Z}_6$ examples below.

Much of the M-theory discussion from Section 2 carries over verbatim to the general case. The fixed seven-planes are again denoted by 7P and they carry a non-perturbative gauge group, G_7, which for a $\mathbf{Z}_N$ fixed point is (at most) SU_N. The intersection six-planes are again I6 (at $x^{11} = 0$) and I6' (at $x^{11} = \pi R_{11}$). There is one I6, I6' pair for each orbifold fixed point. For non-prime orbifolds there are fixed points of different orders. This leads to new features which we will discuss in detail below. But first we present another prime orbifold.

4.2. Analysis of the gauge couplings

$\mathbf{Z}_3$ orbifold

The discussion here follows very closely the discussion of the $\mathbf{Z}_2$ orbifold in Section 2. We will thus be brief. Since this is a prime orbifold, there is only one type of fixed points, namely nine $\mathbf{Z}_3$ fixed points. Consequently there are nine seven-planes. They are bounded by nine I6 planes and nine I6' planes. The I6 are related by symmetry and are thus completely equivalent and likewise for the I6'. Two of the four moduli of the T^4 are invariant under the $\mathbf{Z}_3$ twist. If we choose $\delta = \frac{1}{3}(-2, 1, 1, 0, 0, 0, 0, 0; \frac{5}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ as shift vector, the gauge group is $G \times G' = (E_6 \times SU_3) \times SU_9$. In addition to the states in the corresponding vector-multiplets, there are twist-invariant massless states which comprise hyper-multiplets transforming as $(\underline{27}, \underline{3}; \underline{1})$ and $(\underline{1}, \underline{1}; \underline{84})$. As for the massless twisted matter there is one hyper-multiplet $(\underline{1}, \underline{3}; \underline{9})$ located at each of the nine fixed points. This part of the spectrum and other data pertaining to this model are summarized in Table 1.

Given the massless spectrum, it is straightforward to compute the one-loop beta-function coefficients for each group factor and from this, via (2.3), the $\tilde{v}_\alpha$, cf. (2.2). We find

$$b(E_6) = -3 \quad \rightarrow \quad \tilde{v}(E_6) = -\frac{3}{2},$$

$$b(SU_3) = 51 \quad \rightarrow \quad \tilde{v}(SU_3) = -\frac{3}{2} + 9,$$

$$b(SU_9) = 15 \quad \rightarrow \quad \tilde{v}(SU_9) = +\frac{3}{2}. \tag{4.2}$$

We thus learn that $\tilde{v}_{E_7} = k_1/2$ and $\tilde{v}_{SU_9} = k_2/2$ with $k_1 = -3$ and $k_2 = +3$, but $\tilde{v}_{SU_3} = k_1/2 + 9$. In analogy with the $\mathbf{Z}_2$ example we conclude that the heterotic SU_3 gauge factor is a linear combination of the perturbative SU_3 and the nine non-perturbative SU_3's on the seven-planes; i.e., $(SU_3)_{\text{het}} = \text{diag}[SU_3^{\text{pert}} \times (SU_3^{\text{non-pert}})^9]$. Since the twisted hyper-multiplets are charged under SU_9, they must be located on the I6' planes and they carry

Table 1

The $\mathbf{Z}_3$ orbifold with gauge group $(E_6 \times SU_3) \times SU_9$. The magnetic charges will be computed in Section 4.3, where also the notation Q_{10} etc. will be explained

		Example 2: $\mathbf{Z}_3$ orbifold	
Shift vector		$\left(-\frac{2}{3}, \frac{1}{3}, \frac{1}{3}, 0, 0, 0, 0, 0\right)$	$\left(\frac{5}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}\right)$
Gauge group		$E_6 \times SU_3$	SU_9
Matter H_0		Two moduli	
		$(\underline{27}, \underline{3}; \underline{1})$	$(\underline{1}, \underline{1}; \underline{84})$
H_1		$9 \times (\underline{1}, \underline{3}; \underline{9})$	
$b(G)$		$b(E_6) = -3$, $b(SU_3) = 51$	$b(SU_9) = 15$
$\tilde{v}_G$		$\tilde{v}_{E_6} = -\frac{3}{2}$, $\tilde{v}_{SU_3} = -\frac{3}{2} + 9$	$\tilde{v}_{SU_9} = +\frac{3}{2}$
$k = n - 12$		-3	$+3$
g_{I6}		$-\frac{1}{3}$	$+\frac{1}{3}$
Q_{10}		$\frac{1}{9}\{(\underline{27}, \underline{3}) - (\underline{78}, \underline{1}) - (\underline{1}, \underline{8})\}$	$\frac{1}{9}\{\cdot(\underline{84}) - (\underline{80})\}$
Q_6		$-$	$(\underline{1}, \underline{3}; \underline{9})$
G_7		SU_3	
Q_7		$\frac{1}{2} \cdot \underline{8}$	$-\frac{1}{2} \cdot \underline{8}$

quantum numbers of a non-perturbative SU_3. The boundary conditions of the hyper and the vector components under the 7d $\rightarrow$ 6d decomposition must be chosen such that the vector component enjoys Dirichlet boundary conditions of the type (2.1) on the $E_6 \times SU_3$ side and Neumann conditions on the SU_9 side; i.e., there is one SU_3 adjoint hyper-multiplet on each I6 and one vector-multiplet on each I6′. This allocation of fields leads to local anomaly cancellation on the six-planes as we will demonstrate in Section 4.3.

$\mathbf{Z}_4$ orbifold

This is our first example of a non-prime orbifold. It possesses two types of fixed points: four $\mathbf{Z}_4$ fixed points and 16 $\mathbf{Z}_2 \subset \mathbf{Z}_4$ fixed-points. The latter are obviously also fixed under $\mathbf{Z}_2$ and the twelve remaining $\mathbf{Z}_2$ fixed points lie pair-wise on $\mathbf{Z}_4$ orbits. There are thus four $\mathbf{Z}_4$ and six $\mathbf{Z}_2$ fixed points. All fixed points of a given type are completely equivalent. The two type we have to treat separately, though. The results of the ensuing below of a particular $\mathbf{Z}_4$ orbifold model are collected in Table 2.

We embed the $\mathbf{Z}_4$ twist into the gauge sector via the shift vector $\delta = \frac{1}{4}(-3, 1, 1, 1, 0, 0, 0, 0; -\frac{7}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$, which leads to a breaking of the gauge group $E_8 \times E_8 \rightarrow (SO_{10} \times SO_6) \times (SU_8 \times SU_2) = G \times G'$. The massless charged matter in the un-twisted sector consists of one hyper-multiplet transforming as $(\underline{16}, \underline{4}; \underline{1}, \underline{1})$ and one hyper-multiplet transforming as $(\underline{1}, \underline{1}; \underline{28}, \underline{2})$. These states, which we denote by H_0, clearly live on 10P′.

Table 2
The $\mathbf{Z}_4$ orbifold with gauge group $(SO_{10} \times SU_4) \times (SU_8 \times SU_2)$

		Example 3: $\mathbf{Z}_4$	
Shift vector		$\left(-\frac{3}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, 0, 0, 0, 0\right)$	$\left(-\frac{7}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8}\right)$
Gauge group		$SO_{10} \times SU_4$	$SU_8 \times SU_2$
Matter	H_0 $\Big\{$	Two moduli	
		$(\underline{16}, \underline{4}; \underline{1}, \underline{1})$	$(\underline{1}, \underline{1}; \underline{28}, \underline{2})$
	H_1	$4 \times (\underline{1}, \underline{4}; \underline{8}, \underline{1})$	
	H_2	$10 \times \frac{1}{2}(\underline{1}, \underline{6}; \underline{1}, \underline{2}) + 6 \times \frac{1}{2}(\underline{10}, \underline{1}; \underline{1}, \underline{2})$	
$b(G)$		$b(SO_{10}) = 6, b(SU_4) = 30$	$b(SU_8) = 6, b(SU_2) = 42$
$\tilde{v}_G$		$\tilde{v}_{SO_{10}} = 0, \tilde{v}_{SU_4} = 0 + 4$	$\tilde{v}_{SU_8} = 0, \tilde{v}_{SU_2} = 0 + 6$
$k = n - 12$		0	0
		$\mathbf{Z}_2$ fixed points: see Sections 2 and 3 $\mathbf{Z}_4$ fixed points:	
g_{16}		$-\frac{3}{8}$	$+\frac{3}{8}$
Q_{10}		$-\frac{5}{32}\{(\underline{45}, \underline{1}) + (\underline{1}, \underline{15})\}$ $+\frac{3}{32}(\underline{10}, \underline{6}) + \frac{1}{16}(\underline{16}, \underline{4})$	$-\frac{5}{32}\{(\underline{63}, \underline{1}) + (\underline{1}, \underline{3})\}$ $+\frac{3}{32}(\underline{70}, \underline{1}) + \frac{1}{16}(\underline{28}, \underline{2})$
Q_6		$-$	$(\underline{1}, \underline{4}; \underline{8}, \underline{1}) + \frac{1}{2}(\underline{1}, \underline{6}; \underline{1}, \underline{2})$
G_7		SU_4	
Q_7		$\frac{1}{2} \cdot \underline{15}$	$-\frac{1}{2} \cdot \underline{15}$

Since for non-prime orbifolds there are different types of fixed-points, one has to be careful with the allocation of the twisted matter. The $\mathbf{Z}_4$ orbifold has three types of twisted sectors. States in the first and the third twisted sectors combine into particle and anti-particle pairs and into complete hyper-multiplets, which we denote as H_1. These states are necessarily located at the $\mathbf{Z}_4$ fixed points. Straightforward application of the rules reviewed in [6] gives $H_1 = 4 \times (\underline{1}, \underline{4}; \underline{8}, \underline{1})$. The double-twisted sector contains both particles and their anti-particles. These states we call H_2. One finds $H_2 = 10 \times \frac{1}{2}(\underline{1}, \underline{6}; \underline{1}, \underline{2}) + 6 \times \frac{1}{2}(\underline{10}, \underline{1}; \underline{1}, \underline{2})$. The correct assignment of these states to the various fixed points is as follows. At each $\mathbf{Z}_2$ fixed point: $\frac{1}{2}(\underline{1}, \underline{6}; \underline{1}, \underline{2}) + \frac{1}{2}(\underline{10}, \underline{1}; \underline{1}, \underline{2})$ and at each $\mathbf{Z}_4$ fixed point: $\frac{1}{2}(\underline{1}, \underline{6}; \underline{1}, \underline{2})$.

Note that the states at each of the $\mathbf{Z}_2$ fixed points combine into representations of $SO_{16} \times (E_7 \times SU_2)$ which is the group left unbroken by the shift 2δ: $\frac{1}{2}[(\underline{1}, \underline{6}; \underline{1}, \underline{2}) + (\underline{10}, \underline{1}; \underline{1}, \underline{2})]_{(SO_{10} \times SO_6) \times (SU_8 \times SU_2)} = \frac{1}{2}(\underline{16}; \underline{1}, \underline{2})_{SO_{16} \times (E_7 \times SU_2)}$. Also, locally at the $\mathbf{Z}_2$ fixed points, there are four moduli hyper-multiplets. In fact, the local physics at each of the six $\mathbf{Z}_2$ fixed points of this $\mathbf{Z}_4$ orbifold is identical to that encountered in the $\mathbf{Z}_2$ model

discussed before. Since there are six $\mathbf{Z}_2$ seven-planes, we expect $\tilde{v}_{SU_2} = k_2/2 + 6$. We will verify this shortly.

It remains to discuss the four $\mathbf{Z}_4$ fixed points with twisted massless matter $(\underline{1}, \underline{4}; \underline{8}, \underline{1})$ $+ \frac{1}{2}(\underline{1}, \underline{6}; \underline{1}, \underline{2})$. Each of the $\mathbf{Z}_4$ seven-planes supports a non-perturbative $SU_4 \sim SO_6$ gauge group. This, together with the gauge quantum numbers of the twisted states, suggests that the twisted matter lives on the I6$'$ planes and transforms as $\underline{4}$ and $\underline{6}$ under a non-perturbative SU_4, respectively. Also, in complete analogy to the previous examples, this would imply $\tilde{v}_{SU_4} = (k_1/2) + 4$ and thus $SU_4^{\text{het}} = \text{diag}[SU_4^{\text{pert}} \times (SU_4^{\text{non-pert}})^4]$. That this is correct can be easily verified, given that

$$
\begin{aligned}
b(SO_{10}) &= 6 &\rightarrow& \quad \tilde{v}(SO_{10}) = 0, \\
b(SO_6) &= 30 &\rightarrow& \quad \tilde{v}(SO_6) = 0 + 4, \\
b(SU_8) &= 6 &\rightarrow& \quad \tilde{v}(SU_8) = 0, \\
b(SU_2) &= 42 &\rightarrow& \quad \tilde{v}(SU_2) = 0 + 6.
\end{aligned}
\tag{4.3}
$$

We also find $k_1 = k_2 = 0$, i.e., the 24 instantons are distributed evenly over the two sides. It is now also clear how to distribute the seven-plane fields. A SU_4 adjoint hyper-multiplet lives on each of the $\mathbf{Z}_4$ I6 planes and a SU_4 vector-multiplet on each $\mathbf{Z}_4$ I6$'$ plane.

$\mathbf{Z}_6$ orbifold

As in the previous examples, the data of the particular model we will discuss in this section are collected in a Table 3 which can be found at the end of this subsection.

As $\mathbf{Z}_6$ has two non-trivial subgroups, $\mathbf{Z}_2$ and $\mathbf{Z}_3$, a $\mathbf{Z}_6$ orbifold has fixed points of orders 2, 3 and 6. The Lefshetz fixed point theorem gives one $\mathbf{Z}_6$ fixed point which is of course also fixed under the $\mathbf{Z}_2$ and $\mathbf{Z}_3$ subgroups. The remaining eight $\mathbf{Z}_3$ fixed points lie on four $\mathbf{Z}_6$ orbits. The 15 $\mathbf{Z}_2$ fixed points not fixed under $\mathbf{Z}_6$ lie on 5 $\mathbf{Z}_6$ orbits. A $\mathbf{Z}_6$ orbifold thus has one $\mathbf{Z}_6$, four $\mathbf{Z}_3$ and five $\mathbf{Z}_2$ fixed points. Of the four moduli of T^4, two are invariant under the $\mathbf{Z}_6$ twist, i.e., we have two moduli hyper-multiplets.

To proceed, we need to specify the shift vector. Our choice $\delta = \frac{1}{6}(-5, 1, 1, 1, 1, 1, 0, 0; -5, 1, 1, 1, 1, 1, 1, 1)$ leads to the gauge group $G \times G' = (SU_6 \times SU_3 \times SU_2) \times SU_9$. Locally at the $\mathbf{Z}_3$ fixed points the gauge group is that corresponding to the shift vector 2δ. One finds $(E_6 \times SU_3) \times SU_9$ and thus recovers the situation of the $\mathbf{Z}_3$ orbifold model discussed above. At the $\mathbf{Z}_2$ fixed points the shift 3δ leads to the gauge group $(E_7 \times SU_2) \times E_8$, i.e., the $\mathbf{Z}_2 \subset \mathbf{Z}_6$ subgroup leaves the second E_8 unbroken. We will discuss this $\mathbf{Z}_2$ orbifold in some detail in Section 5. For the present purposes it suffices to state the following facts. The untwisted matter for this $\mathbf{Z}_2$ model is $H_0 = (\underline{56}, \underline{2}; \underline{1})$ and the twisted matter is $H_1 = 16 \times \{\frac{1}{2}(\underline{56}, \underline{1}; \underline{1}) + 2(\underline{1}, \underline{2}; \underline{1})\}$. Since the second E_8 is unbroken, all 24 instantons must sit in the first E_8, i.e., $k = 12$. Finally, the SU_2 gauge-factor is purely perturbative, i.e., it does not mix with the seven-plane gauge group.

The untwisted massless matter of the $\mathbf{Z}_6$ orbifold consists of a single hyper-multiplet $H_0 = (\underline{6}, \underline{3}, \underline{2}; \underline{1})$ which lives on 10P. The twisted matter states are $H_1 = (\underline{6}, \underline{1}, \underline{1}; \underline{\bar{9}})$: this hyper-multiplet is necessarily located at the $\mathbf{Z}_6$ fixed point; $H_2 = 4 \times (\underline{1}, \underline{3}, \underline{1}; \underline{9})$: there is one such hyper-multiplet at each of the four $\mathbf{Z}_4$ fixed points; the states in the third-twisted

Table 3
The $\mathbf{Z}_6$ orbifold with gauge group $(SU_6 \times SU_3 \times SU_2) \times SU_9$

	Example 4: $\mathbf{Z}_6$	
Shift vector	$\left(-\frac{5}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, 0, 0\right)$	$\left(-\frac{5}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}\right)$
Gauge group	$SU_6 \times SU_3 \times SU_2$	SU_9
Matter H_0 $\{$	Two moduli $(\underline{6}, \underline{3}, \underline{2}; \underline{1})$	
H_1	$(\underline{6}, \underline{1}, \underline{1}; \underline{\bar{9}})$	
H_2	$4(\underline{1}, \underline{3}, \underline{1}; \underline{9})$	
H_3	$6 \times \frac{1}{2}(\underline{20}, \underline{1}, \underline{1}; \underline{1}) + 5(\underline{6}, \underline{3}, \underline{1}; \underline{1})$ $+10(\underline{1}, \underline{1}, \underline{2}; \underline{1})$	
$b(G)$	$b(SU_6) = 18,\ b(SU_3) = 36,\ b(SU_2) = 12$	$b(SU_9) = 0$
$\tilde{v}_G$	$\tilde{v}_{SU_6} = 1+1,\ \tilde{v}_{SU_3} = 1+4,\ \tilde{v}_{SU_2} = 1$	$\tilde{v}_{SU_9} = -1$
$k = n - 12$	2	-2
	$\mathbf{Z}_2$ fixed points: see Section 5 $\mathbf{Z}_3$ fixed points: see Example 1 $\mathbf{Z}_6$ fixed point:	
g_{16}	$-\frac{5}{12}$	$+\frac{5}{12}$
Q_{10}	$-\frac{35}{144}\{(\underline{35}, \underline{1}, \underline{1}) + (\underline{1}, \underline{8}, \underline{1}) + (\underline{1}, \underline{1}, \underline{3})\}$ $-\frac{5}{72}(\underline{6}, \underline{3}, \underline{2}) + \frac{13}{72}(\underline{15}, \underline{\bar{3}}, \underline{1}) + \frac{19}{144}(\underline{20}, \underline{1}, \underline{2})$	$\frac{13}{72}(\underline{84}) - \frac{35}{144}(\underline{80})$
Q_6	$-$	$(\underline{6}, \underline{1}, \underline{1}; \underline{\bar{9}}) + \frac{1}{2}(\underline{20}, \underline{1}, \underline{1}; \underline{1})$
G_7	SU_6	
Q_7	$\frac{1}{2} \cdot \underline{35}$	$-\frac{1}{2} \cdot \underline{35}$

sector, $H_3 = 6 \times \frac{1}{2}(\underline{20}, \underline{1}, \underline{1}; \underline{1}) + 5 \times (\underline{6}, \underline{3}, \underline{1}; \underline{1}) + 10(\underline{1}, \underline{1}, \underline{2}; \underline{1})$, are assigned to the different fixed points as follows. At each of the five $\mathbf{Z}_2$ fixed points there is a $\frac{1}{2}(\underline{56}, \underline{1}) + 2(\underline{1}, \underline{2})$ hyper-multiplet of $E_7 \times SU_2$. Under $E_7 \times SU_2 \rightarrow SU_6 \times SU_3 \times SU_2$ it decomposes as $\frac{1}{2}(\underline{20}, \underline{1}, \underline{1}) + (\underline{6}, \underline{3}, \underline{1}) + 2(\underline{1}, \underline{1}, \underline{2})$. This leaves one $\frac{1}{2}(\underline{20}, \underline{1}, \underline{1}; \underline{1})$ half-hyper-multiplet at the $\mathbf{Z}_6$ fixed points. To summarize, the massless twisted matter content at the $\mathbf{Z}_6$ fixed point is $(\underline{6}, \underline{1}, \underline{1}; \underline{\bar{9}}) + \frac{1}{2}(\underline{20}, \underline{1}, \underline{1}; \underline{1})$.

The coefficients of the one-loop beta-functions and the resulting values of $\tilde{v}_\alpha$ are easily found to be

$$b(SU_6) = 18 \quad \rightarrow \quad \tilde{v}(SU_6) = 1 + 1,$$
$$b(SU_3) = 36 \quad \rightarrow \quad \tilde{v}(SU_3) = 1 + 4,$$

$$b(SU_2) = 12 \quad \rightarrow \quad \tilde{v}(SU_2) = 1,$$
$$b(SU_9) = 0 \quad \rightarrow \quad \tilde{v}(SU_9) = -1. \tag{4.4}$$

Since the maximal non-perturbative gauge group is SU_6, we conclude that the value of $\tilde{v}_{SU_9}$ must be that of a perturbative SU_9 factor. Consequently $\tilde{v}_{SU_9} = k_2/2$ and $k_2 = -2$, $k_1 = +2$. From $\tilde{v}_{SU_6} = 2$ it then follows that the non-perturbative SU_6 mixes with the perturbative SU_6, i.e., $SU_6^{\text{het}} = \text{diag}[SU_6^{\text{pert}} \times SU_6^{\text{non-pert}}]$. The fact that $\tilde{v}_{SU_3} = 1 + 4$ is in agreement with our expectation that four non-perturbative SU_3 factors located at the four $\mathbf{Z}_3$ seven-planes mix with the perturbative SU_3. As for $\tilde{v}_{SU_2}$ we would naively expect $\tilde{v}_{SU_2} = k_1/2 + 5$. This would in fact be required if at the $\mathbf{Z}_2$ fixed points the local physics were that of the $\mathbf{Z}_2$ orbifold discussed in Section 2. However, as already mentioned above, for the $\mathbf{Z}_2$ orbifold with gauge group $(E_7 \times SU_2) \times E_8$ we will find in Section 5 that the SU_2 factor is completely perturbative, i.e., there is no mixing with the non-perturbative seven-plane gauge group.

It is now straightforward to give the field content on the $\mathbf{Z}_6$ I6 and I6$'$ planes. On I6 with perturbative gauge group $SU_6 \times SU_3 \times SU_2$ there are no twisted matter states. They are all located at I6$'$. Since they carry SU_6 quantum numbers, the non-perturbative SU_6 vector-multiplet must be free on I6$'$ and the SU_6 adjoint hyper-multiplet is free on I6.

4.3. Local anomaly cancellation

We will now generalize the discussion of Section 3 to the models of the previous subsection. In particular we will confirm the relative normalization of the two contributions in (3.7).

$\mathbf{Z}_3$ orbifold

The discussion of the local anomalies for this model is almost identical to the one given in Section 3. One difference is that now we have only two moduli multiplets. Also, when distributing bulk fields and ten-plane fields over the various six-planes we have to take into account that we now have nine fixed points. With $k_1 = -3$ and $k_2 = +3$, the magnetic charges of the nine I6 and the nine I6$'$ planes are $g_{I6} = -1/3$ and $g_{I6'} = +1/3$, respectively.

It is straightforward to determine the quantum + inflow contribution to the local anomaly on the six-planes to

$$\mathcal{A}(\text{quantum} + \text{inflow on I6})$$
$$= \frac{3}{2} \left(\text{tr } F_{SU_3}^2 - \frac{1}{9} \text{tr } R^2 \right) \cdot \left(\text{tr } F_{E_6}^2 + \text{tr } F_{SU_3}^2 - \frac{1}{2} \text{tr } R^2 \right),$$

$$\mathcal{A}(\text{quantum} + \text{inflow on I6}')$$
$$= \frac{3}{2} \left(\text{tr } F_{SU_3}^2 - \frac{1}{9} \text{tr } R^2 \right) \cdot \left(\text{tr } F_{SU_9}^2 - \frac{1}{2} \text{tr } R^2 \right). \tag{4.5}$$

As in our discussion in Section 3, this anomaly can be cancelled by an intersection-anomaly of the form (3.11), provided that we choose the parameters in Y_4 as $\eta = 1/9$ and $\rho = 1$.

$\mathbf{Z}_4$ orbifold

For this model the discussion is complicated by the fact that we have two different types of fixed points. As we have explained before, the local physics at the $\mathbf{Z}_2$ fixed points is identical to that of the $\mathbf{Z}_2$ orbifold model of Sections 2 and 3. This in particular means that anomaly cancellation on the $\mathbf{Z}_2$ six-planes works in exactly the same way as before. When computing the bulk and the ten-plane contribution to the anomaly on the $\mathbf{Z}_4$ six-planes, we must however first subtract the contribution already accounted for on the $\mathbf{Z}_2$ planes and then distribute the remaining anomaly over the four $\mathbf{Z}_4$ planes.

Before illustrating this for the charged ten-plane fields, we will introduce some convenient notation. We will denote the multiplet content of the charged ten-plane fields which contribute to the anomaly by Q_{10}. This splits into hyper-multiplets and vector-multiplets. Taking into account the opposite chirality of the fermions in these multiplets, we write $Q_{10} = H_{10} - V_{10}$. The net number of states will be denoted by $n_{Q_{10}}$. Let us consider the $SO_6 \times SO_{10}$ side. At the $\mathbf{Z}_2$ fixed six-planes, the local physics is as in the $\mathbf{Z}_2$ model of Sections 2 and 3, i.e., the untwisted states consist of a SO_{16} vector-multiplet and one $(\underline{128})_{SO_{16}}$ hyper-multiplet. We now assume that the contribution to the anomaly of a $\mathbf{Z}_2$ six-plane is exactly that of such a six-plane in the $\mathbf{Z}_2$ orbifold, i.e., that of one sixteenth of a SO_{16} vector-multiplet and of one sixteenth of a $\underline{128}$ hyper-multiplet. Taking into account that there are six I6 of this type and four I6 fixed under $\mathbf{Z}_4$, we get for Q_{10}

$$Q_{10} = \frac{1}{4}\left\{[(\underline{16},\underline{4}) - (\underline{45},\underline{1}) - (\underline{1},\underline{15})]_{SO_{10}\times SO_6} - \frac{6}{16}[(\underline{128}) - (\underline{120})]_{SO_{16}}\right\}$$

$$= -\frac{5}{32}[(\underline{45},\underline{1}) + (\underline{1},\underline{15})] + \frac{3}{32}(\underline{10},\underline{6}) + \frac{1}{16}(\underline{16},\underline{4}). \tag{4.6}$$

In the second step we have decomposed the SO_{16} representation under $SO_{16} \rightarrow SO_6 \times SO_{10}$. Also, $n_{Q_{10}} = 1/4$.

It follows from the construction of the states in the untwisted sector that all components of the decomposition of the $(\underline{248})_{E_8}$ under $E_8 \rightarrow SO_6 \times SO_{10}$ have definite $\mathbf{Z}_4$ eigenvalue, namely $e^{2\pi i \delta \cdot P}$ where P is a E_8 root and δ the shift vector in the E_8. One finds $\underline{248} = (\underline{15},\underline{1})_{+1} + (\underline{1},\underline{45})_{+1} + (\underline{6},\underline{10})_{-1} + (\underline{4},\underline{16})_{+i} + (\overline{\underline{4}},\overline{\underline{16}})_{-i} \equiv \alpha(\underline{248})$. The subscripts are the $\mathbf{Z}_4$ eigenvalues. Here α denotes the $\mathbf{Z}_4$ generator whose action on a root of E_8 is specified by the shift vector. Introducing the function $T(x) = x/8 + x^2/32$, whose argument can be either a complex number or an operator, we can rewrite (4.6) as

$$Q_{10} = -T(\alpha)(\underline{248}). \tag{4.7}$$

The justification for introducing this notation is that one can define a function $T(x)$ for all $\mathbf{Z}_N$ orbifolds and this function is universal for any given N, independent of the choice of shift vector. Specifically,

$$T(x) = \begin{cases} \dfrac{x}{16}, & N = 2, \\[2mm] \dfrac{x}{9}, & N = 3, \\[2mm] \dfrac{x}{8} + \dfrac{x^2}{32}, & N = 4, \\[2mm] \dfrac{x}{6} + \dfrac{x^2}{18} + \dfrac{x^3}{48}, & N = 6. \end{cases} \qquad (4.8)$$

We will need the values $T(1) = \{\frac{1}{16}, \frac{1}{9}, \frac{5}{32}, \frac{35}{144}\}$ and $2\,\mathrm{Re}(T(e^{2\pi i/n})) = \{-\frac{1}{8}, -\frac{1}{9},$ $-\frac{1}{16}, \frac{5}{72}\}$ for $n = \{2, 3, 4, 6\}$, respectively. One checks that for every orbifold model we are considering, $\sum_{\mathrm{I6}} T(1) = \sum_{\mathrm{I6}'} T(1) = 1$ and $-2\sum_{\mathrm{I6}} \mathrm{Re}(T(e^{2\pi i/n})) = -2\sum_{\mathrm{I6}'}$ $\times \mathrm{Re}(T(e^{2\pi i/n})) = \frac{1}{2} \cdot \#(\text{moduli})$, where in the case of non-prime orbifolds different types of fixed points have to be summed over.

For the $SU_8 \times SU_2$ side of the $\mathbf{Z}_4$ orbifold one finds in a similar way

$$\begin{aligned} Q_{10'} &= \frac{1}{4}\Bigg\{[(\underline{28}, \underline{2}) - (\underline{63}, \underline{1}) - (\underline{1}, \underline{3})]_{SU_8 \times SU_2} \\ &\qquad - \frac{6}{16}[(\underline{56}, \underline{2}) - (\underline{133}, \underline{1}) - (\underline{1}, \underline{3})]_{E_7 \times SU_2}\Bigg\} \\ &= -\frac{5}{32}[(\underline{63}, \underline{1}) + (\underline{1}, \underline{3})] + \frac{3}{32}(\underline{70}, \underline{1}) + \frac{1}{16}(\underline{28}, \underline{2}) \\ &= -T(\alpha)(\underline{248}), \end{aligned} \qquad (4.9)$$

with $n_{Q_{10'}} = -1/4$.

In addition to the ten-plane fields, on any six-plane there is also the contribution from the seven-plane fields with free boundary conditions on the six-plane, again with a relative sign between hyper and vector multiplets. In analogy to the notation introduced above, we will denote them as Q_7. The subscript denotes the seven-dimensional origin of these states. Including a factor of $1/2$, which was explained in Section 3 and denoting the fields by their $SU_4^{\text{non-pert}}$ representation, we have for the $\mathbf{Z}_4$ orbifold $Q_7 = 1/2 \cdot \underline{15}$ and $Q_{7'} = -1/2 \cdot \underline{15}$.

As for the six-plane fields, there are none on the I6 planes. All $\mathbf{Z}_4$ twisted matter fields live on the I6′ planes. I.e., in the by now familiar notation, $Q_6 = \emptyset$ and $Q_{6'} = (\underline{1}, \underline{4}; \underline{8}, \underline{1})$ $+ \frac{1}{2}(\underline{1}, \underline{6}; \underline{1}, \underline{2})$.

If we define $Q = Q_{10} + Q_7 + Q_6$ we can summarize the contribution of all charged matter fields to the quantum anomaly as

$$\mathcal{A}(Q) = \mathrm{Tr}_Q F^4 - \frac{1}{4}\,\mathrm{tr}\,R^2\,\mathrm{Tr}_Q F^2 + n_Q\left(\frac{1}{240}\,\mathrm{tr}\,R^4 + \frac{1}{192}(\mathrm{tr}\,R^2)^2\right). \qquad (4.10)$$

With the help of the function T we can also express the contribution to the anomaly of the bulk fields, namely

$$\begin{aligned} \mathcal{A}(\text{SUGRA} + \text{tensor}) &= \frac{1}{2}T(1)\left[-\frac{244}{240}\,\mathrm{tr}\,R^4 + \frac{44}{192}(\mathrm{tr}\,R^2)^2\right], \\ \mathcal{A}(\text{moduli}) &= -2\,\mathrm{Re}\big(T(e^{2\pi i/n})\big)\left[\frac{1}{240}\,\mathrm{tr}\,R^4 + \frac{1}{192}(\mathrm{tr}\,R^2)^2\right]. \qquad (4.11) \end{aligned}$$

Note, e.g., that for $\mathbf{Z}_4$ planes, $T(1) = \frac{1}{4}(1 - 6/16)$ and $-2\,\mathrm{Re}(T_4(e^{2\pi i/n})) = \frac{1}{2\cdot 4}(2 - 4\cdot 6/16)$, where for the latter we have taken into account that a $\mathbf{Z}_4$ orbifold has two moduli whereas a $\mathbf{Z}_2$ orbifold has four moduli. (4.11) holds for both the I6 and the I6′ $\mathbf{Z}_4$ planes.

The total quantum anomaly is thus

$$
\begin{aligned}
\mathcal{A}(\text{quantum}) = {}& \mathrm{Tr}_Q\, F^4 - \frac{1}{4}\,\mathrm{tr}\,R^2\,\mathrm{Tr}_Q\,F^2 \\
& + \frac{1}{240}\big(n_Q - 122\,T(1) - 2\,\mathrm{Re}\,T\big(e^{2\pi i/N}\big)\big)\,\mathrm{tr}\,R^4 \\
& + \frac{1}{192}\big(n_Q + 22\,T(1) - 2\,\mathrm{Re}\,T\big(e^{2\pi i/N}\big)\big)\big(\mathrm{tr}\,R^2\big)^2.
\end{aligned}
\tag{4.12}
$$

This expression is valid for every $\mathbf{Z}_N$ six-plane of any $\mathbf{Z}_N$ orbifold, once Q has been specified.

To compute the inflow contribution (3.7) we need the magnetic charge of the six-planes. It can be determined from the sum rule (3.8). We have to distribute the total charge k over all fixed planes on a given side of the x^{11} interval. For prime orbifolds they are all related by symmetry and carry the same magnetic charge. For non-prime orbifolds some care is required. E.g., on the $SO_{10} \times SO_6$ side of the $\mathbf{Z}_4$ model, with total charge zero ($k_1 = k_2 = 0$), there are six $\mathbf{Z}_2$ fixed planes with local gauge group SO_{16}, each with charge $1/4$. The four $\mathbf{Z}_4$ fixed planes must thus carry a total charge of $-6\cdot 1/4$ or $-3/8$ each. I.e., $g_{I6} = -3/8$ and $g_{I6'} = +3/8$.

Note that there is a minimal magnetic charge any fixed plane must carry. This is obtained if $k = -12$, which corresponds to an unbroken E_8. In this case there are no gauge instantons. It is straightforward to compute the minimal magnetic charges of the various $\mathbf{Z}_n$ planes. They can be conveniently summarized in the formula $g_{\mathbf{Z}_n}^{\min} = (1 - n^2)/2n$. The allowed magnetic charges are then $g_{\mathbf{Z}_n} = g + m/n$ where the non-negative integer m counts the number of $\mathbf{Z}_n$ instantons sitting at the fixed point.

We have now provided all ingredients necessary to compute the quantum and the inflow anomaly for the $\mathbf{Z}_4$ model. A short calculation gives

$$
\begin{aligned}
&\mathcal{A}(\text{quantum} + \text{inflow on I6}) \\
&= \frac{3}{2}\left(\mathrm{tr}\,F_{SU_4}^2 - \frac{5}{32}\,\mathrm{tr}\,R^2\right)\cdot\left(\mathrm{tr}\,F_{SO_{10}}^2 + \mathrm{tr}\,F_{SO_6}^2 - \frac{1}{2}\,\mathrm{tr}\,R^2\right), \\
&\mathcal{A}(\text{quantum} + \text{inflow on I6}') \\
&= \frac{3}{2}\left(\mathrm{tr}\,F_{SU_4}^2 - \frac{5}{32}\,\mathrm{tr}\,R^2\right)\cdot\left(\mathrm{tr}\,F_{SU_8}^2 + \mathrm{tr}\,F_{SU_2}^2 - \frac{1}{2}\,\mathrm{tr}\,R^2\right).
\end{aligned}
\tag{4.13}
$$

We once again find that this anomaly can be cancelled via (3.11) with $\rho = 1$ and $\eta = T(1) = 5/32$.

$\mathbf{Z}_6$ orbifold

We will be very brief here. We only have to check anomaly cancellation on the $\mathbf{Z}_6$ six-planes. The ten-plane gauge groups are $SU_6 \times SU_3 \times SU_2$ on 10P and SU_9 on 10P′. Using $T(x)$ as given in (4.8) it is straightforward to show that

$$Q_{10} = -T(\alpha)(\underline{248})$$
$$= -\frac{35}{144}[(\underline{35}, \underline{1}, \underline{1}) + (\underline{1}, \underline{8}, \underline{1}) + (\underline{1}, \underline{1}, \underline{3})] - \frac{5}{72}(\underline{6}, \underline{3}, \underline{2})$$
$$+ \frac{13}{72}(\underline{15}, \bar{\underline{3}}, \underline{1}) + \frac{19}{144}(\underline{20}, \underline{1}, \underline{2}),$$
$$Q_{10'} = \frac{13}{72} \cdot \underline{84} - \frac{135}{144} \cdot \underline{80}, \tag{4.14}$$

where the decomposition is with respect to the ten-plane gauge groups. Also, the discussion in Section 4.1 gave

$$Q_6 = \emptyset, \qquad Q_{6'} = (\underline{6}, \underline{1}, \underline{1}; \bar{\underline{9}}) + \frac{1}{2}(\underline{20}, \underline{1}, \underline{1}; \underline{1}) \tag{4.15}$$

and

$$Q_7 = \frac{1}{2} \cdot \underline{35}, \qquad Q_{7'} = -\frac{1}{2} \cdot \underline{35}, \tag{4.16}$$

where the latter states are with respect to $SU_6^{\text{non-pert}}$. The magnetic charges are easily determined to $g_{\text{I6}} = -5/12$ and $g_{\text{I6'}} = +5/12$.

For both the I6 and the I6' planes, the quantum + inflow anomaly can be cancelled via an intersection anomaly (3.11) with $\rho = 1$ and $\eta = T(1) = 35/144$.

4.4. Common features

We have demonstrated in all four examples that the local anomaly on each intersection six-plane cancels. This is to say that in the sum of quantum+inflow+intersection anomaly the coefficients of $\text{tr } R^4$, $(\text{tr } R^2)^2$, $\text{tr } R^2$ and of the term without dependence on the Ricci-form vanish separately. The four conditions for this to happen are:

$$n_Q = 122T(1) + T(e^{2\pi i/n}) + T(e^{-2\pi i/n}) + 30g,$$
$$\eta = T(1),$$
$$\frac{1}{3}\text{Tr}_Q F^2 = \rho \text{ tr } F_7^2 + (g + 2\eta) \text{ tr } F_{10}^2,$$
$$\frac{2}{3}\text{Tr}_Q F^4 = \text{tr } F_{10}^2 \left(\frac{g}{2}\text{ tr } F_{10}^2 + \rho \text{ tr } F_7^2\right). \tag{4.17}$$

In particular, local anomaly cancellation fixed the coefficients η and ρ in (3.12) to $\eta = T(1)$ and $\rho = 1$, respectively. There is another way to see why these values are generic, which we will now present. It also uses anomaly arguments and it involves a direct comparison of the heterotic and the M-theory point of views.

In the heterotic theory, anomaly cancellation is of course guaranteed by the well-established consistency of the perturbative heterotic string. However, by realizing that the massless fields which contribute to the heterotic anomaly on a given fixed six-plane are precisely those which, in M-theory, contribute on a I6, I6' pair which is connected by a seven-plane, we can, by comparison, determine the coefficients η and ρ.

The boundary conditions of the seven-plane fields were chosen such that in the limit $R_{11} \to 0$ there are no additional massless states. This is reflected in $Q_7 + Q_7' = 0$. Also,

the magnetic charges satisfy $g_{I6} + g_{I6'} = 0$. We then have $Q_{net} \equiv Q + Q' = Q_{10} + Q'_{10} + Q_6 + Q'_6$ and the first condition in (4.17) gives $n_{Q_{net}} = 244 T(1) + 4 \operatorname{Re} T(e^{2\pi i/n})$, which is correct in the heterotic context without any reference to M-theory.

We can now compute the anomaly on a I6, I6$'$ pair from the heterotic and from the M-theoretic point of view. The former gives (cf. Appendix B)

$$\frac{2}{3}\mathcal{A} = \frac{2}{3} \operatorname{Tr}_{Q_{net}} F^4 - \frac{1}{6} \operatorname{tr} R^2 \operatorname{Tr}_{Q_{net}} F^2 + T(1)\left(\operatorname{tr} R^2\right)^2$$
$$= \left(\operatorname{tr} R^2 - \sum_\alpha \operatorname{tr} F_\alpha^2\right) \wedge \left(T(1) \operatorname{tr} R^2 - \sum_\alpha \tilde{v}_\alpha \operatorname{tr}_\alpha F^2\right). \tag{4.18}$$

Computing the same anomaly in the M-theory picture produces instead (use the last two equations in (4.17))

$$\frac{2}{3}\mathcal{A} = \left(\operatorname{tr} R^2 - \operatorname{tr} F_{10}^2 - \operatorname{tr} F_{10'}^2\right) \wedge \left(\eta \operatorname{tr} R^2 + \frac{g}{2}\left(\operatorname{tr} F_{10}^2 - \operatorname{tr} F_{10'}^2\right) + \rho \operatorname{tr} F_7^2\right). \tag{4.19}$$

Comparison gives once more $\eta = T(1)$ and

$$v_{10} = v_{10'} = 1 \quad \text{and} \quad \tilde{v}_{10} = \frac{1}{2} \sum_{\substack{\text{fixed} \\ \text{planes}}} g = -\tilde{v}_{10'},$$

$$v_7 = 0 \quad \text{and} \quad \tilde{v}_7 = \rho. \tag{4.20}$$

The seven-plane gauge groups G_7 thus have the characteristic of non-perturbatively generated gauge groups. However, in the heterotic dual, which is completely perturbative, they are not visible as additional gauge group factors. This leaves two options: (*i*) $\rho \neq 0$ and ρ is the level of the gauge group G_7. This is the situation we have encountered in all four examples considered so far. The seven-plane gauge group mixes with the ten-plane gauge group, as, e.g., in (2.7). (*ii*) $\rho = 0$: this case will be discussed in Section 5.

So far we have determined the parameters ρ and η using anomaly arguments. However, ultimately these parameters should come from seven-dimensional physics. In fact, we will now give an independent 'derivation' of the values for η and ρ, which does not rely on any anomaly arguments.

In M-theory on K3, the parameters η and ρ enter through the electric coupling of the seven-plane to the three-form potential C of 11-dimensional supergravity via the term $C \wedge Y_4$, cf. (3.12) and [11]. The eleven-dimensional origin of this term are the two CS terms $C \wedge G \wedge G$ and $C \wedge (\operatorname{tr} R^4 - \frac{1}{4}(\operatorname{tr} R^2)^2)$. However, it is easier to discuss these couplings from the dual point of view, exploiting the duality between M-theory on K3 and the heterotic theory on T^3 [1,14]. [12]

This heterotic — M-theory duality in $d = 7$ relates the field strength of C, denoted by G, to the field strength H of the Kalb–Ramond field B of the heterotic theory and vice versa: $H \leftrightarrow *G$. The moduli spaces of the heterotic compactification on T^3 and of M-theory on K3 are isomorphic. At a generic point on the Narain lattice, the gauge symmetry of

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[12] This seven-dimensional heterotic theory is of course completely different from the Hořava–Witten theory we have discussed so far.

the heterotic string compactified on T^3 is $U(1)^{22}$ which is also the gauge group of the M-theory at a generic point of the K3 moduli space. On the heterotic side, the Bianchi identity reads

$$dH \propto \operatorname{tr} R^2 + \sum_{I,J=1}^{22} d_{IJ} F_I F_J, \qquad (4.21)$$

where d_{IJ} is a Lorentzian metric with signature $((+)^3, (-)^{19})$ which is also the signature of the intersection matrix of the K3 homology 2-cycles. Duality now implies $d(*G) \propto dH$ and thus an electric coupling $\propto C \wedge dH$. At special points in the Narain moduli space the heterotic gauge symmetry acquires non-Abelian components which contribute to dH, i.e., (4.21) gets modified to

$$dH \propto \operatorname{tr} R^2 - \sum \operatorname{tr} F_i^2 + \text{Abelian part.} \qquad (4.22)$$

For the M-theory the gauge group enhancement happens at the orbifold points of the K3 moduli space. The additional states which comprise the non-Abelian gauge multiplets are provided by M2-branes wrapping the vanishing cycles (cf. the discussion in Section 1). At the orbifold points the seven-dimensional gauge groups are $SU_2^{16} \times U_1^6$ for the $\mathbf{Z}_2$ orbifold, $SU_3^9 \times U_1^4$ for $\mathbf{Z}_3$, $SU_2^6 \times SU_4^4 \times U_1^4$ for $\mathbf{Z}_4$ and $SU_2^5 \times SU_3^4 \times SU_6 \times U_1^4$ for $\mathbf{Z}_6$.

From the eleven-dimensional point of view the Abelian part lives in the bulk and is completely broken after compactification to six-dimensions. It is thus of no further interest for us. The non-Abelian part, on the other hand, contributes to the local $C \wedge Y_4$ coupling on the seven-planes as $-C \wedge \sum_i \operatorname{tr} F_{7i}^2$. The sum is over all fixed seven-planes. As far as the $C \wedge \operatorname{tr} R^2$ piece of this coupling is concerned, it has to be apportioned between all seven-planes. The apportioning happens in exactly the same manner as with the bulk anomaly in previous sections, namely as $T(1) \operatorname{tr} R^2$. The reason for this is the same as before. The contribution on a $\mathbf{Z}_N$ plane depends only on N, independent of the orbifold model.

We thus find that the relevant coupling is proportional to

$$C \wedge \left(\operatorname{tr} R^2 - \sum_i \operatorname{tr} F_{7i}^2 \right) = C \wedge \sum_i \left(T_i(1) \operatorname{tr} R^2 - \operatorname{tr} F_{7i}^2 \right). \qquad (4.23)$$

Comparing this with $C \wedge \sum_i Y_{4i}$ leads to the identification $\eta = T(1)$ and $\rho = 1$.

We should point out that we have not been careful with the overall normalization of the $C \wedge Y_4$ coupling. In fact, this coupling is somewhat controversial since it is related to the normalization of the two eleven-dimensional CS terms. Nevertheless the normalization must be universal and it can be fixed by considering any orbifold model, such as the $\mathbf{Z}_2$ model of Sections 2 and 3. [13]

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[13] If one changed the *relative* normalization of the two terms in (3.7) one would find that the second of the conditions in (4.17) will be modified (η and $T(1)$ will no longer be proportional to each other), in contradiction with the independent arguments given here.

5. Open problems

All the orbifold models we have discussed thus far followed a common pattern from the M-theory point of view: each $\mathbf{Z}_n$ fixed seven-plane carries a non-perturbative SU_n gauge theory which mixes with a perturbative gauge theory living on one of the two ten-planes. The resulting theory contains an SU_n gauge group which appears to be a subgroup of the $G \subset E_8$ (or $G' \subset E_8'$) but isn't actually confined to one side of the x^{11} interval. Instead, it reaches to the other side along the fixed seven-planes — and that's how the twisted states living on the I6$'$ intersections manage to have charges under both the $SU_n \subset G$ and the G' gauge groups.

However, in many other orbifold models, the non-perturbative gauge groups living on the $\mathbf{Z}_n$ fixed seven-planes turn out to be proper subgroups $G_7 \subset SU_n$ rather that complete SU_n's. Furthermore, such reduced non-perturbative groups — or some of their factors — do not mix with the perturbative gauge group factors but simply decouple from the massless states of the six-dimensional theory.[14] At present, we do not know any M-theoretical rules governing breaking of the non-perturbative gauge groups or their mixing with the perturbative gauge group factors. All we have are the 'experimental data' about the non-perturbative gauge groups implied by the quantum numbers of the twisted states in a score of orbifold models we have studied in some detail.

We see no point in boring the reader with technical details of too many models. Instead, we shall simply present examples of two common problems we have seen in several models. In the following Section 5.1 we discuss models with the unbroken E_8' gauge group. 'Experimentally', all such models have $G_7 = (U_1)^{(n-1)}$, the Cartan subgroup of the SU_n, and none of the non-perturbative U_1 factors mixes with any perturbative gauge groups. Furthermore, the local anomaly cancellation requires the non-perturbative U_1's to have $\rho = 0$, in blatant contradiction with the seven-dimensional arguments of Section 4.4.

In Section 5.2 we present an example of a more complicated model where the combined quantum, inflow and intersection anomalies *do not cancel out* for any G_7 consistent with the twisted states' quantum numbers. This problem occurs in all models with mixed perturbative/non-perturbative Abelian gauge fields, although it may affect the non-Abelian fields as well. We speculate how seven-dimensional Chern–Simons couplings may solve this problem, but a thorough analysis has to be postponed to a later publication.

5.1. Models with an unbroken E_8

To exemplify the problems that arise in this class of models, consider the best known heterotic orbifold, namely the $\mathbf{Z}_2$ orbifold with the standard imbedding of the spin connection into the gauge group, i.e., $\delta = (-\frac{1}{2}, +\frac{1}{2}, 0, 0, 0, 0, 0, 0; 0, 0, 0, 0, 0, 0, 0, 0)$. In this model, the first E_8 is broken down to $G = E_7 \times SU_2$ while the second E_8 remains unbroken. Note that we have already encountered such $\mathbf{Z}_2$ fixed points in the $\mathbf{Z}_6$ orbifold

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[14] Technically, the locking boundary conditions (2.1) at one side of the x^{11} interval are replaced with the simple Dirichlet boundary conditions $A_\mu^{\text{non-pert}}(x^{11} = 0) = 0$.

Table 4
The $\mathbf{Z}_2$ orbifold with the $(E_7 \times SU_2) \times E_8$ gauge group

		Example 5: $\mathbf{Z}_2$ orbifold with unbroken E_8	
Shift vector		$\left(\frac{-1}{2}, \frac{+1}{2}, 0, 0, 0, 0, 0, 0\right)$	$(0, 0, 0, 0, 0, 0, 0, 0)$
Gauge group		$E_7 \times SU_2$	E_8
Matter		Four moduli	
	$H_0 \left\{ \vphantom{\begin{matrix}a\\a\end{matrix}} \right.$	$(\underline{56}, \underline{2}; \underline{1})$	$-$
	H_1	$16 \times \{\frac{1}{2}(\underline{56}, \underline{1}; \underline{1}) + 2(\underline{1}, \underline{2}; \underline{1})\}$	
$b(G)$		$b(E_7) = 42,\ b(SU_2) = 42$	$b(E_8) = -30$
$\tilde{v}_G$		$\tilde{v}_{E_7} = 6,\ \tilde{v}_{SU_2} = 6$	$\tilde{v}_{E_8} = -6$
$k = n - 12$		$+12$	-12
g_{16}		$+\frac{3}{4}$	$-\frac{3}{4}$
Q_{10}		$\frac{1}{16}\{(\underline{56}, \underline{2}) - (\underline{133}, \underline{1}) - (\underline{1}, \underline{3})\}$	$-\frac{1}{16} \cdot (\underline{248})$
Q_6		$\frac{1}{2}(\underline{56}, \underline{1}; \underline{1}^0) + 2(\underline{1}, \underline{2}; \underline{1}^0)$	$-$
G_7		$U_1 \subset SU_2$	
Q_7		$-\frac{1}{2} \cdot \underline{1}^0$	$\frac{1}{2} \cdot \underline{1}^0$

in Section 4, but let us take a closer look now. The hypermultiplet spectrum and other technical details of the 'standard' $\mathbf{Z}_2$ orbifold are summarized in Table 4.

Note that unlike the 'non-standard' $\mathbf{Z}_2$ orbifold discussed in Section 2, the 'standard' $\mathbf{Z}_2$ orbifold has $\tilde{v} = k/2$ for all gauge group factors including SU_2; indeed, on the $E_7 \times SU_2$ side, we have $k = +12$ and $\tilde{v}_{SU_2} = 6$ rather than $\tilde{v}_{SU_2} = 6 + 16 = 22$. According to Eq. (2.7), this indicates that in the standard $\mathbf{Z}_2$ orbifold, the SU_2 gauge factor is purely perturbative and does not mix with any non-perturbative factors. Consequently, no massless states in this model can be simultaneously charged under the SU_2 and the E_8 gauge groups — and indeed there are no such states in the standard orbifold.

The non-perturbative gauge group G_7 on each of the 16 fixed seven-planes must fit inside an SU_2 (bigger groups are not available at A_1 singularities such as $\mathbf{Z}_2$ fixed points), but because it does not mix with the perturbative $SU(2)$, we may have either $G_7 = SU_2$ or $G_7 = U_1$. The choice of G_7 affects the Q_7 contribution to the quantum anomaly at each end of the x^{11}: given our general rule of opposite boundary conditions on opposite ends (two avoid non-perturbative massless states in 6d) and the requirement for free vector fields at either end to form a closed subgroup of the G_7, it follows that $n(Q_7^{(\prime)}) = \mp 3/2$ for $G_7 = SU(2)$ but $n(Q_7^{(\prime)}) = \mp 1/2$ for $G_7 = U_1$. Since the $Q_6^{(\prime)}$ and $Q_{10}^{(\prime)}$ contributions to the anomaly on each side are completely fixed by the massless spectrum of the orbifold, and since $k = \pm 12$ implies $g = \pm 3/4$, the need to cancel the $\mathrm{tr}\,R^4$ term in the local anomaly at each end (cf. first Eq. (4.17)) requires $n(Q_7) = -1/2$ on the $E_7 \times SU_2$ side

and $n(Q_7') = +1/2$ on the E_8 side — and hence $G_7 = U_1 \subset SU_2$ rather than $G_7 = SU(2)$.

Without going into any more details of Eqs. (4.17), let us simply state that for $G_7 = U(1)$, the entire local anomaly polynomials $\mathcal{A}$ on both I6 and I6$'$ planes cancel, provided (1) all the twisted states Q_6 are neutral with respect to the non-perturbative U_1 and (2) $\rho = 0$ in the intersection anomaly term, all the arguments in Section 4.4 for $\rho = 1$ notwithstanding; note however that $\eta = 1/16 = T(1)$ as required by the second Eq. (4.17). At present, we have no explanation for the 'experimental fact' of $\rho = 0$ except that its required to cancel the anomalies locally. Likewise, we have no M-theoretical explanation for the $G_7 = U_1$, only the brute fact that this too is required to cancel the anomalies locally. The theory would have to wait for a later publication.

In lieu of theory, we offer a summary of 'experimental' data to show a common pattern of *invisible* non-perturbative Abelian gauge groups, which do not mix with any perturbative gauge group factors and don't couple to any twisted massless states (i.e., the hypermultiplets living on I6 and I6$'$ are neutral with respect to the invisible groups). In particular, in *all* $\mathbf{Z}_n$ orbifolds ($n = 2, 3, 4, 6$) with an unbroken E_8 group [15] we can cancel all anomalies locally, provided: (1) each $G_7(\mathbf{Z}_n$ plane$) = (U_1)^{(n-1)}$, the Cartan subgroup of the SU_n; (2) none of the non-perturbative U_1's mixes with any of the perturbative groups; (3) all twisted states are neutral under all the non-perturbative U_1's; and (4) all the non-perturbative U_1's have $\rho = 0$. In addition, many $\mathbf{Z}_4$ and $\mathbf{Z}_6$ orbifolds have a locally-unbroken E_8 on the $\mathbf{Z}_2$ or $\mathbf{Z}_3$ fixed planes — and all such fixed planes carry invisible U_1 or $(U_1)^2$ gauge groups, exactly like similar fixed planes in the corresponding $\mathbf{Z}_2$ or $\mathbf{Z}_3$ models. For example, the $\mathbf{Z}_6$ orbifold of Section 4 has such an invisible $G_7 = U_1$ on each of its $\mathbf{Z}_2$ fixed planes.

It is easy to see that any invisible U_1 gauge field must have $\rho = 0$ for the sake of local anomaly cancellation. Indeed, an invisible seven-plane U_1 has no quantum anomalies — since all the chiral fields are neutral — and no inflow anomalies — which involve only the ten-plane groups or the *visible* seven-plane groups that mix with them. Consequently, any invisible U_1 factor should also decouple from the intersection anomaly, which means it must have $\rho = 0$. Unfortunately, this anomaly-counting argument does not tell us how the invisible seven-plane gauge groups differ from the visible seven-plane groups *from the seven-dimensional point of view*, so the M-theoretical origins of the $\rho_{\text{invisible}} = 0$ remain obscure.

5.2. Models with perturbative U_1 factors

The invisible Abelian groups of the previous section are puzzling from the M-theoretical point of view, but as far as the local anomaly cancellation is concerned, the *visible* Abelian groups are much more troublesome. Generally, the combined quantum, inflow and intersection anomalies in such models cancel each other globally but not locally in x^{11},

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[15] There are ten distinct heterotic orbifolds of this type, including the standard $\mathbf{Z}_2$ orbifold, two different $\mathbf{Z}_3$ orbifolds, two $\mathbf{Z}_4$ orbifolds and five $\mathbf{Z}_6$ orbifolds.

$$\mathcal{A}(\text{I}6) + \mathcal{A}(\text{I}6') = 0, \quad \text{but } \mathcal{A}(\text{I}6) \neq 0 \neq \mathcal{A}(\text{I}6'). \tag{5.1}$$

In other words, the anomalies cancel locally in ten dimensions (which always works in any perturbative heterotic orbifold) but not the eleventh dimension.

As an example of such anomaly trouble, let us consider a $\mathbf{Z}_4$ orbifold summarized in Table 5.

Note that all data in Table 5 are completely determined by the perturbative heterotic string theory — except of course for the inherently non-perturbative G_7 and Q_7. However, the model must have $G_7 \supset SU_2 \times U(1)$ since the gauge couplings of the perturbative SU_2

Table 5
A $\mathbf{Z}_4$ orbifold with the $(SO_{10} \times SU_4) \times (E_6 \times SU_2 \times U_1)$ gauge group and anomaly troubles

	Example 6: $\mathbf{Z}_4$ orbifold	
Shift vector	$\left(-\frac{3}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, 0, 0, 0, 0\right)$	$\left(-\frac{1}{2}, \frac{1}{4}, \frac{1}{4}, 0, 0, 0, 0, 0\right)$
Gauge group	$SO_{10} \times SU_4$	$E_6 \times SU_2 \times U_1$
Matter		Two moduli
H_0	$(\underline{16}, \underline{4}; \underline{1}, \underline{1}, 0)$	$\left(\underline{1}, \underline{1}; \underline{27}, \underline{2}, \frac{-1}{\sqrt{12}}\right) + \left(\underline{1}, \underline{1}; \underline{1}, \underline{2}, \frac{+3}{\sqrt{12}}\right)$
H_1	$4 \times \left\{\left(\underline{1}, \underline{4}; \underline{1}, \underline{2}, \frac{+3}{2\sqrt{12}}\right) + 2\left(\underline{1}, \underline{4}; \underline{1}, \underline{1}, \frac{-3}{2\sqrt{12}}\right) + \left(\overline{\underline{16}}, \underline{1}; \underline{1}, \underline{1}, \frac{+3}{2\sqrt{12}}\right)\right\}$	
H_2	$\frac{10}{2} \times (\underline{10}, \underline{1}; \underline{1}, \underline{2}, 0) + \frac{6}{2} \times (\underline{1}, \underline{6}; \underline{1}, \underline{2}, 0)$	
$b(G)$	$b(SO_{10}) = b(SU_4) = +18$	$b(E_6) = -6, b(SU_2) = +54, b(U_1) = +30$
$k = n - 12$	$+4$	-4
$\tilde{v}_G$	$\tilde{v}_{SO_{10}} = \tilde{v}_{SU_4} = +2$	$\tilde{v}_{E_6} = -2, \tilde{v}_{SU_2} = +8, \tilde{v}_{U_1} = +4$
$\delta\tilde{v} = \tilde{v} - \frac{k}{2}$	$\delta\tilde{v}_{SO_{10}} = \delta\tilde{v}_{SU_4} = 0$	$\delta\tilde{v}_{E_6} = 0, \delta\tilde{v}_{SU_2} = 6 + 4, \delta\tilde{v}_{U_1} = 4 \times \frac{3}{2}$
	$\mathbf{Z}_2$ fixed points: see Sections 2 and 3	
	$\mathbf{Z}_4$ fixed points	
g_{16}	$4 + \frac{5}{8}$	$-\frac{5}{8}$
Q_{10}	$-\frac{5}{32}\{(\underline{45}, \underline{1}) + (\underline{1}, \underline{15})\}$ $+ \frac{1}{16}(\underline{16}, \underline{4})$ $+ \frac{3}{32}(\underline{10}, \underline{6})$	$-\frac{5}{32}\{(\underline{78}, \underline{1}, 0) + (\underline{1}, \underline{3}, 0) + (\underline{1}, \underline{1}, 0)\}$ $+ \frac{1}{16}\left\{\left(\underline{27}, \underline{2}, \frac{-1}{\sqrt{12}}\right) + \left(\underline{1}, \underline{2}, \frac{+3}{\sqrt{12}}\right)\right\}$ $+ \frac{3}{32}\left(\underline{27}, \underline{1}, \frac{2}{\sqrt{12}}\right)$
Q_6	$\left(\underline{1}, \underline{4}; \underline{2}, \frac{+3}{2\sqrt{12}}\right) + 2\left(\underline{1}, \underline{4}; \underline{1}, \frac{-3}{2\sqrt{12}}\right)$ $+ \left(\overline{\underline{16}}, \underline{1}; \underline{1}, \frac{+3}{2\sqrt{12}}\right) + \frac{1}{2}(\underline{10}, \underline{1}; \underline{2}, 0)$	–
G_7		$G_7 \supset SU_2 \times U_1; \quad G_7 \subset SU_4$
Q_7	????	????

and U_1 group factors indicates their mixing with the non-perturbative gauge fields living on the $\mathbf{Z}_4$ fixed seven-planes. Actually, the SU_2 factor gets non-perturbative contributions from both $\mathbf{Z}_2$ and $\mathbf{Z}_4$ fixed planes, hence $\tilde{v}_{SU_2} - (k/2) = 6 + 4$. In the U_1 case however, only the $\mathbf{Z}_4$ fixed seven-planes are involved, but there appear to be a non-trivial mixing angle, thus $\tilde{v}_{U_1} = 4 \times 3/2$ rather than simply 4.

The question at this point is whether any consistent choice of G_7 and $Q_7^{(\prime)}$ would lead to a local cancellation of all the anomalies on the I6 and I6$'$ planes. The answer turns out to be negative in an interesting way: the requirement of local anomaly cancellation does have a unique solution

$$Q_7 = -Q_7' = -\frac{1}{2}(\underline{3}, 0) - \left(\underline{2}, \frac{3}{\sqrt{12}}\right). \tag{5.2}$$

In terms of the seven-plane fields and their boundary conditions, this means $G_7 = SU_3 \times U_1' \subset SU_4$ where $SU_3 \supset SU_2 \times U_1$ while the U_1' factor is invisible. In $SU_3 \times U_1'$ terms, $Q_7 = \frac{1}{2}\{\underline{1}^0 - \underline{8}^0\}$, which means free (Neumann) boundary conditions for all eight of SU_3 vector fields at I6 on the $SO_{10} \times SU_4$ end of x^{11}. Consequently, the fields living on the I6 intersection plane should form complete multiplets of the $(SO_{10} \times SU_4)^{\text{pert}} \times SU_3^{\text{n.p.}}$ gauge group visible at I6. Unfortunately, they do not — cf. the Q_6 row of the Table 5 — which means the solution (5.2) to the local anomaly problems is inconsistent with group theory. Group-theoretically, the only possibilities consistent with all the quantum numbers are $G_7 = SU_2 \times U_1$ and $G_7 = SU_2 \times U_1 \times U_1'$. Both choices lead to non-zero net gauge, gravitational and mixed local anomalies on both the I6 and the I6$'$ intersection planes.

This type of local anomaly miscancellation is common in orbifolds with perturbative abelian groups. Generally, the quantum numbers of the twisted states in such models require some fixed seven-planes to carry *Abelian* non-perturbative gauge fields that lock onto the perturbative U_1 fields on one side of the eleventh dimension, say at I6, but have free boundary conditions on the other side at I6$'$. Locally on the I6$'$ plane, there are no $F_{U_1}^4$ inflow or intersection anomalies (since the U_1 does not lock onto the ten-plane gauge fields on *that* side) but there is a non-zero quantum $F_{U_1}^4$ anomaly because there are U_1 charged hyper-multiplets (twisted states) but no charged vector-multiplets (if there were charged vector fields, the group would not be Abelian). Altogether, *the Abelian gauge anomaly $F_{U_1}^4$ does not cancel locally* on the I6$'$ plane.

'Experimentally', in all the models we have studied, a miscancelled Abelian gauge anomaly is accompanied by miscancelled anomalies involving other non-perturbative gauge fields living on the same plane as well as gravity. On the other hand, all anomalies involving the un-mixed purely perturbative gauge fields do cancel out. Algebraically,

$$\mathcal{A}(\text{I6}) = -\mathcal{A}(\text{I6}') = P(F_7, R) \tag{5.3}$$

for some quartic polynomial P in seven-dimensional gauge and gravitational curvature two-forms F_7 and R.

The algebraic structure (5.3) of the net local anomaly suggests that it may ultimately cancel against a Chern–Simons action term in the effective theory on the fixed seven-plane. Indeed, consider

$$\text{Action} \supset \int_{7P} I_7, \tag{5.4}$$

where I_7 is a Chern–Simons 7-form which varies under gauge transformations by a total derivative, $\delta I_7 = d I_6^1$. Consequently,

$$\delta \text{Action} = \int_{\partial(7P)} I_6^1 = \int_{I6'} I_6^1 - \int_{I6} I_6^1, \tag{5.5}$$

which looks *exactly* like local anomalies on the I6 and I6′ boundary six-planes with exactly opposite anomaly polynomials

$$I_8(\text{I6}') = -I_8(\text{I6}) = d I_7. \tag{5.6}$$

Therefore, if $d I_7 = P(F_7, R)$ (cf. Eq. (5.3)) or rather $d I_7 = P/(192\pi^3 i)$, then all anomalies would cancel locally.

We are currently investigating whether such Chern–Simons terms do actually cancel the residual local anomalies of orbifold models with Abelian gauge factors. There are (at least) two issues that must be addressed. First, there is a question of normalization: gauge, gravitational and mixed Chern–Simons terms all have quantized coefficients, which may turn out to be consistent or inconsistent with the local anomaly cancellation in various models. The second open problem is the physical origin of the 7d Chern–Simons terms. All anomalies cancel locally in models considered in Sections 2–4 without any help from the CS terms, so why should the CS terms appear in other models? At the moment the CS terms still seem to pose as many new questions as they are able to answer. We hope to be able to report some progress on this issue in the near future.

6. Summary and discussion

In this paper we have investigated the correspondence between perturbative heterotic $E_8 \times E_8$ orbifolds on $T^4/\mathbf{Z}_N$ ($N = 2, 3, 4, 6$) and their M-theory duals. Our aim was to obtain information about the structure of M-theory orbifolds, in particular the spatial locations of all the charged fields and the gauge groups that act upon them. Our main tools were constraints arising from local consistency of quantum numbers, from the gauge coupling considerations and from the requirements of local anomaly cancellation. We have put forward a scenario which allowed us to resolve an apparent paradox associated with the existence of twisted massless states simultaneously charged under gauge groups which appear to live on two different boundary ten-planes. The resolution relies on the mixing of a factor (or factors) of the gauge group G living on one boundary ten-plane with the G_7 gauge groups living on the fixed seven-planes $R^{5,1} \otimes \{\text{f.p.}\} \otimes S^1/\mathbf{Z}_2$ of the orbifold. The twisted states which live on the I6′ planes where the fixed seven-planes reach the other ten-plane boundary thus have perfectly local $G_7 \times G'$ charges. Thanks to mixing, however, the G_7 charges of those states masquerade as the perturbative G charges from the six-dimensional point of view, hence the appearance of the simultaneous G and G' charges in the heterotic perturbation theory.

Considerations of the gauge couplings confirm that there is a mixing between the non-perturbative gauge groups on the seven-planes and the perturbative gauge group on the boundary planes. Another confirmation comes from the overall 6d anomaly polynomial. The most powerful test however is the local anomaly cancellation. Here the correct allocation of all states — in particular the hyper and vector component of the seven-plane gauge multiplets — to the six-planes where the anomaly is required to cancel is absolutely crucial. A consistent assignment of states was only possible after a judicious choice of boundary conditions on the seven-plane fields which projects out the vector components on one side and the hyper-component on the other. This also guaranteed that there are no additional massless states in the $R_{11} \to 0$ heterotic limit.

There are three contributions to the net local anomaly: quantum anomalies, inflow anomalies and intersection anomalies. Given a correct identification of all the locally relevant fields, the quantum one-loop anomalies are fairly straightforward, but the inflow and intersection anomalies involve un-settled issues of the overall normalization. In principle, the overall normalization of the inflow anomalies follows from that of the Chern–Simons terms in eleven-dimensional SUGRA, but the derivation is rather subtle and the result is controversial. Instead, we simply calibrated the normalization by requiring local anomaly cancellation in the $\mathbf{Z}_2$ model (cf. Section 3), then used the same normalization in all the other models of Section 4; eventually, this normalization ought to be confirmed by a direct derivation.

Likewise, the η and ρ parameters of the intersection anomaly (cf. Eqs. (3.11) and (3.12)) should follow from the seven-dimensional Chern–Simons terms (which in turn follow from the M-theory) and indeed the analysis of Section 4.4 gives specific values $\eta = T(1)$ and $\rho = 1$. Alternatively, we can treat them as free parameters, fixed by the anomaly cancellation requirements of each model. According to Eqs. (4.17), $\eta = T(1)$ is indeed universally valid, but the ρ parameter is more tricky: $\rho = 1$ works for model with $G_7 = SU_n$ (cf. Sections 3 and 4) but models with the invisible Abelian $G_7 = U_1^{(n-1)}$ (cf. Section 5.1) require $\rho = 0$ instead. Unfortunately, we do not know yet the M-theoretic origin of the seven-plane SU_n breakdown to its Cartan subgroup in these models, and we do not know how this Cartan subgroups ends up with $\rho = 0$ either.

A worse anomaly trouble plagues models with Abelian factors in their perturbative gauge groups, cf. Section 5.2. In those models, the local anomalies simply do not cancel for any consistent choices of the seven-plane gauge fields and their boundary conditions at the two ends of the x^{11}. This presumably means one of the two things: either we do not know how to correctly interpret such models in M-theory terms or else there must be additional sources of local anomalies. For example, a Chern–Simons term in the seven-plane effective action has an effect of transferring local 6d anomaly from one boundary of the seven plane to the other boundary, thus possibly helping to cancel the local anomalies on both boundaries. Much work is needed however before we know whether this mechanism really works.

Our main conclusion is that the heterotic orbifolds do have consistent M-theory duals with identical massless spectra and gauge couplings and locally cancelled anomalies. Unfortunately, while the duality and the local anomalies tell us *what* should happen on

every six-, seven- or ten-plane of the dual model, we do not understand *how* it happens in M-theory. The big mystery is the dynamic origin of the boundary conditions for the seven-plane fields. In particular, we know that each seven-plane field always has exactly opposite boundary conditions on the two boundaries of the 7P, but the M-theoretic reasons for this twist remain completely obscure. Another mystery why the $\mathbf{Z}_n$-fixed seven-planes carry full SU_n gauge groups in some models while in other models the SU_n is broken to a subgroup; in fact, we are not even sure of the mechanism of such seven-plane symmetry breaking. We are however quite certain that the eventual resolution of these issues will shed much new light on the basic structure of the M-theory.

We conclude this article with a few comments on M-theory duals of the four-dimensional heterotic orbifolds. Generally, such models live on $\mathbb{R}^{3,1} \otimes (T^6/\Gamma) \otimes (S^1/\mathbf{Z}_2)$ comprising the 11d bulk, fixed seven-planes, fixed five-planes as well as their respective ten-plane, six-plane and four-plane boundaries. The bulk, the seven-planes and their boundaries should behave just as they do in the 6d orbifolds (modulo orbifolding of their extra compact coordinates), it's the five-planes that radically complicate the physics. The problem with the fixed five-planes is that they are poorly understood from the M-theory point of view. We know that the singular five-planes in M-theory carry superconformal theories whose excitations include both massless particles and tensionless strings. Generally, the superconformal theories in five dimensions are associated with infinite-coupling gauge theories, but the M-theory tells us nothing about the gauge group, only the global symmetries (if any) and some gauge-invariant operators — and these data are quite insufficient for the model building purposes. Also, the constraints of local anomaly cancellation on the four-plane boundaries of the five-planes are rather weak, simply because the anomalies in four dimensions are much simpler than in six. Consequently, although we have tentatively identified the M-theory duals of a few 4d heterotic orbifolds, we do not have enough constraints to be confident in our identification at the time of this writing. This work is in progress and we hope to present some interesting results in the near future.

Acknowledgements

Appendix A. Anomaly polynomials for $\mathcal{N} = 1$ multiplets in $d = 6$

This appendix summarizes the six-dimensional gravitational, gauge and mixed anomaly contributions of all relevant six-dimensional $\mathcal{N} = 1$ multiplets, see, e.g., [15]. The representation content of each such multiplet is spelled out in terms of the little group

$SO_4 \simeq SU_2 \times SU_2$ for massless particles in 6d. The correctly normalized anomaly polynomials are

$$I_8 = \frac{-i}{4!(2\pi)^3}\,\mathcal{A}, \tag{A.1}$$

but since we use $\mathcal{A}$ instead of I_8 throughout this paper, the anomalies below are written in terms of $\mathcal{A}$ as well.

• Gravity multiplet $[(3,3) + 2(2,3) + (1,3)]$:

$$\mathcal{A}_{\text{grav}} = -\frac{273}{240}\,\text{tr}\,R^4 + \frac{51}{192}\big(\text{tr}\,R^2\big)^2. \tag{A.2}$$

tr is the trace in the vector representation of $SO(6)$.

• Vector-multiplets $n_V[(2,2) + 2(1,2)]$:

$$\mathcal{A}_{\text{V}} = -\frac{n_{\text{V}}}{240}\,\text{tr}\,R^4 - \frac{n_{\text{V}}}{192}\big(\text{tr}\,R^2\big)^2 + \frac{1}{4}\,\text{tr}\,R^2\,\text{Tr}_{\text{V}}\,F^2 - \text{Tr}_{\text{V}}\,F^4; \tag{A.3}$$

since the vector multiplets comprise the adjoint representation of the gauge group G, Tr_V is the trace in the adjoint representation and $n_V = \dim(G)$.

• Hyper-multiplets $n_H[2(2,1) + 4(1,1)]$:

$$\mathcal{A}_{\text{H}} = \frac{n_{\text{H}}}{240}\,\text{tr}\,R^4 + \frac{n_{\text{H}}}{192}\big(\text{tr}\,R^2\big)^2 - \frac{1}{4}\,\text{tr}\,R^2\,\text{Tr}_{\text{H}}\,F^2 + \text{Tr}_{\text{H}}\,F^4, \tag{A.4}$$

where Tr_H is the trace in the representation of G comprised of all the hypermultiplets (whatever such representation might be in any particular model) and n_H is the net number of hypermultiplets in the model.

• Selfdual-tensor multiplet $[(3,1) + 2(2,1) + (1,1)]$:

$$\mathcal{A}_{\text{tensor}} = \frac{29}{240}\,\text{tr}\,R^4 - \frac{7}{192}\big(\text{tr}\,R^2\big)^2. \tag{A.5}$$

Appendix B. Further anomaly considerations

In Section 2.2, we mentioned that in perturbative K3 compactifications of the heterotic string, the gauge couplings are related to the 6d anomaly polynomial. In this appendix, we explain this relation.

The anomaly we are concerned with here is the overall anomaly of the 6d effective theory rather than local anomalies on some singular six-planes. In perturbative K3 compactifications, there is just one tensor field $B_{\mu\nu}$, hence the Green–Schwarz mechanism of anomaly cancellation works *only* if the net anomaly happens to factorize in the form [8]:

$$\frac{2}{3}\mathcal{A} = \left(\text{tr}\,R^2 - \sum_\alpha v_\alpha\,\text{tr}\,F_\alpha^2\right) \wedge \left(\text{tr}\,R^2 - \sum_\alpha \tilde{v}_\alpha\,\text{tr}\,F_\alpha^2\right) \equiv I_4 \wedge \tilde{I}_4. \tag{B.1}$$

Furthermore, the $\mathcal{N} = 1$, $d = 6$ supersymmetry relates the couplings of the $B_{\mu\nu}$ tensor fields to those of the heterotic dilaton ϕ, hence the factorization coefficient v_α and $\tilde{v}_\alpha$ in this formula also show up in the exact Eqs. (2.2) for the gauge couplings.

The Green–Schwarz mechanism does not cancel the irreducible $\mathrm{tr}\, R^4$ term in the anomaly polynomial, hence anomaly-free theories with just one selfdual-tensor multiplet must have $n_H - n_V = 244$, cf. individual multiplets' anomalies listed in Appendix A. Consequently, the net anomaly can be summarized as

$$\frac{2}{3}\mathcal{A} = \left(\mathrm{tr}\, R^2\right)^2 - \frac{1}{6}\mathrm{tr}\, R^2\, \mathrm{Tr}_{H-V}\, F^2 + \frac{2}{3}\mathrm{Tr}_{H-V}\, F^4, \tag{B.2}$$

were the notation Tr_{H-V} denotes the trace taken over all hyper-multiplets minus the trace over all vector-multiplets. The relative minus-sign follows from opposite chiralities of fermions in $\mathcal{N} = 1$, $d = 6$ hyper- and vector-multiplets. Comparing the mixed gauge/gravitational anomaly terms in Eqs. (B.2) and (B.1), we see that Green–Schwarz anomaly cancellation requires

$$\mathrm{Tr}_{H-V}\, F^2 = \sum_\alpha 6(v_\alpha + \tilde{v}_\alpha)\, \mathrm{tr}\, F_\alpha^2. \tag{B.3}$$

(See Appendix C for exact normalizations of various traces.) Curiously, $\mathrm{Tr}_{H-V}\, F^2$ also appears in beta-function coefficients of $\mathcal{N} = 2$, $d = 4$ gauge theories (including but not limited to toroidal compactifications of the $\mathcal{N} = 1$, $d = 6$ theories); specifically, $\mathrm{Tr}_{H-V}\, F^2 = \sum_\alpha b_\alpha\, \mathrm{tr}\, F_\alpha^2$. Comparing this formula to Eq. (B.3), we immediately arrive at Eq. (2.3).

From the M-theory point of view, there is another interesting way to write the anomaly polynomial as a sum of two factorized terms, each associated with the boundary at the corresponding end of x^{11}. For smooth K3 compactifications (perturbative or otherwise) with instanton numbers $n_1 + n_2 = 24$ (hence exactly one selfdual-tensor multiplet), we have [16]

$$\frac{2}{3}\mathcal{A} = \left(\frac{1}{2}\mathrm{tr}\, R^2 - \sum_\alpha v_{\alpha 1}\, \mathrm{tr}\, F_{\alpha 1}^2\right) \wedge \left(\frac{1}{4}(n_1 - 8)\, \mathrm{tr}\, R^2 - \sum_\alpha \tilde{v}_{\alpha 1}\, \mathrm{tr}\, F_{\alpha 1}^2\right)$$

$$+ \left(\frac{1}{2}\mathrm{tr}\, R^2 - \sum_\alpha v_{\alpha 2}\, \mathrm{tr}\, F_{\alpha 2}^2\right) \wedge \left(\frac{1}{4}(n_2 - 8)\, \mathrm{tr}\, R^2 - \sum_\alpha \tilde{v}_{\alpha 2}\, \mathrm{tr}\, F_{\alpha 2}^2\right). \tag{B.4}$$

There are similar expressions for singular K3's such as orbifolds. For example, the $\mathbf{Z}_2$ model of Section 2 has

$$\frac{2}{3}\mathcal{A} = \left(\frac{1}{2}\mathrm{tr}\, R^2 - \mathrm{tr}\, F_{E_7}^2 - \mathrm{tr}\, F_{SU_2}^2\right) \wedge \left(2\,\mathrm{tr}\, F_{E_7}^2 - 14\,\mathrm{tr}\, F_{SU_2}^2\right)$$

$$+ \left(\frac{1}{2}\mathrm{tr}\, R^2 - \mathrm{tr}\, F_{SO_{16}}^2\right) \wedge \left(2\,\mathrm{tr}\, R^2 - 2\,\mathrm{tr}\, F_{SO_{16}}^2 - 16\,\mathrm{tr}\, F_{SU_2}^2\right). \tag{B.5}$$

We observe that one of the gauge factors, namely SU_2, now appears on both sides but only non-perturbatively ($v = 0$, $\tilde{v} = 16$) on the SO_{16} side. This lends support to the M-theory description of this model that we have put forward in Section 2. We have a non-perturbatively generated gauge group $(SU_2)^{16}$. Each of these SU_2's has $\tilde{v} = 1$ and they mix with the perturbative SU_2 such that only the diagonal SU_2 contributes in the heterotic description of the model, i.e., only one SU_2 is visible. This SU_2 is however visible on both sides of the x^{11} interval.

Appendix C. Some results from group theory

We collect some group theoretical results which are needed to verify anomaly cancellation. The notation is such that Tr_R is always the trace in the representation R whereas $\text{tr}\, F^2$ means $\sum_a F^a F^a$. The normalizations are such that the long roots are normalized to length 1.

SU_2

$$\text{Tr}_{\underline{2}}\, F^2 \equiv \frac{1}{2}\,\text{tr}\, F^2, \qquad \text{Tr}_{\underline{2}}\, F^4 = \frac{1}{8}\big(\text{tr}\, F^2\big)^2,$$
$$\text{Tr}_{\underline{3}}\, F^2 = 2\,\text{tr}\, F^2, \qquad \text{Tr}_{\underline{3}}\, F^4 = 2\big(\text{tr}\, F^2\big)^2, \tag{C.1}$$

SU_3

$$\text{Tr}_{\underline{3}}\, F^2 = \frac{1}{2}\,\text{tr}\, F^2, \qquad \text{Tr}_{\underline{3}}\, F^4 = \frac{1}{8}\big(\text{tr}\, F^2\big)^2,$$
$$\text{Tr}_{\underline{8}}\, F^2 = 3\,\text{tr}\, F^2, \qquad \text{Tr}_{\underline{8}}\, F^4 = \frac{9}{4}\big(\text{tr}\, F^2\big)^2, \tag{C.2}$$

$SU_N,\ N \geqslant 4$

$$\text{Tr}_{\underline{N}}\, F^2 = \frac{1}{2}\,\text{tr}\, F^2, \qquad \text{Tr}_{\underline{N}}\, F^4 = \frac{3}{16}\big(\text{tr}\, F^2\big)^2 - \frac{1}{4}\,\text{tr}\, F^4,$$
$$\text{Tr}_{\text{ad}}\, F^2 = N\,\text{tr}\, F^2, \qquad \text{Tr}_{\text{ad}}\, F^4 = \frac{3}{8}(N+4)\big(\text{tr}\, F^2\big)^2 - \frac{N}{2}\,\text{tr}\, F^4,$$
$$\text{Tr}_{\underline{\text{8}}}\, F^2 = \frac{N-2}{2}\,\text{tr}\, F^2,$$
$$\text{Tr}_{\underline{\text{8}}}\, F^4 = \frac{3}{16}(N-4)\big(\text{tr}\, F^2\big)^2 + \left(2 - \frac{N}{4}\right)\text{tr}\, F^4,$$
$$\text{Tr}_{\underline{\text{8}}}\, F^2 = \frac{1}{4}(N-2)(N-3)\,\text{tr}\, F^2,$$
$$\text{Tr}_{\underline{\text{8}}}\, F^4 = \frac{3}{32}\big(N^2 - 9N + 22\big)\big(\text{tr}\, F^2\big)^2 - \frac{1}{8}\big(N^2 - 17N + 54\big)\text{tr}\, F^4, \tag{C.3}$$

$SO_N,\ N \geqslant 5$

$$\text{Tr}_{\underline{N}}\, F^2 \equiv \text{tr}\, F^2, \qquad\qquad \text{Tr}_{\underline{N}}\, F^4 \equiv \text{tr}\, F^4,$$
$$\text{Tr}_{\text{ad}}\, F^2 = (N-2)\,\text{tr}\, F^2, \qquad \text{Tr}_{\text{ad}}\, F^4 = 3\big(\text{tr}\, F^2\big)^2 + (N-8)\,\text{tr}\, F^4,$$
$$\text{Tr}_{\text{spinor}}\, F^2 = \frac{\text{d}}{8}\,\text{tr}\, F^2, \qquad\quad \text{Tr}_{\text{spinor}}\, F^4 = \frac{\text{d}}{4}\left[\frac{3}{16}\big(\text{tr}\, F^2\big)^2 - \frac{1}{4}\,\text{tr}\, F^4\right], \tag{C.4}$$

where $\text{d} = 2^{\frac{N}{2}-1}$ (N even) and $\text{d} = 2^{\frac{N-1}{2}}$ (N odd) is the dimension of the spinor representation.

E_6

$$\text{Tr}_{\underline{27}}\, F^2 \equiv 3\,\text{tr}\, F^2, \qquad \text{Tr}_{\underline{27}}\, F^4 = \frac{3}{4}\big(\text{tr}\, F^2\big)^2,$$
$$\text{Tr}_{\underline{78}}\, F^2 = 12\,\text{tr}\, F^2, \qquad \text{Tr}_{\underline{78}}\, F^4 = \frac{9}{2}\big(\text{tr}\, F^2\big)^2, \tag{C.5}$$

E_7

$$\mathrm{Tr}_{\underline{56}}\, F^2 \equiv 6\,\mathrm{tr}\, F^2, \qquad \mathrm{Tr}_{\underline{56}}\, F^4 = \frac{3}{2}\big(\mathrm{tr}\, F^2\big)^2,$$

$$\mathrm{Tr}_{\underline{133}}\, F^2 = 18\,\mathrm{tr}\, F^2, \qquad \mathrm{Tr}_{\underline{133}}\, F^4 = 6\big(\mathrm{tr}\, F^2\big)^2, \tag{C.6}$$

E_8

$$\mathrm{Tr}_{\mathrm{ad}}\, F^2 = 30\,\mathrm{tr}\, F^2, \qquad \mathrm{Tr}_{\mathrm{ad}}\, F^4 = 9\big(\mathrm{tr}\, F^2\big)^2. \tag{C.7}$$

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