Black holes in the brane world: Time symmetric initial data

Tetsuya Shiromizu\textsuperscript{1,2,3} and Masaru Shibata\textsuperscript{4,5}

\textsuperscript{1}MPI für Gravitationsphysik, Albert-Einstein-Institut, D-14476 Golm, Germany
\textsuperscript{2}Department of Physics, The University of Tokyo, Tokyo 113-0033, Japan
\textsuperscript{3}Research Center for the Early Universe (RESCEU), The University of Tokyo, Tokyo 113-0033, Japan
\textsuperscript{4}Department of Physics, University of Illinois at Urbana-Champaign, Urbana, IL 61801, USA
\textsuperscript{5}Department of Earth and Space Science, Osaka University, Toyonaka 560-0043, Japan

We numerically construct time-symmetric initial data sets of a black hole in the Randall-Sundrum brane world model, assuming that the black hole is spherical on the brane. We find that the apparent horizon is cigar-shaped in the 5D spacetime.

I. INTRODUCTION

Motivated by Horava-Witten model\textsuperscript{[1]}, the so-called brane world model has been actively investigated\textsuperscript{[2]}. Among several models, a simple, but very attractive model was recently proposed by Randall and Sundrum\textsuperscript{[3]}. According to their scenario, we are living in a 4D domain wall in 5D bulk spacetime. The noteworthy features of their model are that in the linearized theory, the conventional geometry can be recovered on the brane\textsuperscript{[4]}, and that a homogeneous, isotropic universe can be simply described if we consider a 4D domain wall in the 5D Schwarzschild-antidote-Sitter spacetime\textsuperscript{[5]}.

One of the most non-linear objects in the theory of gravity is a black hole, which should be also investigated to understand the nature of the models in strong fields. However, because of the complexity of the equations, any realistic solutions for black holes have not been discovered in the brane world model, even with help of numerical computation so far. We only know that the effective 4D gravitational equation on the brane is different from the Einstein equation\textsuperscript{[6]} (see Appendix A), so that the static solution for a non-rotating black hole should not be identical with the 4D Schwarzschild solution. Indeed, a linear perturbation analysis\textsuperscript{[7]} shows that a solution of gravitational field outside self-gravitating bodies on the brane is slightly different from the 4D Schwarzschild solution. Chamblin et al.\textsuperscript{[8]} conjecture that the topology of black hole event horizons would be spherical with the cigar-shaped surface in the 5D spacetime. However, nothing has been clarified substantially.

In this paper, as a first step toward self-consistent studies for black holes in the brane world, we numerically compute a black hole space using a time symmetric initial value formulation; namely, we solve the 5D Einstein equation only on a spacelike 4D hypersurface. Thus, the black hole obtained here is not static nor the exact solution for the 5D Einstein equation, implying that we cannot identify the event horizon. However, we can investigate the property of the horizon determining the apparent horizon which could give us an insight on the black hole in the brane world. We focus on the Randall-Sundrum's second model\textsuperscript{[3]}, and assume that the black hole is spherical on the brane, but the shape of the horizon is non-trivial in the bulk. We will determine the apparent horizon on the brane and show that the black hole is cigar-shaped as conjectured in\textsuperscript{[10]}.

II. FORMULATION AND RESULTS

We consider time symmetric, spacelike hypersurfaces in the brane world model assuming the vanishing extrinsic curvature; i.e.,

\[ H = ( + t \cdot t )^{(4)} r \cdot t = 0; \]

where \( t \) is the unit normal timelike vector to \( \tau \), and \( (4)^r \) is the covariant derivative with respect to the 4D metric on \( \tau \). In this case, the momentum constraint is satisfied trivially, and the equation of the Hamiltonian constraint becomes

\[ (4)^r R = 16 G_5 ( \cdot + (5)^T t \cdot t ); \]

where \( (4)^r R \) is the Ricci scalar on \( \tau \), and \( G_5 ( = \frac{2}{5} = 8 ) \), and \((5)^T t \) denote the gravitational constant, negative cosmological constant, and energy-momentum tensor in 5D spacetime [cf., Eq. (4.3)]. We choose the line element on \( \tau \) in the form

\[ ds^2 = \frac{1}{2} h_{zz} r^2 dr^2 + 4 (dz^2 + r^2 d\theta^2 ); \]

where \( \zeta = \frac{6}{p} \), \( r = 6, z ( = 1 ) \) denotes the coordinate orthogonal to the brane and \( r ( = 0 ) \) is the radial coordinate on the brane. We assume that the brane is located at \( z = 1 \). Note that we simply choose this line element for convenience of the analysis. In this paper, we focus on a black hole which is spherical on the brane, i.e., \( z = (x; z) \). Then, the explicit form of the Hamiltonian constraint in the bulk (for \( z > 1 \)) is written in the form...
\[
\omega + \frac{2}{r} + \frac{3}{2r^2} \theta_z^2 + \frac{3}{2z} \theta_z^4 + 3(\theta_z^2)^3 = \frac{2}{4} \tag{2.4}
\]

where \( \omega = \theta = \sigma \), and \((a)\) is the energy-momentum tensor in the bulk, as introduced for numerical convenience.

Equation (2.4) is an elliptic type equation and should be solved in posing boundary conditions at \( z = 1, z = \frac{1}{2}, r = 0 \), and \( r = 1 \). The boundary condition at \( z = 1 \) is derived from Israel’s junction condition \((b)\) as (see Appendix A for the derivation)

\[ \theta_z \bigg|_{z=1} = 0 \]  \hspace{1cm} (2.5)

The boundary conditions at \( z = 1 \) and \( r = 1 \) are obtained from the linear perturbation analysis (see Appendix B). For \( r > 'z, z \) becomes

\[ 1 + \frac{M G^4}{2r} + \frac{1}{2} \frac{R^2}{r^2} + O ('z)^4 \tag{2.6} \]

where \( G^4 \equiv G_5 = 'z \), \( M \) is the gravitational mass on the brane, and \( R = (2\pi)^{3/2} \). For \( z = 1 \),

\[ 1 + \frac{3G^4 M}{4R^2} + \frac{r^2}{R^2} = 0 \tag{2.7} \]

To determine the existence of a black hole, we search for the apparent horizon. Here, we determine two horizons \((c)\). One is defined to be the spherical two-surface on the brane on which the expansion of the null geodesic congruence connected on the brane is zero \((d)\), i.e.,

\[ \frac{3}{2} z^2 + \frac{1}{r} = 0 \tag{2.8} \]

The other is the apparent horizon in full 4D space, which is defined with respect to the null geodesic congruence in full 5D spacetime and satisfies \((e)\)

\[ 4 = (4) r = s^2 \tag{2.9} \]

where \( s^2 \) is a unit normal to the surface of the apparent horizon. Explicit equation for determining this apparent horizon is shown in Appendix C.

The procedure of numerical analysis is as follows. First, we artificially put the matter of \( h \) \((a)\) \( r = 0 \) in the bulk. This method is employed because we do not have to consider the inner boundary condition of black holes with this treatment. As long as \( h \) is connected around the brane and inside the horizon, it does not significantly affect the geometry outside the horizon. Then, we solve Eq. (2.4), and try to find the apparent horizon both on the brane and in the bulk. When the distribution of \( h \) is sufficiently compact, the apparent horizon exists. It should be noted that two horizons do not coincidentally appear. In some cases, the apparent horizon on the brane exists although that in the bulk does not.

---

**FIG. 1.** Location of the apparent horizons on the brane (solid circle) and in the 4D space (solid line). An articial attar is connected in the region shown by the dashed line.

**FIG. 2.** Pro le of 1 on the brane (solid line). Location of the apparent horizon on the brane is shown. The dashed line denotes 1 = M/2.

Here, we show one example of numerical results. We set \( G^4 = 1 \). In this example, an artificial attar is put for \( r < 0.2R \) and \( 1 < 1.2 \). Equation (2.4) is solved using a uniform grid with grid size 1200 \( 1200 \) for \( r \) and \( z \) directions, which covers a domain with \( 0 < R < 17 \) and \( 1 < 18 \). In this case, the gravitational mass on the brane is \( M \) = 0.25, and both apparent horizons on the brane and in the bulk exist. We note that the results are essentially the same for \( M = 0.5 \). In Fig. 1, we show the location of apparent horizons in the bulk and on the brane. The apparent horizon in the bulk is apparently cigar-shaped. Due to this cigar-shape the circumferential radius of the apparent horizon is different depending on the choice of the circumferential radius in the bulk. In Fig. 2, we show that the profile of 1 on the brane. For \( r = R \), 1 behaves as \( M = 2r \), implying that the solution approximate agrees with that in the 4D Einstein gravity, i.e., the bulk effect is small. However, the existence of the bulk is significant for \( R \) as expected. Indeed, 1 deviates from \( M = 2R \) with decreasing \( r \). This effect is in particular important for the location and area of the apparent horizon on the brane. In the case of 4D gravity without bulk, the apparent horizon is located at \( r_{AH} = M = 2 \) with the area \( A_{AH} = 16 \ M^2 \). However, in the brane world model, they take different values in general. (In this example, \( r_{AH} \approx 0.9M \) and \( A_{AH} \approx 8.8M^2 \).)
and the coefficients converge to well-known 4D values (0.5 and 16) with increasing M, implying that the effect of the existence of the bulk becomes less important.)

III. SUMMARY

We numerically computed time symmetric initial data sets of a black hole in the brane world model, assuming that the black hole is spherical on the brane. As has been expected, the black hole (apparent horizon) is cigar-shaped in the bulk [9].

We remark that we only present time symmetric initial data of a black hole space. This implies that the black hole is not static and will evolve to other state with time evolution. The qualitative features of the nat state could be different from the present result. Self-consistent analysis for static black holes should be carried out for future to obtain a definite answer with regard to black holes in the brane world. However, we believe that the present result provides us a guideline for such future works.

ACKNOWLEDGMENTS

We thank B. Carter, N. Dadhich, D. Langlois, R. M. Maitens, K. M. Maeda, M. Sasaki and T. Tanaka for discussions. TS is grateful to thank Relativity Group for their hospitality at G MPI near by Potsdam. MS gratefully acknowledges support by JSPS and the hospitality at the Department of Physics of University of Illinois at Urbana-Champaign.

APPENDIX A: THE ESSENCE OF THE BRANE WORLD

We briefly review the covariant form of the brane world [8]. For the matter source of the 5D Einstein equation, \( G = \frac{5}{2}T \), we choose the energy-momentum tensor as
\[
(\text{5})T = (\text{4})T + (\text{5}) T ; \quad (A1)
\]
where \( T \) is the tension of the brane, \( q \) is the induced metric on the brane, and \( (\text{4})T \) is the energy-momentum tensor on the brane. Due to the singular source at \( T = 0 \) and the \( Z_2 \) symmetry, we can derive the Israel’s junction condition at \( T = 0 \) as
\[
K = \frac{1}{6} q \quad \frac{1}{2} (\text{4})T + \frac{1}{3} (\text{4})T ; \quad (A2)
\]
where \( K = q D \) and \( D \) and \( N \) are the covariant derivative with respect to \( q \) and the normal spacelike vector to the brane. In the text, we consider the case in which \( (\text{4})T = 0 \). Using \( (\text{4}) G \) form, the effective 4D equation on the brane has the form
\[
(\text{4}) G = q E ; \quad (A3)
\]
where \( q = \frac{1}{2} \) + 1 and \( E = (\text{5}) \). The above, for simplicity, we set \( (\text{5}) E = 0 \). Equation (A3) implies that we can consider \( G \) as the effective source term of the 4D Einstein equation on the brane, and as long as \( E \) is not vanishing, the geometry on the brane is different from that in the 4D gravity even in the vacuum case. Only for very special case such as for the black string solution [4], **E** = 0 holds.

From Eq. (4), we find that the Minkowski spacetime is realized on the brane when \( E = 0 \) and \( q = 0 \). In this paper, we set \( q = 0 \) to focus on asymptotically at brane. Then, the junction condition at \( T = 0 \) is rewritten to \( K = \frac{1}{2} q \). In the case when we choose the line element as Eq. (2.3), the junction condition reduces to Eq. (2.3).

APPENDIX B: ASYMPTOTIC BOUNDARY CONDITIONS

To specify the boundary condition at infinity, we investigate the linearized equation of Eq. (2.4):
\[
\gamma_{\mu} + \frac{2}{5} r^2 + 1 R^2 \eta^2 \eta \eta \frac{3}{2} \eta' \frac{3}{4} \eta \frac{3}{4} h + (-1 + \chi) \chi = 0 \quad (B1)
\]
where \( \chi = 1 + \chi \) and \( \chi' = 1 \). We can obtain the formal solution with aid of the Green function \( G_s (x; z; x'; z'; 0) \) as
\[
G_s (x; z; x'; z; 0) \sim 2 G_4 \left[ \frac{d^2}{dz} \eta (x; z; x'; z' 0) h (x; z; 0) \right] \quad (B2)
\]
Assuming that \( h \) is non-zero only in the small region around the brane, we can derive the relevant Green function as [3]
\[
G_s (x; z; x'; z; 0) = \frac{Z_1}{Z_2} \eta \int_0^1 \frac{d m}{k} \frac{u_m (0) u_m (1)^i}{k^2 + m^2}, \quad (B3)
\]
where \( u_m (z) \) is the mode function given from the Bessel functions \( J_n \) and \( N_n \) as
\[
u_m (z) = \frac{e^{-\gamma z}}{2} \frac{m R^2}{Z} \quad (B4)
\]
where
\[
\begin{align*}
J_1 (\gamma (R)) & N_2 (\gamma (R)) \quad N_1 (\gamma (R)) J_2 (\gamma (R)) \quad \frac{(J_1 (\gamma (R)))^2 + (N_1 (\gamma (R)))^2}{(\gamma (R))^2},
\end{align*}
\]
where \( R = (2=3)^{1=2} \), \( G_r \) and \( G_{KK} \) are the Green function of zero and KK modes, respectively. From Eq. (B.2) we can derive the asymptotic boundary conditions shown in the text.

**APPENDIX C: APPARENT HORIZON IN THE BULK**

We derive the equation for the apparent horizon in the bulk. After we perform the coordinate transformation from \((x;z)\) to \((x;\varphi)\) as \( z = 1 + x \cdot \text{cos} \varphi \) and \( r = ' \cdot \sin \varphi \), the surface of the apparent horizon is denoted by \( x = h(\varphi) \). Then, the non-zero components of \( g_{ij} \) is written as

\[
s_x = C \quad \text{and} \quad s = C h \; ; \tag{C1}
\]

where \( C \) is a normalization constant calculated from \( s^2 s = 1 \), and \( h_r = \frac{d h}{d \varphi} \). Then, the equation for \( h \) can be written to the following ordinary differential equation of second-order

\[
\frac{d^2 h}{d \varphi^2} = \frac{h^2}{4C^2} \left[ \frac{\theta_x}{h} + \frac{3}{h(1 + h \cdot \text{cos} \varphi)} + \frac{\theta_x C}{h} \right] \cdot \sin^2 \varphi + 4 \cos^2 \varphi \left( \text{cos}^2 \varphi + 4 \sin^2 \frac{h_r}{h} \right)
+ h \left( \frac{\theta_x}{h} + 3 \frac{h \sin \varphi}{1 + h \cdot \text{cos} \varphi} \right) \frac{\cos^2 \varphi + 4 \sin^2 \frac{h_r}{h}}{h} + h \left( \frac{\theta_x}{h} \right) \cos(2 \varphi) + 4h^2 \sin^2 \cos \frac{3}{h} \theta
+ f(1 \cdot \cos(2 \varphi) + 4 \sin^2 \frac{h_r}{h} \; ; \tag{C2}
\]

where

\[
D = C^2 \left( \sin^2 \varphi + 4 \cos^2 \varphi + h \sin \varphi \cos \varphi \right) + 2 \left( \cos + h \sin h_r \right)^2 \); \tag{C3}
\]

Eq. (C2) is solved in imposing boundary conditions at \( \varphi = 0 \) and \( \varphi = \pi \). In the limit \( \varphi = 0 \), we impose the following boundary condition,

\[
h = h_0 + h_2 \cdot \sin \varphi + 0 \left( \cos \varphi \right) ; \tag{C4}
\]

where \( h_2 \) is evaluated at \( x = h_0 \) and \( = 0 \) from the following equation;

\[
h_2 = \frac{h_0^2}{6} \cdot \frac{3 \theta_x}{h_0(1 + h_0)} + \frac{\theta_x C}{C} + \frac{3}{h_0} \left( \sin^2 \varphi \right) \; ; \tag{C5}
\]

At \( \varphi = \pi \), the boundary condition is imposed as \( h = 0 \).

Note that in the limit \( \varphi = \pi \), i.e., on the brane, Eq. (C3) is written in the form

\[
\frac{d^2 h}{d \varphi^2} = h + \frac{h^2}{4h^2} \left( \cos \varphi + 2 \frac{h^2}{h} \right) \; ; \tag{C6}
\]

where we use \( h_2 = 0 \) and the relation \( \theta = D = \theta_x C \). Note that the equation which the apparent horizon on the brane satisfy is \( \theta = 2 \) (see Eq. (2.8)). Thus, unless \( \frac{d^2 h}{d \varphi^2} = 0 \) at \( \varphi = \pi \), the apparent horizon on the brane cannot coincide with that in 4D space. Note that the black string solution [13,14] exceptionally satisfy \( \frac{d^2 h}{d \varphi^2} = 0 \) at \( \varphi = \pi \).