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Open Strings and Non-commutative Geometry of Branes on Group Manifolds [*]

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Abstract

In this contribution we review some recent work on the non-commutative geometry of branes on group manifolds. In particular, we show how fuzzy spaces arise in this context from an exact world-sheet description and we sketch the construction of a low-energy effective action for massless open string modes. The latter is given by a combination of a Yang-Mills and a Chern-Simons like functional on the fuzzy world-volume. It can be used to study condensation on various brane configurations in curved backgrounds.

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1 Introduction

The connection between branes, open strings and non-commutative geometry has received a lot of attention recently. Almost all of these recent studies focused on branes in a flat background with constant B-field. In this case, the brane's world-volume geometry is given by a Moyal-Weyl deformation of the classical algebra of functions on the brane [1, 2, 3] and scattering amplitudes of massless open string modes give rise to a non-commutative Yang-Mills theory [4].

It is of obvious interest to generalize these findings to branes in non-trivial backgrounds. But even though the perturbative analysis of [3] suggests a very general relation between brane geometry and quantization theory, not much progress has been made in terms of background independent investigations. At the moment, new insights can be expected only from those backgrounds that admit an exact world-sheet description. For branes in flat space the latter involves 2-dimensional free field theories. Non-trivial backgrounds, however, require the use of conformal field theory (CFT) techniques.

In this paper, we review results from a series of papers on the geometry of branes on group manifolds G [5, 6, 7]. Our discussion will deal primarily with $G = \mathrm{SU}(2)$, but all statements have an obvious generalization to other groups. While $G = \mathrm{SU}(2) \cong S^3$ appears as part of the NS5-brane geometry and in $AdS_3 \times S^3 \times M_4$, the interest in other group manifolds is more indirect and rests mainly on CFT model building which has its roots in the WZW model.

There is one aspect of branes on group manifolds that deserves to be stressed here. Namely, group manifolds are curved and carry a non-vanishing NSNS 3-form field $H = dB$. This does not only imply that all branes carry non-vanishing B-fields (so that one expects to find non-commutative geometries) but it also represents an interesting deformation of brane geometry in flat space. As we shall show below, the presence of a NS-field H has intriguing consequences on brane dynamics which are similar to the phenomena found in [8]. The main advantage of our scenario below is that it admits a full perturbative string theory description so that stringy effects can be taken into account.

This short note is organized as follows: In the next section we collect some basic mathematical background on the fuzzy (non-commutative) geometry of certain group orbits. Based on a semi-classical analysis, we shall then argue in Section 3 that these fuzzy orbits arise from branes on group manifolds in a particular decoupling limit. Even

though this discussion makes (illegitimate) use of some formulas that were established for branes in flat space, the results can be confirmed by an exact conformal field theory construction. We review this in Section 4 before sketching the computation of gauge theories on the branes' world-volumes in the final section.

2   Orbits, quantization, and fuzzy geometry

Our discussion of branes on group manifolds below will lead us to some famous quantization problem that has been analyzed extensively. It is useful to recall some of this mathematical background before showing how it arises in a string theoretic context. To make our discussion as explicit as possible, we shall restrict our presentation to the example of SU(2). The generalization to other compact groups is mostly obvious (but see [9] for details).

Consider the 3-dimensional Lie algebra $\mathcal{G} = \mathrm{su}(2)$. Its structure constants $f_{ab}{}^c$ define a linear Poisson structure on the space $\mathbb{R}^3$. In terms of the coordinate functions y_a on $\mathbb{R}^3$ it is given by

$$\{\, y_a \,,\, y_b \,\} \;=\; f_{ab}{}^c \, y_c \;\; . \tag{2.1}$$

This Poisson algebra has a large center. In fact, any function of $c(y) = \sum_a y_a^2$ has vanishing Poisson bracket with any other function on $\mathbb{R}^3$ so that formula (2.1) induces a Poisson structure on the 2-spheres

$$c(y) \;:=\; \sum_a y_a^2 \;\overset{!}{=}\; \mathrm{c} \tag{2.2}$$

of points which have the same distance c from the origin in $\mathbb{R}^3$. These 2-spheres are the Poisson spaces we want to consider. In the case c $= 0$ the 2-sphere degenerates to a single point.

Quantization of this structure requires to find some operators Y_a acting on a state space V such that

$$[\,\mathrm{Y}_a \,,\, \mathrm{Y}_b\,] \;=\; i\, f_{ab}{}^c \, \mathrm{Y}_c \tag{2.3}$$

$$\mathrm{C} \;:=\; \sum \mathrm{Y}_a^2 \;=\; \mathrm{c}\, \mathbf{1} \tag{2.4}$$

where $\mathbf{1}$ denotes the identity operator on the state space V. These two requirements are the quantum analogues of the classical relations (2.1, 2.2). The quantization problem posed in rel. (2.3,2.4) is easy to solve. By the commutation relation (2.3), the

operators Y_a have to form a representation of su(2). Condition (2.4) states that in this representation, the quadratic Casimir element C must be proportional to the identity $\mathbf{1}$. This is true if and only if the representation is irreducible. Hence, the irreducible representations of su(2) provide quantizations of our 2-spheres.

Irreducible representations of su(2) are labeled by one discrete parameter $\alpha = 0, 1/2, 1,$ .... This implies that only a discrete set of 2-spheres in $\mathbb{R}^3$ can be quantized with their radii being related to the value of the quadratic Casimir in the corresponding irreducible representation. For each quantizable 2-sphere $S_\alpha^2 \subset \mathbb{R}^3$ we obtain a state space V^α of dimension $\dim V^\alpha = 2\alpha + 1$ equipped with an action of the quantized coordinate functions Y_a on V^α. The latter generate the matrix algebra $\mathrm{Mat}(2\alpha + 1)$.

It is useful to go a bit deeper into exploring these quantized 2-spheres. Let us start by recalling that the space $\mathrm{Fun}(S^2)$ of functions on a 2-sphere is spanned by spherical harmonics $Y_m^J \in \mathrm{Fun}(S^2)$ where J runs through all integer isospins. A product of any two spherical harmonics is again a function on the 2-sphere and hence it can be written as a linear combination of spherical harmonics,

$$Y_l^I \, Y_m^J \;=\; \sum_{K,n} c_{IJK} \left[\begin{smallmatrix} I & J & K \\ l & m & n \end{smallmatrix}\right] Y_n^K \tag{2.5}$$

with [:::] denoting the Clebsch-Gordan coefficients of su(2). The structure constants c_{IJK} can be found e.g. in [9].

The algebra $\mathrm{Fun}(S^2)$ admits an action of su(2) which is obtained from the adjoint action $t_a : y_b \mapsto f_{ab}{}^c y_c$ of su(2) on $\mathbb{R}^3$. It is easy to see that this action preserves the constraint $c(y) = c$ and hence it descends to $\mathrm{Fun}(S^2)$. Since the spherical harmonics $Y_a^1 \in \mathrm{Fun}(S^2)$ are obtained by restricting the coordinate functions y_a from $\mathbb{R}^3$ to $S^2 \in \mathbb{R}^3$, they transform as $Y_b^1 \mapsto f_{ab}{}^c Y_c^1$ under the action of the generator $t_a \in$ su(2). Spherical harmonics Y_m^J form multiplets with respect to the su(2) action on $\mathrm{Fun}(S^2)$. This classical action survives the quantization, i.e. there exists an analogous action of su(2) on $\mathrm{Mat}(2\alpha + 1)$. It is defined by

$$t_a : \; A \;\mapsto\; [\, t_a^\alpha \,, A\,] \quad \text{for all} \quad A \in \mathrm{Mat}(2\alpha + 1) \,,$$

where t_a^α are the generators of su(2) evaluated in the $(2\alpha + 1)$-dimensional irreducible representation. In particular, it follows from the identification $Y_b^1 := Y_b \sim t_b^\alpha$ that $t_a : Y_b^1 \mapsto i f_{ab}{}^c Y_c^1$ as in the classical case.

Under the su(2)-action we have just described, the space $\mathrm{Mat}(2\alpha+1)$ decomposes into multiplets which are spanned by matrices Y_m^J where $J = 0, 1, \ldots, 2\alpha$. The product of any two such matrices can be expressed as a linear combination of matrices Y_n^K,

$$\mathrm{Y}_l^I \, \mathrm{Y}_m^J \;=\; \sum_{K \leq 2\alpha, n} \left\{ {}_{\alpha}^{I} \, {}_{\alpha}^{J} \, {}_{\alpha}^{K} \right\} \left[{}_{l}^{I} \, {}_{m}^{J} \, {}_{n}^{K} \right] \mathrm{Y}_n^K \quad . \tag{2.6}$$

Here $\{:::\}$ denote the recoupling coefficients (or 6J-symbols) of su(2). This relation can be considered as a quantization of the expansions (2.5) and the classical expression is recovered from eq. (2.6) upon taking the limit $\alpha \to \infty$. Hence, the matrices Y_m^J in the quantized theories are a proper replacement for spherical harmonics. Note, however, that the angular momentum $J \leq 2\alpha$ is bounded from above. This may be interpreted as 'fuzzyness' of the quantized 2-spheres on which short distances cannot be resolved [10]. We shall eventually refer to Y_m^J as 'fuzzy spherical harmonics'.

3 Branes on Lie groups – semi-classical geometry

Strings moving on a 3-sphere S^3 of radius $R \sim \sqrt{\mathsf{k}}$ are described by the SU(2) WZW model at level k. By the string equations of motion the 3-sphere comes equipped with a constant NS 3-form field strength $H \sim 1/\sqrt{\mathsf{k}} \, \Omega$ where Ω denotes the volume form of the unit sphere.

The world-sheet swept out by an open string in S^3 is parametrized by a map $g : \mathrm{H} \to \mathrm{SU}(2)$ from the upper half-plane H into the group manifold $\mathrm{SU}(2) \cong S^3$. We shall be interested in maximally symmetric D-branes on SU(2) that are characterized by imposing the condition $J(z) = \bar{J}(\bar{z})$ on chiral currents all along the boundary $z = \bar{z}$, i.e.

$$-\mathsf{k} \, (\partial g) g^{-1} \;=:\; J(z) \;\overset{!}{=} \Big|_{z=\bar{z}} \; \bar{J}(\bar{z}) \;:=\; \mathsf{k} \, g^{-1} \bar{\partial} g \quad . \tag{3.7}$$

As we shall discuss below, there exists an exact solution of the boundary WZW model with rel. (3.7). Even though the latter essentially goes back to Cardy [13], the geometrical interpretation of the condition (3.7) was only found in [5] (see also [11],[12]).

To describe the findings of [5], we split $\partial, \bar{\partial}$ into derivatives ∂_x, ∂_y tangential and normal to the boundary and rewrite eq. (3.7) in the form

$$(\mathrm{Ad}(g) - 1) \, g^{-1} \partial_y g \;=\; \Big|_{z=\bar{z}} \; i \, (\mathrm{Ad}(g) + 1) \, g^{-1} \partial_x g \quad . \tag{3.8}$$

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Here, $\mathrm{Ad}(g)$ denotes the adjoint action (i.e. action by conjugation) of $\mathrm{SU}(2)$ on the Lie algebra su(2). Let us now decompose the tangent space $T_h\mathrm{SU}(2)$ at each point $h \in \mathrm{SU}(2)$ into a part $T_h^{\parallel}\mathrm{SU}(2)$ tangential to the conjugacy class through h and its orthogonal complement $T_h^{\perp}\mathrm{SU}(2)$ (with respect to the Killing form). Using the simple fact that $\mathrm{Ad}_g|_{T_g^{\perp}} = 1$ we can now see that with condition (3.7)

1. the endpoints of open strings on $\mathrm{SU}(2)$ are forced to move along conjugacy classes, i.e.

$$(g^{-1}\partial_x g)^{\perp} = 0 \ .$$

   Except for two degenerate cases, namely the points e and $-e$ on the group manifold, the conjugacy classes are 2-spheres in $\mathrm{SU}(2)$.

2. the branes wrapping conjugacy classes of $\mathrm{SU}(2)$ carry a B-field which is given by

$$B \sim \frac{\mathrm{Ad}(g) + 1}{\mathrm{Ad}(g) - 1} \ . \tag{3.9}$$

   The associated 2-form is obtained as $\mathrm{tr}(g^{-1}dg\, B\, g^{-1}dg)$ and it is easily seen to provide a potential for the NSNS 3-form H.

The second statement follows from eq. (3.8) by comparison with the usual condition $g_{\nu\mu}\, \partial_y X^{\mu} = i\, B_{\nu\mu}\, \partial_x X^{\mu}$. It was shown in [14], [15] that the spherical branes are stabilized by the NSNS background field H.

We know from the theory of branes in flat space that the relevant object that determines the geometry of branes is not B but another anti-symmetric object Θ [1, 2, 3]. Employing the standard relation between Θ and B from flat space, we obtain

$$\Theta = \frac{2}{B - B^{-1}} = \frac{1}{2}\left(\mathrm{Ad}(g^{-1}) - \mathrm{Ad}(g)\right) .$$

Let us have a closer look at Θ in the limit $\mathsf{k} \to \infty$ where the 3-sphere grows and approaches flat 3-space $\mathbb{R}^3$. One can parametrize points on $\mathrm{SU}(2)$ by an object X taking values in the Lie algebra su(2), such that $g \approx 1 + X$. Insertion into our formula for Θ gives

$$\Theta = -\mathrm{ad}(X) \tag{3.10}$$

where ad denotes the adjoint action of su(2) on itself. If we expand $X = y_c t^c$ we can evaluate the matrix elements of Θ more explicitly,

$$\Theta_{ab} = -\left(t_a\,, \mathrm{ad}(X)t_b\right) = -y_c\left(t_a\,, f^c{}_b{}^d\, t_d\right) = f_{ab}{}^c\, y_c \ ,$$

with $(\cdot, \cdot)$ being the Killing form on su(2). Hence, the Poisson structure defined by Θ is determined by the linear Poisson bracket (2.1) that we discussed in the previous section.

Recall that the Moyal-Weyl products that show up for brane geometry in flat space with constant B-field are obtained from the constant Poisson bracket $\{x_\mu, x_\nu\} = \Theta_{\mu\nu}$ on $\mathbb{R}^d$ through quantization. When combined with our semi-classical analysis for branes on $SU(2) \cong S^3$ this suggests that the quantization of 2-spheres in $\mathbb{R}^3$ with Poisson bracket (2.1) becomes relevant for the geometry of branes on $SU(2)$ in the limit where $k \to \infty$. Consequently, we expect the world-volume of branes on $SU(2)$ to be fuzzy 2-spheres. This will be confirmed by the exact construction of open string backgrounds in the next section.

4   Branes in WZW models – the exact solution

As explained in [3, 6], the geometry of branes can be read off from the operator product of tachyonic open string vertex operators; see also [16] for earlier such proposals in the closed string context. This procedure requires to solve the underlying world-sheet theory. In the theory of branes in flat space, Wick's theorem applies so that all information about the boundary theory is encoded in the propagator. This is no longer true for non-trivial backgrounds such as group manifolds. To deal with such more general situations, one employs techniques of boundary conformal field theory. We will give a short survey of the constructions relevant for WZW models.

The chiral fields of the $SU(2)$ WZW model form an affine Kac-Moody algebra denoted by $\widehat{SU}(2)_k$. It is generated by fields $J_a(z)$, $a = 1, 2, 3$, which possess the following operator product expansions

$$J_a(x_1) \, J_b(x_2) \;=\; \frac{k}{2} \, \frac{\delta_{ab}}{(x_1 - x_2)^2} \;+\; \frac{i \, f_{ab}{}^c}{(x_1 - x_2)} \, J_c(x_2) \;+\; \ldots \; . \tag{4.11}$$

The current $J(z)$ from Section 3 is given by $J = J_a t^a$. For the situation we are dealing with (gluing conditions $J = \bar{J}$ in a "parent" CFT on the full complex plane with diagonal modular invariant partition function), Cardy [13] was able to list all possible boundary conditions. There exist $k + 1$ of them, differing in the bulk field one-point functions (brane charges) and labeled by an index $\alpha = 0, \frac{1}{2}, \ldots, \frac{k}{2}$. Without entering a detailed description of these boundary theories [13], we recall that their state spaces

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have the form

$$\mathcal{H}_\alpha = \bigoplus_J N^J_{\alpha\alpha} \, \mathcal{H}^J \tag{4.12}$$

where $\mathcal{H}^J$, $J = 0, 1/2, \ldots, \mathsf{k}/2$, denote irreducible highest weight representations of the affine Lie algebra $\widehat{\mathrm{SU}}(2)_\mathsf{k}$, and where N^K_{IJ} are the associated fusion rules. Note that only integer spins J appear on the right hand side of (4.12).

There exists a variant of the state-field correspondence which assigns a boundary field $V(x)$ to each element $|V\rangle \in \mathcal{H}_\alpha$ (see e.g. [17]). Each representation J contains ground states of lowest possible energy, which form $(2J + 1)$-dimensional multiplets spanned by some basis vectors Y^J_m with $|m| < J$. To these states we assign vertex operators $V[\mathrm{Y}^J_m](x)$. They are similar to the tachyonic vertex operators $V[e_k] = \exp(-ikX)$ for open strings in flat space. While the latter are associated with eigenfunctions of linear momentum k, our vertex operators $V[\mathrm{Y}^J_m]$ carry definite 'angular' momentum, i.e. the obey

$$J_a(x_1) \, V[\mathrm{Y}^J_m](x_2) = \frac{1}{x_1 - x_2} \, V[L_a \mathrm{Y}^J_m](x_2) + \ldots \tag{4.13}$$

where $L_a \mathrm{Y}^J_m = (t^J_a)_{mn} \mathrm{Y}^J_n$ denotes the action of su(2) on the multiplet (Y^J_j). Eq. (4.12) shows that the angular momentum is cut off at $J = \min(2\alpha, \mathsf{k} - 2\alpha)$. In the limit $\mathsf{k} \to \infty$, this cut-off agrees with the one that we encountered in our discussion of fuzzy 2-spheres. In other words, for $\mathsf{k} \to \infty$ the tachyonic open string vertex operators in the boundary theory labeled by α are in one-to-one correspondence with fuzzy spherical harmonics of the fuzzy 2-sphere S^2_α. This can be regarded as the first confirmation of the expectation formulated at the end Section 3.

The operator product expansion of these open string vertex operators are subject to certain factorization constraints (associativity). They were formulated first by Lewellen [18] and solved for minimal models by Runkel in [19]. In case of the WZW boundary theories, the solution reads [19, 6]

$$V[\mathrm{Y}^I_m](x_1) \, V[\mathrm{Y}^J_n](x_2) = \sum_{K,k} x_{12}^{h_K - h_I - h_J} \left[\begin{smallmatrix} I & J & K \\ i & j & k \end{smallmatrix}\right] F_{\alpha K} \left[\begin{smallmatrix} J & I \\ \alpha & \alpha \end{smallmatrix}\right] V[\mathrm{Y}^K_k](x_2) + \ldots \tag{4.14}$$

where $h_J = J(J + 1)/(\mathsf{k} + 2)$ is the conformal dimension of $V[\mathrm{Y}^J]$ and F stands for the fusing matrix of the WZW model. In the limit $\mathsf{k} \to \infty$, the fusing matrix elements approach the $6J$ symbols of the classical Lie algebra su(2). At the same time the conformal dimensions $h_J = J(J + 1)/(\mathsf{k} + 2)$ tend to zero so that the OPE (4.14) of boundary fields becomes regular as in a topological model. In this sense, the limit $\mathsf{k} \to \infty$ is similar to the decoupling limit considered in [4].

Using the last two observations about the limiting theory we obtain a much more elegant way of writing the operator product expansion (4.14). In fact, the comparison of eq. (4.14) with eq. (2.6) shows that the operator product expansion is encoded in the multiplication of matrices if we think of Y_m^J as elements of $\mathrm{Mat}(2\alpha + 1)$, i.e. for $\mathsf{k} \to \infty$ we find

$$V[Y_m^I](x_1)\, V[Y_n^J](x_2)\ =\ V[Y_m^I\, Y_n^J](x_2) \quad \text{for} \quad Y_m^I, Y_n^J \in \mathrm{Mat}(2\alpha + 1)$$

up to subleading terms. The emergence of the matrix product in this relation confirms the results from Section 3 since it shows that the world-volumes of branes in S^3 become fuzzy two-spheres when we send the level to infinity. We can now rewrite the operator product in a way which does no longer refer to a particular choice of basis in $\mathrm{Mat}(2\alpha+1)$. For an arbitrary matrix $\mathrm{A} \in \mathrm{Mat}(2\alpha + 1)$ with $\mathrm{A} = \sum a_{Jm} Y_m^J$ we introduce $V[\mathrm{A}] = \sum a_{Jm} V[Y_m^J]$ and obtain

$$V[\mathrm{A}_1](x_1)\, V[\mathrm{A}_2](x_2)\ =\ V[\mathrm{A}_1\, \mathrm{A}_2](x_2) \tag{4.15}$$

for all $\mathrm{A}_1, \mathrm{A}_2 \in \mathrm{Mat}(2\alpha+1)$. This product allows us to compute arbitrary correlations functions of such vertex operators,

$$\langle V[\mathrm{A}_1](x_1)\, V[\mathrm{A}_2](x_2)\ \cdots\ V[\mathrm{A}_n](x_n)\rangle\ =\ \mathrm{tr}(\mathrm{A}_1\, \mathrm{A}_2\ \cdots\ \mathrm{A}_n)\ . \tag{4.16}$$

The trace appears because the vacuum expectation value is SU(2) invariant and the trace maps matrices to their SU(2) invariant component. Formulas (4.11,4.13,4.16) provide a complete solution of the boundary WZW models in the decoupling limit $\mathsf{k} \to \infty$. The generalization to branes localized along conjugacy classes of other Lie groups is straightforward (see [20] for details).

5   Open strings and gauge theory on fuzzy orbits

Equipped with the exact solution of the boundary WZW model we are finally prepared to calculate the low-energy effective action for massless open string modes. Compared to the flat space case [4] there are two important changes in the computation. First, it follows from eq. (4.16) that Moyal-Weyl products get replaced by matrix multiplication. Second, there appears a new term $f_{ab}{}^c J_c$ in the operator product expansion of currents (4.11). This term leads to an extra contribution of the form $f_{abc}\mathrm{A}^a \mathrm{A}^b \mathrm{A}^c$ in the scattering amplitude of three massless open string modes. Consequently, the resulting effective

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action is not given by Yang-Mills theory on a fuzzy 2-sphere but involves also a Chern-Simons like term.

For M branes of type α on top of each other, the results of the complete computation [7] can be summarized in the following formula,

$$\mathcal{S}_{(M,\alpha)} \; = \; \mathcal{S}_{\mathrm{YM}} + \mathcal{S}_{\mathrm{CS}} \; = \; \frac{1}{4}\,\mathrm{tr}\left(\mathrm{F}_{ab}\,\mathrm{F}^{ab}\right) - \frac{i}{2}\,\mathrm{tr}\left(f^{abc}\,\mathrm{CS}_{abc}\right) \tag{5.17}$$

where we defined the 'curvature form' F_{ab} by the expression

$$\mathrm{F}_{ab}(\mathrm{A}) \; = \; i\,\mathrm{L}_a\mathrm{A}_b - i\,\mathrm{L}_b\mathrm{A}_a + i\,[\mathrm{A}_a\,,\,\mathrm{A}_b] + f_{abc}\mathrm{A}^c \tag{5.18}$$

and a non-commutative analogue of the Chern-Simons form by

$$\mathrm{CS}_{abc}(\mathrm{A}) \; = \; \mathrm{L}_a\mathrm{A}_b\,\mathrm{A}_c + \frac{1}{3}\,\mathrm{A}_a\,[\mathrm{A}_b\,,\,\mathrm{A}_c] - \frac{i}{2}\,f_{abd}\,\mathrm{A}^d\,\mathrm{A}_c \quad . \tag{5.19}$$

The three fields $\mathrm{A}^a = \sum a^a_{Jm}\mathrm{Y}^J_m$ on the fuzzy 2-sphere S^2_α take values in $\mathrm{Mat}(M)$, i.e. $a^a_{Jm} \in \mathrm{Mat}(M)$. Hence the fields A^a are elements of $\mathrm{Mat}(M) \otimes \mathrm{Mat}(2\alpha + 1)$. Gauge invariance of (5.17) under the gauge transformations

$$\mathrm{A}_a \; \rightarrow \; L_a\Lambda \; + \; i\,[\,A_a\,,\,\Lambda\,] \quad \text{for} \quad \Lambda \in \mathrm{Mat}(M) \otimes \mathrm{Mat}(2\alpha + 1)$$

follows by straightforward computation. Note that the 'mass term' in the Chern-Simons form (5.19) guarantees the gauge invariance of $\mathcal{S}_{\mathrm{CS}}$. On the other hand, the effective action (5.17) is the unique combination of $\mathcal{S}_{\mathrm{YM}}$ and $\mathcal{S}_{\mathrm{CS}}$ in which mass terms cancel. As we shall see below, it is this special feature of our action that allows solutions describing translations of the branes on the group manifold. The action $\mathcal{S}_{\mathrm{YM}}$ was already considered in the non-commutative geometry literature [21, 23, 22], where it was derived from a Connes spectral triple and viewed as describing Maxwell theory on the fuzzy sphere. Arbitrary linear combinations of non-commutative Yang-Mills and Chern-Simons terms were considered in [24].

From eq. (5.17) we obtain the following equations of motion for the elements $\mathrm{A}^a \in \mathrm{Mat}(M) \otimes \mathrm{Mat}(2\alpha + 1)$

$$\left[\,A^a\,,\,[\,A_a\,,\,A_b\,] - i\,f_{abc}\,A^c\,\right] \; = \; 0 \quad . \tag{5.20}$$

Solutions of these equations (5.20) describe condensates on a stack of M branes of type α. It is easy to find two very different types of solutions. The first one is given by a set

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of 3 pairwise commuting $M \times M$ matrices A_a. It comes as an $M \cdot 3$ parameter family of solutions corresponding to the number of eigenvalues appearing in $\{A_a\}$. The same kind of solutions appears also for branes in flat backgrounds. They describe rigid translations of the M branes on the group manifold. Since each brane's position is specified by 3 coordinates, the number of parameters matches nicely with the interpretation. Moving branes around in the background is a rather trivial symmetry and the corresponding conformal field theories are easy to construct (see also [25]). Note, however, that the existence of these solutions is guaranteed by the absence of the mass term in the full effective action.

There exists a second type of solutions to eqs. (5.20) which is a lot more interesting. In fact, any $M(2\alpha+1)$-dimensional representation of the Lie algebra su(2) can be used to solve the equations of motion. Their interpretation was found in [7]. Let us describe the answer for a stack of M branes of type $\alpha = 0$, i.e. of M point-like branes at the origin of SU(2). In this case, $A_a \in \text{Mat}(M) \otimes \text{Mat}(1) \cong \text{Mat}(M)$ so that we need an M-dimensional representation of su(2) to solve the equations of motion. Let us choose the M-dimensional irreducible representation σ. Our claim then is that this drives the initial stack of M point-like branes at the origin into a final configuration containing only a single brane wrapping the sphere of type $\alpha = (M-1)/2$, i.e.

$$(M, \, \alpha = 0) \; \xrightarrow{\sigma} \; (1, \, \alpha = (M-1)/2) \; .$$

Support for this statement comes from both the open string sector and the coupling to closed strings (see [7]). In the open string sector one can study small fluctuations δA_a of the fields $A_a = \Lambda_a + \delta A_a \in \text{Mat}(M)$ around the stationary point $\Lambda_a \in \text{Mat}(M)$. If Λ_a form an irreducible representation of su(2), one finds,

$$\mathcal{S}_{(M,0)}(\Lambda_a + \delta A_a) \; = \; \mathcal{S}_{(1,(M-1)/2)}(\delta A_a) \; + \; \text{const} \quad .$$

In the closed string channel one can show, along the lines of [26], that the leading term (in the $1/\mathsf{k}$-expansion) from the exact mass formula of a brane wrapping the sphere of type $\alpha = (M-1)/2$ coincides with the value of the action $\mathcal{S}_{(M,0)}(\Lambda_a)$ at the stationary point Λ_a [7]. Note that the mass of the final state is lower than the mass of the initial configuration. This means that a stack of M point-like branes on a 3-sphere is unstable against decay into a single spherical brane. Stationary points of the action (5.17) and the formation of spherical branes on S^3 were also discussed recently in [27, 28]. Similar effects have been described for branes in RR-background fields [8]. The advantage of

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our scenario with NSNS-background fields is that it can be treated in perturbative string theory so that string effects may be taken into account (see [30]).

6   Conclusion and Outlook

The analysis we have reviewed here generalizes the well-known relation between open strings and non-commutative geometry of branes in flat space to a large class of branes on group manifolds. In particular, we have described how fuzzy geometries and gauge theories on these spaces appear in this context. Our main result (5.17,5.18,5.19) was formulated for su(2) but its generalization to other Lie algebras is essentially obvious.

The condensation processes that are encoded in the effective action were used in [29] to argue that the charges of branes on S^3 are only defined modulo some integer. This seems to fit nicely [30] with a proposal of Bouwknegt and Mathai in [31] according to which the charges of branes in a background X with non-vanishing NSNS H-field $H \in H^3(X, \mathbb{Z})$ generate certain twisted K-theory groups $K_H(X)$.

Let us finally recall that WZW models provide the most important starting point for CFT model building, such as coset and orbifold constructions. Hence, one can hope to use the results reported above to analyze brane dynamics in many other backgrounds, such as minimal models, various Kazama-Suzuki quotients and Gepner models. The concrete realization of this program for $N = 2$ supersymmetric minimal models is sketched in [32].

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