Jet coordinates for local BRST cohomology

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Abstract

The construction of appropriate jet space coordinates for calculating local BRST cohomology groups is discussed. The relation to tensor calculus is briefly reviewed too.

Introduction

The BRST formalism has its roots in quantum field theory where it was first established for gauge theories of the Yang-Mills type [1, 2, 3, 4, 5]. Since then the construction was generalized in several ways. The best known generalization is the so-called antifield formalism [6, 7] (for reviews see [8, 9]) which generalizes the original construction to Lagrangean gauge theories of any kind (with irreducible or reducible gauge symmetries, whose commutator algebra closes off-shell or only on-shell). More recently it has been generally proved that the construction can be generalized so as to include also global symmetries of the Lagrangian or 'higher order symmetries' [10]. Moreover, the formalism is by no means restricted to Lagrangean field theories but can be established already at the level of the equations of motion, whether or not these equations are Euler-Lagrange equations [11, 8, 12].[1] The present work applies to any of these generalizations.

The central object of the BRST formalism and its generalizations is a differential s which contains – through the s-transformations of the fields and antifields – the equations of motion, the nontrivial gauge symmetries (if any), possibly global or higher order symmetries, as well as related structure, such as the commutator algebra of the symmetries contained in it. The local BRST cohomology concerns cohomology groups of s defined on jet-spaces associated with the fields and antifields[2]; of particular interest in physics are $H(s)$ and $H(s|d)$, the cohomology of s and of s modulo d in the space

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[1]Particular features of Lagrangean models, such as the antibracket [6], have no counterpart in a general non-Lagrangean model.

[2]These are spaces whose coordinates are the coordinates x^μ of the base space, the fields, antifields and derivatives of the fields and antifields. In the context of local BRST cohomology it is useful to count also the differentials dx^μ among the jet coordinates. This is done here.

of differential forms on these jet spaces, see [13] for a recent review and for applications. $H(s|d)$ is related to $H(s)$ through the so-called descent equations, though the precise relation can be quite involved [14, 13]. Furthermore the physically important cohomology groups $H^{*,n}(s|d)$ in maximal form-degree n (= base space dimension) are directly related to $H(s+d)$ through the descent equations [15, 16, 13]. We denote the combination $s+d$ by $\tilde{s}$,

$$\tilde{s} := s + d. \tag{1}$$

An important technique within the calculation of $H(s)$ or $H(\tilde{s})$ are contracting homotopies which eliminate pairs of variables from the respective cohomology. In case of $H(s)$, these pairs are s-doublets (u^ℓ, v^ℓ),

$$su^\ell = v^\ell \quad (\Rightarrow \ sv^\ell = 0). \tag{2}$$

Accordingly they are $\tilde{s}$-doublets $(\tilde{u}^{\tilde{\ell}}, \tilde{v}^{\tilde{\ell}})$ in case of $H(\tilde{s})$,

$$\tilde{s}\tilde{u}^{\tilde{\ell}} = \tilde{v}^{\tilde{\ell}} \quad (\Rightarrow \ \tilde{s}\tilde{v}^{\tilde{\ell}} = 0). \tag{3}$$

However, (2) or (3) alone do not guarantee the existence of contracting homotopies that eliminate the u's and v's, or $\tilde{u}$'s and $\tilde{v}$'s, not even locally (simple counterexamples are the $\tilde{s}$-doublets (x^μ, dx^μ) in Yang-Mills theory, see below). For (2), a suitable supplementary requirement [16, 17] is the existence of variables w^I which complete the set $\{u^\ell, v^\ell\}$ to a new jet coordinate system and are such that sw^I is a function which can be expressed solely in terms of the w's,

$$sw^I = r^I(w). \tag{4}$$

Furthermore the w^I are required to be local functions of the fields and antifields[3]. (2) and (4) imply that s leaves separately invariant the subspaces of local functions $f(u,v)$ and $f(w)$ depending only on the u's and v's, and w's, respectively. As an immediate consequence, $H(s)$ factorizes by the Künneth formula (see, e.g., [18]) into the s-cohomology in these subspaces,

$$H(s) = H(s, \mathcal{F}_{u,v}) \otimes H(s, \mathcal{F}_w), \quad \mathcal{F}_{u,v} = \{f(u,v)\}, \quad \mathcal{F}_w = \{f(w)\}. \tag{5}$$

Hence, $H(s)$ can be obtained by computing separately $H(s, \mathcal{F}_{u,v})$ and $H(s, \mathcal{F}_w)$. Furthermore (2) implies in many (though not in all) cases that $H(s, \mathcal{F}_{u,v})$ is trivial (represented just by constants) so that the u's and v's disappear from the cohomology and $H(s)$ is given just by $H(s, \mathcal{F}_w)$.[4]

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[3]The precise definition of 'local function' depends on the context, i.e., on the model and applications one is studying. Often a local function is required to live on a finite dimensional jet space, or to have an expansion in antifields or coupling constants such that each term in the expansion lives on a finite dimensional jet space.

[4]Whether or not $H(s, \mathcal{F}_{u,v})$ is trivial depends on the precise properties of $\mathcal{F}_{u,v}$ (in particular on topological features) and is not to be discussed here (the focus of the present work is on the construction of the w's rather than on contracting homotopies for u's and v's). An instructive example where $H(s, \mathcal{F}_{u,v})$ is not trivial can be found in section 5 of [19], see theorem 5.1 there.

Analogous results hold for $H(\tilde{s})$ if there is a jet coordinate system $\{\tilde{u}^{\tilde{\ell}}, \tilde{v}^{\tilde{\ell}}, \tilde{w}^{\tilde{I}}\}$ such that

$$\tilde{s}\tilde{w}^{\tilde{I}} = \tilde{r}^{\tilde{I}}(\tilde{w}). \tag{6}$$

There are three types of u's and $\tilde{u}$'s which one typically meets (see, e.g., the Yang-Mills example below): (a) antighost fields and their derivatives, (b) antifields and a subset of their derivatives, (c) gauge fields and a subset of their derivatives. In reducible gauge theories there are normally additional u's and $\tilde{u}$'s given by a subset of derivatives of ghosts, ghosts for ghosts etc. (4) and (6) pose no problem in connection with the antighost fields and the corresponding v's (these v's are Nakanishi-Lautrup fields used for gauge fixing) or $\tilde{v}$'s because the s-transformations and $\tilde{s}$-transformations of the other fields and antifields do not involve these fields[5]. (4) and (6) are much more challenging – and much more interesting – in connection with the s-doublets and $\tilde{s}$-doublets associated with the gauge fields and antifields. The construction of corresponding w's can be quite a nontrivial matter. The existence of such w's is related to a tensor calculus and a gauge covariant algebra. This was discussed in [16, 17] and will be briefly reviewed at the end of this letter.

The purpose of this letter is to discuss how and when one can construct w's and $\tilde{w}$'s fulfilling (4) and (6), respectively. We shall describe an iterative construction which can be used as an algorithm to construct the w's explicitly or to prove their existence if only one can show that the iteration terminates or, at least, that it results in meaningful (local) expressions for the w's (the algorithm applies to the construction of $\tilde{w}$'s as well). Such a proof is model dependent. To give an example, we shall prove that the algorithm terminates for Yang-Mills theory. We shall also briefly comment on how one can master a typical complication in supersymmetric models related to the supersymmetry ghosts.

If one can prove in this manner the existence of the w's, then the algorithm also shows that the functions r^I in (4) can be already read off from the original s-transformations, i.e., it is not necessary to construct the w's explicitly in order to determine their s-transformations. Hence, one may then analyse $H(s)$ even without having to construct the w's explicitly. This can have practical use because the explicit construction of the w's can be rather involved. Furthermore, the functions r^I provide directly the tensor calculus and gauge covariant algebra associated with the w's and thus even this algebra can be obtained without an explicit construction of the w's. Analogous comments apply to the functions $\tilde{r}^{\tilde{I}}$ in (6) and to $H(\tilde{s})$. The relation to the tensor calculus may be used in two ways. If an appropriate tensor calculus is already known for a particular model under study, it may be useful for the finding or construction of the w's. Conversely one may derive a previously unknown tensor calculus for a model from (4) or (6).

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[5]We refer here to the so-called "classical basis" of the antifields. It is related to the "gauge fixed" basis simply by a "canonical transformation" which is just a local (anti)field redefinition, see e.g. section 2.6 of [13]. Of course, the cohomology does not depend on the basis used to compute it.

Iterative construction of w's

Since s-doublets are obvious from the s-transformations of the fields and antifields, our starting point is the assumption that there is jet coordinate system $\{u^\ell, v^\ell, w^I_{(0)}\}$ such that (2) holds and that $sw^I_{(0)}$ is a power series in the u's and v's,

$$sw^I_{(0)} = r^I(w_{(0)}) + O(1), \tag{7}$$

where $O(1)$ collects all terms that are at least linear in u's or v's. The aim is to complete the $w^I_{(0)}$ to local functions $w^I = w^I_{(0)} + O(1)$ which fulfill (4).[6] Assume that we constructed already a jet coordinate system $\{u^\ell, v^\ell, w^I_{(m)}\}$ with $w^I_{(m)} = w^I_{(0)} + O(1)$ such that

$$sw^I_{(m)} = r^I(w_{(m)}) + h^I_{m+1}(u, v, w_{(m)}) + O(m + 2) \tag{8}$$

where the non-vanishing $h^I_{m+1}(u, v, w_{(m)})$ have degree $m+1$ in the u's and v's and $O(m+2)$ collects all terms of higher degrees. Note that we refer here to the jet coordinates $(u, v, w_{(m)})$; expressed in terms of the coordinates $(u, v, w_{(0)})$, h^I_{m+1} has, in general, no definite degree in the u's and v's but decomposes into terms of various degrees $\geq m+1$, owing to $w_{(m)} = w_{(0)} + O(1)$. We now define, again referring to the coordinates $(u, v, w_{(m)})$,

$$\rho_{(m)} := u^\ell \frac{\partial}{\partial v^\ell}. \tag{9}$$

Using (8), one easily computes the anticommutator of $\rho_{(m)}$ and s. On the u^ℓ, v^ℓ and $w^I_{(m)}$ one gets, respectively,

$$(\rho_{(m)}s + s\rho_{(m)})u^\ell = u^\ell, \quad (\rho_{(m)}s + s\rho_{(m)})v^\ell = v^\ell,$$
$$(\rho_{(m)}s + s\rho_{(m)})w^I_{(m)} = \rho_{(m)}sw^I_{(m)} = Y^I_{m+1} + O(m+2) \tag{10}$$

where

$$Y^I_{m+1} := \rho_{(m)}h^I_{m+1}(u, v, w_{(m)}). \tag{11}$$

Note that one has $Y^I_{m+1} = Y^I_{m+1}(u, v, w_{(m)})$ and that $Y^I_{m+1}(u, v, w_{(m)})$ has degree $m+1$ in the u's and v's. Since both $\rho_{(m)}$ and s are antiderivations, (10) implies, on functions $f(u, v, w_{(m)})$,

$$\rho_{(m)}s + s\rho_{(m)} = N_{u,v} + [Y^I_{m+1} + O(m+2)] \frac{\partial}{\partial w^I_{(m)}}, \tag{12}$$

where $N_{u,v}$ is the counting operator for the u's and v's,

$$N_{u,v}f(u, v, w_{(m)}) = \left[u^\ell \frac{\partial}{\partial u^\ell} + v^\ell \frac{\partial}{\partial v^\ell}\right] f(u, v, w_{(m)}).$$

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[6]The notation anticipates that the algorithm yields the same functions r^I in (4) as in (7).

We now evaluate $(\rho_{(m)}s + s\rho_{(m)})sw_{(m)}^I$ in two ways. On the one hand, we get, using that s is a differential $(s^2 = 0)$,

$$(\rho_{(m)}s + s\rho_{(m)})sw_{(m)}^I \quad \overset{s^2=0}{=} \quad s\rho_{(m)}sw_{(m)}^I$$

$$\overset{(8)}{=} \quad s\rho_{(m)}[r^I(w_{(m)}) + h_{m+1}^I(u, v, w_{(m)}) + O(m+2)]$$

$$\overset{(9),(11)}{=} \quad sY_{m+1}^I + O(m+2). \tag{13}$$

On the other hand we get, using (12) and $N_{u,v}h_{m+1}^I(u, v, w_{(m)}) = (m+1)h_{m+1}^I(u, v, w_{(m)})$,

$$(\rho_{(m)}s + s\rho_{(m)})sw_{(m)}^I \overset{(8),(12)}{=} Y_{m+1}^J \frac{\partial r^I(w_{(m)})}{\partial w_{(m)}^J} + (m+1)h_{m+1}^I(u, v, w_{(m)}) + O(m+2). \tag{14}$$

(13) and (14) imply

$$h_{m+1}^I(u, v, w_{(m)}) = \tfrac{1}{m+1}\left[sY_{m+1}^I - Y_{m+1}^J \frac{\partial r^I(w_{(m)})}{\partial w_{(m)}^J}\right] + O(m+2).$$

Using this in (8), the latter yields

$$s\left[w_{(m)}^I - \tfrac{1}{m+1}Y_{m+1}^I\right] = r^I(w_{(m)}) - \tfrac{1}{m+1}Y_{m+1}^J \frac{\partial r^I(w_{(m)})}{\partial w_{(m)}^J} + O(m+2)$$

$$= r^I(w_{(m)} - \tfrac{1}{m+1}Y_{m+1}) + O(m+2). \tag{15}$$

Defining now

$$w_{(m+1)}^I := w_{(m)}^I - \tfrac{1}{m+1}Y_{m+1}^I \,, \tag{16}$$

equation (15) gives

$$sw_{(m+1)}^I = r^I(w_{(m+1)}) + h_{m+2}^I(u, v, w_{(m+1)}) + O(m+3) \tag{17}$$

for some $h_{m+2}^I(u, v, w_{(m+1)})$ which has degree $m+2$ in the u's and v's. This is an equation as (8), but for $m+1$ in place of m. Furthermore (8) holds for $m = 0$ by assumption (eq. (7)). Hence, the algorithm provides indeed an inductive existence proof for a set $\{w^I\}$ fulfilling (4) if one can prove that it terminates or that it results in closed local expressions for the w's. Such a proof can often be accomplished by means of appropriate degrees that can be assigned to the variables, such as the ghost number and the (mass) dimension of the variables, see the example below. Note that the iteration does not modify the functions r^I and thus one can read off the resulting functions r^I in (4) already from the transformations of the $w_{(0)}$'s in (7), as promised.

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Yang-Mills theory as an example

To get an idea how one may prove that the algorithm terminates or results in local expressions let us assume for a moment that one can assign a dimension to each of the variables u^ℓ, v^ℓ, $w_{(0)}^I$ such that (i) s has dimension 0 when all (coupling) constants are dimensionless; (ii) all u's and v's have positive dimensions; (iii) there are only finitely many $w_{(0)}$'s with negative dimensions and all of these variables are Grassmann odd (anticommuting). Let us also assume that local functions are power series' in all variables with nonvanishing dimensions and that the spectrum of dimensions of the variables is quantized in integer of half-integer units (which is the standard case). Then it is easy to prove that the inductive construction of the w's terminates: because of (i), the algorithm implies that each Y_m^I which appears in the construction has the same dimension as the corresponding $w_{(0)}^I$; furthermore, each Y_m^I is a local function (by assumption, s is a local operator which implies that h_m^I is local and thus Y_m^I is local too); because of (ii) and (iii) a local function with definite dimension cannot have arbitrarily high degree in the u's and v's and thus there is a bound m_I for each value of I such that $Y_m^I = 0 \ \forall m > m_I$ (recall that $Y_m^I(u, v, w_{(0)})$ contains only terms with degrees $\geq m$ in the u's and v's); hence, one gets $w^I = w_{(0)}^I - \sum_{m=1}^{m_I}(1/m)Y_m^I$.

An example to which this argument applies is Yang-Mills theory with standard Lagrangian

$$L = -\tfrac{1}{4} g_{ab} F_{\mu\nu}^a F^{\mu\nu b}, \quad F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + f_{bc}{}^a A_\mu^b A_\nu^c \tag{18}$$

where g_{ab} and $f_{ab}{}^c$ are the Cartan-Killing metric and structure constants of the Lie algebra of the gauge group, respectively. We denote the gauge fields and ghost fields by A_μ^a and C^a, their antifields by $A_a^{*\mu}$ and C_a^*, respectively (the inclusion of antighost fields and Nakanishi-Lautrup fields is trivial and therefore we neglect them). 'Local functions' are in this example required to depend polynomially on the fields, antifields and their derivatives. The s-transformations of these fields and antifields are

$$
\begin{aligned}
sA_\mu^a &= D_\mu C^a \\
sC^a &= \tfrac{1}{2} f_{bc}{}^a C^c C^b \\
sA_a^{*\mu} &= g_{ab} D_\nu F^{\nu\mu b} + C^b f_{ba}{}^c A_c^{*\mu} \\
sC_a^* &= -D_\mu A_a^{*\mu} + C^b f_{ba}{}^c C_c^*
\end{aligned}
\tag{19}
$$

where D_μ is the covariant derivative,

$$D_\mu X^a = \partial_\mu X^a + A_\mu^b f_{bc}{}^a X^c \ , \ \ D_\mu X_a = \partial_\mu X_a - A_\mu^b f_{ba}{}^c X_c \ .$$

The s-transformations of the coordinates and differentials vanish,

$$sx^\mu = 0 \ , \ \ sdx^\mu = 0. \tag{20}$$

The s-transformations of derivatives of the fields and antifields are obtained from (19) simply through prolongation, using $[s, \partial_\mu] = 0$. One verifies readily that (i) is fulfilled

with the following standard dimension assignments:

$$
\begin{array}{c|c|c|c|c}
 & x^\mu, dx^\mu & C^a & A^a_\mu, \partial_\mu & A^{*\mu}_a & C^*_a \\
\text{dimension}: & -1 & 0 & 1 & 3 & 4
\end{array}
\tag{21}
$$

Let us now identify s-doublets and a corresponding jet coordinate system $\{u^\ell, v^\ell, w^I_{(0)}\}$. The antifields C^*_a and their independent derivatives are u's; the corresponding v's, given by $-\partial_\mu A^{*\mu}_a + \dots$ and derivatives thereof, are taken as elements of the sought jet coordinate system $\{u^\ell, v^\ell, w^I_{(0)}\}$. $A^{*\mu}_a$ and their remaining derivatives of are also u's; their v's, given by $g_{ab}\partial_\nu(\partial^\nu A^{\mu b} - \partial^\mu A^{\nu b}) + \dots$ and derivatives thereof, are also taken as new jet coordinates. The remaining u's are the undifferentiated gauge fields and their symmetrized derivatives $\partial_{(\mu_1} \dots \partial_{\mu_k} A^a_{\nu)}$; the corresponding v's are new jet coordinates substituting for all derivatives of the ghost fields C^a. A possible set $\{w^I_{(0)}\}$ contains thus only the undifferentiated C^a, the dx^μ and x^μ, and derivatives of the gauge fields which complete those derivatives contained already in $\{u^\ell, v^\ell\}$ to a basis for all derivatives of the gauge fields. Property (ii) is then fulfilled too. Property (iii) is not fulfilled because the x^μ are bosonic variables with negative dimension. However, this does not spoil the argument since the x^μ are s-singlets which do not occur in the s-transformations of the fields, antifields and their derivatives; hence, one can simply disregard the x^μ when applying the algorithm and concludes that the w's exist, without having to construct them explicitly.

Of course, since we know the tensor calculus of Yang-Mills theory, it is not difficult to find the set of w's explicitly in this case and thus a formal existence proof is not needed: obviously $\{w^I\} = \{C^a, dx^\mu, x^\mu, F^a_{\mu\nu}, \dots\}$ fulfills all requirements where the nonwritten elements are first and higher order covariant derivatives of $F^a_{\mu\nu}$ corresponding to the second and higher order derivatives of A^a_μ contained in $\{w^I_{(0)}\}$.

However, when one now considers $\tilde{s}$ rather than s, then one gets already an example where an existence proof is simpler than an explicit construction. Indeed, the existence of $\tilde{w}$'s for pure Yang-Mills theory can be proved along exactly the same lines as the existence of w's, but the explicit construction of the $\tilde{w}$'s is much more involved than the construction of the w's. This is seen as follows. Using that $\tilde{s}$ is dimensionless with the assignments (21), one readily checks that all arguments go through for $\tilde{s}$ in place of s, with the same u's and $w_{(0)}$'s (i.e., one can use $\{\tilde{u}^{\tilde{\ell}}\} = \{u^\ell\}$ and $\{\tilde{w}^{\tilde{I}}_{(0)}\} = \{w^I_{(0)}\}$). One concludes the existence of a set $\{\tilde{w}^I\} = \{\tilde{C}^a, dx^\mu, x^\mu, \tilde{F}^a_{\mu\nu}, \dots\}$ such that (6) holds. The explicit form of these variables is quite involved, though. For instance, $\tilde{F}^a_{\mu\nu}$ contains $F^a_{\mu\nu}$ but also antifield dependent terms proportional to $g^{ab}dx_{[\mu}A^*_{\nu]b}$ and $g^{ab}dx_\mu dx_\nu C^*_b$, as can be verified by applying the algorithm explicitly (the origin of these antifield dependent terms is the presence of $D^\rho F^a_{\rho\nu}$ in $(s+d)F^a_{\mu\nu}$).

Note that we have counted x^μ and dx^μ among the $\tilde{w}$'s in spite of the fact that they form $\tilde{s}$-doublets,

$$
\tilde{s}x^\mu = dx^\mu.
\tag{22}
$$

The reason is that the algorithm would not terminate if one included x^μ in $\{\tilde{u}^{\tilde{\ell}}\}$. The proof sketched above breaks down because property (ii) does not hold anymore (x and

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dx have negative dimension). What happens, is the following. $\tilde{s}\tilde{w}_{(0)}$ contains $dx^\mu \partial_\mu \tilde{w}_{(0)}$ and therefore the algorithm would give, among others, a contribution $x^\mu \partial_\mu \tilde{w}_{(0)}$ to $\tilde{Y}_1$ (as dx^μ would be a $\tilde{v}$ when x^μ is a $\tilde{u}$). This contribution creates a term $x^\mu x^\nu \partial_\mu \partial_\nu \tilde{w}_{(0)}$ in $\tilde{Y}_2$ and so forth; one gets $\tilde{Y}_k$ containing $x^{\mu_1} \ldots x^{\mu_k} \partial_{\mu_1} \ldots \partial_{\mu_k} \tilde{w}_{(0)}$ with arbitrarily large k.

The Yang-Mills example provides a simple example how one may keep control of locality. Other models may be treated similarly, using appropriate properties that substitute for (i)–(iii). For instance, when s contains global or local supersymmetries (with constant ghosts in the case of global supersymmetries), the undifferentiated supersymmetry ghosts are Grassmann even variables with dimension $-1/2$ (as follows from the supersymmetry algebra, assuming standard dimension assignments). Hence, property (iii) is not fulfilled in that case, even when one disregards x^μ. Nevertheless, even in supersymmetric models one may often use arguments similar to those above by taking the ghost numbers of the variables into account. For instance, assume that all relevant variables with negative ghost numbers (i.e., the antifields) have dimensions ≥ 1 and that the undifferentiated supersymmetry ghosts and the x^μ are the only Grassmann even variables with negative dimensions (this is the standard case in supersymmetric models, for the dx^μ and the translation or diffeomorphism ghosts are Grassmann odd). Then one cannot construct local x-independent functions with a fixed ghost number and a fixed dimension containing arbitrarily high powers of supersymmetry ghosts, i.e., effectively the degree in variables with negative dimensions is still bounded from above.

Gauge covariant algebras and tensor calculus

In the following we review very briefly the relation to gauge covariant algebras and tensor calculus, referring to [16, 17] for details and further discussion. For illustrative purpose we assume that there is a jet coordinate system $\{u^\ell, v^\ell, w^I\}$ with the properties described above, and that the set of w's contains variables of ghost number 0 and 1 only,

$$\{w^I\} = \{C^M, T^A\}, \quad gh(C^M) = 1, \quad gh(T^A) = 0. \tag{23}$$

Such a set of w's arises typically in irreducible gauge theories, such as Yang-Mills theory or gravity (in fact we have seen above that it arises in pure Yang-Mills theory). The reason is the following. An irreducible gauge theory contains only fields with ghost numbers ≤ 1 (there are no 'ghosts for ghosts'). Assuming that appropriate regularity conditions are fulfilled (see e.g., [20, 13]), all antifields and their derivatives appear in s-doublets. The same holds of course for the antighost fields. Hence, all field and antifield variables with negative ghost numbers will then give rise to s-doublets. Assuming that corresponding w's exist, one gets (23). The C^M will be appropriate ghost variables, typically corresponding to the undifferentiated ghost fields and (possibly) a subset of derivatives of the ghost fields which do not appear in s-doublets with the gauge fields and derivatives thereof. The T^A, or their antifield independent part, may be interpreted as

tensor fields. The typical situation in reducible gauge theories is similar, the difference being that then there are also w's with ghost numbers > 1 corresponding to ghost fields with higher ghost numbers ('ghosts for ghosts'). In the extended antifield formalism [10], the set $\{C^M\}$ contains in addition constant ghosts corresponding, for instance, to global symmetries.

Since s has ghost number 1 (i.e., it raises the ghost number by one unit), (23) and (4) imply that

$$sT^A = C^M R_M^A(T), \quad sC^M = \tfrac{1}{2} \, (-)^{\epsilon_K+1} C^K C^L F_{LK}{}^M(T), \tag{24}$$

for some functions $R_M^A(T)$ and $F_{KL}{}^M(T)$ of the T's. $(\epsilon_K + 1)$ denotes the Grassmann parity of C^K and has been introduced to make the following formulae nicer (it can of course be absorbed into $F_{KL}{}^M(T)$ by redefining the latter). Owing just to $s^2 T^A = 0$ (and thus, in particular, regardless whether or not the commutator algebra of the gauge transformations closes off-shell), (24) implies an algebra

$$[\nabla_M, \nabla_N] = -F_{MN}{}^K(T)\nabla_K \tag{25}$$

where $[\ ,\ ]$ is the graded commutator and ∇_M are the graded derivations

$$\nabla_M = R_M^A(T)\frac{\partial}{\partial T^A} \ . \tag{26}$$

$s^2 C^M = 0$ gives the consistency conditions for the algebra (25) equivalent to $(-)^{\epsilon_K \epsilon_N}[\nabla_K, [\nabla_M, \nabla_N]] + \text{cyclic} = 0$,

$$(-)^{\epsilon_K \epsilon_N}[\nabla_K F_{MN}{}^L + F_{MN}{}^R F_{RK}{}^L] + \text{cyclic} = 0. \tag{27}$$

In the Yang-Mills example discussed above, one simply gets:

$$\text{Yang-Mills:} \quad \{C^M\} \equiv \{C^a\}, \quad \{T^A\} \equiv \{dx^\mu, x^\mu, F_{\mu\nu}^a, \ldots\},$$
$$\{\nabla_M\} \equiv \{\delta_a\}, \quad \{F_{MN}{}^K\} \equiv \{-f_{ab}{}^c\}$$

where δ_a are the elements of the Lie algebra of the gauge group in a basis with structure constants $f_{ab}{}^c$.

Similar consequences arise when there is a jet coordinate system $\{\tilde{u}^\ell, \tilde{v}^\ell, \tilde{w}^I\}$ with analogous properties. The analogue of (23) is

$$\{\tilde{w}^{\tilde{I}}\} = \{\tilde{C}^{\tilde{M}}, \tilde{T}^{\tilde{A}}\}, \quad tot(\tilde{C}^{\tilde{M}}) = 1, \quad tot(\tilde{T}^{\tilde{A}}) = 0 \tag{28}$$

where tot is the degree associated to $\tilde{s}$, i.e., the sum of the ghost number and form-degree,

$$tot = gh + form\text{-}degree.$$

The implications of (28) are analogous to (24) through (27). However, there is an additional structure hidden in the resulting equations which comes from the presence of

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d in $\tilde{s}$. To exhibit it, one decomposes $\tilde{C}^{\tilde{M}}$ into terms with different antifield numbers. Since $\tilde{C}^{\tilde{M}}$ has total degree 1, its antifield independent piece contains in general two pieces: one piece with form-degree 1 and ghost number 0, and another one with form-degree 0 and ghost number 1,

$$\tilde{C}^{\tilde{M}} = dx^\mu A_\mu^{\tilde{M}} + C^{\tilde{M}} + \text{terms with antifields}, \quad gh(A_\mu^{\tilde{M}}) = 0, \quad gh(C^{\tilde{M}}) = 1. \quad (29)$$

The antifield independent piece of $\tilde{T}^{\tilde{A}}$ is denoted by $T^{\tilde{A}}$,

$$\tilde{T}^{\tilde{A}} = T^{\tilde{A}} + \text{terms with antifields}. \quad (30)$$

Using this decomposition in $\tilde{s}\tilde{T}^{\tilde{A}} = \tilde{C}^{\tilde{M}}\tilde{R}_{\tilde{M}}^{\tilde{A}}(\tilde{T})$ which is the analogue of the first equation in (24), one gets by comparing the coefficients of dx^μ of the antifield independent pieces on both sides of the equation,

$$\partial_\mu T^{\tilde{A}} \approx A_\mu^{\tilde{M}} \nabla_{\tilde{M}} T^{\tilde{A}}, \quad \nabla_{\tilde{M}} := R_{\tilde{M}}^{\tilde{A}} \frac{\partial}{\partial T^{\tilde{A}}} \quad (31)$$

where $R_{\tilde{M}}^{\tilde{A}}$ is the antifield independent part of $\tilde{R}_{\tilde{M}}^{\tilde{A}}(\tilde{T})$ and $\approx$ denotes equality on-shell (in general one only has equality on-shell because the piece in $\tilde{T}^{\tilde{A}}$ with antifield number 1 gives rise to antifield independent terms in $s\tilde{T}^{\tilde{A}}$ which vanish on-shell). Normally (31) indicates that the set $\{\nabla_{\tilde{M}}\}$ contains covariant derivatives $\mathcal{D}_m$ corresponding to a subset $\{A_\mu^m\}$ of $\{A_\mu^{\tilde{M}}\}$ which defines an invertible (generally field dependent) $n \times n$ matrix,

$$\{A_\mu^{\tilde{M}}\} = \{A_\mu^m, A_\mu^I\}, \quad \mu, m \in \{1, \ldots, n\}, \quad (32)$$
$$\{\nabla_{\tilde{M}}\} = \{\mathcal{D}_m, \nabla_I\}, \quad \mathcal{D}_m = A_m^\mu(\partial_\mu - A_\mu^I \nabla_I), \quad A_m^\mu A_\mu^k = \delta_m^k. \quad (33)$$

In the Yang-Mills example, one gets:

$$\text{Yang-Mills:} \quad \{\tilde{C}^{\tilde{M}}\} \equiv \{dx^\mu, \tilde{C}^a\}, \quad \{A_\mu^{\tilde{M}}\} \equiv \{\delta_\mu^m, A_\mu^a\},$$
$$\{\tilde{T}^{\tilde{A}}\} \equiv \{x^\mu, \tilde{F}_{\mu\nu}^a, \ldots\}, \quad \{T^{\tilde{A}}\} \equiv \{x^\mu, F_{\mu\nu}^a, \ldots\},$$
$$\{\nabla_I\} \equiv \{\delta_a\}, \quad \mathcal{D}_m = \delta_m^\mu(\partial_\mu - A_\mu^a \delta_a).$$

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